BLUEPRINT FOR BROUWER FIXED POINT THEOREM

Remark 0.1. This text is meant as a blueprint for a proof of Brouwer's Fixed Point Theorem.

We will follow [Cas17] throughout.

1. Sperner's Lemma

Definition 1.1. Given a simplicial complex S, and a a coloring f on it, we say that f is a Sperner coloring of S if $f(x) \neq i$ if x belongs to the i-th face (that with zero i-th coordinate).

Lemma 1.2. *A simplicial complex with a Sperner coloring on it, contains a panchromatic simplex.*

2. Brouwer's Fixed Point Theorem

2.1. Given $f \colon \Delta \to \Delta$ (where Δ is triangulated), we construct a coloring of the triangulation.

Definition 2.1. Let Δ be a simplex. Given an ordered list of its vertices, and a function $f : \Delta \to \Delta$, the induced color of a point $x \in \Delta$ is $i = \min\{j \mid f(x)_j < x_j\}$.

Lemma 2.2. *The coloring above is a Sperner coloring.*

Apply the above to a triangulation of Δ of diameter $< \epsilon$, to get a panchromatic triangle of diameter ϵ.

Lemma 2.3. *For all $\varepsilon > 0$ exists $\delta > 0$, such that given a panchromatic simplex $X \subseteq \Delta$ of a diameter $\leq \delta$ then $|f(x) - x| < \varepsilon,\ \forall x \in X$.*

Proof. By compactness, f is uniformly continuous in Δ. Then for all $\frac{\varepsilon}{2n} > 0$ there exists $\delta_0 > 0$ such that $diam(f(X)) < \frac{\varepsilon}{2n}$. We define $\delta := \min(\delta_0, \frac{\varepsilon}{4n})$, that preserves the properties of δ.
Let $p_0, \dots, p_n$ be the vertices of X. Then, since X is panchromatic, for each $i = 0, \dots, n-1$, $f(p_i)_i < (p_i)_i$ and $f(p_{i+1})_i \geq (p_{i+1})_i$. Then, for all $x \in X$,

$$f(x)_i \leq f(p_i)_i + diam(f(X)) < (p_i)_i + diam(f(X)) \leq (x)_i + \delta + diam(f(X))$$

and

$$f(x)_i \geq f(p_{i+1})_i - diam(f(X)) \geq (p_{i+1})_i - diam(f(X)) \geq (x)_i - \delta - diam(f(X))$$

Putting them together,

$$|f(x)_i - (x)_i| \leq \delta + diam(f(X)), \quad i \in \{0, ..., n-1\}$$

Therefore, for all $x \in X$,

$$|f(x)_n - (x)_n| = |(1 - f(x)_0 - ... - f(x)_{n-1}) - (1 - (x)_0 - ... - (x)_{n-1})| =$$
$$= |((x)_0 - f(x)_0) + ... + ((x)_n - f(x)_n)| \leq$$
$$\leq |(x)_0 - f(x)_0| + ... + |(x)_n - f(x)_n| \leq (n-1)(\delta + diam(f(X)))$$

Date: September 16, 2021.

Finally, for all $x \in X$,

$$|f(x) - x| = \sqrt{(f(x)_0 - (x)_0)^2 + ... + (f(x)_{n-1} - (x)_{n-1})^2 + (f(x)_n - (x)_n)^2}$$
$$\leq \sqrt{(1 + ... + 1 + (n-1)^2)(\delta + diam(f(X))^2} =$$
$$= \sqrt{n(n-1)}(\delta + diam(f(X)) < \frac{\varepsilon}{n}\sqrt{n(n-1)} < \varepsilon \iff |f(x) - x| < \varepsilon$$

□

Lemma 2.4. *$f \colon X \to X$ continuous and X compact metric space. Suppose that $\forall \epsilon > 0$, there exists $x \in X$ such that $d(f(x), x) < \epsilon$. Then f has a fixed point.*

Proof. Using the epsilons property, we can construct a sequance verifying the following inequality

$$|f(x_n) - x_n| \leq \frac{1}{n}$$

Because of the Bolzano-Weierstrass theorem, we know there exists a convergent subsequance x_{k_n} of x_n satisfying,

$$\left|f(x_{n_k}) - x_{n_k}\right| \leq \frac{1}{n_k}$$

Applying the limits in the inequality, we obtain the equality

$$\left|\lim_{x\to\infty} f(x_{n_k}) - x\right| \leq 0 \iff \lim_{x\to\infty} f(x_{n_k}) - x = 0 \iff \lim_{x\to\infty} f(x_{n_k}) = x$$

Using the continuity property of limits we can conclude,

$$f(x) = f\left(\lim_{x\to\infty} x_{n_k}\right) = \lim_{x\to\infty} f(x_{n_k}) = x$$

□

Now we apply Sperner's lemma to deduce that there is a panchromatic face in the subdivision.

Definition 2.5. The baricenter of points $p_1, \dots, p_k$ is $B(p_1, \dots, p_k) = \frac{1}{k}\sum_{i=1}^{k} p_i$

Definition 2.6. Let $S \subseteq \mathbb{R}^n$ be a n-dimensional simplex with barycentric coordinates $x_0, \dots, x_n$. The barycentric subdivison is a map form permutations of the $n+1$ vertices of S to n-dimensional simplexes defined as follows

$$(p_0, p_1, \dots, p_n) \mapsto Simplex(B(p_0), B(p_0, p_1), \dots, B(p_0, p_1, \dots, p_n))$$

See end of `https://github.com/leanprover-community/mathlib/blob/sperner-again/src/combinatorics/`

Lemma 2.7. *Let S be a k-dimensional simplex with vertices $v_0, \dots, v_k$. The diameter of S $Diam(S) = \max_{x,y\in S} |x - y|$ equals $\max |v_i - v_j|$.*

Proof. Since $v_i, v_j \in S$, we have $Diam(S) \geq \max |v_i - v_j|$.

Let $l \max |v_i - v_j|$, and for each vertex consider the closed ball centered at v_i and radius l.

$$B_l(v_i) = \{x : |x - v_i| \leq l\}.$$

Since $B_l(v_i)$ is convex, and it contains every vertex of S, we have $|x - v_i| \leq l$ for all $x \in S$.

Finally we show $|x - y| \leq l$ for all $x, y \in S$. Given $x \in S$, $B_l(x)$ contains all vertices of S, and since $B_l(x)$ is convex, it contains any $y \in S$. Therefore $|x - y| \leq l$ for all $x, y \in S$. This is $diam(S) \leq l$.

□

Lemma 2.8. *Let $\hat{S} = B(v_0, v_1, \ldots, v_k)$ be the barycenter of S. We bound the distance of any vertex v_i of S to $\hat{S}$. Given a simplex S of diameter D, its barycentric subdivision has diameter at most $\frac{n}{n+1}D$*

Proof. The dinstance of any vertex of S to its barycenter is bounded by

$$|v_i - \hat{S}| = |v_i - \sum_{k=0}^{k} \frac{1}{n+1} v_k| \leq \sum_{t=0}^{k} \frac{1}{k+1} |v_i - v_t| \leq \frac{1}{n+1} D.$$

Therefore, the closed ball centered at $\hat{S}$, of radius $\frac{p}{p+1}D$, contains the vertices of S, by convexity it contains S.

Finally we proof the result by strong induction, for $k = 0$ the result is trivial. If s and s' are faces of S such that s is a proper face of s',

$$|\hat{s} - \hat{s}'| \leq (\frac{k}{k+1}) diam(S).$$

□

Lemma 2.9. *For all $\epsilon > 0$, there exists a subdivision of Δ of diameter $< \epsilon$.*

Proof. □

References

[Cas17] Natàlia Castellana. Hi havia una vegada un punt fix... *Butlletí de la Societat Catalana de Matemàtiques*, pages 99–120, 2017.