GRAM SCHMIDT PROCESS

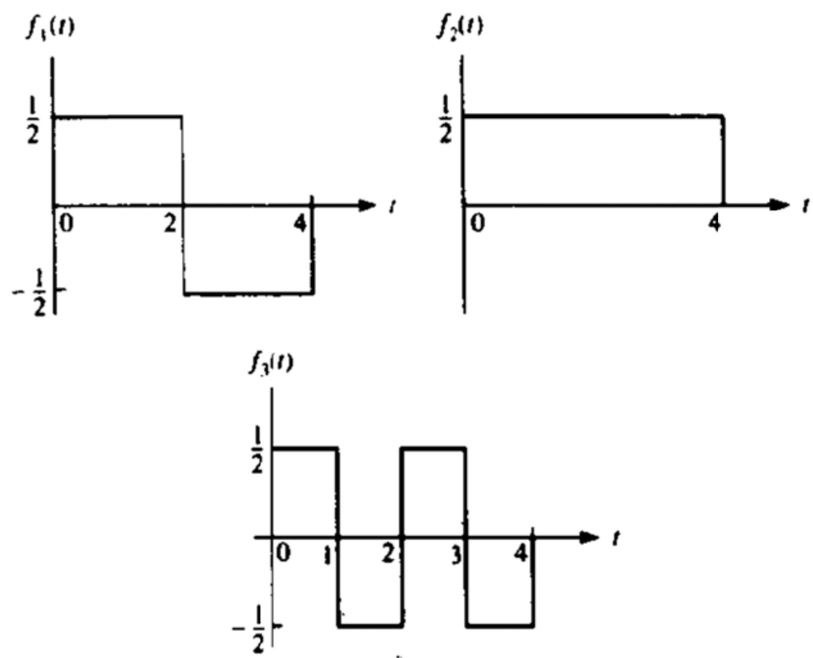
4-10 Consider the three waveforms $f_n(t)$ shown in Fig. P4-10.

a Show that these waveforms are orthonormal.

b Express the waveform $x(t)$ as a weighted linear combination of $f_n(t)$, $n = 1, 2, 3$, if

$$x(t) = \begin{cases} -1 & (0 \leq t < 1) \\ 1 & (1 \leq t < 3) \\ -1 & (3 \leq t < 4) \end{cases}$$

and determine the weighting coefficients.



(a) To show that the waveforms $f_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t) f_n(t) dt = 0, \qquad m \neq n$$

Clearly:

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \int_0^4 f_1(t) f_2(t) dt \\
&= \int_0^2 f_1(t) f_2(t) dt + \int_2^4 f_1(t) f_2(t) dt \\
&= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\
&= 0
\end{aligned}$$

Similarly:

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t) dt = \int_0^4 f_1(t) f_3(t) dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

and :

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t) dt = \int_0^4 f_2(t) f_3(t) dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1, 2, 3$$

Hence, they are orthonormal.

(b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t) dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t) f_1(t) dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t) f_2(t) dt = \frac{1}{2}\int_0^4 x(t) dt = 0 \\
x_3 &= \int_0^4 x(t) f_3(t) dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

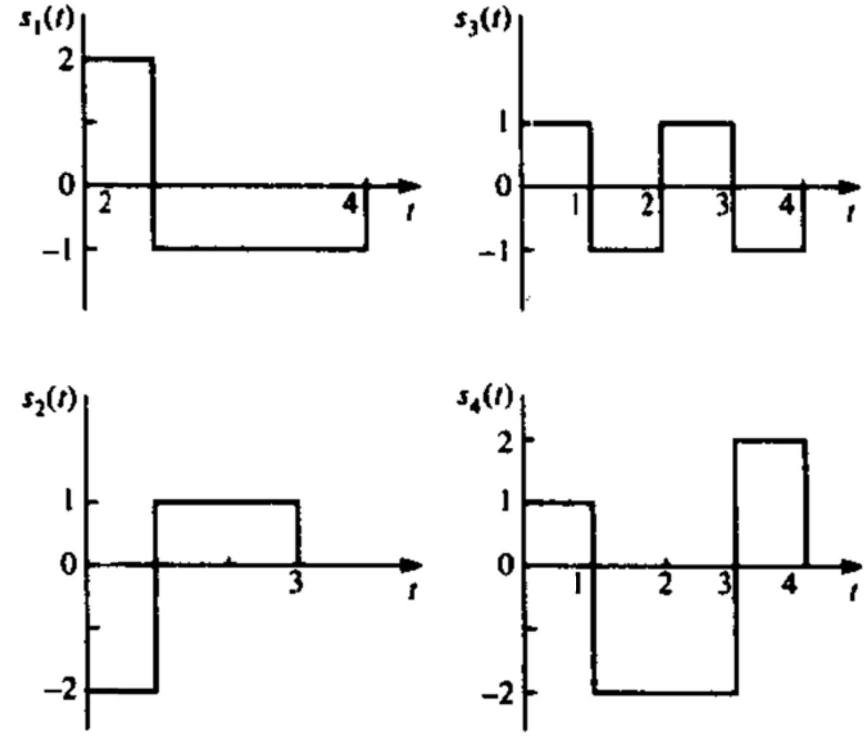
As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

4-11 Consider the four waveforms shown in Fig. P4-11.

a Determine the dimensionality of the waveforms and a set of basis functions.

b Use the basis functions to represent the four waveforms by vectors $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, and $\mathbf{s}_4$.

c Determine the minimum distance between any pair of vectors.



(a) As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

(b) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

(c) The distance between the first and the second vector is:

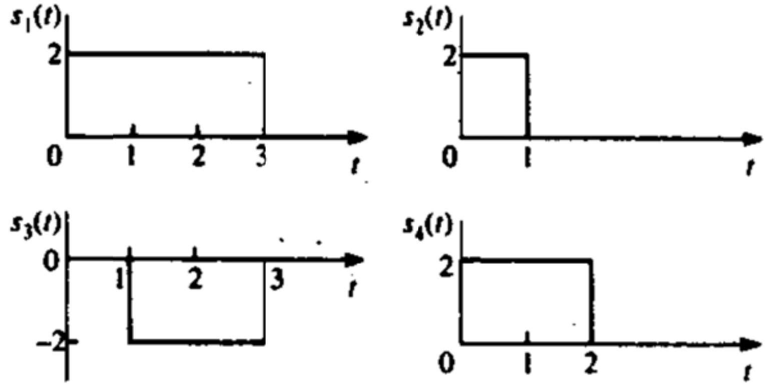
$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

4-12 Determine a set of orthonormal functions for the four signals shown in Fig. P4-12.



As a set of orthonormal functions we consider the waveforms

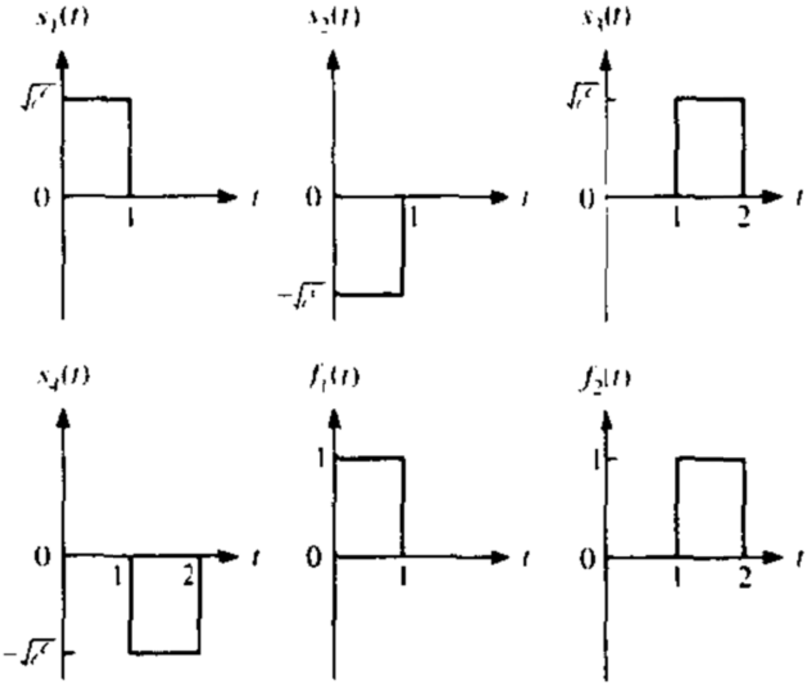
$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

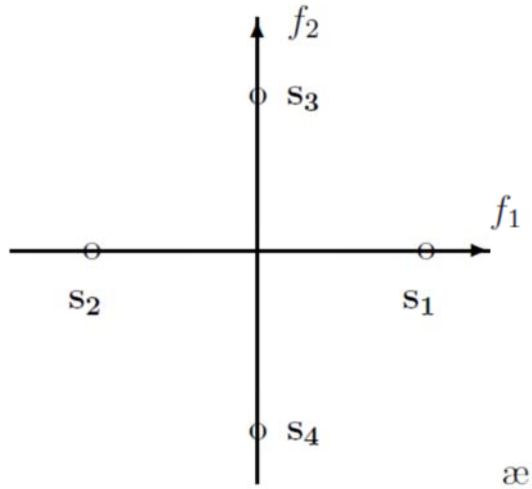
$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

4-18 Determine the signal space representation of the four signals $s_k(t)$, $k = 1, 2, 3, 4$, shown in Fig. P4-18, by using as basis functions the orthonormal functions $f_1(t)$ and $f_2(t)$. Plot the signal space diagram and show that this signal set is equivalent to that for a four-phase PSK signal.



$$\mathbf{s}_1 = \left(\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_2 = \left(-\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_3 = \left(0, \sqrt{\mathcal{E}}\right)$$
$$\mathbf{s}_4 = \left(0, -\sqrt{\mathcal{E}}\right)$$



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As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

MATCHED FILTER AND CORRELATOR RECEIVER

5-1 A matched filter has the frequency response

$$H(f) = \frac{1 - e^{-j2\pi fT}}{j2\pi f}$$

a Determine the impulse response $h(t)$ corresponding to $H(f)$.
b Determine the signal waveform to which the filter characteristic is matched.

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

where $sgn(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

(b) The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

5-2 Consider the signal

$$s(t) = \begin{cases} (A/T)t\cos 2\pi f_c t & (0 \leq t \leq T) \\ 0 & \text{(otherwise)} \end{cases}$$

a Determine the impulse response of the matched filter for the signal.
b Determine the output of the matched filter at $t = T$.
c Suppose the signal $s(t)$ is passed through a correlator that correlates the input $s(t)$ with $s(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in (b).

(a) The impulse response of the matched filter is :

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

(b) The output of the matched filter at $t = T$ is :

$$\begin{aligned}
g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\
&\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}$$

(c) The output of the correlator at $t = T$ is :

$$\begin{aligned}
q(T) &= \int_0^T s^2(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)d\tau
\end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

5-4 A binary digital communication system employs the signals

$$s_0(t) = 0, \quad 0 \leqslant t \leqslant T$$
$$s_1(t) = A, \quad 0 \leqslant t \leqslant T$$

for transmitting the information. This is called *on–off signaling*. The demodulator cross-correlates the received signal $r(t)$ with $s(t)$ and samples the output of the correlator at $t = T$.

a Determine the optimum detector for an AWGN channel and the optimum threshold, assuming that the signals are equally probable.

b Determine the probability of error as a function of the SNR. How does on–off signaling compare with antipodal signaling?

(a) The correlation type demodulator employes a filter :

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{T}} & 0 \leq \mathrm{t} \leq \mathrm{T} \\ 0 & \mathrm{o.w} \end{array} \right\}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable witl variance :

$$\sigma_n^2 \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \begin{array}{c} s_0 \\ > \\ < \\ s_1 \end{array} 1$$

or equivalently :

$$r \begin{array}{c} s_1 \\ > \\ < \\ s_0 \end{array} \frac{1}{2} A\sqrt{T}$$

The optimum threshold is $\frac{1}{2} A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}$$

where

$$\mathrm{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

5-6 Consider the equivalent lowpass (complex-valued) signal $s_l(t)$, $0 \leq t \leq T$, with energy

$$\mathscr{E} = \frac{1}{2}\int_0^T |s_l(t)|^2\,dt$$

Suppose that this signal is corrupted by AWGN, which is represented by its equivalent lowpass form $z(t)$. Hence, the observed signal is

$$r_l(t) = s_l(t) + z(t), \quad 0 \leq t \leq T$$

The received signal is passed through a filter that has an (equivalent lowpass) impulse response $h_l(t)$. Determine $h_l(t)$ so that the filter maximizes the SNR at its output (at $t = T$).

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E\left[|n(T)|^2\right]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{array}{rcl} E\left[|n(T)|^2\right] & = & \int_0^T \int_0^T E\left[z(a)z^*(b)\right] h_l(T-a)h_l^*(T-b)dadb \\ & = & 2N_0 \int_0^T |h_l(T-t)|^2 \, dt \end{array}$$

so we want to maximize :

$$SNR = \frac{\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2}{2N_0 \int_0^T |h_l(T-t)|^2 \, dt}$$

From Schwartz inequality :

$$\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2 \le \int_0^T |h_l(T-t)|^2 \, dt \int_0^T |s_l(t)|^2 \, dt$$

Hence :

$$SNR \le \frac{1}{2N_0} \int_0^T |s_l(t)|^2 \, dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$

5-8 The two equivalent lowpass signals shown in Fig. P5-8 are used to transmit a binary sequence over an additive white gaussian noise channel. The received signal can be expressed as

$$r_i(t) = s_i(t) + z(t), \quad 0 \leqslant t \leqslant T, \quad i = 1, 2$$

where $z(t)$ is a zero-mean gaussian noise process with autocorrelation function

$$\phi_{zz}(\tau) = \tfrac{1}{2}E[z^*(t)z(t+\tau)] = N_0\delta(\tau)$$

a Determine the transmitted energy in $s_1(t)$ and $s_2(t)$ and the cross-correlation coefficient ρ_{12}.

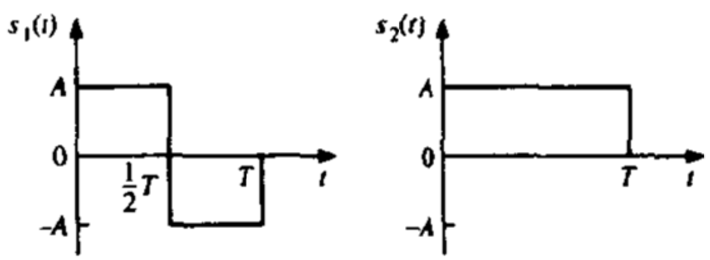
b Suppose the receiver is implemented by means of coherent detection using two matched filters, one matched to $s_1(t)$ and the other to $s_2(t)$. Sketch the equivalent lowpass impulse responses of the matched filters.

c Sketch the noise-free response of the two matched filters when the transmitted signal is $s_2(t)$.

d Suppose the receiver is implemented by means of two cross-correlators (multipliers followed by integrators) in parallel. Sketch the output of each integrator as a function of time for the interval $0 \leqslant t \leqslant T$ when the transmitted signal is $s_2(t)$.

e Compare the sketches in (c) and (d). Are they the same? Explain briefly.

f From your knowledge of the signal characteristics, give the probability of error for this binary communications system.

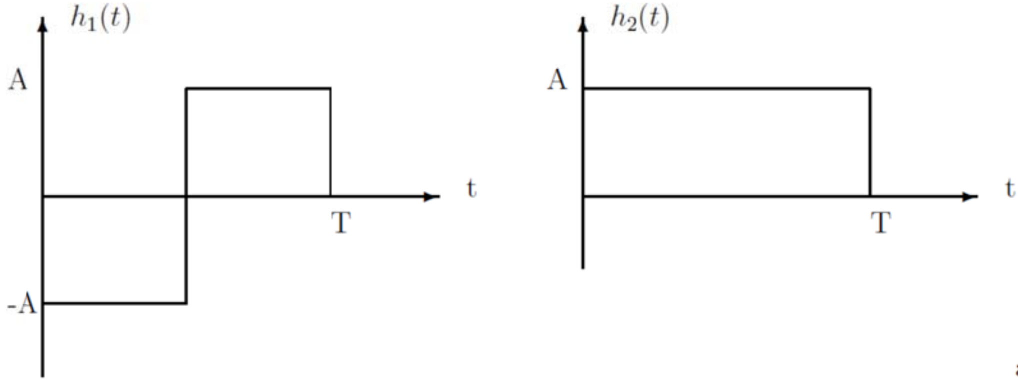


(a) Since the given waveforms are the equivalent lowpass signals :

$$\mathcal{E}_1 = \frac{1}{2}\int_0^T |s_1(t)|^2\,dt = \frac{1}{2}A^2\int_0^T dt = A^2T/2$$
$$\mathcal{E}_2 = \frac{1}{2}\int_0^T |s_2(t)|^2\,dt = \frac{1}{2}A^2\int_0^T dt = A^2T/2$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}}\int_0^T s_1(t)s_2^*(t)dt = 0.$

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T - t)$. The following figure shows $h_i(t)$:



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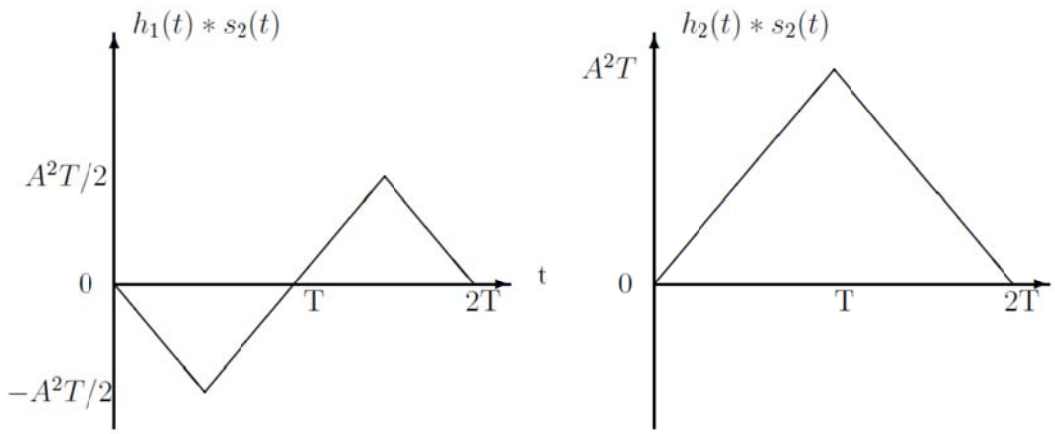
(c)



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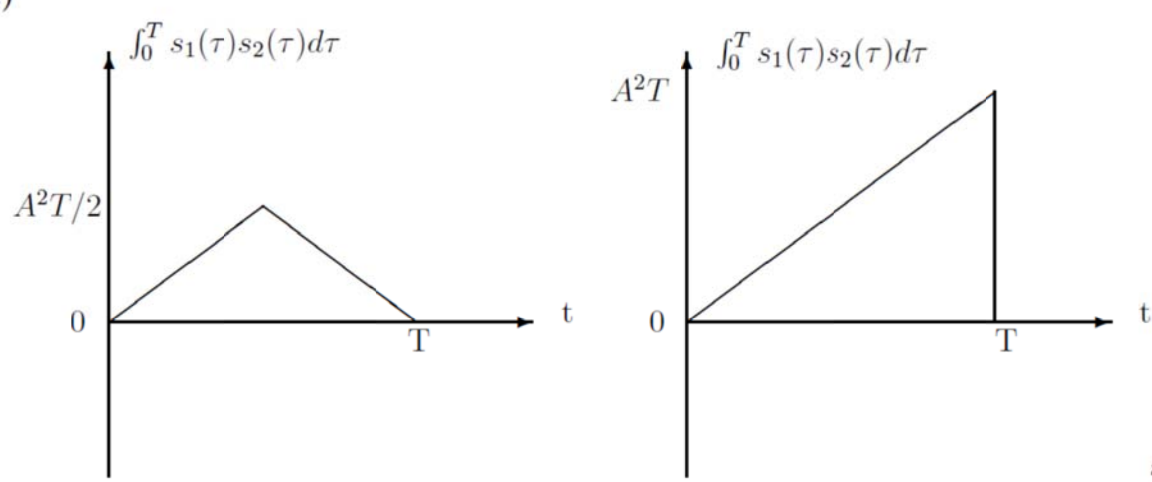
(d)



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(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

PROBABILITY OF ERROR

5-10 A ternary communication system transmits one of three signals, $s(t)$, 0, or $-s(t)$, every T seconds. The received signal is either $r_l(t) = s(t) + z(t)$, $r_l(t) = z(t)$, or $r_l(t) = -s(t) + z(t)$, where $z(t)$ is white gaussian noise with $E[z(t)] = 0$ and $\phi_{zz}(\tau) = \frac{1}{2}E[z(t)z^*(\tau)] = N_0\delta(t-\tau)$. The optimum receiver computes the correlation metric

$$U = \text{Re}\left[\int_0^T r(t)s^*(t)\,dt\right]$$

and compares U with a threshold A and a threshold $-A$. If $U > A$, the decision is made that $s(t)$ was sent. If $U < -A$, the decision is made in favor of $-s(t)$. If $-A < U < A$, the decision is made in favor of 0.

a Determine the three conditional probabilities of error: P_e given that $s(t)$ was sent, P_e given that $-s(t)$ was sent, and P_e given that 0 was sent.

b Determine the average probability of error P_e as a function of the threshold A, assuming that the three symbols are equally probable a priori.

c Determine the value of A that minimizes P_e.

(a) $U = Re\left[\int_0^T r(t)s^*(t)dt\right]$, where $r(t) = \left\{ \begin{array}{c} s(t)+z(t) \\ -s(t)+z(t) \\ z(t) \end{array} \right\}$ depending on which signal was sent. If we assume that $s(t)$ was sent :

$$U = Re\left[\int_0^T s(t)s^*(t)dt\right] + Re\left[\int_0^T z(t)s^*(t)dt\right] = 2E + N$$

where $E = \frac{1}{2}\int_0^T s(t)s^*(t)dt$, and $N = Re\left[\int_0^T z(t)s^*(t)dt\right]$ is a Gaussian random variable with zero mean and variance $2EN_0$ (as we have seen in Problem 5.7). Hence, given that $s(t)$ was sent, the probability of error is :

$$P_{e1} = P(2E + N < A) = P(N < -(2E - A)) = Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right)$$

When $-s(t)$ is transmitted : $U = -2E + N$, and the corresponding conditional error probability is :

$$P_{e2} = P(-2E + N > -A) = P(N > (2E - A)) = Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right)$$

and finally, when 0 is transmitted : $U = N$, and the corresponding error probability is :

$$P_{e3} = P(N > A \text{ or } N < -A) = 2P(N > A) = 2Q\left(\frac{A}{\sqrt{2N_0E}}\right)$$

(b)

$$P_e = \frac{1}{3}(P_{e1} + P_{e2} + P_{e3}) = \frac{2}{3}\left[Q\left(\frac{2E - A}{\sqrt{2N_0E}}\right) + Q\left(\frac{A}{\sqrt{2N_0E}}\right)\right]$$

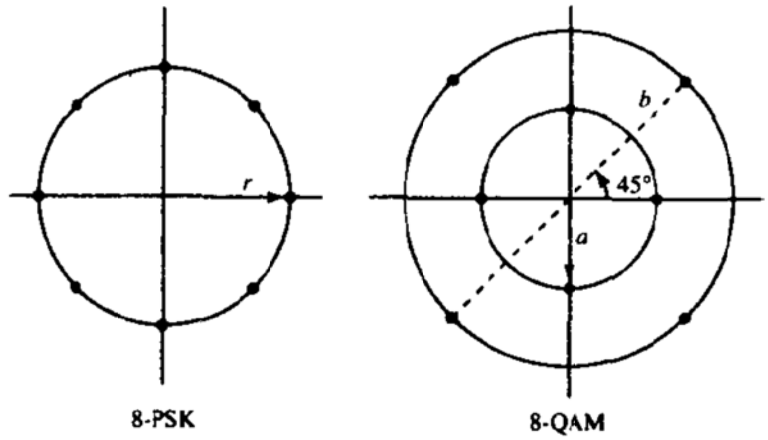
(c) In order to minimize P_e :

$$\frac{dP_e}{dA} = 0 \Rightarrow A = E$$

where we differentiate $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)dt$ with respect to x, using the Leibnitz rule : $\frac{d}{dx}\left(\int_{f(x)}^\infty g(a)da\right) = -\frac{df}{dx}g(f(x))$. Using this threshold :

$$P_e = \frac{4}{3}Q\left(\frac{E}{\sqrt{2N_0E}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{E}{2N_0}}\right)$$

5-16 Consider the octal signal point constellations in Fig. P5-16.



a The nearest-neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles.

b The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.

c Determine the average transmitter powers for the two signal constellations and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable.)

(a) Consider the QAM constellation of Fig. P5-16. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

(b) If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

(c) The average transmitted power of the PSK constellation is:

$$P_{\rm PSK} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\rm PSK} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\rm QAM} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\rm QAM} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\rm PSK}}{P_{\rm QAM}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

5-17 Consider the 8-point QAM signal constellation shown in Fig. P5-16.

a Is it possible to assign three data bits to each point of the signal constellation such that nearest (adjacent) points differ in only one bit position?

b Determine the symbol rate if the desired bit rate is 90 Mbits/s.

(a) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P5-16. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$\mathrm{B} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad \mathrm{C} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

(b) Since each symbol conveys 3 bits of information, the resulted symbol rate is :

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

5-18 Suppose that binary PSK is used for transmitting information over an AWGN with a power spectral density of $\frac{1}{2}N_0 = 10^{-10}$ W/Hz. The transmitted signal energy is $\mathscr{E}_b = \frac{1}{2}A^2T$, where T is the bit interval and A is the signal amplitude. Determine the signal amplitude required to achieve an error probability of 10^{-6} when the data rate is (a) 10 kbits/s, (b) 100 kbits/s, and (c) 1 Mbit/s.

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

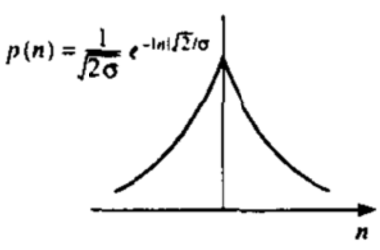
5-19 Consider a signal detector with an input

$$r = \pm A + n$$

where $+A$ and $-A$ occur with equal probability and the noise variable n is characterized by the (Laplacian) pdf shown in Fig. P5-19.

a Determine the probability of error as a function of the parameters A and σ

b Determine the SNR required to achieve an error probability of 10^{-5}. How does the SNR compare with the result for a Gaussian pdf?



(a) The PDF of the noise n is :

$$p(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

where $\lambda = \frac{\sqrt{2}}{\sigma}$ The optimal receiver uses the criterion :

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \begin{array}{c} A \\ > \\ < \\ -A \end{array} 1 \Longrightarrow r \begin{array}{c} A \\ > \\ < \\ -A \end{array} 0$$

The average probability of error is :

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\ &= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\ &= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\ &= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\ &= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{\frac{-\sqrt{2}A}{\sigma}} \end{aligned}$$

(b) The variance of the noise is :

$$\begin{aligned} \sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\ &= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2 \end{aligned}$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2SNR}}$$

For $P(e) = 10^{-5}$ we obtain:

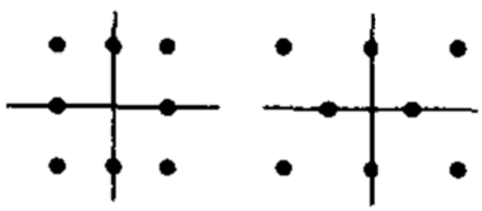
$$\ln(2 \times 10^{-5}) = -\sqrt{2SNR} \Longrightarrow SNR = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{SNR}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{SNR} = 4.26$ and therefore SNR $= 18.1476 = 12.594$ dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

5-20 **Consider the two 8-point QAM signal constellations shown in Fig. P5-20. The minimum distance between adjacent points is $2A$. Determine the average transmitted power for each constellation, assuming that the signal points are equally probable. Which constellation is more power-efficient?**



(a) (b)

The constellation of Fig. P5-20(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is:

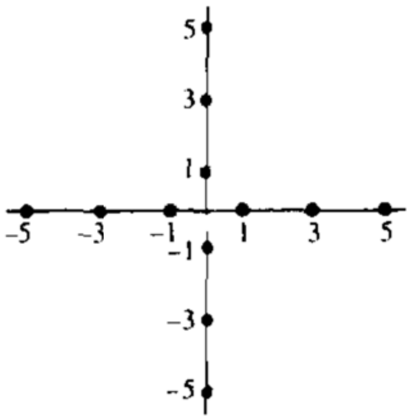
$$P_a = \frac{1}{8}\left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2\right] = 6A^2$$

The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is:

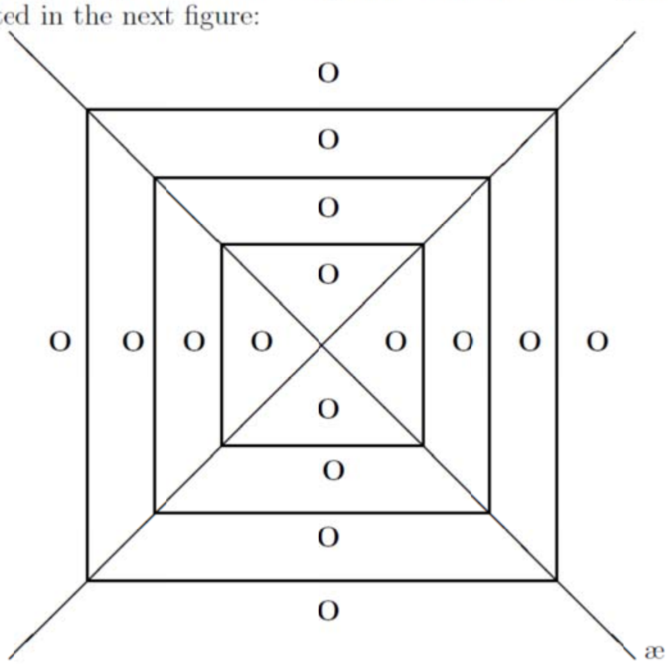
$$P_b = \frac{1}{8}\left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

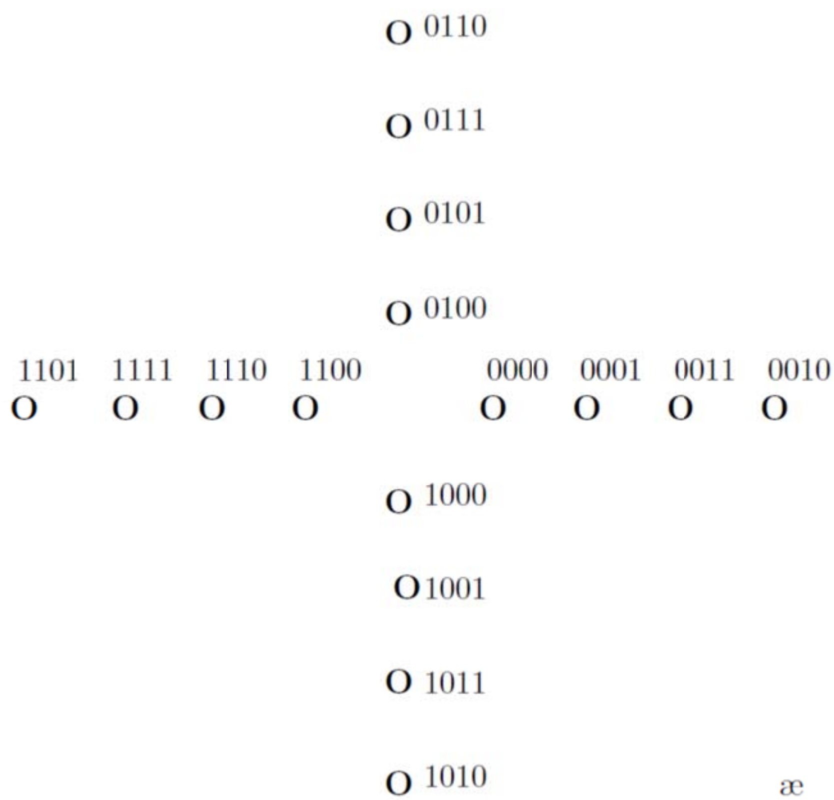
5-21 For the QAM signal constellation shown in Fig. P5-21, determine the optimum decision boundaries for the detector, assuming that the SNR is sufficiently high so that errors only occur between adjacent points.



The optimum decision boundary of a point is determined by the perpedicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:



5-22 Specify a Gray code for the 16-QAM signal constellation shown in Fig. P5-21.



5-24 Three messages m_1, m_2, and m_3 are to be transmitted over an AWGN channel with noise power spectral density $\frac{1}{2}N_0$. The messages are

$$s_1(t) = \begin{cases} 1 & (0 \leqslant t \leqslant T) \\ 0 & (\text{otherwise}) \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & (0 \leqslant t \leqslant \frac{1}{2}T) \\ -1 & (\frac{1}{2}T \leqslant 0 \leqslant T) \\ 0 & (\text{otherwise}) \end{cases}$$

a What is the dimensionality of the signal space?
b Find an appropriate basis for the signal space. [*Hint:* You can find the basis without using the Gram–Schmidt procedure.]
c Draw the signal constellation for this problem.
d Derive and sketch the optimal decision regions R_1, R_2, and R_3.
e Which of the three messages is more vulnerable to errors and why? In other words, which of $P(\text{error} \mid m_i \text{ transmitted})$, $i = 1, 2, 3$, is larger?

(a) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

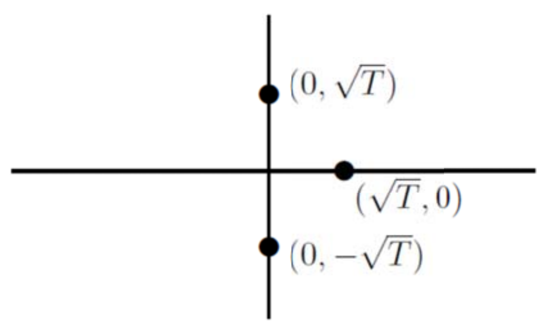
(b) As a basis of the signal space we consider the functions:

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \qquad f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is:

$$\begin{aligned} \mathbf{m}_1 &= [\sqrt{T}, \ 0] \\ \mathbf{m}_2 &= [0, \ \sqrt{T}] \\ \mathbf{m}_3 &= [0, \ -\sqrt{T}] \end{aligned}$$

(c) The signal constellation is depicted in the next figure :



(d) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T}+n_1, n_2)$, $(n_1, \sqrt{T}+n_2)$ and $(n_1, -\sqrt{T}+n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric (see 5-1-44):

$$C(\mathbf{r}, \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

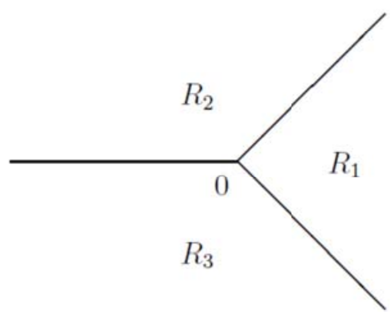
or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r}, \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.



(e) If the signals are equiprobable then:

$$P(e|\mathbf{m}_1) = P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_2|^2 | \mathbf{m}_1) + P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_3|^2 | \mathbf{m}_1)$$

When $\mathbf{m}_1$ is transmitted then $\mathbf{r} = [\sqrt{T} + n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as:

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0. Hence:

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] = 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When $\mathbf{m}_2$ is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore:

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain:

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is momononically decreasing, we obtain:

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message $\mathbf{m}_1$ is more vulnerable to errors. The reason for that is that it has both threshold lines close to it, while the other two signals have one of the their threshold lines further away.

5-26 Consider a digital communication system that transmits information via QAM over a voice-band telephone channel at a rate 2400 symbols/s. The additive noise is assumed to be white and gaussian.

- **a Determine the $\mathscr{E}_b/N_0$ required to achieve an error probability of 10^{-5} at 4800 bits/s.**
- **b Repeat (a) for a rate of 9600 bits/s.**
- **c Repeat (a) for a rate of 19 200 bits/s.**
- **d What conclusions do you reach from these results?**

(a) The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 5 \times 10^{-6} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 9.7682$$

(b) If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{4}\right) Q\left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}}\right]\right)^2$$

Thus,

$$Q\left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}}\right] = \frac{1}{3} \times 10^{-5} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 25.3688$$

(c) If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{16}\right) Q\left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

(d) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.