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UNIT II

Gauss elimination method

Solution of a System of Linear Equations

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The above are 3 equations in 3 unknowns

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \\ a_3 x + b_3 y + c_3 z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $AX = B$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the coefficient matrix

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is called column matrix of unknown

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is called column matrix of constants

$[A|B] = [\;:\;]$ is called the augmented matrix

(2)

If $d_1 = d_2 = d_3 = 0$ i.e. $B = 0$, then $AX = 0$

Such a system is called a system of homogeneous linear equations.

If at least one of d_1, d_2, d_3 is non-zero, then $B \neq 0$

Such a system of equations is called a system of non-homogeneous linear equations.

Solution (Gauss Jordan - elimination method) Solving the matrix equation $AX = B$ means finding X, i.e. finding a column matrix $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ such that $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. Then $x = \alpha$, $y = \beta$, $z = \gamma$.

The matrix $AX = B$ need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

A system of equations having no solution is called an ~~in~~ inconsistent system of equations.

A system of equations having one or more solutions is called a consistent system of equations.

The ~~Argm~~ matrix $[A:B]$ given by

$$[A:B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$ is called the augmented matrix.

For a systems of ~~non-homogeneous~~ linear equations

$AX = B$

(i) if $\rho[A:B] \neq \rho(A)$, the system is inconsistent

(ii) if $\rho[A:B] = \rho(A)$, = no. of unknowns, the system has a unique solution.

(iii) if $\rho[A:B] = \rho(A) <$ no. of unknowns, the system has an infinite no. of solns.

no. of equations can be less than or also greater than no. of unknowns

The values of unknowns which simultaneously satisfy all the equations is called soln. of simul. eqns.

How to get the soln.?

① $AX = B$

premultiplying both sides by A^{-1}

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

which is the required soln.

Note: If A is a non-sing. matrix then the system will have unique soln. ($\because A^{-1}$ is unique)

But if A is singular then this method is not applicable.

② Rank method.

Solving a non-homogeneous system of $AX = B$ of linear equations by rank method:

Step I: Obtain the coefficient matrix A & column matrix of constants B. (Write the given eqns in matrix form & obtain the coeff matrix, augmented matrix & unknown column)

Step II: Write the augmented matrix $[A:B]$

Step III: Reduce the augmented matrix to reduced row echelon form by applying a sequence of elementary row-operations

Step IV: Determine the no. of non-zero rows in A & $[A:B]$ to determine the ranks of A & $[A:B]$ respectively.

Step V: If $\rho(A) \neq \rho(A:B)$ then the system is inconsistent. STOP, else goto step VI

Step VI: If $\rho(A) = \rho(A:B)$ = no. of unknowns, then the system has unique solution which can be obtained by back substitution.

If $\rho(A) = \rho(A:B) <$ no. of unknowns, then the system has an infinite no. of solutions which can also be obtained by back substitution.

Ex. (RD 11.2) Discuss the consistency of the following systems of linear equations.

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$

Soln: The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$$

or $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$\sim R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

which is in echelon form.

$\rho(A:B) = 3$

Also $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

Since $\rho(A) \neq \rho(A:B)$

$\therefore$ the given system of equations is inconsistent.

Ex. Show that the following system of linear equations is consistent and solve it.
RD-M22

$3x - y - 2z = 2$

$2x - z = -1$

$3x - 5y = 3$

Soln The given eqn can be written in matrix form as

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

or $AX = B$

where $A =$, $B =$, $C =$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 2 & 0 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 2 & 0 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & 2 & 1 & -7 \\ 0 & -2 & 3 & -6 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & 2 & 1 & -7 \\ 0 & 0 & 4 & -13 \end{array}\right] \quad \text{--- (1)}$$

which is in echelon form

& $\rho(A:B) = \rho(A) = 3 =$ no. of unknowns

Hence the system is consistent (∴ unique soln.)

Rewriting the equations from the matrix (1), we obtain

$$x - y - z = 3$$
$$2y + z = -7$$
$$4z = -13$$

$\Rightarrow z = -13/4$

$2y + z = -7 \Rightarrow 2y = -7 + \frac{13}{4}$

$\Rightarrow y = -15/8$

Putting the values of y & z in

$x - y - z = 3$

we get $x = -17/8$

$\therefore x = \frac{-17}{8}, \quad y = \frac{-15}{8}, \quad z = \frac{-13}{4}$

Ex.
NP 37

Solve the following systems of equations using [illegible]

$2x_1 + x_2 + 2x_3 + x_4 = 6$

[illegible]

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

Soln.

The augmented matrix is

$$[A:B] = \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1, \quad R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_2 \to \left(-\frac{1}{9}\right) R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & 5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -3 & 1 & 6 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right] \longrightarrow (1)$$

which is Echelon form.

$\therefore \rho(A:B) = 4$

$\rho(A) = 4$

$\therefore \rho(A) = \rho(A:B) = 4 =$ no. of unknowns.

Rewriting the equations from matrix (1), we obtain

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$x_2 - x_4 = -2$$
$$-x_3 - 4x_4 = -11$$
$$13x_4 = 39$$

On solving we get

$x_1 = 2, \; x_2 = 1, \; x_3 = -1, \; x_4 = 3$

Ex.
RD.11.22

Show that the system of equations

$$x - 3y - 8z = -10$$
$$3x + y - 4z = 0$$
$$2x + 5y + 6z = 13$$

is consistent and hence solve it.

Soln. The given system of linear equations can be written as

$$\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 13 \end{bmatrix}$$

or $AX = B$

where $A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} -10 \\ 0 \\ 13 \end{bmatrix}$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{array}\right]$$

~~$R_3 \to R_3 - \frac{11}{10} R_2$~~ $R_2 \to \frac{1}{10} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 11 & 22 & 33 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 11R_2 \\ \text{——— (1)} \end{array}$$

$\rho(A) = \rho(A:B) = 2 < 3$ (no. of unknowns)

So, the given system of equations is consistent with infinitely many solutions.

Rewriting the equations from the matrix (1), we obtain

$$x - 3y - 8z = -10 \quad \text{——— (i)}$$
$$y + 2z = 3 \quad \text{——— (ii)}$$

Taking $z = k$ (where k is any arbitrary real no.)

we obtain

$$y = 3 - 2k$$

putting $z = k$ & $y = 3-2k$ in (i) we get

$$x - 3(3-2k) - 8k = -10$$
$$\Rightarrow x - 9 + 6k - 8k = -10$$
$$\Rightarrow x = 2k - 1$$

Hence $x = 2k-1$, $y = 3-2k$, $z = k$, where k is arbitrary real no.

is the req. soln.

Note: If no. of unknowns − rank = 1, we take one unknown as constant

If no. of unknowns − rank = 2, we take two unknowns as arbitrary constants

& so on.

Ex. 52, HP

(11)

For what values of a & b do the equations

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

have (i) no solution, (ii) a unique soln. (iii) more than one solution.?

Soln. Writing the given system of equations in matrix notation

$$AX = B$$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$

Augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$$

Case I when $a=8$, $b \neq 15$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\because$ $\rho(A) \neq \rho(A:B)$

$\therefore$ The system has no solution.

Case II

If $a \neq 8$, b may have any value

$\rho(A) = \rho(A:B) = 3 =$ no. of unknowns

$\therefore$ The system has unique soln.

Case III

If $a = 8$, $b = 15$

$\rho(A) = \rho(A:B) = 2 <$ no. of unknowns.

$\therefore$ The system will have an infinite no. of solutions.

Ex. NP 47 Show that the equations.

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no solution unless $a+b+c=0$. In which case they have infinitely many solutions? Find these solutions when $a=1$, $b=1$, $c=-2$

Soln: Augmented matrix

$$[A:B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 3 & -3 & a+2c \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & c \\ 0 & -3 & 3 & b-c \\ 0 & 0 & 0 & a+b+c \end{array}\right]$$

Case I If $a+b+c \neq 0$

$\rho(A:B) = 3$, $\rho(A) = 2$

$\therefore \rho(A) \neq \rho(A:B)$. Hence no soln.

Case II: If $a+b+c = 0$

$\rho(A:B) = 2 = \rho(A) < 3$

Hence the system has infinite no. of solutions

~~If a=1, b=1, c=-2~~

Equivalent system of equations is

$x + y - 2z = -2 \quad (c = -2)$

$-3y + 3z = 3 \quad (\cancel{}\ b - c = 1+2 = 3)$

Let $z = k$, k being an arbitrary constant.

$$-3y + 3k = 3$$

$$\Rightarrow y = k - 1$$

& $x + (k-1) - 2k = -2$

$$\Rightarrow x - k - 1 = -2$$

$$\Rightarrow x = k - 1$$

Hence a solution is

$$x = k-1, \quad y = k-1, \quad z = k.$$

System of homogeneous linear equations $AX = 0$

i) For homogeneous sys. $X = 0$ is always a solution. This solution in which each unknown has the value zero is called the Null solution or the Trivial solution.

Thus a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite no. of solutions.

ii) If $\rho(A) =$ no. of unknowns, the system will have only the trivial solution.

iii) If $\rho(A) <$ no. of unknowns, the system has an infinite no. of non-trivial solutions.

In case of a homogeneous system of linear equations the rank of the augmented matrix is always same as that of the coefficient matrix. So a homogeneous system of linear equations is always consistent.

(So here we don't need to find the augmented matrix)

Algorithm for solving a homogeneous system of linear equations

Step I Write the given system of equations in matrix form $AX = 0$ and obtain the coeff matrix A.

Step II Reduce the coeff matrix (A) to echelon form by applying a sequence of elementary row operations.

Step III: Determine the no. of non-zero rows to get the rank of A [$\rho(A)$]

Step IV: If $\rho(A) =$ the no. of unknowns then the system will have a trivial solution. (i.e $x = y = z = 0$) STOP, else goto step V.

Step V If $\rho(A) <$ no. of unknowns, then the system will have infinitely no. of solutions. For finding these solutions put $z = k$ (any arbitrary real no.) and solve any two equations for x & y in terms of k. The values of x & y so obtained with $z = k$ give a solution of the given system of equations.

Ex. Solve the following system of equations

$$2x+3y-z=0$$
$$x-y-2z=0$$
$$3x+y+3z=0$$

Soln: The given system of homogeneous equations can be written as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = O$

where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & -5 & -3 \\ 0 & -11 & 0 \end{bmatrix}$$

$R_2 \to \left(-\frac{1}{5}\right) R_2$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 3/5 \\ 0 & -11 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 11 R_2$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 3/5 \\ 0 & 0 & -\frac{33}{5} \end{bmatrix}$$

$\rho(A) = 3 =$ no. of unknowns

$\therefore$ system will have trivial soln.

i.e. $x = y = z = 0$

Ex. Solve the following system of equations

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

soln. The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = 0$, where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \& \ O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no. of unknowns

$\therefore$ the given system of equations possesses an infinite no. of solutions.

Now since the rank of A is 2 & there are 3 unknowns

$\therefore 3-2=1$ variable can be assigned ~~with~~ arbitrary values

The given system of equations is equivalent to the system of equations

$$x + 3y - 2z = 0$$

$$-7y + 8z = 0$$

let $z = k$

$\therefore$ ~~y~~ $-7y + 8k = 0$

$\Rightarrow y = \frac{8}{7}k$

& $x + 3 \cdot \frac{8k}{7} - 2k = 0$

$\Rightarrow x = 2k - \frac{24}{7}k = -\frac{10}{7}k$

$\therefore$ Req. soln. is

$$x = -\frac{10}{7}k, \; y = \frac{8}{7}k, \; z = k.$$

Ex.
RNR 53

Find the values of λ for which the equations

$$(11-\lambda)x - 4y - 7z = 0$$
$$7x - (\lambda+2)y - 5z = 0$$
$$10x - 4y - (6+\lambda)z = 0$$

possess a non-trivial soln.
For these values of λ, find the soln. also.

<u>soln.</u> For the given system of equations to have a non-trivial solution,

$\rho(A) < 3$ where $A = \begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -(\lambda+2) & -5 \\ 10 & -4 & -(6+\lambda) \end{bmatrix}$

For this $|A| = 0$

$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -(\lambda+2) & -5 \\ 10 & -4 & -(6+\lambda) \end{vmatrix} = 0$$

Operating. $C_1 \to C_1 + (C_2 + C_3)$

If the elements of any row (or col.) be added to any constant multiple of the corresponding elements of any other row (or col.) resp., the values of the determinant remains unchanged
i.e. C_1 can be changed as
$C_1 \to C_1 + l C_2 + m C_3$

$$\begin{vmatrix} -\lambda & -4 & -7 \\ -\lambda & -(\lambda+2) & -5 \\ -\lambda & -4 & -(6+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} 1 & -4 & -7 \\ 1 & -(\lambda+2) & -5 \\ 1 & -4 & -(6+\lambda) \end{vmatrix} = 0$$

Operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Rightarrow -\lambda \begin{vmatrix} 1 & -4 & -7 \\ 0 & -\lambda+2 & 2 \\ 0 & 0 & -\lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \left[1(-\lambda+2)(-\lambda+1) \right] = 0$$

$\lambda = 0, 2, 1$

when $\lambda = 0$

$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 7R_1$, $R_3 \to R_3 - 10R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 2 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

linear eqn. are

$x - z = 0$

$-2y + 2z = 0$

let $z = k$

$\therefore -2y + 2k = 0$

$\Rightarrow y = k$

& $x = k$

$\therefore x = k, \; y = k, \; z = k$

when $\lambda = 1$

$$A = \begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix}$$

$R_3 \to R_3 - R_1, \quad R_2 \to R_2 - \frac{7}{10} R_1$

$$\sim \begin{bmatrix} 10 & -4 & -7 \\ 0 & -1/5 & -1/10 \\ 0 & 0 & 0 \end{bmatrix}$$

$-3 +$

$= -3$

$-5 +$

$-\frac{1}{10}$

echelon form

eqv. linear equation

$10x - 4y - 7z = 0$

$-\frac{1}{5}y - \frac{1}{10}z = 0$

let $z = k$

$\therefore -\frac{1}{5}y = \frac{1}{10}k \Rightarrow y = -k/2$

$10x = 4 \cdot (-\frac{k}{2}) + 7 \cdot k \Rightarrow x = \frac{1}{2}k$

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when $\lambda = 2$

$$A = \begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} -1 & 0 & 1 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix}$$

$R_1 \to (-1) R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix}$$

$R_2 \to R_2 - 7R_1, \quad R_3 \to R_3 - 10 R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 2 \\ 0 & -4 & 2 \end{bmatrix}$$

$R_2 \to R_3 \cancel{+} - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - z = 0$

$-4y + 2z = 0$

let $y = k$

$\Rightarrow -4k + 2z = 0$

$\Rightarrow z = 2k$

$x = 2k$

Ex solve
NP 43

$x_1 + x_2 + 2x_3 + 3x_4 = 0$

$3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$

$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$

$6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$

$\rho(A) = 4$

trivial soln.

Solution of System of linear equations by L.U Decomposition (Crouts method)

Consider the system of Equations

$$A\underline{X} = \underline{B} \quad \text{--- (1)}$$

where $\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1st ~~Put~~ $A = LU$ --- (2)

[Decompose the matrix A into two matrices L & U as $A = L \cdot U$]

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

(lower triangular matrix of above form)

and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

From (1) & (2), we get

$$LU\underline{X} = \underline{B} \quad \text{--- (3)}$$

put $U\underline{X} = \underline{Y}$ --- (4), where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then (3) becomes

$$L\underline{Y} = \underline{B} \quad \text{--- (5)}$$

solving (5) for $\underline{Y}$

put the value of $\underline{Y}$ in (4) and solve it for $\underline{X}$

From this method it is quite easy to solve (5) and then (4).

(*)

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

We need to keep track of the elementary row operations to write A as a upper triangular matrix

U

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 3 & 4 & 7 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2\ \boxed{*} & 1 & 0 \\ \boxed{*} & \boxed{*} & 1 \end{bmatrix}$$

$+3 \qquad -\frac{1}{2}$

$R_3' \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & -8 & 16 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 15.5 \end{bmatrix} = U$$

$\therefore A = LU$

$$\begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 15.5 \end{bmatrix}$$

$*$ $A = \begin{bmatrix} 2 & 4 & -4 \\ 1 & -4 & 3 \\ -6 & -9 & 5 \end{bmatrix}$

LU

$R_2 \to R_2 \boxed{-\frac{1}{2}} R_1$

$\sim \begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ -6 & -9 & 5 \end{bmatrix}$

$R_3 \to R_3 + \boxed{3} R_1$

$\sim \begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 3 & -7 \end{bmatrix}$

$R_3 \to R_3 + \boxed{\frac{1}{2}} R_2$

$\sim \begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix}$

$\therefore \quad A = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} \leftarrow \boxed{*} & 1 & 0 \\ -3 \leftarrow \boxed{*} & \underset{\downarrow \; -\frac{1}{2}}{\boxed{*}} & 1 \end{bmatrix}$$

Ex. Solve by LU Decomposition method

$$x_1 - x_2 + x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix}; \quad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 90 \\ 80 \end{bmatrix}$$

$$A\underline{x} = \underline{B}$$

(interchange of rows are not allowed)

Let $A = L \cdot U$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 30 & -20 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \end{bmatrix}.$$

(*) [Using the opp. of the multipliers used in the row operations to obtain U, we can build L.]

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{0} & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 20 & * & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 20 & 3 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 20 & 3 & 1 \end{bmatrix}$$

$$AX = B$$

$$LUX = B$$

$$U\underline{X} = \underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$L\underline{Y} = \underline{B}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 20 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ 80 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ 20y_1 + 3y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ 80 \end{bmatrix}$$

$$\Rightarrow y_1 = 0, \quad y_2 = 90, \quad y_3 = -190$$

$$\therefore \underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ -190 \end{bmatrix}$$

$$U\underline{X} = \underline{Y} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ -190 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 - x_2 + x_3 \\ 10x_2 + 25x_3 \\ -95x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ -190 \end{bmatrix}$$

$$\Rightarrow x_3 = 2$$

$$x_2 = 4$$

$$x_1 = 2$$

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Solve by LU Decomposition

Ex.

$x_1 + x_2 - x_3 = 4$

$x_1 - 2x_2 + 3x_3 = -6$ $\qquad$ $A\underline{x} = \underline{B}$

$2x_1 + 3x_2 + x_3 = 7$

$$A = \begin{bmatrix} 1 & +1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

Let $A = LU$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{3} R_2$

$3 + \frac{4}{3}$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix} = U$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1/3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ 0 & -3 & 4\\ 0 & 0 & 13/3\end{bmatrix}$$

$AX = B$

$LU\underline{X} = B$

$L\underline{Y} = \underline{B}$; $\underline{Y} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

$$\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 2 & -1/3 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = \begin{bmatrix}4\\ -6\\ 7\end{bmatrix}$$

$\cancel{y_1 + y_2 - y_3 = 4}$

$\cancel{-3y_2 + 4y_3 = -6}$

$\cancel{\frac{10}{3}y_3 = 7}$

$y_1 = 4$

$y_1 + y_2 = -6 \Rightarrow y_2 = -6-4 = -10$

$2y_1 - \frac{1}{3}y_2 + y_3 = 7 \Rightarrow y_3 = \frac{-13}{3}$

$\cancel{\Rightarrow y_3 = \frac{21}{10}}$; $\cancel{-3y_2 + 4\cdot\frac{21}{10} = -6 \Rightarrow y_2 = \frac{24}{5}}$

$\cancel{y_1 + \frac{24}{5} - \frac{21}{10} = 4 \Rightarrow y_1 = \frac{13}{10}}$

$$U\underline{X} = \underline{Y} \Rightarrow \begin{bmatrix}1 & 1 & -1\\ 0 & -3 & 4\\ 0 & 0 & 13/3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}4\\ -10\\ -\frac{13}{3}\end{bmatrix}$$

$x_1 + x_2 - x_3 = 4$

$-3x_2 + 4x_3 = -10$

$\frac{13}{3}x_3 = -\frac{13}{3} \Rightarrow x_3 = -1$; $x_2 = \cancel{\frac{8}{5}}\ 2$, $x_1 = \cancel{\frac{-11}{10}}\ 1$

Prerequisites for spectral decomposition

Orthogonal ~~sets~~ vectors

Defn: We say that two vectors are orthogonal if they are perpendicular to each other, i.e. the dot product of the two vectors is zero.

Defn: We say that a set of vectors $\{v_1, v_2, \ldots, v_n\}$ are mutually orthogonal if every pair of vectors is orthogonal to o.

$$v_i \cdot v_j = 0, \text{ for all } i \neq j$$

Ex. The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \right\}$ is mutually orthogonal.

- $(1, 0, -1) \cdot (1, \sqrt{2}, 1) = 0$
  $(1, 0, -1) \cdot (1, -\sqrt{2}, 1) = 0$
  $(1, \sqrt{2}, 1) \cdot (1, -\sqrt{2}, 1) = 0$

Defn: A set of vectors S is orthonormal if every vector in S has magnitude 1 and the set of vectors are mutually orthogonal.

Ex. We checked that the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

are mutually orthogonal. The vectors however are are not normalized (i.e. vectors are not of magnitude 1).

Let
$$\underline{q}_1 = \frac{\underline{v}_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underline{q}_2 = \frac{\underline{v}_2}{\|v_2\|} = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

$$\underline{q}_3 = \frac{\underline{v}_3}{\|v_3\|} = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

The set of vectors $\{\underline{q}_1, \underline{q}_2, \underline{q}_3\}$ is <u>orthonormal</u>

How to obtain orthonormal set from a given set of vectors?

Let $\{\underline{u}_1\}$ be a set of vector

then a unit vector (~~orthogonal to~~ for) $\underline{u}_1$ is given by

$$\boxed{\underline{q}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|}}$$

e.g. $\underline{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$$\underline{q}_1 = \frac{1}{\sqrt{2^2+3^2}} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

Let $\{\underline{u}_1, \underline{u}_2\}$ be a set of vectors, then (by gram-schmidt method) second vector orthogonal to u_1 is given by

$$\boxed{\underline{v}_2 = \underline{u}_2 - (\underbrace{\underline{q}_1 \cdot \underline{u}_2}_{\text{or } q_1^T \cdot u_2})\, q_1}$$

e.g. $\left\{ \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}}_{u_2} \right\}$

$$\text{then } \underline{v}_2 = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} - \left(\underbrace{\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}_{\text{or } [2/\sqrt{13}\ \ 3/\sqrt{13}\ \ 0]} \cdot \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \right) \begin{pmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} - 4.16 \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \\ 0 \end{bmatrix} = \begin{bmatrix} 3.7 \\ -2.45 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\underline{q}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|}}$$

Let $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ be a set of vectors, then vector orthogonal to $\underline{u}_1 \,\&\, \underline{u}_2$ is obtained as

$$\boxed{\underline{v}_3 = \underline{u}_3 - (\underline{q}_1 \cdot \underline{u}_3)\, \underline{q}_1 - (\underline{q}_2 \cdot \underline{u}_3)\, \underline{q}_2}$$

$$\underline{q}_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|}$$

(unit vector along $\underline{v}_3$)

~~$\{\underline{v}_1, \underline{v}_2\}$~~

$\{\underline{q}_1, \underline{q}_2, \underline{q}_3\}$ → orthonormal Basis

x———————x

Diagonalization

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is diagonalizable
2. All eigenvalues of A are real
3. If λ is an eigenvalue of A with multiplicity K, then λ has K linearly independent eigenvectors. That is, the eigenspace of λ has dimension K.

Orthogonal Matrix:

A square matrix P is called orthogonal when it is invertible and when $P^{-1} = P^T$ (i.e. $PP^T = I$)

Eg.
$$P = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}; \quad P^{-1} = P^T = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

Properties

Theorem: An $n \times n$ matrix P is orthogonal if & only if its column vectors form an orthonormal set.

Theorem: Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\underline{x}_1$ & $\underline{x}_2$ are orthogonal if $\underline{x}_1 \cdot \underline{x}_2 = 0$

Orthogonal Diagonalization

A matrix A is orthogonally diagonalizable when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

Fundamental Theorem of Symmetric Matrices

Let A be an $n \times n$ matrix. Then A, is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric.

(To verify that A is diagonalizable & has real eigenvalues verify if A is symmetric)

Ex. Which matrices are orthogonally diagonalizable?

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \checkmark, \quad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix} \times$$

$$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \times, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \checkmark$$

(not even square matrix)

(By above fundamental theorem)

Orthogonal Diagonalization of a Symmetric Matrix

1. Find all eigenvalues of A and determine the multiplicity of each.

2. For each eigenvalue of multiplicity 1, find a unit eigenvector. (Find any eigenvector and then normalize it)

3. For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, then apply Gram-Schmidt orthonormalization process.

4. The results ~~in~~ of steps 2 and 3 produce an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal. (The main diagonal entries of D are the eigenvalues of A.)
{P orthogonally diagonalizes A}

6'

Ex. Find the matrix P that diagonalizes

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Soln First find the eigen values

C.E. is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = 2$$

Now find vector corresponding to $\lambda = -3$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{4} R_1$

$$\sim \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = t$

$$x_1 + \frac{1}{2} x_2 = 0 \Rightarrow x_1 = -\frac{1}{2} t$$

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$\therefore \underline{x} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = \frac{t}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore u_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (since we are having single vector (not a vector set $\begin{pmatrix} -t_1+t_2 \\ t_2 \end{pmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

$\therefore \underline{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore$ unit vector for $\underline{v}_1$ is $\underline{q}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

for $\lambda = 2$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In A

$R_1 \to (-1)R_1$

$\sim \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

let $x_2 = t$

$x_1 - 2x_2 = 0$

$\Rightarrow x_1 = 2t$

$\therefore \underline{x} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

6'''

$$u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{q}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$\therefore$ Finally using $\underline{q}_1, \underline{q}_2$ as column vectors, we obtain

$$P = [\,\underline{q}_1 \mid \underline{q}_2\,]$$

$$= \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

P is the required matrix which diagonalizes A.

<u>Verification</u> $P^T A P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$$= \begin{bmatrix} -3/\sqrt{5} & -6/\sqrt{5} \\ 4/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

which is a diagonal matrix ~~which~~ with main diagonal elements as the eigen values (-3 & 2).

Spectral Decomposition

Defn. The **Spectrum** of a matrix A is the set of eigenvalues of A.

e.g. $A = \begin{bmatrix} 1 & 5 & -2 \\ 5 & 4 & 5 \\ -2 & 5 & 1 \end{bmatrix}$ Spectrum $= \{9, -6, 3\}$

↑ eigenvalues of A

$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & -2 \\ 4 & -2 & 1 \end{bmatrix}$ Spectrum $= \{6, -3\}$

Theorem: (The spectrum theorem for symmetric matrices)

An $n \times n$ symmetric matrix A has the following properties.

a) A has n real eigenvalues (counting multiplicities) (check)

b) Dimension of the eigenspace for λ = multiplicity of λ as a root in characteristic equation.

c) Eigenspaces are mutually orthogonal.

d) A is orthogonally diagonalizable.

Spectral Decomposition

Let A be a symmetric matrix ~~of~~ with orthonormal diagonalization $A = PDP^{-1}$ with $P = [\underline{q}_1, \underline{q}_2, \ldots \underline{q}_n]$

$A = PDP^T$ ($\because$ when P has orthonormal column, then $P^T = P^{-1}$)

$$A = PDP^T = [\underline{q}_1 \ldots, \underline{q}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix}$$

$$= [\lambda_1 \underline{q}_1 + \cdots + \lambda_n \underline{q}_n] \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix}$$

$$= \lambda_1 \underline{q}_1 \underline{q}_1^T + \cdots + \lambda_n \underline{q}_n \underline{q}_n^T.$$

Definition: $A = \lambda_1 \underline{u}_1 \underline{u}_1^T + \cdots + \lambda_n \underline{u}_n \underline{u}_n^T$ is called the spectral decomposition of A because it decomposes A into pieces determined by its eigenvalues.

The matrix $\underline{u}_i \underline{u}_i^T$ is called projection matrix, for any $\underline{x} \in \mathbb{R}^n$.

~~$\underline{u}_i \underline{u}_i^T \underline{x} = \underline{u}_i (\underline{u}_i^T$~~

~~$\underline{u}_i \underline{u}_i^T \underline{x}$~~

Basis

[orthogonal vector, orthonormal Basis, Diagonalization (orthogonal matrix), spectral decomposition]

Ex. Find a spectral decomposition of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Soln: $\lambda_1, \lambda_2 \to$ eigenvalues

$q_1, q_2 \to$ unit orthogonal vectors

spectral decomposition

Find eigen values

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-2-\lambda)-4 = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -3, \text{ or } \lambda = 2$$

Find Eigen vector corresponding to $\lambda = -3$ (order does not matter)

$$\left[\begin{array}{cc|c} 4 & 2 & \cancel{0} \\ 2 & 1 & \cancel{0} \end{array}\right]$$

For reduced row echelon form

perform $R_1 \to \frac{1}{4} R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 1/2 & \cancel{0} \\ 2 & 1 & \cancel{0} \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 1/2 & \cancel{0} \\ 0 & 0 & \cancel{0} \end{array}\right]$$

Let $x_2 = t$

$x_1 + \frac{1}{2} x_2 = 0 \Rightarrow x_1 = \frac{1}{2} t$

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$$\therefore \underline{x} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = \frac{t}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\therefore u_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, (Since we are having single vector)

$\therefore \underline{v_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\underset{\text{(unit vector) for } v_1}{q_1} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

$R_1 \to -1R_1$

$$\sim \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = t$

$x_1 - 2x_2 = 0$

$\Rightarrow x_1 = 2t$

$$\therefore x = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{q_2} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

Spectral decomposition

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$$

$$\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} = -3 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$+ 2 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\boxed{\therefore A = -3 \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} + 2 \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}}$$ Ans

what spectral decomposition tell us?

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} x = -\left(3 \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} x\right) + 2\left(\begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \underline{x}\right)$$

$$= -3 \operatorname{proj}_{q_1} \underline{x} + 2 \operatorname{proj}_{q_2} x$$

(projection of $\underline{x}$ onto q_1) (proj. of $\underline{x}$ onto $\underline{q_2}$)

$$\begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} x_1 - \frac{2}{5} x_2 \\ -2/5\, x_1 + \frac{4}{5} x_2 \end{bmatrix}$$

$$\operatorname{proj}_{\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-\frac{1}{\sqrt{5}} x_1 + \frac{2}{\sqrt{5}} x_2}{1} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} x_1 - \frac{2}{5} x_2 \\ -2/5\, x_1 + \frac{4}{5} x_2 \end{bmatrix}$$

(c.)

(12)

(46)

Ex. Find spectral decomposion of $A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & -2 \\ 4 & -2 & 1 \end{bmatrix}$

Soln.

Find Eigen values

$$\begin{vmatrix} 1-\lambda & -2 & 4 \\ -2 & -2-\lambda & -2 \\ 4 & -2 & 1-\lambda \end{vmatrix} = -\lambda^3 + 27\lambda + 54 = 0$$

$$\Rightarrow -(\lambda-6)(\lambda+3)^2 = 0$$

$$\therefore \lambda = 6, -3$$

For each eigenvalue, find the corresponding eigen vector (eigen space)

$\lambda = 6$

$$\begin{bmatrix} -5 & -2 & 4 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -5 & -2 & 4 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} -1 & -4 & -1 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{bmatrix}$$

$R_1 \to (-1) R_1$

$$\begin{bmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & -18 & -9 \end{bmatrix}$$

$R_3 \to R_3/-18$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ let $x_3 = t$

$$x_2 + \frac{1}{2} x_3 = 0$$

$$x_1 - x_3 = 0$$

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$\therefore x_1 = t,\ x_2 = -\frac{1}{2}t,\ x_3 = t$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} = t_1\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \therefore \boxed{\underline{q_1} = \begin{bmatrix} 2/\sqrt{9} \\ -1/\sqrt{9} \\ 2/\sqrt{9} \end{bmatrix}} \text{ ——— } (1)$$

For $\lambda = -3$

$$\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

$R_1 \to \frac{1}{4}R_1$

$$\sim \begin{bmatrix} 1 & -1/2 & 1 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$; $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = K_1$, $x_2 = K_2$

and $x_1 - \frac{1}{2}x_2 + x_3 = 0$

$\Rightarrow x_1 = \frac{1}{2}K_2 - K_1$

$$\therefore \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}K_2 - K_1 \\ K_2 \\ K_1 \end{bmatrix}$$

$$= K_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + K_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= K_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + K_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now $u_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$; $u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$q_1 = \frac{u_1}{\|u_1\|}$ $\quad q_1 = \begin{bmatrix} 1/\sqrt5 \\ 2/\sqrt5 \\ 0 \end{bmatrix}$ —— (2)

$q_2 = \frac{v_2}{\|v_2\|}$, where $v_2 = u_2 - (q_1 \cdot u_2)\, q_1$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt5 \\ 2/\sqrt5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt5 \\ 2/\sqrt5 \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(-1/\sqrt5 + 0 + 0 \right) \begin{bmatrix} 1/\sqrt5 \\ 2/\sqrt5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/5 \\ -2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

$$\therefore\ q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{45}/5} \begin{bmatrix} -4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$ —— (3)

∴ From (1), (2) & (3) we can form the matrix P as

$$P = \begin{bmatrix} 2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ -1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

∴ Spectrum decomposition is given as follows:

$$A = \lambda_1 \underline{q}_1 \underline{q}_1^T + \lambda_2 \underline{q}_2 \underline{q}_2^T + \lambda_3 \underline{q}_3 \underline{q}_3^T$$

$$\therefore A = 6 \begin{bmatrix} 2/\sqrt{9} \\ -1/\sqrt{9} \\ 2/\sqrt{9} \end{bmatrix}_{3\times1} \begin{bmatrix} 2/\sqrt{9} & -1/\sqrt{9} & 2/\sqrt{9} \end{bmatrix}_{1\times3} + (-3) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix}$$

$$+ (-3) \begin{bmatrix} -4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} -4/\sqrt{45} & 2/\sqrt{45} & 5/\sqrt{45} \end{bmatrix}$$

Ex:

Find the spectral decomposition of

$$A = \begin{bmatrix} 1 & 5 & -2 \\ 5 & 4 & 5 \\ -2 & 5 & 1 \end{bmatrix}$$

Singular Value Decomposion (SVD)

Consider a matrix A of order m×n. ~~as A~~

Using SVD the matrix A can be written as

$A = U \Sigma V^T$

U & V are orthogonal matrices

Σ is a diagonal matrix

entries of Σ are called singular values (σ_1, σ_2 ----)

where $\sigma_1 \geq \sigma_2 \geq ---- \geq \sigma_r$.

r is the rank of A.

Let

$A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$

$= V \Sigma^T \underbrace{U^T U}_{} \Sigma V^T$

$= V \Sigma^T \Sigma V^T$ (∵ U is orthogonal matrix i.e. $U^T U = I$)

$A^T A = V \Sigma^2 V^T$ (∵ Σ is a diagonal matrix i.e. $\Sigma = \Sigma^T$)

• V has eigenvector of $A^T A$. ——— (1)

Also

$\sqrt{A^T A} = V \Sigma V^T$

• Σ has eigenvalues of $\sqrt{A^T A}$ ——— (2)

(Σ^2 is eigenvalues of $A^T A$. Σ is eigen values of $\sqrt{A^T A}$. i.e. if eigen value of $(A^T A)^{1/2} = (\Sigma^2)^{1/2} = \Sigma$)

Now

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T$$

$$= U\Sigma V^T V \Sigma^T U^T$$

$$= U\Sigma\Sigma^T U^T$$

$$= U\Sigma^2 U^T$$

$\therefore$ U contains eigenvectors of AA^T. —— 3

Summary

Find SVD of $A_{m\times n}$

$A = U\Sigma V^T$ (V is orthogonal)

(1) $V_{n\times n}$ contains eigenvectors of A^TA. (Also we have to normalize it)

(2) $\Sigma_{m\times n}$ contains eigenvalues of $\sqrt{A^TA}$.

(3) U contains eigenvectors of AA^T. (")
(U is orthogonal)

REMARK 1: "The matrices U and V found using this procedure are <u>not unique</u>. We also note that changing the signs of the column vectors in U and V also produces orthogonal matrices that diagonalize AA^T and A^TA. <u>As a result, finding an SVD of A may require changing the signs of certain columns of U or V.</u>"

(Reference: Intr. to Lin. Algebra with App. by Jim DeFranza & D. Gagliardi)

Ex. Find SVD of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}_{2\times 3}$$

Soln. $A = U\Sigma V^T$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now to find eigen values and eigen vectors of A^TA

C.E. is $|\overset{A^TA}{A} - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\{(1-\lambda)^2 - 0\} - 0\{\ \} + 1\{(1-\lambda)\} = 0$$

$\Rightarrow (1-\lambda)^3 - (1-\lambda) = 0 \Rightarrow (1-\lambda)\{(1-\lambda)^2 - 1\} = 0$

$\Rightarrow (1-\lambda)\{1+\lambda^2 - 2\lambda - 1\} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 2\lambda) = 0$

$\Rightarrow \lambda(1-\lambda)(\lambda-2) = 0$

$\Rightarrow \lambda = 0, 1, 2$

Corresponding to $\lambda = 0$

$[A - \lambda I]\underline{x}_1 = \underline{0}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$

$x_2 = 0$

$x_1 + x_3 = 0 \Rightarrow x_1 = -t$

$$\therefore \underline{x}_1 = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Normalizing

$$\underline{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

for $\lambda = 1$

$[A - \lambda I]\underline{x}_2 = \underline{0}$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_2 = t$

$x_3 = 0$

$x_1 = 0$

$$\underline{x}_2 = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Normalizing

$$\underline{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For
$\lambda = 2$

$[A - \lambda I]\, \underline{x_3} = \underline{0}$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$R_1 \to (-1) R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to (-1) R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = t$

$x_2 = 0$

$x_1 - x_3 = 0$

$\Rightarrow x_1 = t$

$$\underline{x_3} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalizing.

$$\underline{q_3} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Now as V has eigenvectors of $A^T A$ (h in sequence of column in ~~increasing~~ decreasing order of eigen values) ($\because$ in Σ values of singular values are sorted in ~~incr~~ decreasing order)

$$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Now Σ contains eigen values of $\sqrt{A^T A}$ (Dimension of Σ will be 2×3, same as A)

$$\therefore \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ (Since only 2 columns were linearly independent)

(c)

U contains eigenvectors of AA^T.

$$AA^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Now for the eigen values of AA^T, we have C.E as

$$|(AA^T) - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 1$$

for $\lambda = 1$, eigen vectors can be found as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

let $x_2 = t$

$x_1 = 0$

$$\therefore X_4 = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Normalizing

$$q_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow (-1) R_1$

$$\sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

free variable P.C

let $x_1 = t$

$x_2 = 0$

$$\therefore \underline{x_5} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Normalizing.

$$q_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore$ U can be written as

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Singular value decomposition of A is

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

To justify the equality changing U to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (See Remark (1)) we get

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Quadratic form

In System of linear equations $A\underline{x}=\underline{b}$

e.g. $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}x=\begin{bmatrix}5\\ 6\end{bmatrix} \Rightarrow \begin{matrix} x_1+2x_2=5 \\ 3x_1+4x_2=6\end{matrix}$ } linear eqns since degrees of ~~~~ variables are at most one

Defn. A quadratic form is a polynomial in n-variables where the terms are all degree 2.

e.g. $Q_1=x_1^2$ (quadratic form in 1 variable)

$Q_2=x_1^2+x_2^2$ (" " " 2 ")

$Q_3=x_1^2+\underbrace{2x_1x_2}_{\text{cross-product term}}+3x_2^2$ (" 3 ")

$Q_4=\underbrace{x_1^2}_{\text{non-cross product term}}+x_3^2\underbrace{-5x_1x_2+7x_2x_3}_{\text{cross product term}}$

Remark. Any quadratic term can be expressed as $Q(\underline{x})=\underline{x}^TA\underline{x}$ where A is an $n\times n$ symmetric matrix. This matrix A is called matrix of the quadratic form

Ex. What is the quadratic form?

$$Q(\underline{x})=\underline{x}^T I_2\underline{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=x_1^2+x_2^2$$

$$Q(\underline{x}) = \underline{x}^T I_3 \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + x_2^2 + x_3^2$$

$$Q(\underline{x}) = \underline{x}^T \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Symm. matrix

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 - 3x_2 \\ -3x_1 + 4x_2 \end{bmatrix} = x_1(2x_1 - 3x_2) + x_2(-3x_1 + 4x_2)$$

$$= 2x_1^2 - 6x_1x_2 + 4x_2^2$$

$$Q(\underline{x}) = \underline{x}^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

$$= x_1(ax_1 + bx_2) + x_2(bx_1 + cx_2)$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

Coeff. of x_1^2 & x_2^2 are elements at ~~at~~ position (1,1) & (2,2) resp. & crossproduct is the sum of off-diagonal terms.

(non-cross product term)

$$Q(\underline{x}) = \underline{x}^T \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ bx_1 + dx_2 + ex_3 \\ cx_1 + ex_2 + fx_3 \end{bmatrix}$$

$$= x_1(ax_1 + bx_2 + cx_3) + x_2(bx_1 + dx_2 + ex_3) + x_3(cx_1 + ex_2 + fx_3)$$

$$= ax_1^2 + 2b\,x_1x_2 + 2c\,x_1x_3 + dx_2^2 + 2e\,x_2x_3 + f\,x_3^2$$

Non-cross-product term have coeff. a, d & f are diagonal elements.

Cross-product term $2b\,x_1x_3$ is the sum of $(1,2)$ entry & $(2,1)$ entry.

c-p term $2c\,x_1x_3$ is the sum of $(1,3)$ & $(3,1)$ entry

C-p ,, $2e\,x_2x_3$,, ,, ,, $(2,3)$ & $(3,2)$,, .

Ex: Find the matrix of the following quadratic form.

$Q(\underline{x}) = x_1^2 + 2x_1x_2 + 3x_2^2$

Ex. For the following Q.F. write associated matrix

i) $2x^2 - 4xy$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$$

ii) $x^2 + y^2$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(iii) $2xy$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(iv) $x^2 + 2xy - y^2$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(v) $5x_1^2 + x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 - 4x_2x_3$

$$A = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix}$$

Definiteness of Quadratic forms

For quadratic form of one variable $Q(x) = a \cdot x^2$

If $a > 0$ then $Q(x) > 0$ for each nonzero x.

If $a < 0$ then $Q(x) < 0$ for each nonzero x.

So the sign of the coefficient 'a' determines the sign of one variable quadratic forms.

The quadratic form $Q(x, y) = x^2 + y^2$ is +ve for all nonzero arguments (x, y). Such forms are called +ve definite.

The quadratic form $Q(x, y) = -x^2 - y^2$ is -ve for all nonzero arguments (x, y). Such forms are called ~~+ve def~~ -ve definite

The quadratic form $Q(x,y) = (x-y)^2$ is <u>non-negative</u>. This means that $Q(x,y) = (x-y)^2$ is either positive or zero for nonzero arguments. So (for e.g. $x=3, y=3$). Such forms are called +ve ~~semi~~ semidefinite.

The quadratic form $Q(x,y) = -(x-y)^2$ is <u>non positive</u>. This means that $Q(x,y) = -(x-y)^2$ is either -ve or zero for nonzero arguments. Such forms are called negative semidefinite.

The quadratic form $Q(x,y) = x^2 - y^2$ is called indefinite since it can take both +ve or -ve values, for e.g.

$Q(3,1) = 9-1 = 8 > 0$, $Q(1,3) = 1-9 = -8 < 0$.

<u>Definitions</u>

<u>Defn.</u> A quadratic form $Q(x) = x'Ax$ (equivalently a <u>Symmetric</u> matrix A) is (A should be a Symm matrix i.e. we have to change it to Symm form)

(a) +ve definite if $Q(x) > 0$ $\forall x \neq 0 \in \mathbb{R}^n$.

(b) +ve semidefinite if $Q(x) \geq 0$ $\forall x \neq 0 \in \mathbb{R}^n$.

(c) -ve definite if $Q(x) < 0$ $\forall x \neq 0 \in \mathbb{R}^n$

(d) -ve semidefinite if $Q(x) \leq 0$ $\forall x \neq 0 \in \mathbb{R}^n$.

(e) indefinite if $Q(x) > 0$ for some x and $Q(x) < 0$ for some other x.

Testing for Definiteness

[Principal minors: Let A be an $n\times n$ matrix. A $K\times K$ submatrix of A obtained by deleting any $n-K$ columns and the same $n-K$ rows from A is called a K^{th} order principal submatrix of A. The determinant of a $K\times K$ principal submatrix is called a K^{th} order principal minor of A.

Ex. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- There is one 3^{rd} order principal minor of A, which is $|A|$
- There are three second order principal minors

$A_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $A_2 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $A_3 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

(by deleting 3^{rd} row & col.) del. 2^{nd} row & 2^{nd} col. del. first 1^{st} row & col.

- There are also three first order principal minors:

a_{11} (by deleting the last two rows & columns)

a_{22} (,, ,, ,, first & last ,, ,, ,,)

a_{33} (,, ,, ,, first two rows & columns)

Leading principal minors: The leading principal minor is the determinant of the leading principal submatrix obtained by deleting the last $n-K$ rows & columns of an $n\times n$ matrix A.

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

leading principal minors

$A_1 = |a_{11}|$, $A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Scalar i.e. element a_{11}

we can test for the definitness of the matrix in the following manner.

1. ~~Let A~~ ~~consider~~ A quadratic form $Q(x) = x'Ax$ (equivalently a symmetric matrix A) is

1: +ve definite iff all of its leading principal minors of A are strictly +ve. (i.e. $|A_1| > 0, |A_2| > 0, \ldots$)

2: -ve definite iff all of its leading principal minors of A ~~strictly~~ are alternate in sign
$|A_1| < 0, |A_2| > 0, |A_3| < 0, \ldots$
(starting from $|A_1| < 0$)

3. indefinite if it does not fit either of the above sign patterns.
e.g. if $|A_1| > 0, |A_2| < 0$
if $|A_1| < 0, |A_2| < 0$.

4. +ve semidefinite iff every principal minor of A is ≥ 0.

5. -ve semidefinite iff every principal minor of A of odd order is ≤ 0 & every principal minor of even order is ≥ 0.