

# BLUEPRINT FOR BROUWER FIXED POINT THEOREM

**Remark 0.1.** This text is meant as a blueprint for a proof of Brouwer's Fixed Point Theorem.

We will follow [Cas17] throughout.

## 1. SPERNER'S LEMMA

**Definition 1.1.** Given a simplicial complex  $S$ , and a coloring  $f$  on it, we say that  $f$  is a Sperner coloring of  $S$  if  $f(x) \neq i$  if  $x$  belongs to the  $i$ -th face (that with zero  $i$ -th coordinate).

**Lemma 1.2.** *A simplicial complex with a Sperner coloring on it, contains a panchromatic simplex.*

## 2. BROUWER'S FIXED POINT THEOREM

**2.1.** Given  $f: \Delta \rightarrow \Delta$  (where  $\Delta$  is triangulated), we construct a coloring of the triangulation.

**Definition 2.1.** Let  $\Delta$  be a simplex. Given an ordered list of its vertices, and a function  $f: \Delta \rightarrow \Delta$ , the induced color of a point  $x \in \Delta$  is  $i = \min\{j \mid f(x)_j < x_j\}$ .

**Lemma 2.2.** *The coloring above is a Sperner coloring.*

Apply the above to a triangulation of  $\Delta$  of diameter  $< \epsilon$ , to get a panchromatic triangle of diameter  $\epsilon$ .

**Lemma 2.3.** *For all  $\epsilon > 0$  exists  $\delta > 0$ , such that given a panchromatic simplex  $X \subseteq \Delta$  of a diameter  $\leq \delta$  then  $|f(x) - x| < \epsilon$ ,  $\forall x \in X$ .*

*Proof.* By compactness,  $f$  is uniformly continuous in  $\Delta$ . Then for all  $\frac{\epsilon}{2n} > 0$  there exists  $\delta_0 > 0$  such that  $\text{diam}(f(X)) < \frac{\epsilon}{2n}$ . We define  $\delta := \min(\delta_0, \frac{\epsilon}{4n})$ , that preserves the properties of  $\delta$ . Let  $p_0, \dots, p_n$  be the vertices of  $X$ . Then, since  $X$  is panchromatic, for each  $i = 0, \dots, n-1$ ,  $f(p_i)_i < (p_i)_i$  and  $f(p_{i+1})_i \geq (p_{i+1})_i$ . Then, for all  $x \in X$ ,

$$f(x)_i \leq f(p_i)_i + \text{diam}(f(X)) < (p_i)_i + \text{diam}(f(X)) \leq (x)_i + \delta + \text{diam}(f(X))$$

and

$$f(x)_i \geq f(p_{i+1})_i - \text{diam}(f(X)) \geq (p_{i+1})_i - \text{diam}(f(X)) \geq (x)_i - \delta - \text{diam}(f(X))$$

Putting them together,

$$|f(x)_i - (x)_i| \leq \delta + \text{diam}(f(X)), \quad i \in \{0, \dots, n-1\}$$

Therefore, for all  $x \in X$ ,

$$\begin{aligned} |f(x)_n - (x)_n| &= |(1 - f(x)_0 - \dots - f(x)_{n-1}) - (1 - (x)_0 - \dots - (x)_{n-1})| = \\ &= |((x)_0 - f(x)_0) + \dots + ((x)_n - f(x)_n)| \leq \\ &\leq |(x)_0 - f(x)_0| + \dots + |(x)_n - f(x)_n| \leq (n-1)(\delta + \text{diam}(f(X))) \end{aligned}$$

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Finally, for all  $x \in X$ ,

$$\begin{aligned} |f(x) - x| &= \sqrt{(f(x)_0 - (x)_0)^2 + \dots + (f(x)_{n-1} - (x)_{n-1})^2 + (f(x)_n - (x)_n)^2} \\ &\leq \sqrt{(1 + \dots + 1 + (n-1)^2)(\delta + \text{diam}(f(X))^2} = \\ &= \sqrt{n(n-1)}(\delta + \text{diam}(f(X))) < \frac{\varepsilon}{n} \sqrt{n(n-1)} < \varepsilon \iff |f(x) - x| < \varepsilon \end{aligned}$$

□

**Lemma 2.4.**  $f: X \rightarrow X$  continuous and  $X$  compact metric space. Suppose that  $\forall \epsilon > 0$ , there exists  $x \in X$  such that  $d(f(x), x) < \epsilon$ . Then  $f$  has a fixed point.

*Proof.* Using the epsilons property, we can construct a sequence verifying the following inequality

$$|f(x_n) - x_n| \leq \frac{1}{n}$$

Because of the Bolzano-Weierstrass theorem, we know there exists a convergent subsequence  $x_{k_n}$  of  $x_n$  satisfying,

$$|f(x_{k_n}) - x_{k_n}| \leq \frac{1}{n_{k_n}}$$

Applying the limits in the inequality, we obtain the equality

$$\left| \lim_{x \rightarrow \infty} f(x_{k_n}) - x \right| \leq 0 \iff \lim_{x \rightarrow \infty} f(x_{k_n}) - x = 0 \iff \lim_{x \rightarrow \infty} f(x_{k_n}) = x$$

Using the continuity property of limits we can conclude,

$$f(x) = f\left(\lim_{x \rightarrow \infty} x_{k_n}\right) = \lim_{x \rightarrow \infty} f(x_{k_n}) = x$$

□

Now we apply Sperner's lemma to deduce that there is a panchromatic face in the subdivision.

**Definition 2.5.** The baricenter of points  $p_1, \dots, p_k$  is  $B(p_1, \dots, p_k) = \frac{1}{k} \sum_{i=1}^k p_i$

**Definition 2.6.** Let  $S \subseteq \mathbb{R}^n$  be a  $n$ -dimensional simplex with barycentric coordinates  $x_0, \dots, x_n$ . The barycentric subdivision is a map from permutations of the  $n+1$  vertices of  $S$  to  $n$ -dimensional simplexes defined as follows

$$(p_0, p_1, \dots, p_n) \mapsto \text{Simplex}(B(p_0), B(p_0, p_1), \dots, B(p_0, p_1, \dots, p_n))$$

See end of <https://github.com/leanprover-community/mathlib/blob/sperner-again/src/combinatorics/>

**Lemma 2.7.** Let  $S$  be a  $k$ -dimensional simplex with vertices  $v_0, \dots, v_k$ . The diameter of  $S$   $\text{Diam}(S) = \max_{x, y \in S} |x - y|$  equals  $\max |v_i - v_j|$ .

*Proof.* Since  $v_i, v_j \in S$ , we have  $\text{Diam}(S) \geq \max |v_i - v_j|$ .

Let  $l = \max |v_i - v_j|$ , and for each vertex consider the closed ball centered at  $v_i$  and radius  $l$ .

$$B_l(v_i) = \{x : |x - v_i| \leq l\}.$$

Since  $B_l(v_i)$  is convex, and it contains every vertex of  $S$ , we have  $|x - v_i| \leq l$  for all  $x \in S$ .

Finally we show  $|x - y| \leq l$  for all  $x, y \in S$ . Given  $x \in S$ ,  $B_l(x)$  contains all vertices of  $S$ , and since  $B_l(x)$  is convex, it contains any  $y \in S$ . Therefore  $|x - y| \leq l$  for all  $x, y \in S$ . This is  $\text{diam}(S) \leq l$ . □

**Lemma 2.8.** *Let  $\hat{S} = B(v_0, v_1, \dots, v_k)$  be the barycenter of  $S$ . We bound the distance of any vertex  $v_i$  of  $S$  to  $\hat{S}$ . Given a simplex  $S$  of diameter  $D$ , its barycentric subdivision has diameter at most  $\frac{n}{n+1}D$*

*Proof.* The distance of any vertex of  $S$  to its barycenter is bounded by

$$|v_i - \hat{S}| = |v_i - \sum_{k=0}^k \frac{1}{n+1} v_k| \leq \sum_{t=0}^k \frac{1}{k+1} |v_i - v_t| \leq \frac{1}{n+1} D.$$

Therefore, the closed ball centered at  $\hat{S}$ , of radius  $\frac{p}{p+1}D$ , contains the vertices of  $S$ , by convexity it contains  $S$ .

Finally we proof the result by strong induction, for  $k = 0$  the result is trivial. If  $s$  and  $s'$  are faces of  $S$  such that  $s$  is a proper face of  $s'$ ,

$$|\hat{s} - \hat{s}'| \leq \left(\frac{k}{k+1}\right) \text{diam}(S).$$

□

**Lemma 2.9.** *For all  $\epsilon > 0$ , there exists a subdivision of  $\Delta$  of diameter  $< \epsilon$ .*

*Proof.* □

## REFERENCES

- [Cas17] Natàlia Castellana. Hi havia una vegada un punt fix... *Butlletí de la Societat Catalana de Matemàtiques*, pages 99–120, 2017.