Part I. Fundamentals

1 Gravity Before Relativity

In the first session we talked about the history of gravity and at the end we briefly discussed Mach's ideas on the physical structure of space and time. I followed the first chapter of [1] for the historical part and [2] for Mach's principle.

- For axiomatic approach to Newton's laws I recommend you to visit this page.
- One may think that a sufficient far point on a rotating disk have a velocity higher than c and that violates special relativity. The point is that you need an infinite amount of energy to speed-up such a disk. Also, this rotation would cause an infinite force acting on an infinitesimal part of the the disk with $v = c$. To see this, we can calculate the proper time of an arbitrary point on the disk. Assume that (t', x', y', z') is the coordinate label of rotating frame.

$$t' = t$$
$$r' = r$$
$$\phi' = \phi - \omega t$$
$$z' = z$$
$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta}$$

  the proper length in rotating frame is

$$ds^2 = (1 - \frac{\omega^2 r'^2}{c^2}) c^2 dt'^2 - 2\omega r'^2 d\phi' dt' - dz'^2 - dr'^2 - r'^2 d\phi'^2$$

  so the proper time of an arbitrary observer at distance r from origin is

$$d\tau = \sqrt{1 - \frac{\omega^2 r^2}{c^2}} dt$$

  Also, the geodesic equation tells us that $\frac{d^2x}{d\tau^2}$ and $\frac{d^2y}{d\tau^2}$ are proportional to $\frac{dt'}{d\tau}$. This means the acceleration is infinite in a point with $r = c/\omega$.

- It was proposed that Hoofman experiment may supports Mach's ideas. It seems that this experiment demonstrates frame-dragging effect and this can by fully understood by general relativity using Kerr metric. Therefore, it doesn't justify a "new" interpretation of gravity. You can visit this page for more information about frame-dragging effect.

2 Equivalence Principle

I used the third chapter of [1] and [3] for the statements and consequences of equivalence principle. It is shown in section 2.4 of [4] that why equivalence principle forbid us to describe gravity as an electromagnetic field theory. At the end, Pooya Farokhi talked about different versions of equivalence principle. This talk was based on [5].

3 Manifolds and Tensor Fields I

The basic definitions and theorems of general topology and manifolds is discussed. We introduced tangent space and proved "Identification Lemma" and arrived at the beginning of topological and differential structure on tangent bundle. I followed part D, chapter 4 of [6].

4 Manifolds and Tensor Fields II

We used "Sum Topology" to construct a topology on $TM = \bigsqcup_{x \in M} T_xM$. We stated and partially proved the theorem that the tangent bundle is itself a manifold. We also proved that the set of all derivatives at one point in a manifold is a vector space with the same dimension as manifold. At the end, we discussed a bit about the intuition of curvature on a two dimensional surface. All mathematical concepts discussed throughout the lecture can be found in [6].

- Given an arbitrary metric tensor $g : TM \times TM \to \mathbb{R}$, we can find an orthogonal basis for tangent space, at specific point on manifold, which it takes a diagonal form. This is possible since it is a symmetric second rank tensor or equivalently a matrix. We normalize the eigenvectors so that $g(v_\mu, u_\nu) = \pm\delta_{\mu\nu}$. The number of + and - signs are called the signature of metric. It is evident that trace of a matrix is independent of chosen orthogonal basis and so is the signature of metric. We will show later that at each point, there is a specific coordinate called Riemann normal coordinate which the metric tensor is flat $g_{\mu\nu} = diag(-1, 1, 1, 1)$. Now If our metric is diagonal or If we know that there is an orthogonal transformation between Riemann normal coordinate and our previous coordinate, we conclude that the signature of metric is three + and one -. As an example consider the Schwarzschild metric

  $$ds^2 = -\big(1 - \frac{r_s}{r}\big)dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2 d\Omega^2$$

  It is clear that even by passing the event horizon the signature of metric, in the way we defined it, doesn't change.

- Consider a vector field X on a two dimensional surface $M \subset \mathbb{R}^3$ and a smooth curve $\gamma : I \to M$ passing from p at $t = 0$. If we walk from p, a tiny bit, along γ and measure the change of X, $\frac{dX \circ \gamma(t)}{dt}$ we will end up with a vector. Now project this vector onto the tangent space of surface at that point. We call the final vector, which lies in tangent space at p, the covariant derivative of vector field X along γ at p and denote it by $D_{\gamma'(0)}X$. We talked intuitively about the concept of parallel transport. Now we can formulate it mathematically. Let w be a tangent vector at point p on the surface. Then there exist a unique vector field $W(t)$ along γ with $W(0) = w$ such that $D_{\gamma'(t)}W(t) = 0$ for all $t \in I$. This is an immediate consequence of existence and uniqueness theorem of differential equations. You can check standard differential geometry textbooks for more information.

5 Curvature

We discussed the third chapter of the book and arrived at the beginning of tetrad orthogonal basis section.

- Does a covariant derivative exist on a manifold?

  Yes. There is a trivial zero map. We can also construct many nontrivial covariant derivatives. Consider an arbitrary tensor field $T = T_{\mu\nu}\ dx^\mu \otimes dx^\nu$ on manifold. For a chart $U \subset M$, one can create the following operator
  $$\nabla_a T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^a}.$$
  Note that we assume the coordinate of derivation is *fixed* and does not change, like treating it as a function $f(x)$ as if by changing the coordinate becomes $f(x(x'))$. Now using the partition of unity in manifold, you can extend this to a covariant derivative over the whole manifold. There is another more natural covariant derivative on a manifold that we can construct. We know that every smooth manifold has a Riemannian metric. Use this metric to construct Christoffel symbols, compatible with the metric, and define
  $$\nabla_a \omega_\mu = \partial_a \omega_{\mu\nu} - \Gamma^\sigma_{a\mu}\omega_\sigma$$
  It is clear that this is a well defined covariant derivative.

- I have used the notion of Frobenius theorem many times to show the fact that Lie brackets of a couple of coordinate vector fields vanish. Of course, Frobenius theorem is more general than this fact, this can be proved easily. For coordinate vector fields $X = \frac{\partial}{\partial x^\mu}$ and $Y = \frac{\partial}{\partial y^\mu}$, doesn't need to be orthogonal, we have
  $$[X, Y] = \left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right] = 0.$$

6 Einstein's Equations I

We followed sections 4.1 to 4.3 and arrived at the beginning of linearized gravity, section 4.4.

- There is a paper by Geroch and Jang, proving that isolated, gravitating body in general relativity moves approximately along a geodesic. Which merely means that geodesic equations could be derived from Einstein's equations. You can find the paper on the main page.

7 Einstein's Equations II

We discussed gravitational waves and finished chapter 4.

- Why there is no dipole radiation of gravitational waves in general relativity?

  The derivation rules out the contribution of dipole moments, because $\gamma_{\mu\nu}$ is a second rank tensor, but $Q^\mu = \int T^{00}(x)x^\mu d^3x$ is a vector. Also, like electromagnetism, if we assume that the radiation is proportional to $\ddot{p}^\mu$, where p^μ is the dipole moment, then the contribution of dipole radiation would be zero, for we know that
  $$\frac{d^2Q^\beta}{dt^2} = \partial_0 \int \partial_0 T^{00} x^\beta d^3x = -\partial_0 \int \partial_\alpha T^{\alpha 0} x^\beta d^3x = \partial_0 \int T^{\beta 0} d^3x = 0 \quad (\alpha, \beta = 1, 2, 3)$$

8 The Schwarzschild Solution

We finished chapter 6 and briefly discussed Frobenius theorem.

- What are the black hole solutions in higher dimensions?

  For spherically symmetric solutions, due to Birkhoff's theorem, the only solution is Schwarzschild.

  $$ds^2 = -\left(1 - \frac{2M}{r^{d-3}}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r^{d-3}}\right)} dr^2 + r^2 \, d\Omega_{d-2}^2$$

  The same method for $d = 4$ can be used to derive the above formula. For rotating black holes or only static spacetimes, the problem is more subtle and one needs more sophisticated methods to solve Einstein's equations. For instance, there are static asymptotically flat (black hole) solutions called "black ring" in five dimensional spacetime that possess $S^1 \times S^2$ topology in the horizon. It seems that black holes have rich structures in higher dimensions [7], [8].

- Do we need staticity for constructing the coordinate system orthogonal to timelike killing vector ξ_a ?

  Yes. All other types of coordinates, like Gaussian normal coordinate, are defined locally, sufficiently small, near a point. In order to define the surface orthogonal to ξ_a as a hypersurface, one needs to make sure that these hypersurfaces are manifolds. This can be achieved by using the dual form of Frobenius theorem.

- Geodesic equation and null curves.

  The general form of a geodesic equation, with an arbitrary parametrization is

  $$u^b \nabla_b u^a = \alpha u^a$$

  This is also true for null paths. It is easy to see that you can always find a reparametrization of the curve, such that the rhs of the above equation is zero. So, basically, null paths satisfy both $ds = 0$ and geodesic equation.

- How could we know that the spacetime is extendable or not?

  This is the exact notion of "maximal geodesic" spacetime. We postpone this question and come back to it after we have finished chapter 9.

9 The Principle of General Covariance and Mach's Principle

In this session, Pooya Farokhi gave us a brief and interesting timeline of ideas, yielded to general relativity theory. Then he went on to the hole argument, physical meaning of general covariance, diff invariance, their resolutions and Mach's principles. You can find his presentation file on the main page.

- Is it possible to have a spacetime with vanishing Ricci tensor everywhere, but with nonzero Riemann curvature? In two and three dimensions, Riemann tensor completely determined by Ricci tensor. But in four dimensions, it is not the case.

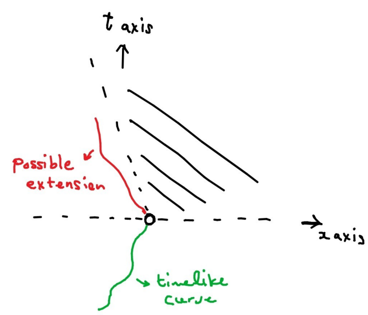
Part II. Advanced Topics

10 Causal Structure of Spacetime I

we introduced the basic definitions and theorems and finished the first part of chapter 8.

- Why is it not possible to extend a smooth curve beyond its future endpoint p, while preserving smoothness of the curve?

  Consider Minkowski spacetime. Remove all the x axis at $t = 0$ but the origin. Also, take out the region $0 \leq \theta \leq 120^0$ with $r > 0$. Now we can construct a curve, which has the desired properties.



It is evident that if you attach green and red curve, you will end up with a causal curve which is not differentiable.

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