# BLUEPRINT FOR BROUWER FIXED POINT THEOREM

**Remark 0.1.** This text is meant as a blueprint for a proof of Brouwer's Fixed Point Theorem.

We will follow [Cas17] throughout.

## 1. Sperner's Lemma

**Definition 1.1.**

**Lemma 1.2.**

## 2. Brouwer's Fixed Point Theorem

**2.1.** Given $f\colon \Delta \to \Delta$ (where $\Delta$ is triangulated), we construct a coloring of the triangulation.

**Definition 2.1.** Let $\Delta$ be a simplex. Given an ordered list of its vertices, and a function $f : \Delta \to \Delta$, the induced color of a point $x \in \Delta$ is $i = \min\{j \mid f(x)_j < x_j\}$.

**Lemma 2.2.** *The coloring above is a Sperner coloring.*

Apply the above to a triangulation of $\Delta$ of diameter $< \epsilon$, to get a panchromatic triangle of diameter $\epsilon$.

**Lemma 2.3.** *For all $\varepsilon > 0$ exists $\delta > 0$, such that given a panchromatic simplex $X \subseteq \Delta$ of a diameter $\leq \delta$ then $|f(x) - x| < \varepsilon,\ \forall x \in X$.*

*Proof.* By compactness, $f$ is uniformly continuous in $\Delta$. Then for all $\frac{\varepsilon}{2n} > 0$ there exists $\delta_0 > 0$ such that $diam(f(X)) < \frac{\varepsilon}{2n}$. We define $\delta := \min(\delta_0, \frac{\varepsilon}{4n})$, that preserves the properties of $\delta$.
Let $p_0, \dots, p_n$ be the vertices of $X$. Then, since $X$ is panchromatic, for each $i = 0, \dots, n-1$, $f(p_i)_i < (p_i)_i$ and $f(p_{i+1})_i \geq (p_{i+1})_i$. Then, for all $x \in X$,

$$f(x)_i \leq f(p_i)_i + diam(f(X)) < (p_i)_i + diam(f(X)) \leq (x)_i + \delta + diam(f(X))$$

and

$$f(x)_i \geq f(p_{i+1})_i - diam(f(X)) \geq (p_{i+1})_i - diam(f(X)) \geq (x)_i - \delta - diam(f(X))$$

Putting them together,

$$|f(x)_i - (x)_i| \leq \delta + diam(f(X)), \quad i \in \{0, ..., n-1\}$$

Therefore, for all $x \in X$,

$$|f(x)_n - (x)_n| = |(1 - f(x)_0 - ... - f(x)_{n-1}) - (1 - (x)_0 - ... - (x)_{n-1})| =$$
$$= |((x)_0 - f(x)_0) + ... + ((x)_n - f(x)_n)| \leq$$
$$\leq |(x)_0 - f(x)_0| + ... + |(x)_n - f(x)_n| \leq (n-1)(\delta + diam(f(X)))$$

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Finally, for all $x \in X$,

$$|f(x) - x| = \sqrt{(f(x)_0 - (x)_0)^2 + ... + (f(x)_{n-1} - (x)_{n-1})^2 + (f(x)_n - (x)_n)^2}$$
$$\leq \sqrt{(1 + ... + 1 + (n-1)^2)(\delta + diam(f(X))^2} =$$
$$= \sqrt{n(n-1)}(\delta + diam(f(X)) < \frac{\varepsilon}{n}\sqrt{n(n-1)} < \varepsilon \Longleftrightarrow |f(x) - x| < \varepsilon$$

□

**Lemma 2.4.** *$f\colon X \to X$ continuous and $X$ compact metric space. Suppose that $\forall \epsilon > 0$, there exists $x \in X$ such that $d(f(x), x) < \epsilon$. Then $f$ has a fixed point.*

*Proof.* Using the epsilons property, we can construct a sequance verifying the following inequality

$$|f(x_n) - x_n| \leq \frac{1}{n}$$

Because of the Bolzano-Weierstrass theorem, we know there exists a convergent subsequance $x_{k_n}$ of $x_n$ satisfying,

$$\left|f(x_{n_k}) - x_{n_k}\right| \leq \frac{1}{n_k}$$

Applying the limits in the inequality, we obtain the equality

$$\left|\lim_{x\to\infty} f(x_{n_k}) - x\right| \leq 0 \Longleftrightarrow \lim_{x\to\infty} f(x_{n_k}) - x = 0 \Longleftrightarrow \lim_{x\to\infty} f(x_{n_k}) = x$$

Using the continuity property of limits we can conclude,

$$f(x) = f\left(\lim_{x\to\infty} x_{n_k}\right) = \lim_{x\to\infty} f(x_{n_k}) = x$$

□

Now we apply Sperner's lemma to deduce that there is a panchromatic face in the subdivision.

**Definition 2.5.** The baricenter of points $p_1, \dots, p_k$ is $B(p_1, \dots, p_k) = \frac{1}{k}\sum_{i=1}^{k} p_i$

**Definition 2.6.** Let $S \subseteq \mathbb{R}^n$ be a $n$-dimensional simplex with barycentric coordinates $x_0, \dots, x_n$. The barycentric subdivison is a map form permutations of the $n+1$ vertices of $S$ to $n$-dimensional simplexes defined as follows

$$(p_0, p_1, \dots, p_n) \mapsto Simplex(B(p_0), B(p_0, p_1), \dots, B(p_0, p_1, \dots, p_n))$$

See end of `https://github.com/leanprover-community/mathlib/blob/sperner-again/src/combinatorics/`

**Lemma 2.7.** *Let $S$ be a $k$-dimensional simplex with vertices $v_0, \dots, v_k$. The diameter of $S$ $Diam(S) = \max_{x,y\in S} |x - y|$ equals $\max |v_i - v_j|$.*

*Proof.* Since $v_i, v_j \in S$, we have $Diam(S) \geq \max |v_i - v_j|$.

Let $l \max |v_i - v_j|$, and for each vertex consider the closed ball centered at $v_i$ and radius $l$.

$$B_l(v_i) = \{x : |x - v_i| \leq l\}.$$

Since $B_l(v_i)$ is convex, and it contains every vertex of $S$, we have $|x - v_i| \leq l$ for all $x \in S$.

Finally we show $|x - y| \leq l$ for all $x, y \in S$. Given $x \in S$, $B_l(x)$ contains all vertices of $S$, and since $B_l(x)$ is convex, it contains any $y \in S$. Therefore $|x - y| \leq l$ for all $x, y \in S$. This is $diam(S) \leq l$.

□

**Lemma 2.8.** *Let $\hat{S} = B(v_0, v_1, \ldots, v_k)$ be the barycenter of $S$. We bound the distance of any vertex $v_i$ of $S$ to $\hat{S}$. Given a simplex $S$ of diameter $D$, its barycentric subdivision has diameter at most $\frac{n}{n+1}D$*

*Proof.* The dinstance of any vertex of $S$ to its barycenter is bounded by

$$|v_i - \hat{S}| = |v_i - \sum_{k=0}^{k} \frac{1}{n+1} v_k| \leq \sum_{t=0}^{k} \frac{1}{k+1} |v_i - v_t| \leq \frac{1}{n+1} D.$$

Therefore, the closed ball centered at $\hat{S}$, of radius $\frac{p}{p+1}D$, contains the vertices of $S$, by convexity it contains $S$.

Finally we proof the result by strong induction, for $k = 0$ the result is trivial. If $s$ and $s'$ are faces of $S$ such that $s$ is a proper face of $s'$,

$$|\hat{s} - \hat{s}'| \leq (\frac{k}{k+1}) diam(S).$$

□

**Lemma 2.9.** *For all $\epsilon > 0$, there exists a subdivision of $\Delta$ of diameter $< \epsilon$.*

*Proof.* □

## References

[Cas17] Natàlia Castellana. Hi havia una vegada un punt fix... *Butlletí de la Societat Catalana de Matemàtiques*, pages 99–120, 2017.