GRAPH THEORY: Modeling, Applications, and Algorithms,
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Pearson Prentice Hall, 1st printing, (2007)[1]

Selected solutions (or drafts thereof) to the Exercises
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Chapter 1

1.5:
By (a) we have that both $P(1)$ and $P(2)$ are true. If now $r \in \{1, 2\}$ and $P(3k+r)$ is true, then so is $P(3(k+1)+r)$ by (b). By induction we have that $P(n)$ is true for all $n \geq 1$ that are not divisible by 3.

Since n is divisible by 3 if, and only if, $n+3$ is divisible by 3, and neither 1 nor 2 are divisible by 3, then we cannot conclude from (a) and (b) alone that $P(3k)$ is true for any k.

If we have additionally that $P(3)$ is true (which we do not have in this problem) then by (b) we obtain that $P(3k)$ is true for all $k \geq 1$ and hence $P(n)$ is true for all $n \in \mathbb{N}$.

1.6:
Since $Q(1001)$ is true by (a), then so are $Q(998), Q(995), \ldots, Q(5), Q(2)$ by (b). In fact, if $Q(n)$ is true, then so is $Q(n-3\ell)$ for any ℓ such that $n-3\ell$ is positive. By (c) we therefore have that $Q(4)$ is true and so by (b) $Q(1)$ is true.

We proceed by induction on m: Assume that $Q(3k+1)$ and $Q(3k+2)$ are true. By (c) we have $Q(6k+2)$ and $Q(6k+4)$ are true, and hence by (b) $Q(3(k+1)+2)) = Q((6k+2)-3(k-1))$ is true and also $Q(3(k+1)+1) = Q((6k+4)-3k)$. By induction we have that $Q(n)$ is true for any number that is not divisible by 3.

However, 1001 is not divisible by 3. If n is not divisible by 3, then neither is $n-3$. Finally, if n is not divisible by 3, then neither is $2n$. Hence, we cannot conclude from (a), (b) and (c) alone that $Q(n)$ is true for any n that is divisible by 3.

1.9:
The graph in Figure 1.4 has 11 edges that we can label $\{e_1, e_2, \ldots, e_{11}\}$ from left-to-right along a horizontal line just above the A_i-vertices. So here $G = (V, E, \phi)$ can be given by

$$\begin{aligned} V &= \{A_1, A_2, \ldots, A_7\} \cup \{J_1, J_2, \ldots, J_5\}, \\ E &= \{e_1, e_2, \ldots, e_{11}\}, \end{aligned}$$

[1]The book actually appeared in September of 2006.

and the edge map $\phi : E \to \mathbb{P}(V)$ is given by

$$\begin{array}{lll} \phi(e_1) = \{A_1, J_5\}, & \phi(e_2) = \{A_2, J_1\}, & \phi(e_3) = \{A_2, J_2\}, \\ \phi(e_4) = \{A_2, J_4\}, & \phi(e_5) = \{A_3, J_2\}, & \phi(e_5) = \{A_3, J_3\}, \\ \phi(e_7) = \{A_4, J_5\}, & \phi(e_8) = \{A_5, J_2\}, & \phi(e_9) = \{A_5, J_3\}, \\ \phi(e_{10}) = \{A_6, J_5\}, & \phi(e_{11}) = \{A_7, J_5\}. & \end{array}$$

Here we note that ϕ is an injective map, since G is a simple graph, so ϕ is unnecessary here, since we may as well let each edge e_i be its image under ϕ. (The graph of Figure 1.5 is similar).

1.18:

To prove Lemma 1.23, first note that

$$S \subseteq S' \Rightarrow N_G(S) \subseteq N_G(S'). \tag{1}$$

Therefore by (1) we have $N_G(S_1), N_G(S_2) \subseteq N_G(S_1 \cup S_2)$ and hence $N_G(S_1) \cup N_G(S_2) \subseteq N_G(S_1 \cup S_2)$.

For the other way, pick $u \in N_G(S_1 \cup S_2)$. This means that $u \in N_G(s)$ for some $s \in S_1 \cup S_2$. If $s \in S_1$, then we have $u \in N_G(S_1)$ and if $s \in S_2$, then we have $u \in N_G(S_2)$, in either case we have $u \in N_G(S_1) \cup N_G(S_2)$. This proves the first equality: $N_G(S_1) \cup N_G(S_2) = N_G(S_1 \cup S_2)$.

For the closed neighborhoods we therefore have by the above that

$$\begin{aligned} N_G[S_1 \cup S_2] &= N_G(S_1 \cup S_2) \cup (S_1 \cup S_2) \\ &= (N_G(S_1) \cup N_G(S_2)) \cup (S_1 \cup S_2) \\ &= (N_G(S_1) \cup S_1) \cup (N_G(S_2) \cup S_2) \\ &= N_G[S_1] \cup N_G[S_2]. \end{aligned}$$

For the intersection we have by (1) that $N_G(S_1 \cap S_2) \subseteq N_G(S_1), N_G(S_2)$ and hence $N_G(S_1 \cap S_2) \subseteq N_G(S_1) \cap N_G(S_2)$, which proves the third containment.

For the last containment we have by the above that $N_G(S_1 \cap S_2) \subseteq N_G(S_1) \cap N_G(S_2) \subseteq N_G[S_1] \cap N_G[S_2]$. Also we clearly have $S_1 \cap S_2 \subseteq N_G[S_1] \cap N_G[S_2]$. Combining we therefore have

$$N_G[S_1 \cap S_2] = N_G(S_1 \cap S_2) \cup (S_1 \cap S_2) \subseteq N_G[S_1] \cap N_G[S_2],$$

which proves the last containment.

Note that the last two can be strict: If $G = P_3$ is the 3-path on u_1, u_2 and u_3 in this order and $S_i = \{u_i\}$, then $N_G(S_1 \cap S_2) = N_G[S_1 \cap S_2] = \emptyset$, but $N_G(S_1) \cap N_G(S_2) = N_G[S_1] \cap N_G[S_2] = \{u_2\}$.

1.20:

By assumption we have that $d_G(u) \leq \frac{30}{100}(n-1)$, and hence, by the Hand-Shaking Theorem, we have

$$\begin{aligned} |E(G)| &= \frac{1}{2} \sum_{u \in V(G)} d_G(u) \\ &\leq \frac{1}{2} \sum_{u \in V(G)} \frac{30}{100}(n-1) \\ &= \frac{3}{20} n(n-1) \\ &= p(n), \end{aligned}$$

where $p(x) = \frac{3}{20}x^2 - \frac{3}{20}x$. This bound can be reached for infinitely many n: Consider the cycle C_n where $n = 100k + 1$ on vertices $u_0, \ldots, u_{n-1}$ ordered in a cyclic manner. For each i modulo n, connect u_i to $u_{i+\ell}$ for $1 \leq \ell \leq 15k$ (modulo n). In this way each vertex has degree $30k$ which is precisely 30% of the remaining $n-1$ vertices. In this case the total number of edges is $15k(100k+1)$.

1.21:
A simple graph on n vertices can have vertices of degree at most $n - 1$. So, if the n vertices have distinct degrees, then, in particular, there is a vertex u of degree $n-1$ and a vertex v of degree 0. Since $d(u) = n-1$ it must be connected to all other vertices and hence also v, which means that $d(v) \geq 1$. This is a contradiction. Hence, there are at least two vertices of the same degree in a simple graph.

If the graph is not necessarily simple, it is quite possible for the vertices to have distinct degrees:

Consider the path P_{4n+3} on $4n + 3$ vertices $u_1, \ldots u_{4n+3}$. Label the edges $e_1, \ldots, e_{4n+2}$. For each $i \in \{1, \ldots, 2n + 1\}$ make both the edges e_{2i-1} and e_{2i} in P_{2n+1} into two i-fold edges. Finally add $n + 1$ loops at u_{4n+3}. In this way, $d(u_k) = k$ for all k.

1.27:
Clearly, $\vec{G} = \vec{G}_0$ in this setting. By assumption $\vec{G}_k = \vec{G}_0$, and hence $\vec{G}_{x+yk} = \vec{G}_x$ for any integers $x, y \in \mathbb{Z}$. Also, since $\vec{G}_n = \vec{G}_0$ we have likewise $\vec{G}_{x+yn} = \vec{G}_x$ for any integers $x, y \in \mathbb{Z}$. Letting $x, y \in \mathbb{Z}$ be such that $xk + yn = \gcd(k, n) = d$, then we obtain

$$\vec{G} = \vec{G}_0 = \vec{G}_{xk} = \vec{G}_{xk+yn} = \vec{G}_d,$$

proving our assertion.

In general, one can in the same way show that for any $a, b \in \mathbb{Z}$ we have that $\vec{G}_a = \vec{G}_b = \vec{G}$ implies $\vec{G}_{\gcd(a,b)} = \vec{G}$.

1.28:
Assume we have distinct $i, j \in \{0, 1, \ldots, p - 1\}$ such that $\vec{G}_i = \vec{G}_j$. We may further assume $i < j$. Rotating each digraph i "backwards" we obtain that $\vec{G}_0 = \vec{G}_{j-i}$. Since p is a prime and $j - i \in \{1, \ldots, p - 1\}$, we must have $\gcd(j - i, p) = 1$, and hence, by previous exercise (Ch-1, 27) we have that

$$\vec{G}_1 = \vec{G}_{\gcd(j-i,p)} = \vec{G},$$

which can only happen when $\vec{G}$ is a directed cycle. Hence, all the digraphs $\vec{G}_0, \ldots, \vec{G}_{p-1}$ are distinct.

1.29:
Clearly, $\mathcal{C}(p)$ contains exactly $2^p - 2$ digraphs, the underlying graph of each being the n-cycle C_n. For each $\vec{G} \in \mathcal{C}(p)$ let

$$\mathcal{C}(\vec{G}) = \{\vec{G}_0, \ldots, \vec{G}_{p-1}\}.$$

By previous Exercise 1.28, each $\mathcal{C}(\vec{G})$ contains precisely p elements. Since two distinct subsets $\mathcal{C}(\vec{G})$ and $\mathcal{C}(\vec{G}')$ are disjoint, we have that $\mathcal{C}(p)$ is a finite disjoint