1 Gravity Before Relativity

In the first session we talked about the history of gravity and at the end we briefly discussed Mach's ideas on the physical structure of space and time. I followed the first chapter of [1] for the historical part and [2] for Mach's principle.

- For axiomatic approach to Newton's laws I recommend you to visit this page.
- One may think that a sufficient far point on a rotating disk have a velocity higher than c and that violates special relativity. The point is that you need an infinite amount of energy to speed-up such a disk. Also, this rotation would cause an infinite force acting on an infinitesimal part of the the disk with $v = c$. To see this, we can calculate the proper time of an arbitrary point on the disk. Assume that (t', x', y', z') is the coordinate label of rotating frame.

$$t' = t$$
$$r' = r$$
$$\phi' = \phi - \omega t$$
$$z' = z$$
$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta}$$

  the proper length in rotating frame is

$$ds^2 = (1 - \frac{\omega^2 r'^2}{c^2})c^2 dt'^2 - 2\omega r'^2 d\phi' dt' - dz'^2 - dr'^2 - r'^2 d\phi'^2$$

  so the proper time of an arbitrary observer at distance r from origin is

$$d\tau = \sqrt{1 - \frac{\omega^2 r^2}{c^2}}dt$$

  Also, the geodesic equation tells us that $\frac{d^2x}{d\tau^2}$ and $\frac{d^2y}{d\tau^2}$ are proportional to $\frac{dt'}{d\tau}$. This means the acceleration is infinite in a point with $r = c/\omega$.

- It was proposed that Hoofman experiment may supports Mach's ideas. It seems that this experiment demonstrates frame-dragging effect and this can by fully understood by general relativity using Kerr metric. Therefore, it doesn't justify a "new" interpretation of gravity. You can visit this page for more information about frame-dragging effect.

2 Equivalence Principle

I used the third chapter of [1] and [3] for the statements and consequences of equivalence principle. It is shown in section 2.4 of [4] that why equivalence principle forbid us to describe gravity as an electromagnetic field theory. At the end, Pouya Farokhi talked about different versions of equivalence principle. This talk was based on [5].

3 Manifolds and Tensor Fields I

The basic definitions and theorems of general topology and manifolds is discussed. We introduced tangent space and proved "Identification Lemma" and arrived at the beginning of topological and differential structure on tangent bundle. I followed part D, chapter 4 of [6].

4 Manifolds and Tensor Fields II

We used "Sum Topology" to construct a topology on $TM = \bigsqcup_{x \in M} T_x M$. We stated and partially proved the theorem that the tangent bundle is itself a manifold. We also proved that the set of all derivatives at one point in a manifold is a vector space with the same dimension as manifold. At the end, we discussed a bit about the intuition of curvature on a two dimensional surface. All mathematical concepts discussed throughout the lecture can be found in [6].

- Given an arbitrary metric tensor $g : TM \times TM \to \mathbb{R}$, we can find an orthogonal basis for tangent space, at specific point on manifold, which it takes a diagonal form. This is possible since it is a symmetric second rank tensor or equivalently a matrix. We normalize the eigenvectors so that $g(v_\mu, u_\nu) = \pm\delta_{\mu\nu}$. The number of + and - signs are called the signature of metric. It is evident that trace of a matrix is independent of chosen orthogonal basis and so is the signature of metric. We will show later that at each point, there is a specific coordinate called Riemann normal coordinate which the metric tensor is flat $g_{\mu\nu} = diag(-1, 1, 1, 1)$. Now If our metric is diagonal or If we know that there is an orthogonal transformation between Riemann normal coordinate and our previous coordinate, we conclude that the signature of metric is three + and one -. As an example consider the Schwarzschild metric

  $$ds^2 = -\big(1 - \frac{r_s}{r}\big)dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2 d\Omega^2$$

  It is clear that even by passing the event horizon the signature of metric, in the way we defined it, doesn't change.

- Consider a vector field X on a two dimensional surface $M \subset \mathbb{R}^3$ and a smooth curve $\gamma : I \to M$ passing from p at $t = 0$. If we walk from p, a tiny bit, along γ and measure the change of X, $\frac{dX \circ \gamma(t)}{dt}$ we will end up with a vector. Now project this vector onto the tangent space of surface at that point. We call the final vector, which lies in tangent space at p, the covariant derivative of vector field X along γ at p and denote it by $D_{\gamma'(0)}X$. We talked intuitively about the concept of parallel transport. Now we can formulate it mathematically. Let w be a tangent vector at point p on the surface. Then there exist a unique vector field $W(t)$ along γ with $W(0) = w$ such that $D_{\gamma'(t)}W(t) = 0$ for all $t \in I$. This is an immediate consequence of existence and uniqueness theorem of differential equations. You can check standard differential geometry textbooks for more information.

5 Curvature

We discussed the third chapter of the book and arrived at the beginning of tetrad orthogonal basis section.

- Does a covariant derivative exist on a manifold?

  Yes. There is a trivial zero map. We can also construct many nontrivial covariant derivatives. Consider an arbitrary tensor field $T = T_{\mu\nu}\ dx^\mu \otimes dx^\nu$ on manifold. For a chart $U \subset M$, one can create the following operator

$$\nabla_a T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^a}.$$

Note that we assume the coordinate of derivation is *fixed* and does not change, like treating it as a function $f(x)$ as if by changing the coordinate becomes $f(x(x'))$. Now using the partition of unity in manifold, you can extend this to a covariant derivative over the whole manifold. There is another more natural covariant derivative on a manifold that we can construct. We know that every smooth manifold has a Riemannian metric. Use this metric to construct Christoffel symbols, compatible with the metric, and define

$$\nabla_a \omega_\mu = \partial_a T_{\mu\nu} - \Gamma^\sigma_{a\mu}\omega_\sigma$$

It is clear that this is a well defined covariant derivative.

- I have used the notion of Frobenius theorem many times to show the fact that Lie bracket of a couple of coordinate vector fields commute. Of course, Frobenius theorem is more general than this fact, this can be proved easily. For coordinate vector fields $X = \frac{\partial}{\partial x^\mu}$ and $Y = \frac{\partial}{\partial y^\mu}$, doesn't need to be orthogonal, we have

$$[X, Y] = \left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right] = 0.$$

References

[1] Steven Weinberg. *Gravitation and Cosmology.* Wiley, 1972.

[2] Bahram Mashhon. *Video Lectures on General Theory of Relativity.* Maktabkhone.

[3] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Cambridge University Press, 2004.

[4] Norbert Straumann. *General Relativity.* Springer, 2012.

[5] Eolo Di Casola, Stefano Liberati, and Sebastiano Sonego. Nonequivalence of equivalence principles. arXiv:1410.5093.

[6] Siavash Shahshahani. *An Introductory Course on Differentiable Manifolds.* Dover Publications, 2016.