

BLUEPRINT FOR BROUWER FIXED POINT THEOREM

Remark 0.1. This text is meant as a blueprint for a proof of Brouwer's Fixed Point Theorem.

We will follow [Cas17] throughout.

1. SPERNER'S LEMMA

Definition 1.1. Given a simplicial complex S , and a coloring f on it, we say that f is a Sperner coloring of S if $f(x) \neq i$ if x belongs to the i -th face (that with zero i -th coordinate).

Lemma 1.2. *A simplicial complex with a Sperner coloring on it, contains a panchromatic simplex.*

2. BROUWER'S FIXED POINT THEOREM

2.1. Given $f: \Delta \rightarrow \Delta$ (where Δ is triangulated), we construct a coloring of the triangulation.

Definition 2.1. Let Δ be a simplex. Given an ordered list of its vertices, and a function $f: \Delta \rightarrow \Delta$, the induced color of a point $x \in \Delta$ is $i = \min\{j \mid f(x)_j < x_j\}$.

Lemma 2.2. *The coloring above is a Sperner coloring.*

Apply the above to a triangulation of Δ of diameter $< \epsilon$, to get a panchromatic triangle of diameter ϵ .

Lemma 2.3. *For all $\epsilon > 0$ exists $\delta > 0$, such that given a panchromatic simplex $X \subseteq \Delta$ of a diameter $\leq \delta$ then $|f(x) - x| < \epsilon$, $\forall x \in X$.*

Proof. By compactness, f is uniformly continuous in Δ . Then for all $\frac{\epsilon}{2n} > 0$ there exists $\delta_0 > 0$ such that $\text{diam}(f(X)) < \frac{\epsilon}{2n}$. We define $\delta := \min(\delta_0, \frac{\epsilon}{4n})$, that preserves the properties of δ . Let p_0, \dots, p_n be the vertices of X . Then, since X is panchromatic, for each $i = 0, \dots, n-1$, $f(p_i)_i < (p_i)_i$ and $f(p_{i+1})_i \geq (p_{i+1})_i$. Then, for all $x \in X$,

$$f(x)_i \leq f(p_i)_i + \text{diam}(f(X)) < (p_i)_i + \text{diam}(f(X)) \leq (x)_i + \delta + \text{diam}(f(X))$$

and

$$f(x)_i \geq f(p_{i+1})_i - \text{diam}(f(X)) \geq (p_{i+1})_i - \text{diam}(f(X)) \geq (x)_i - \delta - \text{diam}(f(X))$$

Putting them together,

$$|f(x)_i - (x)_i| \leq \delta + \text{diam}(f(X)), \quad i \in \{0, \dots, n-1\}$$

Therefore, for all $x \in X$,

$$\begin{aligned} |f(x)_n - (x)_n| &= |(1 - f(x)_0 - \dots - f(x)_{n-1}) - (1 - (x)_0 - \dots - (x)_{n-1})| = \\ &= |((x)_0 - f(x)_0) + \dots + ((x)_n - f(x)_n)| \leq \\ &\leq |(x)_0 - f(x)_0| + \dots + |(x)_n - f(x)_n| \leq (n-1)(\delta + \text{diam}(f(X))) \end{aligned}$$

Finally, for all $x \in X$,

$$\begin{aligned} |f(x) - x| &= \sqrt{(f(x)_0 - (x)_0)^2 + \dots + (f(x)_{n-1} - (x)_{n-1})^2 + (f(x)_n - (x)_n)^2} \\ &\leq \sqrt{(1 + \dots + 1 + (n-1)^2)(\delta + \text{diam}(f(X))^2} = \\ &= \sqrt{n(n-1)}(\delta + \text{diam}(f(X))) < \frac{\varepsilon}{n} \sqrt{n(n-1)} < \varepsilon \iff |f(x) - x| < \varepsilon \end{aligned}$$

□

Lemma 2.4. $f: X \rightarrow X$ continuous and X compact metric space. Suppose that $\forall \epsilon > 0$, there exists $x \in X$ such that $d(f(x), x) < \epsilon$. Then f has a fixed point.

Proof. Using the epsilons property, we can construct a sequence verifying the following inequality

$$|f(x_n) - x_n| \leq \frac{1}{n}$$

Because of the Bolzano-Weierstrass theorem, we know there exists a convergent subsequence x_{k_n} of x_n satisfying,

$$|f(x_{k_n}) - x_{k_n}| \leq \frac{1}{n_{k_n}}$$

Applying the limits in the inequality, we obtain the equality

$$\left| \lim_{x \rightarrow \infty} f(x_{k_n}) - x \right| \leq 0 \iff \lim_{x \rightarrow \infty} f(x_{k_n}) - x = 0 \iff \lim_{x \rightarrow \infty} f(x_{k_n}) = x$$

Using the continuity property of limits we can conclude,

$$f(x) = f\left(\lim_{x \rightarrow \infty} x_{k_n}\right) = \lim_{x \rightarrow \infty} f(x_{k_n}) = x$$

□

Now we apply Sperner's lemma to deduce that there is a panchromatic face in the subdivision.

Definition 2.5. The baricenter of points p_1, \dots, p_k is $B(p_1, \dots, p_k) = \frac{1}{k} \sum_{i=1}^k p_i$

Definition 2.6. Let $S \subseteq \mathbb{R}^n$ be a n -dimensional simplex with barycentric coordinates x_0, \dots, x_n . The barycentric subdivision is a map from permutations of the $n+1$ vertices of S to n -dimensional simplexes defined as follows

$$(p_0, p_1, \dots, p_n) \mapsto \text{Simplex}(B(p_0), B(p_0, p_1), \dots, B(p_0, p_1, \dots, p_n))$$

See end of <https://github.com/leanprover-community/mathlib/blob/sperner-again/src/combinatorics/>

Lemma 2.7. Let S be a k -dimensional simplex with vertices v_0, \dots, v_k . The diameter of S $\text{Diam}(S) = \max_{x, y \in S} |x - y|$ equals $\max |v_i - v_j|$.

Proof. Since $v_i, v_j \in S$, we have $\text{Diam}(S) \geq \max |v_i - v_j|$.

Let $l = \max |v_i - v_j|$, and for each vertex consider the closed ball centered at v_i and radius l .

$$B_l(v_i) = \{x : |x - v_i| \leq l\}.$$

Since $B_l(v_i)$ is convex, and it contains every vertex of S , we have $|x - v_i| \leq l$ for all $x \in S$.

Finally we show $|x - y| \leq l$ for all $x, y \in S$. Given $x \in S$, $B_l(x)$ contains all vertices of S , and since $B_l(x)$ is convex, it contains any $y \in S$. Therefore $|x - y| \leq l$ for all $x, y \in S$. This is $\text{diam}(S) \leq l$. □

Lemma 2.8. *Let $\hat{S} = B(v_0, v_1, \dots, v_k)$ be the barycenter of S . We bound the distance of any vertex v_i of S to \hat{S} . Given a simplex S of diameter D , its barycentric subdivision has diameter at most $\frac{n}{n+1}D$*

Proof. The distance of any vertex of S to its barycenter is bounded by

$$|v_i - \hat{S}| = |v_i - \sum_{k=0}^k \frac{1}{n+1} v_k| \leq \sum_{t=0}^k \frac{1}{k+1} |v_i - v_t| \leq \frac{1}{n+1} D.$$

Therefore, the closed ball centered at \hat{S} , of radius $\frac{p}{p+1}D$, contains the vertices of S , by convexity it contains S .

Finally we proof the result by strong induction, for $k = 0$ the result is trivial. If s and s' are faces of S such that s is a proper face of s' ,

$$|\hat{s} - \hat{s}'| \leq \left(\frac{k}{k+1}\right) \text{diam}(S).$$
□

Lemma 2.9. *For all $\epsilon > 0$, there exists a subdivision of Δ of diameter $< \epsilon$.*

Proof. □

REFERENCES

- [Cas17] Natàlia Castellana. Hi havia una vegada un punt fix... *Butlletí de la Societat Catalana de Matemàtiques*, pages 99–120, 2017.