Part I. Fundamentals

1 Gravity Before Relativity

In the first session we talked about the history of gravity and at the end we briefly discussed Mach's ideas on the physical structure of space and time. I followed the first chapter of [1] for the historical part and [2] for Mach's principle.

- For axiomatic approach to Newton's laws I recommend you to visit this page.
- One may think that a sufficient far point on a rotating disk have a velocity higher than c and that violates special relativity. The point is that you need an infinite amount of energy to speed-up such a disk. Also, this rotation would cause an infinite force acting on an infinitesimal part of the the disk with $v = c$. To see this, we can calculate the proper time of an arbitrary point on the disk. Assume that (t', x', y', z') is the coordinate label of rotating frame.

$$\begin{aligned} t' &= t \\ r' &= r \\ \phi' &= \phi - \omega t \\ z' &= z \\ g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} \end{aligned}$$

  the proper length in rotating frame is

$$ds^2 = (1 - \frac{\omega^2 r'^2}{c^2})c^2 dt'^2 - 2\omega r'^2 d\phi' dt' - dz'^2 - dr'^2 - r'^2 d\phi'^2$$

  so the proper time of an arbitrary observer at distance r from origin is

$$d\tau = \sqrt{1 - \frac{\omega^2 r^2}{c^2}}dt$$

  Also, the geodesic equation tells us that $\frac{d^2x}{d\tau^2}$ and $\frac{d^2y}{d\tau^2}$ are proportional to $\frac{dt'}{d\tau}$. This means the acceleration is infinite at a point with $r = c/\omega$.
- It was proposed that Hoofman experiment may supports Mach's ideas. It seems that this experiment demonstrates frame-dragging effect and this can by fully understood by general relativity using Kerr metric. Therefore, it doesn't justify a "new" interpretation of gravity. You can visit this page for more information about frame-dragging effect.

2 Equivalence Principle

I used the third chapter of [1] and [3] for the statements and consequences of equivalence principle. It is shown in section 2.4 of [4] that why equivalence principle forbid us to describe gravity as an electromagnetic field theory. At the end, Pooya Farokhi talked about different versions of equivalence principle. This talk was based on [5].

3 Manifolds and Tensor Fields I

The basic definitions and theorems of general topology and manifolds is discussed. We introduced tangent space and proved ”Identification Lemma” and arrived at the beginning of topological and differential structure on tangent bundle. I followed part D, chapter 4 of [6].

4 Manifolds and Tensor Fields II

We used ”Sum Topology” to construct a topology on $TM = \bigsqcup_{x \in M} T_x M$. We stated and partially proved the theorem that the tangent bundle is itself a manifold. We also proved that the set of all derivatives at one point in a manifold is a vector space with the same dimension as manifold. At the end, we discussed a bit about the intuition of curvature on a two dimensional surface. All mathematical concepts discussed throughout the lecture can be found in [6].

- Given an arbitrary metric tensor $g : TM \times TM \to \mathbb{R}$, we can find an orthogonal basis for tangent space, at specific point on manifold, which it takes a diagonal form. This is possible since it is a symmetric second rank tensor or equivalently a matrix. We normalize the eigenvectors so that $g(v_\mu, u_\nu) = \pm\delta_{\mu\nu}$. The number of + and - signs are called the signature of metric. It is evident that trace of a matrix is independent of chosen orthogonal basis and so is the signature of metric. We will show later that at each point, there is a specific coordinate called Riemann normal coordinate which the metric tensor is flat $g_{\mu\nu} = diag(-1, 1, 1, 1)$. Now If our metric is diagonal or If we know that there is an orthogonal transformation between Riemann normal coordinate and our previous coordinate, we conclude that the signature of metric is three + and one -. As an example consider the Schwarzschild metric

  $$ds^2 = -\big(1 - \frac{r_s}{r}\big)dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2 d\Omega^2$$

  It is clear that even by passing the event horizon the signature of metric, in the way we defined it, doesn’t change.

- Consider a vector field X on a two dimensional surface $M \subset \mathbb{R}^3$ and a smooth curve $\gamma : I \to M$ passing from p at $t = 0$. If we walk from p, a tiny bit, along γ and measure the change of X, $\frac{dX \circ \gamma(t)}{dt}$ we will end up with a vector. Now project this vector onto the tangent space of surface at that point. We call the final vector, which lies in tangent space at p, the covariant derivative of vector field X along γ at p and denote it by $D_{\gamma'(0)} X$. We talked intuitively about the concept of parallel transport. Now we can formulate it mathematically. Let w be a tangent vector at point p on the surface. Then there exist a unique vector field $W(t)$ along γ with $W(0) = w$ such that $D_{\gamma'(t)} W(t) = 0$ for all $t \in I$. This is an immediate consequence of existence and uniqueness theorem of differential equations. You can check standard differential geometry textbooks for more information.

5 Curvature

We discussed the third chapter of the book and arrived at the beginning of tetrad orthogonal basis section.

- Does a covariant derivative exist on a manifold?

  Yes. There is a trivial zero map. We can also construct many nontrivial covariant derivatives. Consider an arbitrary tensor field $T = T_{\mu\nu}\ dx^\mu \otimes dx^\nu$ on manifold. For a chart $U \subset M$, one can create the following operator
  $$\nabla_a T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^a}.$$
  Note that we assume the coordinate of derivation is *fixed* and does not change, like treating it as a function $f(x)$ as if by changing the coordinate becomes $f(x(x'))$. Now using the partition of unity on manifold, you can extend this to a covariant derivative over the whole manifold. There is another more natural covariant derivative on a manifold that we can construct. We know that every smooth manifold has a Riemannian metric. Use this metric to construct Christoffel symbols, compatible with the metric, and define
  $$\nabla_a \omega_\mu = \partial_a \omega_{\mu\nu} - \Gamma^\sigma_{a\mu}\omega_\sigma$$
  It is clear that this is a well defined covariant derivative.

- I have used the notion of Frobenius theorem many times to show the fact that Lie brackets of a couple of coordinate vector fields vanish. Of course, Frobenius theorem is more general than this fact, this can be proved easily. For coordinate vector fields $X = \frac{\partial}{\partial x^\mu}$ and $Y = \frac{\partial}{\partial y^\mu}$, doesn't need to be orthogonal, we have
  $$[X, Y] = \left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right] = 0.$$

6 Einstein's Equations I

We followed sections 4.1 to 4.3 and arrived at the beginning of linearized gravity, section 4.4.

- There is a paper by Geroch and Jang, proving that isolated, gravitating body in general relativity moves approximately along a geodesic. Which merely means that geodesic equations could be derived from Einstein's equations. You can find the paper on the main page.

7 Einstein's Equations II

We discussed gravitational waves and finished chapter 4.

- Why there is no dipole radiation of gravitational waves in general relativity?

  The derivation rules out the contribution of dipole moments, because $\gamma_{\mu\nu}$ is a second rank tensor, but $Q^\mu = \int T^{00}(x)x^\mu d^3x$ is a vector. Also, like electromagnetism, if we assume that the radiation is proportional to $\ddot{p}^\mu$, where p^μ is the dipole moment, then the contribution of dipole radiation would be zero, for we know that
  $$\frac{d^2Q^\beta}{dt^2} = \partial_0 \int \partial_0 T^{00} x^\beta d^3x = -\partial_0 \int \partial_\alpha T^{\alpha 0} x^\beta d^3x = \partial_0 \int T^{\beta 0} d^3x = 0 \quad (\alpha, \beta = 1, 2, 3)$$

8 The Schwarzschild Solution

We finished chapter 6 and briefly discussed Frobenius theorem.

- What are the black hole solutions in higher dimensions?

  For spherically symmetric solutions, due to Birkhoff's theorem, the only solution is Schwarzschild.

  $$ds^2 = -\left(1 - \frac{2M}{r^{d-3}}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r^{d-3}}\right)} dr^2 + r^2 \, d\Omega^2_{d-2}$$

  The same method for $d = 4$ can be used to derive the above formula. For rotating black holes or only static spacetimes, the problem is more subtle and one needs more sophisticated methods to solve Einstein's equations. For instance, there are static asymptotically flat (black hole) solutions called "black ring" in five dimensional spacetime that possess $S^1 \times S^2$ topology in the horizon. It seems that black holes have rich structures in higher dimensions [7], [8].

- Do we need staticity for constructing the coordinate system orthogonal to timelike killing vector ξ_a ?

  Yes. All other types of coordinates, like Gaussian normal coordinate, are defined locally, sufficiently small, near a point. In order to define the surface orthogonal to ξ_a as a hypersurface, one needs to make sure that these hypersurfaces are manifolds. This can be achieved by using the dual form of Frobenius theorem.

- Geodesic equation and null curves.

  The general form of a geodesic equation, with an arbitrary parametrization is

  $$u^b \nabla_b u^a = \alpha u^a$$

  This is also true for null paths. It is easy to see that you can always find a reparametrization of the curve, such that the rhs of the above equation is zero. So, basically, null paths satisfy both $ds = 0$ and geodesic equation.

- How could we know that the spacetime is extendable or not?

  This is the exact notion of "maximal geodesic" spacetime. We postpone this question and come back to it after we have finished chapter 9.

9 The Principle of General Covariance and Mach's Principle

In this session, Pooya Farokhi gave us a brief and interesting timeline of ideas, yielded to general relativity theory. Then he went on to the hole argument, physical meaning of general covariance, diff invariance, their resolutions and Mach's principles. You can find his presentation file on the main page.

- Is it possible to have a spacetime with vanishing Ricci tensor everywhere, but with nonzero Riemann curvature? In two and three dimensions, Riemann tensor completely determined by Ricci tensor. But in four dimensions, it is not the case.

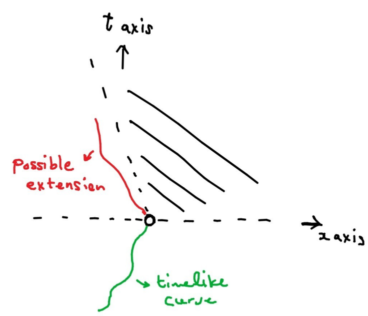
Part II. Advanced Topics

10 Causal Structure of Spacetime I

we introduced the basic definitions and theorems and finished the first part of chapter 8.

- Why is it not always possible to extend a smooth curve beyond its future endpoint p, while preserving smoothness of the curve?

  Consider Minkowski spacetime. Remove all the x axis at $t = 0$ but the origin. Also, take out the region $0 \leq \theta \leq 120^0$ with $r > 0$. Now we can construct a curve, which has the desired properties.

  It is evident that if you attach green and red curve, you will end up with a causal curve which is not differentiable.

11 Causal Structure of Spacetime II

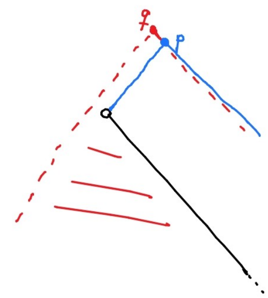
We finished section 8.2 and arrived at page 203.

- Construction of a finite volume measure on a manifold.

  The volume form that I introduced in the class was not correct. It was ill-defined when one compute it in the intersection of two open sets. I don't have a rigorous candidate for it right now, but the following idea may work. There exist countable compact sets K_i such that $K_i \subseteq int(K_{i+1})$ and $M = \bigcup K_i$. Now we can define function f on M with the property that $f|_{int(K_{i+1}-K_i)} = \frac{1}{(i+1)^2 Vol(K_i)}$. It may be not possible to make this function differentiable on boundary of these sets. But we can get around this issue with redefining f on a small tube near each of these boundaries and smear it with some bump functions. In the end, $\mu(A) = \int_A f\omega$ with ω the usual standard volume form, should work as our desirable measure.

- Continuity of the function defined in theorem 8.2.2.

  The continuity of volume form doesn't relate to the continuity of function 8.2.4. Below picture, shows a situation that $F(p) = \mu[I^-(p)]$ is not continuous at point p.



  Here we removed a line (but not its edge point) from Minkowski spacetime. It is clear that point q has more accessible space by causal curves than p, even though q could be arbitrary close to p.

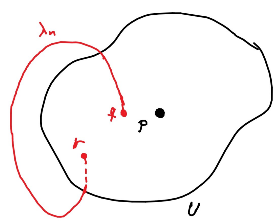
12 Causal Structure of Spacetime III

We finished chapter 8.

- The last part of theorem 8.3.5.

  We shall show that endpoint of null geodesic curve λ lies in $edge(S)$. If the endpoint $p \notin edge(S)$, then by previous arguments of this theorem, we can find a past inextendible null geodesic λ' through p (note that the condition $p \in S$ can be removed in the previous part of the theorem). Which means that we can transcend beyond the endpoint and that is a contradiction because we assumed that λ is extended as much as possible.

- Closed or inextendible property of limit curve in lemma 8.3.8.



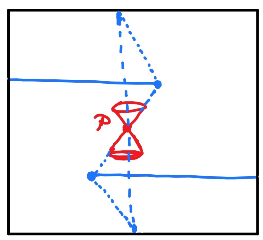
  Assume that strong causality violates at point $p \in M$. Then we can find causal curves λ_n, intersecting small open neighborhood U_n more than once. If we remove the start and end point of λ_n, we will have inextendible causal curves and λ_n have limit point p. Name the limit curve λ. Now we put back all the removed points from λ_n. λ would be inextendible or extendible. Suppose the former occurs. Say λ has an future endpoint q, which also belongs to the future or past endpoint of λ_i for some i. By definition, for any open neighborhood containing q, infinite λ_n penetrate it. Using the famous lemma again, we can construct a future inextendible curve passing through q. This is a contradiction. Hence, there are no future endpoints. Similarly there are no past endpoints.

13 Singularities I

- Every metric space can be completed using limit points of all Cauchy sequences. Also, every Riemannian manifold, adopt a metric structure and hence can be Cauchy completed. Now, if this extended space is itself a manifold, then using Hopf-Rinow theorem, we can conclude that the manifold is geodesic completed.
- One can show that the following metric has conical singularity at $r = 0$ in four dimensions.

$$ds^2 = -dt^2 + dr^2 + (1 - 4\mu^2)r^2\, d\phi^2 + dz^2$$

  With $0 < \mu < 1/2$, $\mu \neq \frac{1}{4}$. So basically, there could be non-curvature singularities in 4 dimensions.
- Here you can see a spacetime which is strongly causal but not stably causal. This is a Minkowski spacetime



  with blue lines removed (not the edge points) and two black border lines at the bottom and top are identified. At point $p \in M$ by expanding the light cones, you can build a closed timelike curve, but in the small neighborhood of p there is no timelike curve which passes the open set more than once, because it gets trapped.

14 Singularities II

We arrived at the middle of section 9.3.

- It is clear that the deviation vector of timelike geodesic congruence X^a satisfies geodesic deviation equation and is a Jacobi field. Additionally, we have $[X, T] = 0$, where T^a is the tangent vector of timelike geodesics. The converse is not obvious, whether every Jacobi field comes from a congruence. Fortunately, there is a theorem that assures us a Jacobi field does come from a geodesic congruence and hence, the Lie derivative of X^a vanishes along T^a.
- We found that $\theta = \frac{1}{detA}\frac{d(detA)}{d\tau}$. Now if $detA \to 0$, two cases occur. It might be positive with negative derivative, or negative with positive derivative. In both cases, the value of θ is negative. This completes the proof that having conjugate points is equivalent to negative infinite expansion.

15 Singularities III

We finished chapter 9.

- Full proof of "only if" part of theorem 9.3.3.

  This proof is based on [9]. Let γ be the timelike geodesic from p to q that locally maximize the proper length. It is proved that γ should be a geodesic. For the second part, suppose we have a conjugate point r between q and p. This means that there is a Jacobi field W^a along T^a, where T^a is the tangent vector of γ and W^a is zero at q and r. Now let K^a be another deviation vector orthogonal to T^a (this means $K_aT^a = 0$, because geodesic deviation equation depends on the part of the deviation vector that is orthogonal to the geodesic itself) with the condition that
  $$K_aT^b\nabla_bW^a = -1 \quad at \quad r.$$
  We found the second variation of arc length in an arbitrary geodesic congruence. More precisely,
  $$L(W,W) = \frac{d^2\tau}{d\alpha^2}|_{\alpha=0} = \int_p^q W^a(\mathcal{O}W_a)$$
  Where
  $$\mathcal{O}W_a = T^c\nabla_cT^b\nabla_bW_a + R_{abc}{}^dW^cT^bT_d.$$
  But we know that W^a is a Jacobi field, so $\mathcal{O}W^a = 0$. Define $Z^a := \epsilon K^a + \frac{1}{\epsilon}W^a$. It is easy to show that
  $$L(Z,Z) = \frac{1}{\epsilon^2}L(W,W) + 2L(K,W) + \epsilon^2 L(K,K)$$
  Of course, $L(W,W) = 0$. Also,
  $$\begin{aligned}
  L(K,W) &= \int_q^p W^a\mathcal{O}K_a \\
  &= \int_q^r W^a\left(T^c\nabla_cT^b\nabla_bK_a + R_{abc}{}^dK^cT^bT_d\right) \\
  &= \int_q^r \left(T^c\nabla_c\left(W^aT^b\nabla_bK_a\right) - T^c\nabla_cW^a\ T^b\nabla_bK_a + R_{abc}{}^dK^cT^bT_dW^a\right) \\
  &= \int_q^r \left(-T^c\nabla_cW^a\ T^b\nabla_bK_a + R_{abc}{}^dK^cT^bT_dW^a\right) \\
  &= \int_q^r \left(-T^b\nabla_b\left(K_aT^c\nabla_cW^a\ \right) + K_a\mathcal{O}W^a\right) \\
  &= 1
  \end{aligned}$$
  The last equality comes from the constraint that we imposed on K^a. Therefore, we have
  $$L(Z,Z) = 2 + \epsilon^2L(K,K).$$
  By taking ϵ sufficiently small, we have $L(Z,Z) > 0$ and this is a contradiction, since we assumed that γ locally maximize the proper length between p and q.

- Compact and orientable condition on surface K in theorem 9.3.11 is necessary to guarantee that the curve has a endpoint on K and the notion of $\partial I^+(K)$ is well-defined.

- In preposition 9.4.1, we assumed that the hypersurfaces $\tau = constant$ foliate the open set around γ.

  We can convince ourselves that the desired foliation exists. On a neighborhood of $p \in \gamma$, the spacetime is flat, if one uses the normal coordinate. We can map the curve into the normal coordinate. In this coordinate, geodesics are straight lines. The curvature of γ is bounded and if we think of it like a curve in three dimensional space, then on every point of it, all orthogonal vectors to $T\gamma$ with length less than $\sup_{x\in\gamma} 1/k_x$ (where k_x is the "curvature" of the curve at x) foliate a sufficiently small neighborhood around γ.

- In FRW metric using singularity theorems, we can show that an incomplete geodesic exists. This merely means, our universe has a singularity. Note that just the fact that in FRW metric, $a(\tau) \to 0$ doesn't tell us anything about singularities, because it may take infinite amount of time (affine parameter) to reach that point. One can show that in FRW metric, the Raychaduri equation becomes

  $$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - (3P + \rho) + \Lambda$$

  Now if we assume that t is small enough, so that the universe is matter dominated, then we can neglect Λ and conclude that θ is negative at least at a single point of the spacetime. Using Hawking Penrose theorem, we conclude that at least one incomplete timelike or null geodesic exists.

16 The Initial Value Formulation I

We finished initial value formulation of Maxwell equations.

17 The Initial Value Formulation II

Chapter finished.

- More about Gauss-Codacci equation .

  This equation relates the elements of first fundamental form to second fundamental form and it is a constrain on the metric and the orthogonal vector, and they are sufficient to identify a hypersurface. More precisely, we have the following theorem. Let $\Omega \subset \mathbb{R}^n$ be an open connected and simply connected subset (e.g. an open ball or cube), and suppose that we are given two smooth n by n matrix valued functions g_{ab} and K_{ab} on W such that g_{ab} and K_{ab} assign to every point a symmetric matrix, g_{ab} gives the matrix of a positive definite bilinear form. In this case, if the functions Γ^a_{bc} derived from the components of g_{ab} satisfy the Gauss-Codacci equations, then there exists a regular parameterized hypersurface $\mathbf{r} : \Omega \to \mathbb{R}^{n+1}$ for which the matrix representations of the first and second fundamental (metric and curvature tensor) forms are g_{ab} and K_{ab} respectively. Furthermore, this hypersurface is unique up to rigid motions of the whole space. Namely, if $\mathbf{r}_1$ and $\mathbf{r}_2$ are two such hypersurfaces, then there exists an isometry $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for which $\mathbf{r}_2 = \Phi \circ \mathbf{r}_1$.

- Does a spacelike Cauchy surface with constant curvature exist for any solution of Einstein's equations?

  It is not always true. See [10].

- How to simulate two emerging black holes?

  Consider a spatial metric h_{ab} on spacetime manifold. Define a new metric $h'_{ab} = \phi^4 h_{ab}$ and $K'_{ab} = \phi^{-2} K_{ab}$. The first constraint from Einstein's equations implies that $D' K'_{ab} = 0$ if $Tr(K) = 0$. The second constraint results in Laplace equation $D^a D_a \phi = 0$. This equation has a classical potential solution $\phi = 1 + \frac{M}{2r}$. This is the spatial and conformal form of Schwarzschild metric. We can construct an interesting metric, where there are two black holes at fixed points x_1 and x_2, by using the superposition principle for Laplace equation, exactly the same as classical electrodynamics. Having the initial value of first and second fundamental form, we can set up the initial value formalism and use the dynamical equations to simulate black hole merging.

- Initial value formulation of Einstein's equations in presence of matter.

  Matter changes the RHS of Einstein's equations, and the constraints for initial value metric and second fundamental on the Cauchy surface will differ from vacuum space, as it is written in 10.2.41 and 10.2.42. If T_{ab} depends on the metric and its first derivative, then the constraint equation $n^\mu G_{\mu\nu} = T_{\mu\nu} n^\mu$ contains information only on the Cauchy surface. So our first step remains valid, provided that T_{ab} contains matters that their initial value formulation is well defined i.e. their equation of motion is of the form 10.1.21. The remaining steps are the same as before and therefore, Einstein's equations with matter have well-posed initial value formulation.

18 Asymptotic Flatness I

Arrived at the beginning of writing null infinity metric.

- Symmetries of metric 11.1.14.

  This metric describes a portion of FRW spacetime. It is conformally flat by construction. That means, the Weyl curvature tensor is zero (since Weyl tensor is conformally invariant). Moreover, there is no time dependent scale factor $a(\tau)$, which means that the Ricci scalar is constant. Now using the usual decomposition of Riemann tensor into conformally and non conformally invariant parts, we arrive at a simple Riemann curvature tensor.

  $$R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad})$$

  This is precisely the metric of maximally symmetric spacetime. Hence, it contains all possible killing vectors.

- The Lie algebra of Killing fields.

  Killing vector fields can also be defined on any (possibly nonmetric) manifold M if we take any Lie group G acting on it instead of the group of isometries. In this broader sense, a Killing vector field is the pushforward of a right invariant vector field on G by the group action. If the group action is effective, then the space of the Killing vector fields is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

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