EML 6934 Verification, Validation and Uncertainty Qualification and Uncertainty Quantification

Paper helicopter project: Phase 2

Report by:

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-Mohit Israni

Contents

Introduction

This document serves as a report on our experiments with using paper helicopters for VVUU course project phase 2. Phase 2 of the project follows phase 1 which is summarized below.

What did we learn in Phase 1?

In the first phase of the project, we build 3 helicopters and analyzed them for uncertainty quantification in fall time and drag coefficient. We calibrated the drag coefficient to find the posterior distribution of its mean from the experiment, and also obtained the model that the helicopter followed, i.e. quadratic drag law or the linear drag law by varying the mass of helicopters by changing the number of pins attached.

Introduction

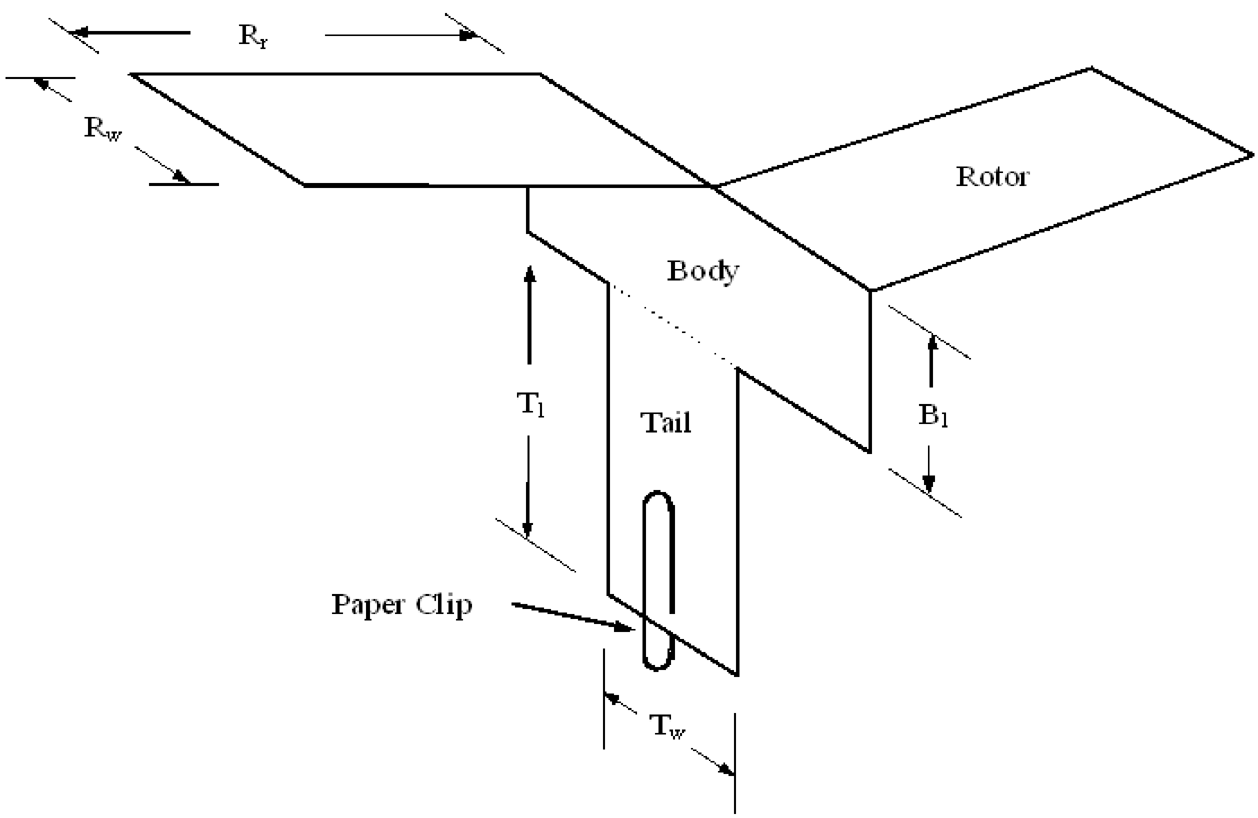


Figure 1: Paper Helicopter

When we release the paper helicopter, shown in the above figure, it starts to autorotate. This is a result of drag and we can express the force relation using Newton's second law, $F = ma$.

Drag can be expressed as $Drag = 0.5\rho C_D A V^2$

Hence the net Force, $ma = mg - 0.5\rho C_D A V^2$

As can be seen in the equation above, drag was assumed to be proportional to the square of velocity. However, there can be cases where we can find a linear dependence of velocity on drag.

Then, the equation of net force becomes , $ma = mg - 0.5\rho C_D A V_0 V$

V_0 is a constant term, which is assigned the value 3 ft/sec^2 for this project.

When the drag force and the mass of helicopter becomes equal, the helicopter achieves a steady state, which on solving differential equations can be expressed as:

Quadratic Model:

$$V_s = \sqrt{g/c} \text{ and } c = \frac{\rho_{\text{air}} A C_D}{2m}$$

Linear Model:

$$V_s = g/c \text{ and } c = \frac{\rho_{\text{air}} A C_D V_0}{2m}$$

To make the calculations simple, a reasonable assumption has been made, that the helicopter reaches steady state instantaneously after the drop. Then the experimental fall time and height can be used to calculate the V_S as:

$$V_S = \frac{height}{fall\ time}$$

The type of model followed by helicopter can be identified by taking helicopter with different masses and recording the fall times.

For a quadratic model the time and mass relation is :

$$\frac{t_1}{t_2} = \sqrt{\frac{m_2}{m_1}}$$

Whereas for a linear model,

$$\frac{t_1}{t_2} = \frac{m_2}{m_1}$$

For the project, we built 3 nominally identical helicopters and dropped each 10 times from same height and noted their fall times. The results of which were used for uncertainty calculations. The same experiment was performed with different masses by varying the paper pins as 1, 2 and 3. Which was used to identify the model of helicopter.

For calibration of Drag Coefficient (C_D), we used a non-informative prior for the mean and standard deviation, and used experiments to obtain the posterior distribution of the drag coefficients. We assumed that the observed drag coefficients from the tests followed a normal distribution.

Since the prior is non-informative the likelihood function equals the posterior distribution which is a joint probability density. To compute the likelihood, we calculate the probability distribution value at each point for each C_D value and take their product. This is represented as surface and the contour plot in the 2D space of Mean of C_D and Standard deviation of C_D. These were plotted for every helicopter for all different masses.

Conditions and Tools used:

- The experiments were carried out in New Engineering Building at University of Florida.
- The Height of fall used for experiments was 226 inches.
- However, the terminal velocity was obtained from the last 70 inches of fall.
- Fall times were measured with stop watch with resolution of 0.01 sec.
- Average mass of the helicopter made with stiffer paper was 1.9 grams each.
- Pins of average mass 0.5 grams each were used.

What changes were made in the helicopter from phase 1 to phase 2?

The dimensions of the helicopter remained same, however a stiffer card paper was used here instead of the standard Xerox paper.

Certain other changes were also made, based on observations from phase 1.

Instead of assuming the steady state velocity to be the average velocity for entire fall, we considered it to be equal to the terminal velocity, which was obtained by estimating the fall time for last 70 inches of complete 226 inches of fall.

Also, as the mass of stiffer card paper was higher, 1 or 2 paper pins were not sufficient to sufficient to keep the helicopter from swaying. Hence the experiments were performed for helicopter with 2, 3, 4 and 5 pins, however the results from the helicopter with 2 pins were very non-uniform and hence were dropped, while analyzing the overall results.

1.

To decrease the variability in the results, stiffer card paper was used and also the steady state speed was now approximated to the terminal velocity instead of the average speed of the fall.

Summary measures are used to describe the amount of variability or spread in a set of data. The most common measures of variability are the range, the interquartile range (IQR), variance, and standard deviation. We here use the range and standard deviation to understand the variability of the steady state speed.

The table below provides the fall speed of helicopter with 3 pins and compares the variability in fall speed of phase 1 to phase 2.

	Phase 1			Phase 2		
	Copter 1	**Copter 2**	**Copter 3**	**Copter 1**	**Copter 2**	**Copter 3**
1	46.31148	49.13043	46.31148	47.61905	44.58599	46.35762
2	48.91775	48.7069	47.47899	50.35971	45.45455	44.02516
3	49.23747	47.67932	45.10978	47.61905	50.35971	46.66667
4	47.78013	51.24717	45.47284	48.95105	46.35762	47.61905
5	47.08333	47.67932	46.98545	47.2973	46.05263	45.75163
6	48.08511	48.394	45.93496	48.61111	47.94521	46.66667
7	46.88797	50.9009	46.59794	48.61111	45.45455	44.58599
8	47.88136	50	49.02386	50	47.94521	47.2973
9	48.8121	51.36364	48.08511	47.2973	46.35762	48.95105
10	46.21677	50.78652	47.57895	47.94521	51.47059	47.61905
Mean	47.69945	49.55054	46.82967	48.40941	47.10633	46.51163
Standard Deviation	1.07673	1.37602	1.15379	1.03163	2.12137	1.39802

2.

To obtain the Predictive distribution we first need to obtain the type of model(linear or quadratic) the phase 2 helicopters follow, before calculating their C_D.

Mass of Helicopters:

Mass of individual Helicopter without pins = 1.9 grams

Average mass of a pin $= 0.5\ gram$

Assuming all the helicopters with the same number of pins weigh the same, as the uncertainty in their weights can be neglected. For determining the model, comparing the mass and fall times of all Helicopters with 3,4 and 4,5 pin pairs.

i) Helicopter 1,

m5/m4 = 1.128

sqrt(m5/m4) = 1.062

<t4/t5> = $\frac{1.376}{1.328} = 1.0361$

m4/m3 =1.147

sqrt(m4/m3) =1.071

<t3/t4> = $\frac{1.446}{1.376} = 1.0509$

ii) Helicopter 2,

m5/m4 = 1.128

sqrt(m5/m4) = 1.062

<t4/t5> = $\frac{1.382}{1.28} = 1.0796$

m4/m3 =1.147

sqrt(m4/m3) =1.071

<t3/t4> = $\frac{1.486}{1.382} = 1.0752$

iii) Helicopter 3,

m5/m4 = 1.128

sqrt(m5/m4) = 1.062

<t4/t5> = $\frac{1.394}{1.294} = 1.07228$

m4/m3 =1.147

sqrt(m4/m3) =1.071

<t3/t4> = $\frac{1.505}{1.394} = 1.0792$

<t3/t4> ~ square root(m4/m3) & **<t4/t5> ~ square root(m5/m4)** for all the three helicopters. Thus, all the three helicopters follow **Quadratic Model.**

Now Drag coefficient for the quadratic model helicopters can be calculated as follows:

$$V_S = sqrt\left(\frac{g}{c}\right)$$

$$c = \frac{\rho_{air} A C_D}{2m}$$

$$C_D = \frac{2mg}{\rho_{air} A V_S^2} * 2402.490\ (\,for\ unit\ conversion)$$

After C_D is obtained for all experimental values. Its mean and standard deviation is obtained assuming the drag coefficient to be normally distributed.

Copter 1					
			2 Clips		
Mass:	1.9	1		2.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	3.17	4.64	1.47	47.61904762	0.978517

2	3.27	4.22	0.95	73.68421053	0.408678
3	3.39	4.63	1.24	56.4516129	0.696269
4	3.25	4.34	1.09	64.22018349	0.538006
5	3.46	4.87	1.41	49.64539007	0.900269
6	3.24	4.46	1.22	57.37704918	0.67399
7	3.44	4.6	1.16	60.34482759	0.609326
8	3.83	4.72	0.89	78.65168539	0.358686
9	4.37	5.65	1.28	54.6875	0.741914
10	3.43	4.69	1.26	55.55555556	0.718911
Average	3.485	4.682	1.197	58.47953216	0.648817

Copter 1

3 Clips

Mass:	1.9	1.5		3.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.93	4.4	1.47	47.61904762	1.147227
2	2.98	4.37	1.39	50.35971223	1.025757
3	3.04	4.51	1.47	47.61904762	1.147227
4	2.96	4.39	1.43	48.95104895	1.085643
5	2.72	4.2	1.48	47.2972973	1.162889
6	3.02	4.46	1.44	48.61111111	1.100879
7	3.01	4.45	1.44	48.61111111	1.100879
8	2.86	4.26	1.4	50	1.040569
9	2.87	4.35	1.48	47.2972973	1.162889
10	2.85	4.31	1.46	47.94520548	1.131672
Average	2.924		1.446	48.40940526	1.110073

Copter 1

4 Clips

Mass:	1.9	2		3.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.8	4.2	1.4	50	1.193594
2	2.89	4.29	1.4	50	1.193594
3	2.96	4.27	1.31	53.4351145	1.045064
4	2.79	4.2	1.41	49.64539007	1.210706
5	2.96	4.29	1.33	52.63157895	1.077218
6	2.94	4.24	1.3	53.84615385	1.02917
7	2.74	4.08	1.34	52.23880597	1.093478
8	2.82	4.19	1.37	51.09489051	1.142988
9	2.83	4.27	1.44	48.61111111	1.262773
10	2.76	4.22	1.46	47.94520548	1.298094
Average	2.849		1.376	50.87209302	1.153021

Copter 1

5 Clips

Mass:	1.9	2.5		4.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.51	3.81	1.3	53.84615385	1.161115
2	2.57	3.92	1.35	51.85185185	1.252149
3	2.6	3.83	1.23	56.91056911	1.039438
4	2.51	3.88	1.37	51.09489051	1.289525

5	2.6	3.96	1.36	51.47058824	1.270768
6	2.56	3.89	1.33	52.63157895	1.215323
7	2.5	3.85	1.35	51.85185185	1.252149
8	2.52	3.89	1.37	51.09489051	1.289525
9	2.62	3.9	1.28	54.6875	1.125663
10	2.59	3.93	1.34	52.23880597	1.233668
Average	2.558		1.328	52.71084337	1.211671

For Helicopter 1:

- 2 pins
  - mu = 0.662457 [0.522964, 0.80195]
  - sigma = 0.194998 [0.134126, 0.35599]
- 3 pins
  - mu = 1.11056 [1.07559, 1.14554]
  - sigma = 0.0488876 [0.0336266, 0.0892496]
- 4 pins
  - mu = 1.15452 [1.09592, 1.21311]
  - sigma = 0.0872197 [0.0609419, 0.153065]
- 5 pins
  - mu = 1.21293 [1.15502, 1.27085]
  - sigma = 0.0809598 [0.055687, 0.147801]

Copter 2					
2 Clips					
Mass:	1.9	1		2.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	4.13	5.55	1.42	49.29577465	0.913084
2	3.17	4.46	1.29	54.26356589	0.753552
3	3.29	4.55	1.26	55.55555556	0.718911
4	3.36	5.14	1.78	39.3258427	1.434742
5	3.38	4.71	1.33	52.63157895	0.801009
6	3.69	4.87	1.18	59.3220339	0.630519
7	4.6	5.69	1.09	64.22018349	0.538006
8	2.94	4	1.06	66.03773585	0.508798
9	3.53	4.92	1.39	50.35971223	0.87491
10	3.54	4.86	1.32	53.03030303	0.789009
Average	3.563	4.875	1.312	53.35365854	0.779474

Copter 2					
3 Clips					
Mass:	1.9	1.5		3.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	3	4.57	1.57	44.58598726	1.308622
2	2.84	4.38	1.54	45.45454545	1.259088

3	2.98	4.37	1.39	50.35971223	1.025757
4	2.81	4.32	1.51	46.35761589	1.210511
5	2.8	4.32	1.52	46.05263158	1.226597
6	2.96	4.42	1.46	47.94520548	1.131672
7	2.97	4.51	1.54	45.45454545	1.259088
8	2.84	4.3	1.46	47.94520548	1.131672
9	2.97	4.48	1.51	46.35761589	1.210511
10	2.95	4.31	1.36	51.47058824	0.981957
Average	2.912		1.486	47.10632571	1.172337
Copter 2					
4 Clips					
Mass:	1.9	2		3.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.94	4.28	1.34	52.23880597	1.093478
2	2.95	4.35	1.4	50	1.193594
3	2.94	4.27	1.33	52.63157895	1.077218
4	2.91	4.3	1.39	50.35971223	1.176603
5	2.77	4.25	1.48	47.2972973	1.333902
6	2.94	4.31	1.37	51.09489051	1.142988
7	2.87	4.21	1.34	52.23880597	1.093478
8	2.84	4.21	1.37	51.09489051	1.142988
9	2.98	4.35	1.37	51.09489051	1.142988
10	2.88	4.31	1.43	48.95104895	1.245296
Average	2.902		1.382	50.6512301	1.163099
Copter 2					
5 Clips					
Mass:	1.9	2.5		4.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.58	3.77	1.19	58.82352941	0.972932
2	2.47	3.91	1.44	48.61111111	1.424668
3	2.55	3.78	1.23	56.91056911	1.039438
4	2.6	3.87	1.27	55.11811024	1.108143
5	2.61	3.87	1.26	55.55555556	1.090761
6	2.57	3.81	1.24	56.4516129	1.056409
7	2.58	3.91	1.33	52.63157895	1.215323
8	2.67	3.98	1.31	53.4351145	1.179047
9	2.55	3.84	1.29	54.26356589	1.14332
10	2.56	3.8	1.24	56.4516129	1.056409
Average	2.574		1.28	54.6875	1.125663

For Helicopter 2:

- 2 pins

  mu = 0.796254 [0.609685, 0.982823]

  sigma = 0.260805 [0.179391, 0.476129]

- 3 pins

mu = 1.17455 [1.09896, 1.25013]
sigma = 0.105658 [0.0726753, 0.19289]

- 4 pins

mu = 1.16425 [1.10817, 1.22033]
sigma = 0.078395 [0.0539228, 0.143119]

- 5 pins

mu = 1.12865 [1.03866, 1.21863]
sigma = 0.125784 [0.0865186, 0.229632]

Copter 3					
			2 Clips		
Mass:	1.9	1		2.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	3.61	5.04	1.43	48.95104895	0.925989
2	3.32	5.02	1.7	41.17647059	1.308675
3	4.12	5.59	1.47	47.61904762	0.978517
4	3.43	4.92	1.49	46.97986577	1.005325
5	3.4	4.82	1.42	49.29577465	0.913084
6	3.57	5.02	1.45	48.27586207	0.952072
7	3.66	4.92	1.26	55.55555556	0.718911
8	3.46	4.87	1.41	49.64539007	0.900269
9	3.35	4.85	1.5	46.66666667	1.018864
10	4.12	5.6	1.48	47.2972973	0.991876
Average	3.604	5.065	1.461	47.91238877	0.966572

Copter 3					
			3 Clips		
Mass:	1.9	1.5		3.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.97	4.48	1.51	46.35761589	1.210511
2	2.83	4.42	1.59	44.02515723	1.342175
3	3.08	4.58	1.5	46.66666667	1.194531
4	2.87	4.34	1.47	47.61904762	1.147227
5	2.94	4.47	1.53	45.75163399	1.24279
6	3.21	4.71	1.5	46.66666667	1.194531
7	*2.91*	*4.48*	1.57	44.58598726	1.308622
8	2.87	4.35	1.48	47.2972973	1.162889
9	3.05	4.48	1.43	48.95104895	1.085643
10	2.89	4.36	1.47	47.61904762	1.147227
Average	2.962		1.505	46.51162791	1.202507

Copter 3					
			4 Clips		
Mass:	1.9	2		3.9	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.87	4.25	1.38	50.72463768	1.159735
2	2.9	4.35	1.45	48.27586207	1.280373
3	2.97	4.31	1.34	52.23880597	1.093478

4	2.94	4.37	1.43	48.95104895	1.245296
5	2.8	4.27	1.47	47.61904762	1.315937
6	2.84	4.31	1.47	47.61904762	1.315937
7	3.03	4.4	1.37	51.09489051	1.142988
8	2.94	4.28	1.34	52.23880597	1.093478
9	2.85	4.21	1.36	51.47058824	1.126363
10	2.94	4.27	1.33	52.63157895	1.077218
Average	2.908		1.394	50.21520803	1.183385
Copter 3					
5 Clips					
Mass:	1.9	2.5		4.4	
Trial	Lap Time	Total Time	Terminal fall time	Velocity(inch/sec)	CD
1	2.54	3.9	1.36	51.47058824	1.270768
2	2.58	3.91	1.33	52.63157895	1.215323
3	2.48	3.87	1.39	50.35971223	1.32745
4	2.71	3.97	1.26	55.55555556	1.090761
5	2.67	3.91	1.24	56.4516129	1.056409
6	2.54	3.8	1.26	55.55555556	1.090761
7	2.58	3.8	1.22	57.37704918	1.022606
8	2.47	3.75	1.28	54.6875	1.125663
9	2.54	3.91	1.37	51.09489051	1.289525
10	2.57	3.8	1.23	56.91056911	1.039438
Average	2.568		1.294	54.09582689	1.150422

For Helicopter 3:

- 2 pins
  mu = 0.971358 [0.866799, 1.07592]
  sigma = 0.146164 [0.100537, 0.266838]
- 3 pins
  mu = 1.20351 [1.15419, 1.25284]
  sigma = 0.0734226 [0.0513016, 0.128852]
- 4 pins
  mu = 1.18508 [1.11712, 1.25304]
  sigma = 0.0950039 [0.065347, 0.17344]
- 5 pins
  mu = 1.15287 [1.07217, 1.23357]
  sigma = 0.112817 [0.0775994, 0.205959]

Our objective now is to estimate unknown two parameters of normal distribution based on our observations of experimental C_D's. That is to find likelihood function, obtained as joint distribution, a function of mean and standard deviation. Since the prior given is non-informative, it will be equal to the joint posterior distribution for the Drag Coefficient.

$$p\left(\mu,\sigma^2\right) \propto \left(\sigma^2\right)^{-1}$$

- Joint posterior distribution:

$$p\left(\theta \mid y\right) \propto \prod_{i=1}^{n} p\left(y_i \mid \theta\right) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\mu\right)^2\right)$$

$$\text{where} \quad s^2 = \frac{1}{n-1}\sum\left(y_i-\mu\right)^2$$

- With some algebra,

$$p\left(\mu,\sigma^2 \mid y\right) = \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\left(n-1\right)s^2 + n\left(\bar{y}-\mu\right)^2\right]\right)$$

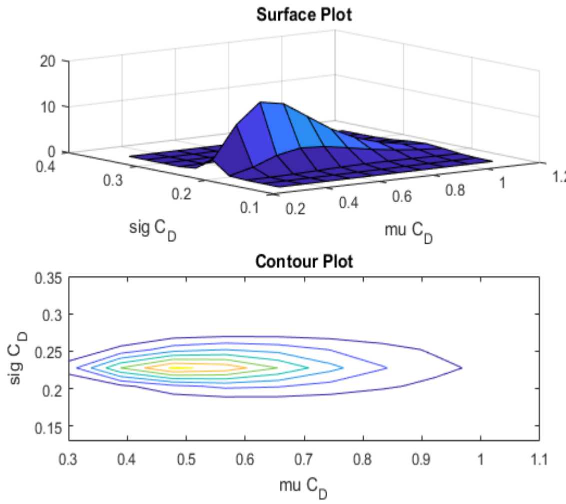
The posterior distribution of Drag coefficient is obtained separately based on all helicopters for all different cases of masses, which is plotted as a surface and a contour.

The MATLAB code to calculate Likelihood is given below:

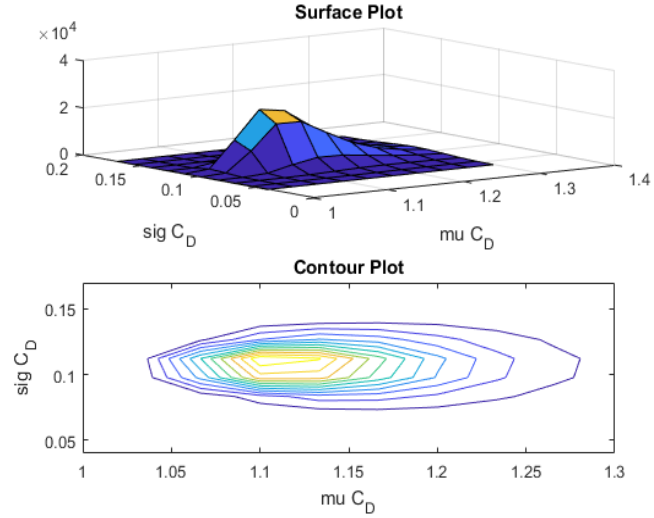
```
%% Matlab Fuction to calculate Likelihood


H1P1=[2.019847709,1.904070563,2.092793478,2.167033185,2.047050739,1.9393303
8,2.167033185,2.019847709,2.261652419,1.96598753]';
k_=length(H1P1);
pd=fitdist(H1P1,'Normal')

mu=1.5:0.1:3.2;
x_=length(mu);
sig=0.07:0.01:0.24;
y_=length(sig);
[X,Y]=meshgrid(mu,sig);

L=meshgrid(mu,sig);
%%For Non-informative Prior
for i=1:x_
    for j=1:y_
        L(i,j)=1;
    end
end

%Likelihood Function
for i=1:x_
    for j=1:y_
        for k_=1:k_
            L(i,j)=L(i,j)*normpdf(Cd1(k_),mu(i),sig(j));
        end
    end
end

surf(X,Y,L)
%contour(X,Y,L)
xlabel('mu C_D');
ylabel('sig C_D');
```

Helicopter 1

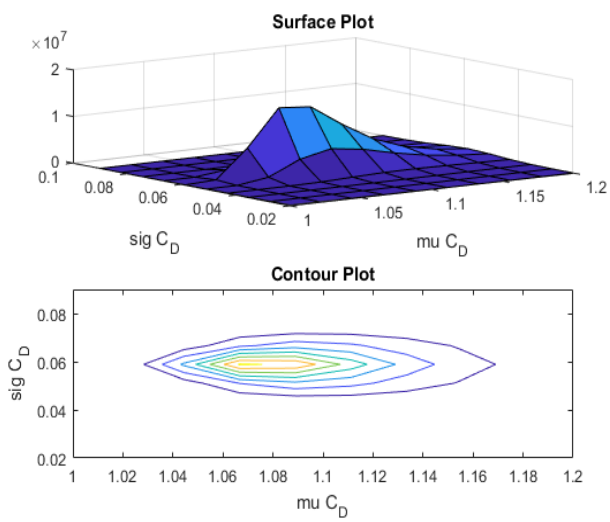
#Pins 2:



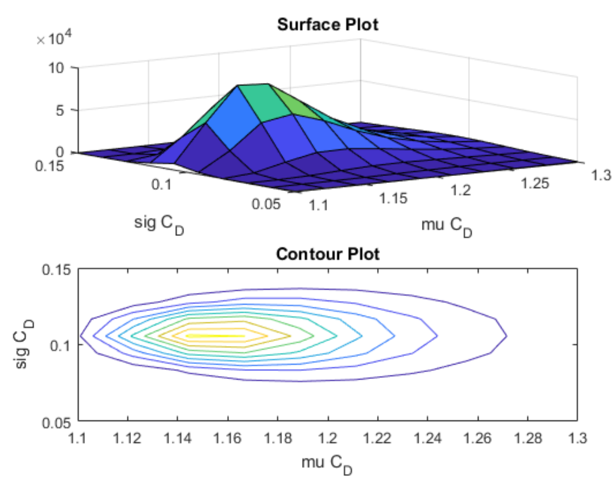
#Pins 4:



#Pins 3:

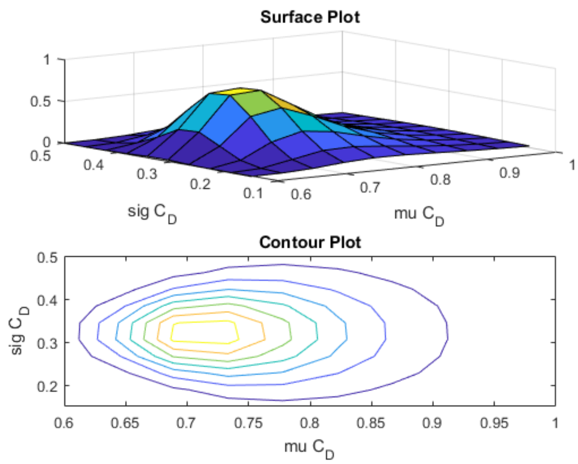


#Pins 5:

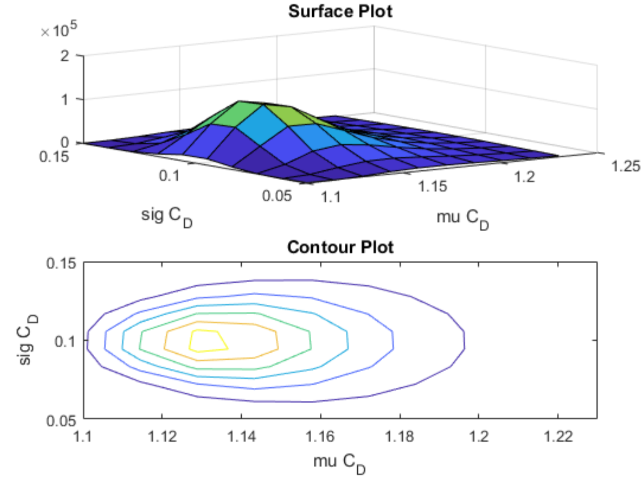


Helicopter 2

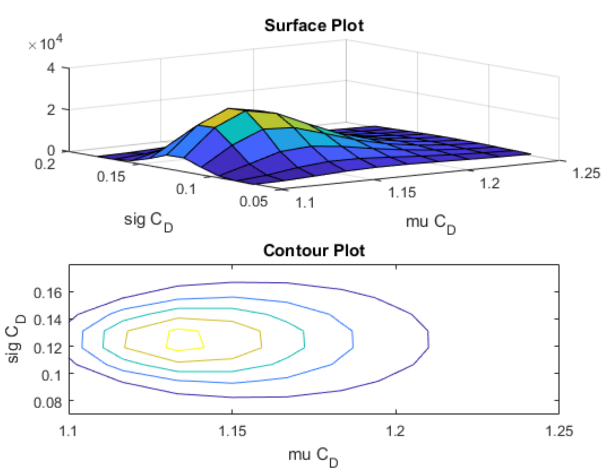
#Pins 2:



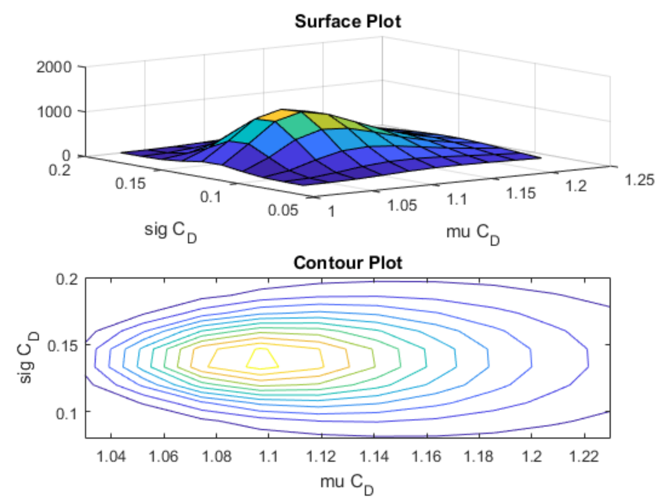
#Pins 4:



#Pins 3:

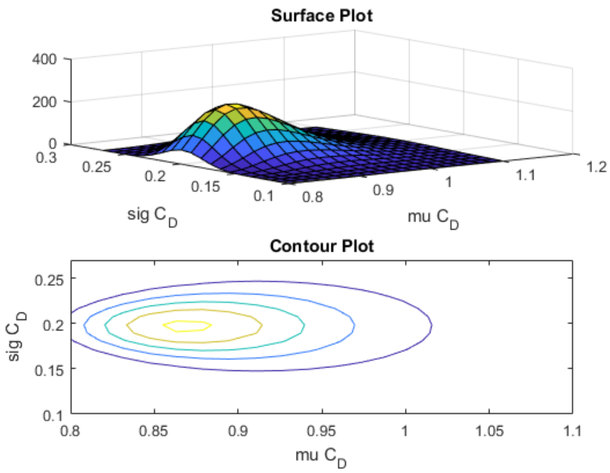


#Pins 5:

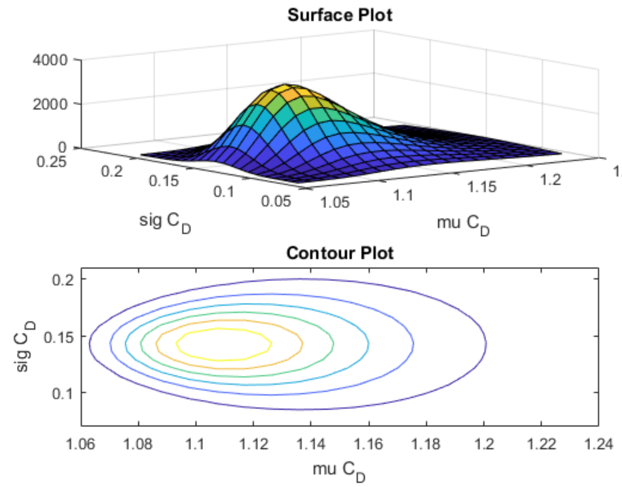


Helicopter 3

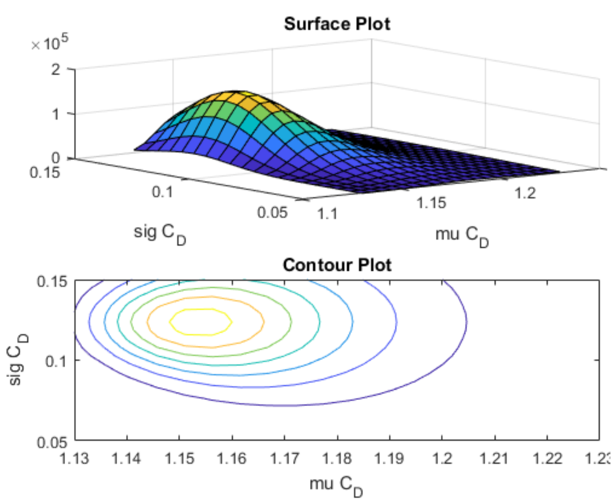
#Pins 2:



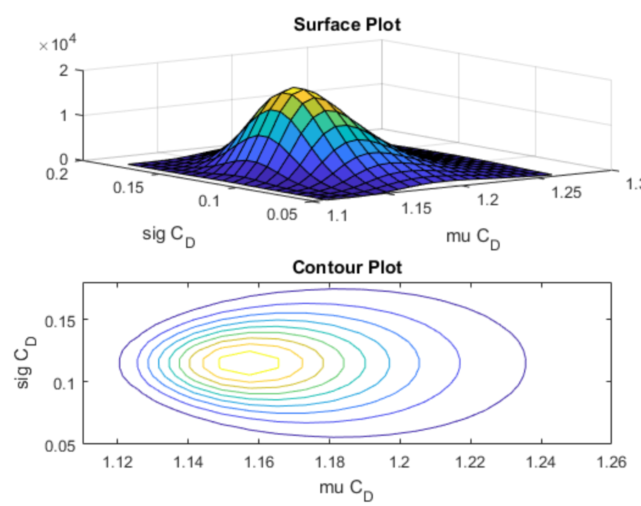
#Pins 4:



#Pins 3:

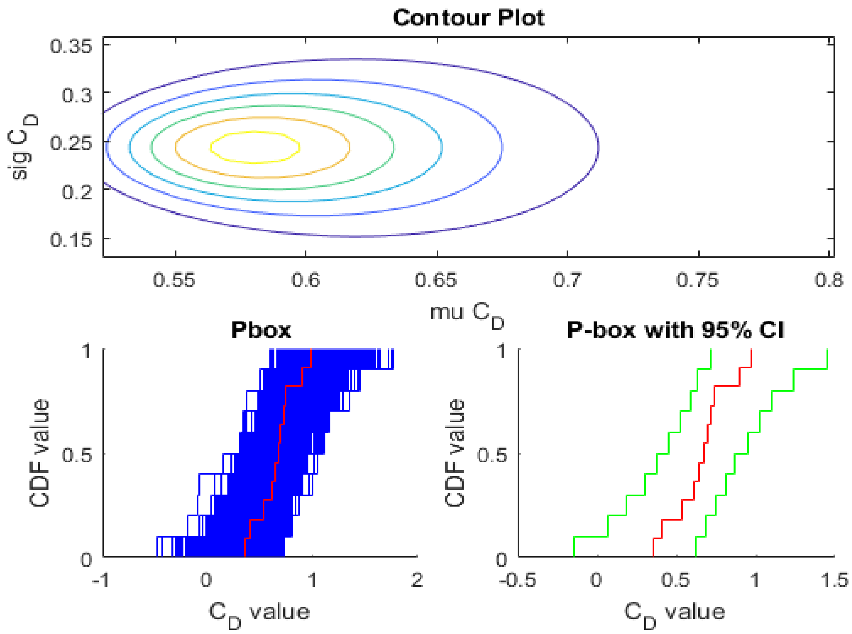


#Pins 5:



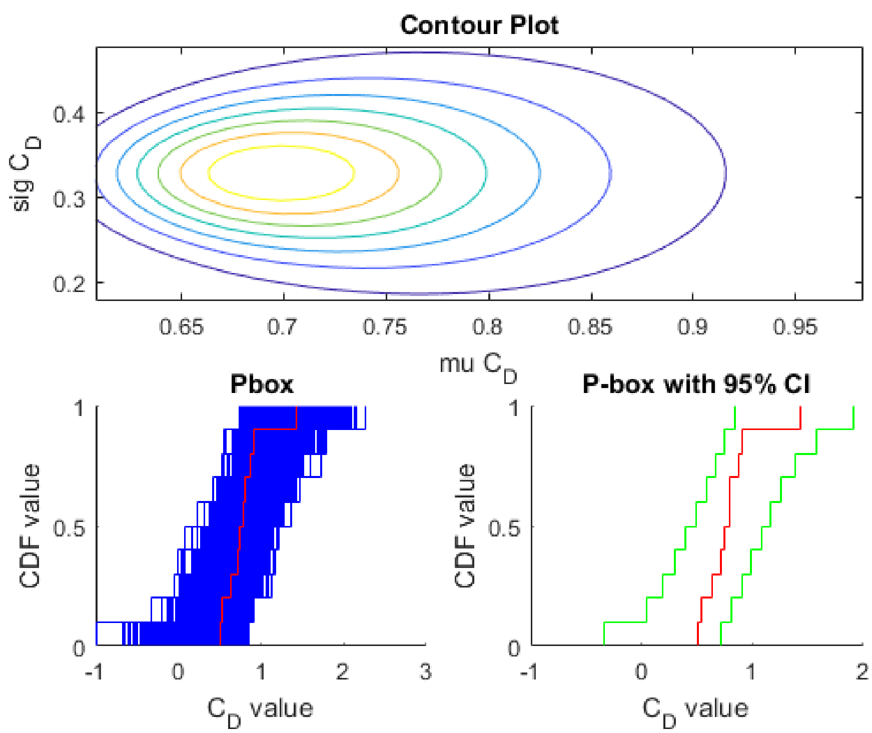
3. **Calibration of Helicopters with 2 pins:**

Helicopter 1:



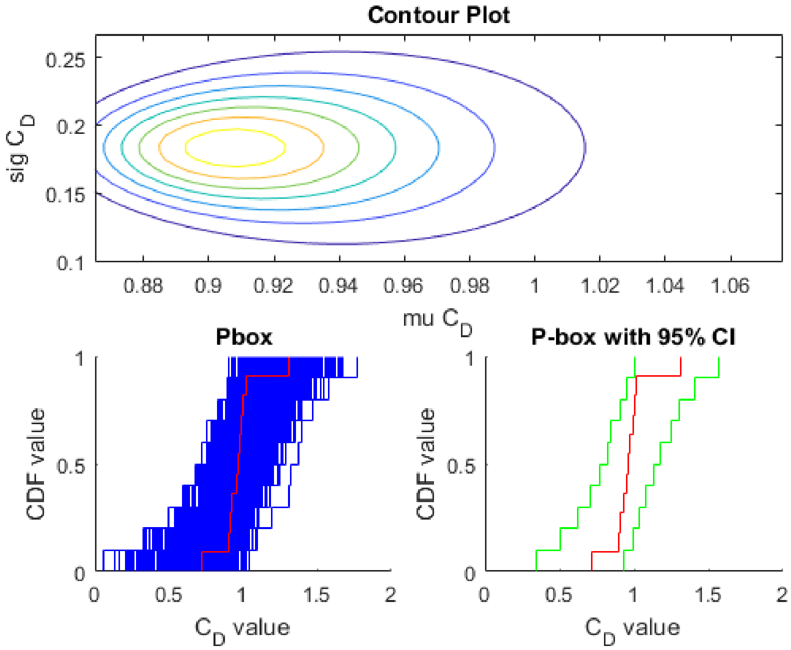
Area Metric = 0

Helicopter 2:



Area Metric = 0

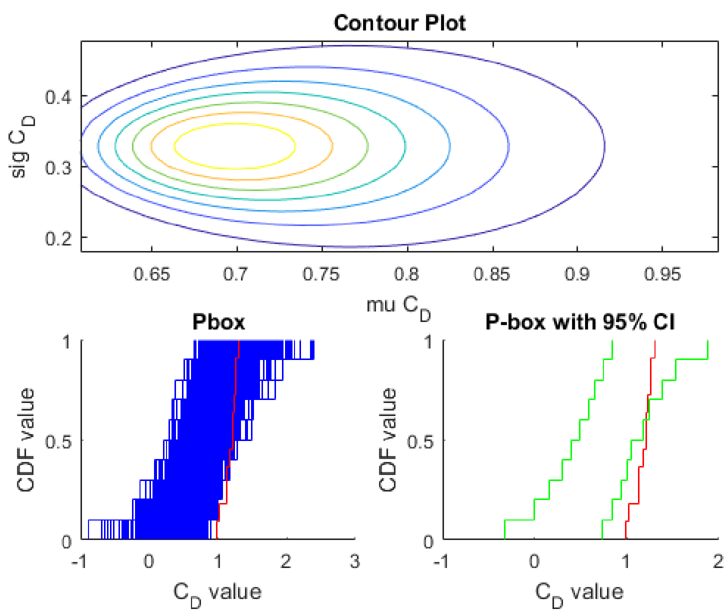
Helicopter 3:



Area Metric = 0

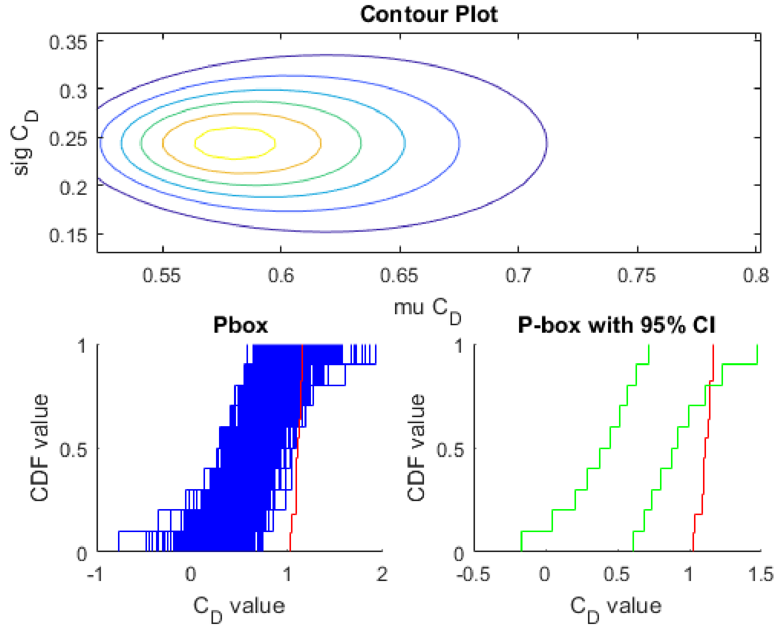
4. Validation of Helicopters with other masses with 2-pin calibrated C_D

Helicopter 1



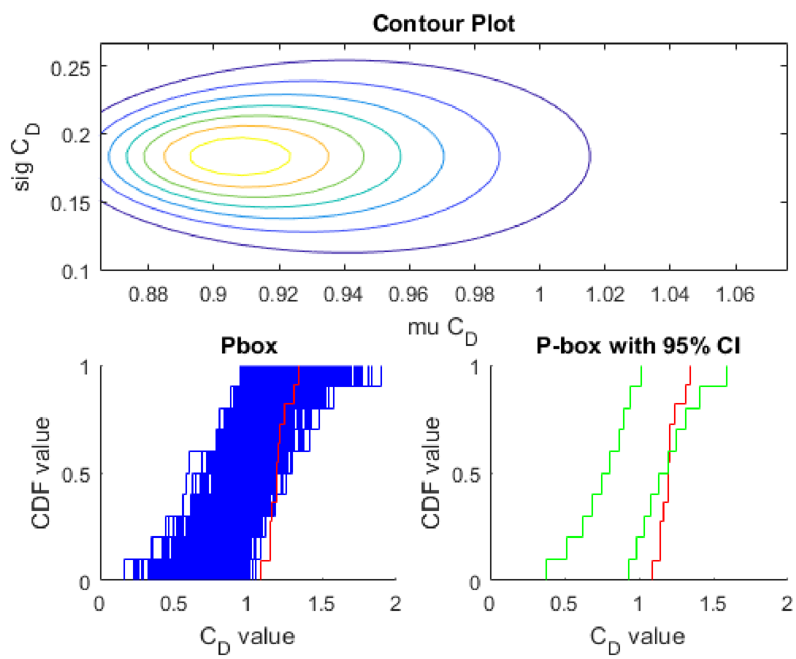
Area Metric= 0.2225

Helicopter 2



Area Metric= 0.1035

Helicopter 3

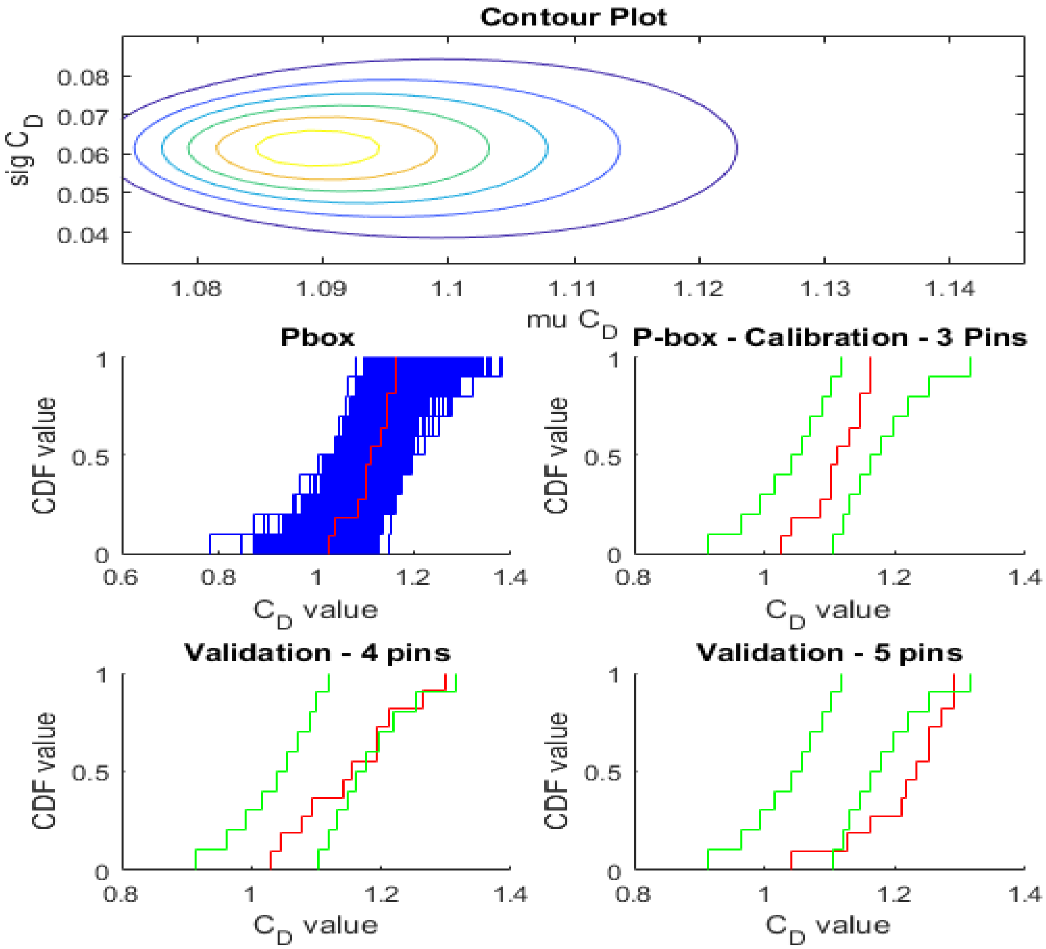


Area Metric= 0.0646

As was mentioned earlier in introduction section, that, as the mass of the helicopter increased due to selection of stiffer card paper, 2 pins were not sufficient to balance the helicopter and hence , it is also NOT USEFUL to calibrate the C_D and further Validation with helicopter with 2 pins.

Therefore the C_D values were calibrated again with helicopter with 3 Pins and then Validated for helicopters with 4 and 5 pins.

Helicopter 1

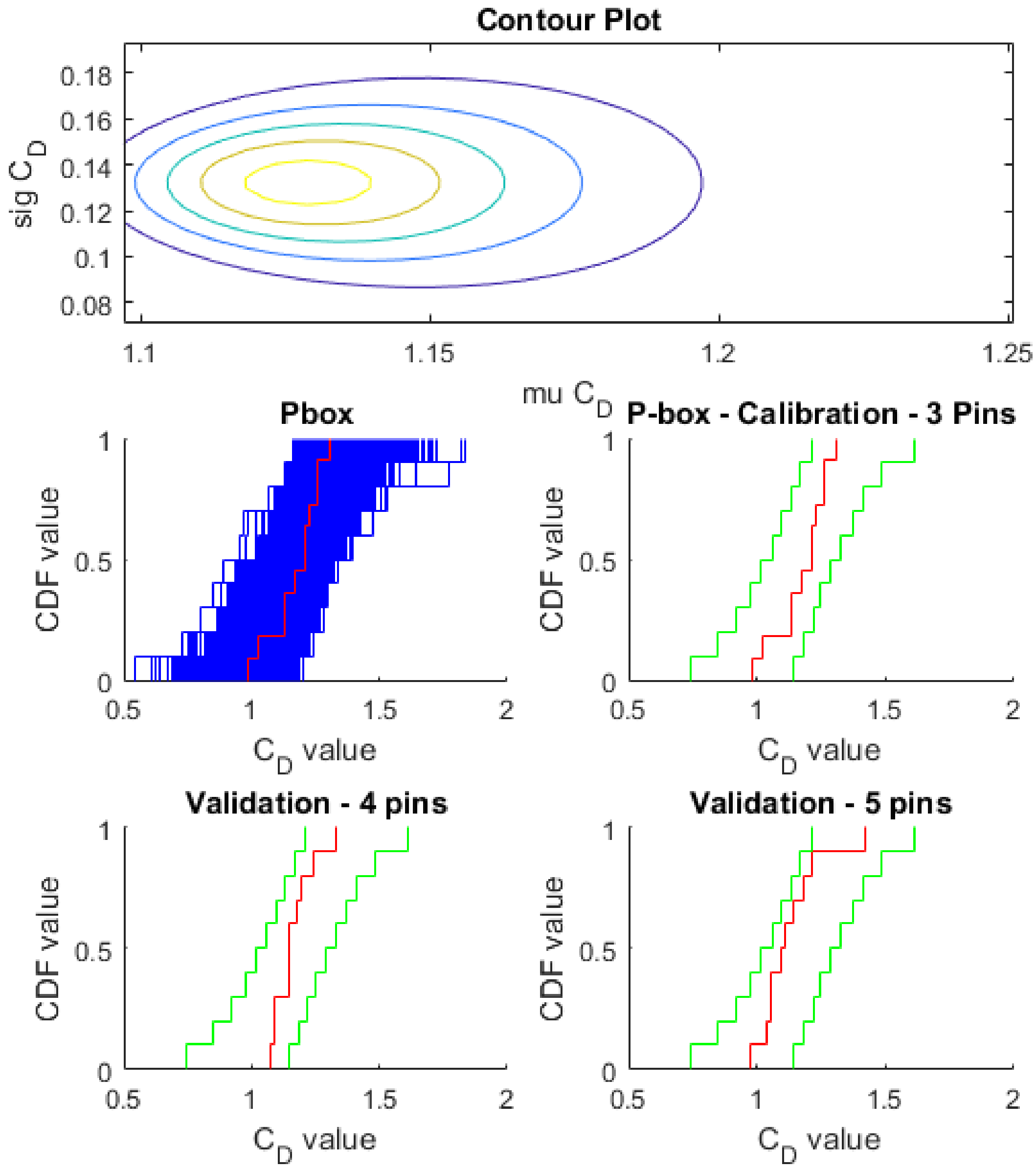


Area Metric for 4 pins= 1.7394e-04

Area Metric for 5 pins= 0.0347

As can be seen, Area metric for helicopter 1 fits well for pins 3 and 4 however it doesn't fit well for 5 pins, when considering **Quadratic Model**, but it validates correctly when considering **Linear Model** in the later part.

Helicopter 2:

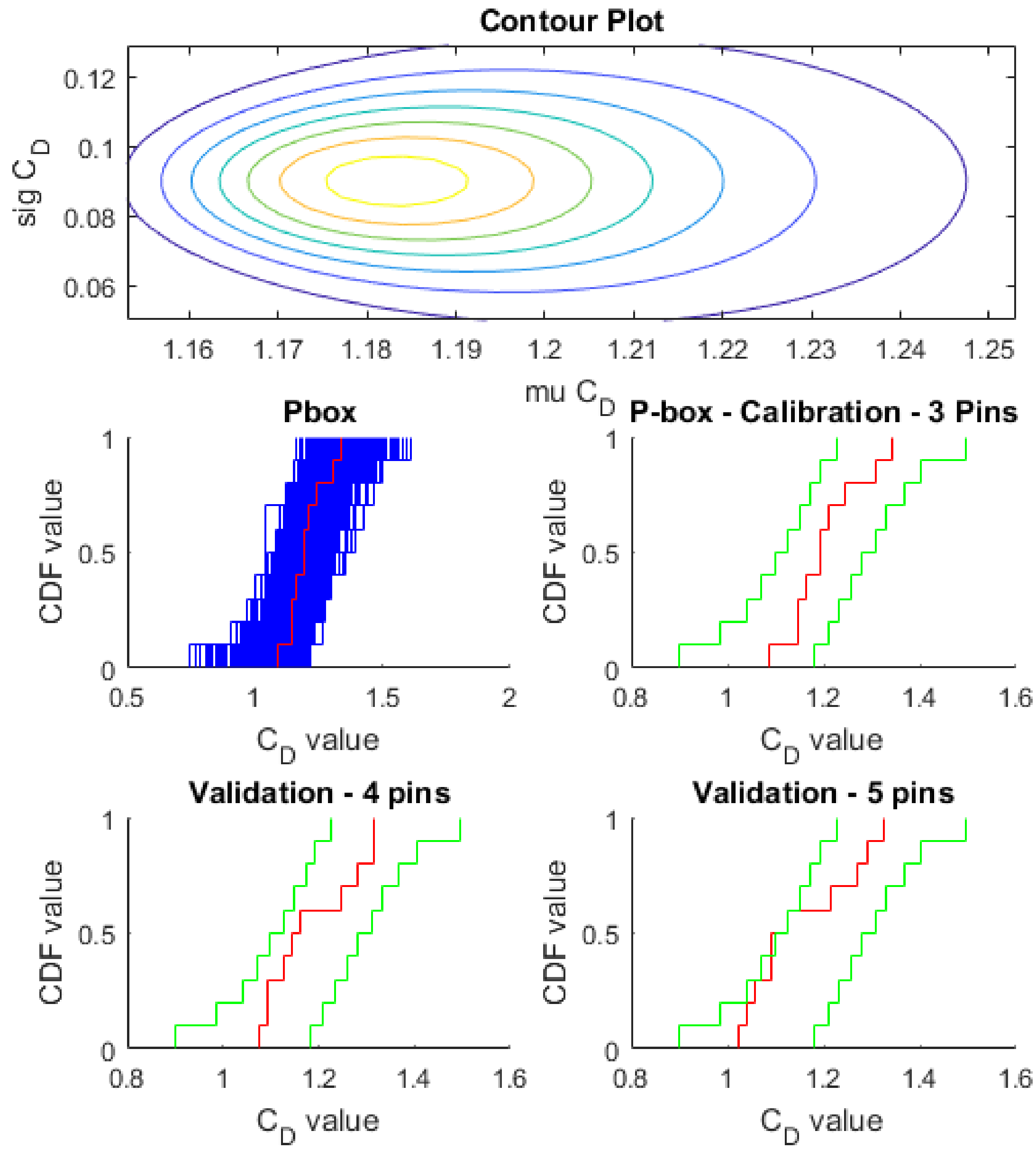


Area Metric for 4 pins = 0

Area Metric for 5 pins = 0

When Helicopter 2 was calibrated using 3 pins, it validated perfectly for pins with 4 and 5, which states that the experiment results are accurate considering the helicopter is truly Quadratic in nature.

Helicopter 3



Area Metric for 4 pins = 0

Area Metric for 5 pins = 0

Like helicopter 2, helicopter 3 when calibrated using 3 pins, validated perfectly for 5 pins. Therefore, experiment results are accurate considering the helicopter is truly Quadratic in nature.

5. Repeating the above process of calibration and validation for assumed linear dependence on speed.

Drag Coefficient (C_D) values assuming linear dependence :

Drag coefficient for the linear model helicopters can be calculated as follows:

$$V_S = \left(\frac{g}{c}\right)$$

$$c = \frac{\rho_{air} A C_D V_0}{2m}$$

$$C_D = \frac{2mg}{\rho_{air} A (V_S V_0)} * 2402.490\ (\mathit{for\ unit\ conversion})$$

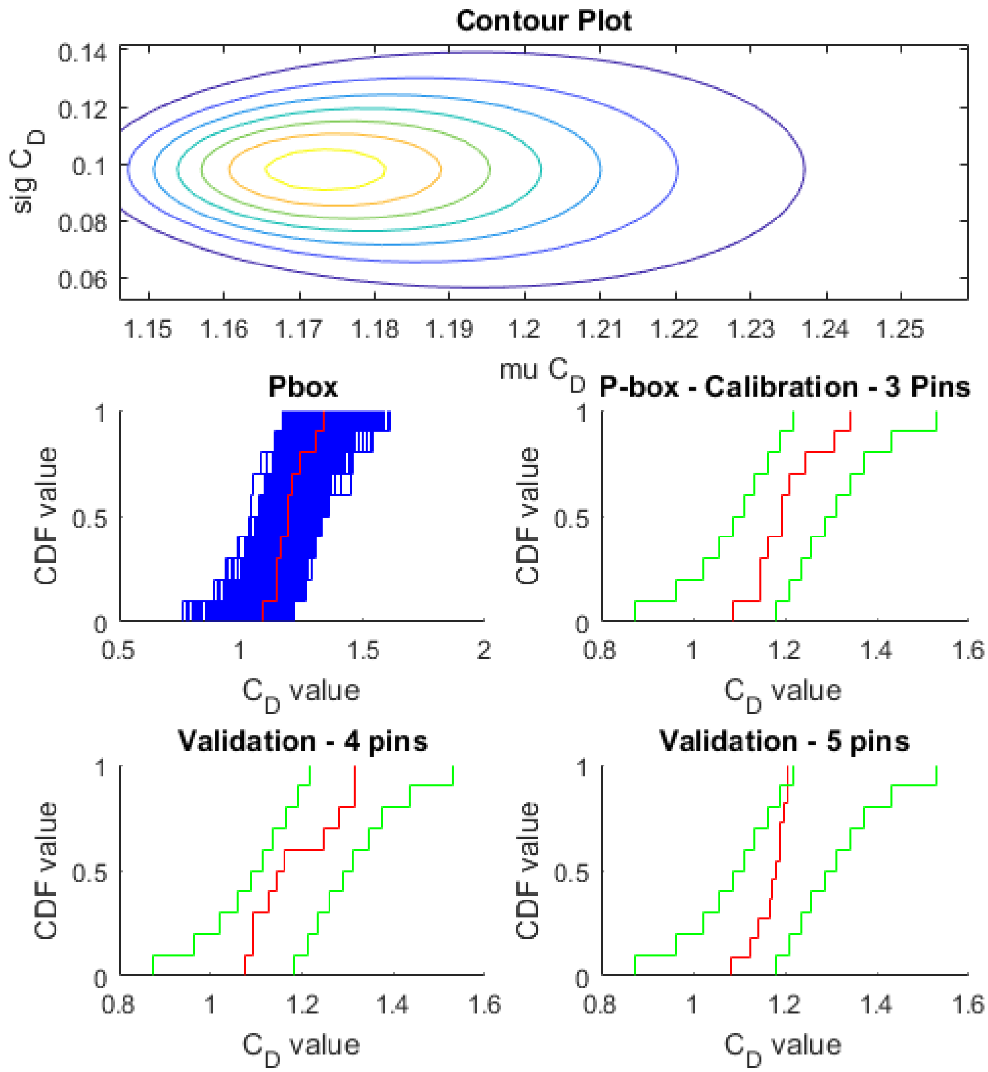
After C_D is obtained for all experimental values. Its mean and standard deviation are obtained assuming the drag coefficient to be normally distributed.

C_D values assuming Linear dependences on speed

2 pins			3 pins		
H1	H2	H3	H1	H2	H3
1.294335	1.25031	0.925989	1.294335	1.382385	1.329555
0.836475	1.135845	1.308675	1.223895	1.35597	1.399995
1.09182	1.10943	0.978517	1.294335	1.223895	1.32075
0.959745	1.56729	1.005325	1.259115	1.329555	1.294335
1.241505	1.171065	0.913084	1.30314	1.33836	1.347165
1.07421	1.03899	0.952072	1.26792	1.28553	1.32075
1.02138	0.959745	0.718911	1.26792	1.35597	1.382385
0.783645	0.93333	0.900269	1.2327	1.28553	1.30314
1.12704	1.223895	1.018864	1.30314	1.329555	1.259115
1.10943	1.16226	0.991876	1.28553	1.19748	1.294335
1.053959	1.155216	0.966572	1.273203	1.308423	1.325153

4 pins			5 pins		
H1	H2	H3	H1	H2	H3
1.2327	1.17987	1.21509	1.14465	0.972932	1.19748
1.2327	1.2327	1.276725	1.188675	1.424668	1.171065
1.153455	1.171065	1.17987	1.083015	1.039438	1.223895
1.241505	1.223895	1.259115	1.206285	1.108143	1.10943
1.171065	1.30314	1.294335	1.19748	1.090761	1.09182
1.14465	1.206285	1.294335	1.171065	1.056409	1.10943
1.17987	1.17987	1.206285	1.188675	1.215323	1.07421
1.206285	1.206285	1.17987	1.206285	1.179047	1.12704
1.26792	1.206285	1.19748	1.12704	1.14332	1.206285
1.28553	1.259115	1.171065	1.17987	1.056409	1.083015
1.211568	1.216851	1.227417	1.169304	1.125663	1.139367

Helicopter 1

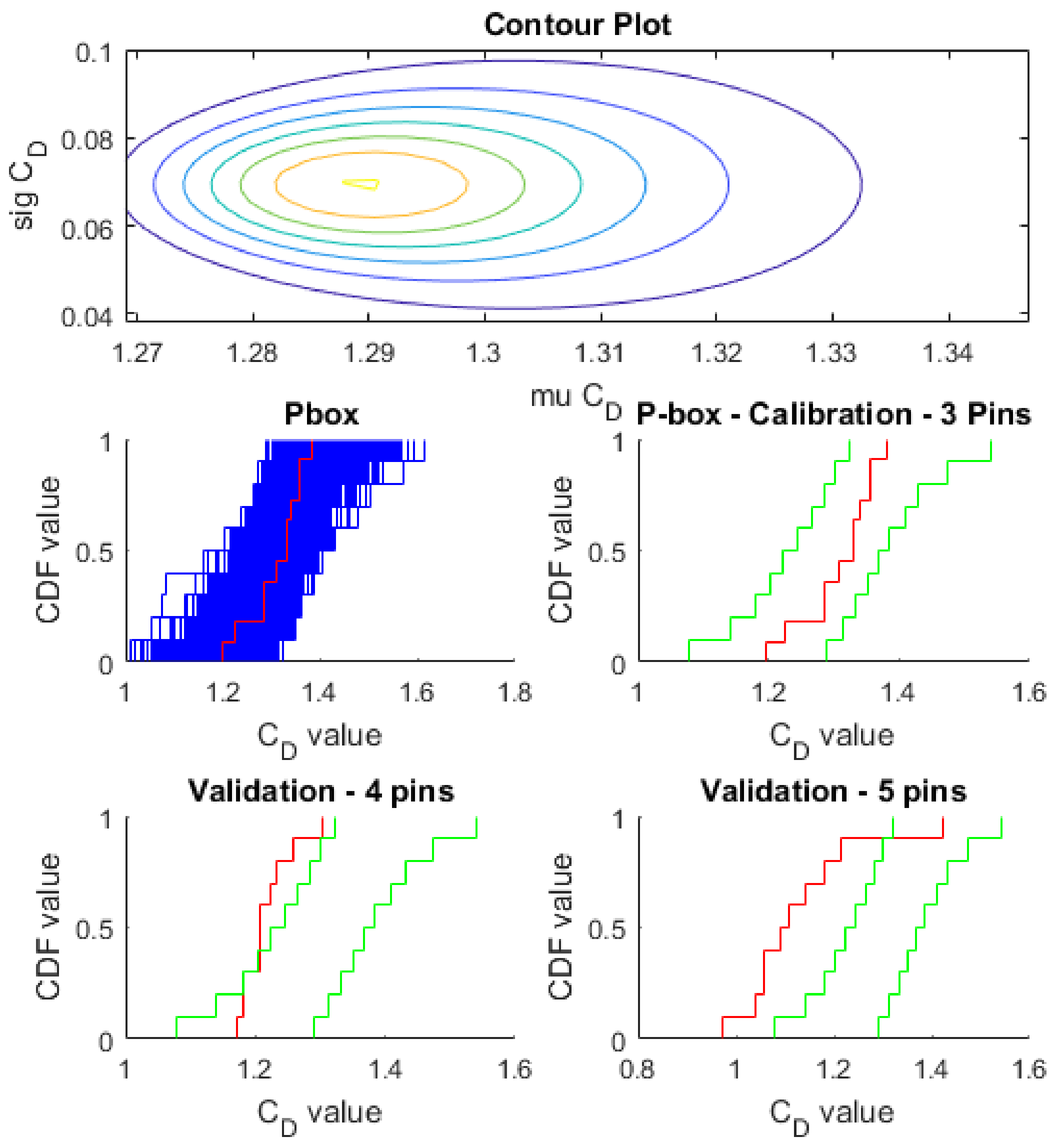


Area Metric for 4 pins = 0

Area Metric for 5 pins = 3.44631e-04

Helicopter 1 when assumed to be **linearly dependent on speed**, validated more accurately than the quadratic model for the calibrated C_D values, as can be seen from the negligible area metric for the experimental values with the p-box.

Helicopter 2

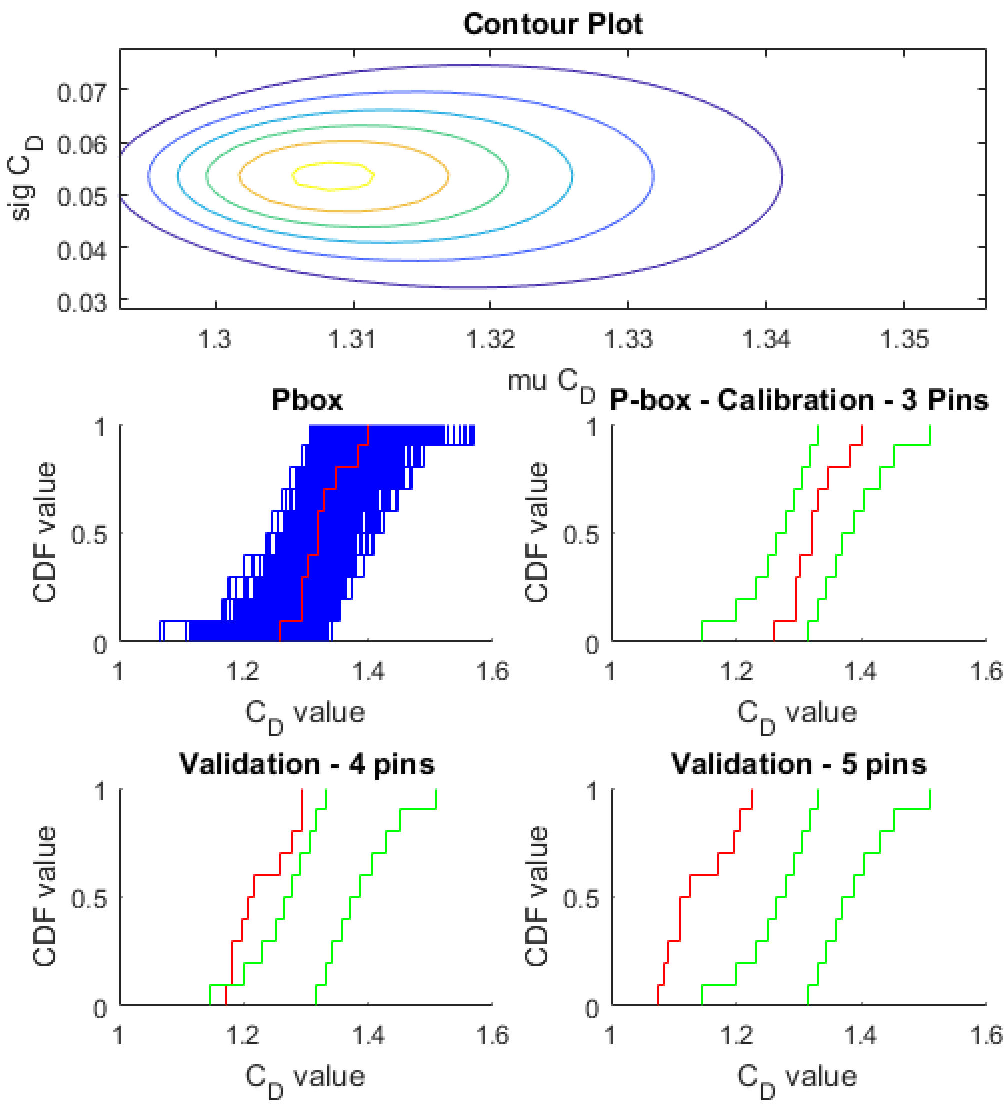


Area Metric for 4 pins = 0.0221

Area Metric for 5 pins = 0.108

From the very large area metric for helicopter 2 C_D values with the calibrated p-box, it can be concluded that the assumed linear dependence on speed was wrong for helicopter 2.

Helicopter 3



Area Metric for 4 pins = 0.0350

Area Metric for 5 pins = 0.1206

Similar to helicopter 3, from the very large area metric for helicopter 3 C_D values with the calibrated p-box, it can be concluded that the assumed linear dependence on speed was wrong for helicopter 3.

6. Comparison with results in the paper by Park, Choi and Haftka:

Data from paper, performed by student A for helicopter is considered for comparison:

Student A Copter 1				
		1 Clip		
Mass:	2.05	Height = 124 inch		
Trial	Fall time	Velocity(inch/sec)	CD_quadratic	CD_linear
1	2.917	42.50942749	0.867991135	1.024939
2	2.9	42.75862069	0.857903475	1.018966
3	3.034	40.87013843	0.93901728	1.066049
4	2.666	46.51162791	0.72504126	0.936746
5	2.951	42.01965435	0.888343339	1.036886
6	2.85	43.50877193	0.828575622	1.001397
7	2.867	43.25078479	0.838489864	1.007371
8	2.833	43.76985528	0.818720341	0.995424
9	2.867	43.25078479	0.838489864	1.007371
10	2.548	48.66562009	0.662279449	0.895284
Average	2.8433	43.61129673	0.824684442	0.999043

Student A Copter 1				
		2 Clip		
Mass:	2.78	Height = 124 inch		
Trial	Fall time	Velocity(inch/sec)	CD_quadratic	CD_linear
1	2.076	59.73025048	0.596194636	0.98919
2	2.193	56.54354765	0.665289439	1.04494
3	2.109	58.79563774	0.615299448	1.004915
4	2.243	55.28310299	0.695972233	1.068764
5	2.126	58.32549389	0.625258906	1.013015
6	1.975	62.78481013	0.539594564	0.941065
7	2.142	57.8898226	0.634705555	1.020639
8	2.076	59.73025048	0.596194636	0.98919
9	2.193	56.54354765	0.665289439	1.04494
10	2.548	48.66562009	0.898115545	1.214093
Average	2.1681	57.19293391	0.650267405	1.033075

Student A Copter 1				
		1 Clip		
Mass:	2.05	Height = 139 inch		
Trial	Fall time	Velocity(inch/sec)	CD_quadratic	CD_linear
1	3.364	41.31985731	0.918688279	1.429932
2	3.415	40.70278184	0.946755021	1.451611
3	3.364	41.31985731	0.918688279	1.429932
4	3.482	39.91958644	0.984268836	1.48009
5	3.381	41.11209701	0.927996937	1.437158
6	3.549	39.16596224	1.022511497	1.50857
7	3.498	39.73699257	0.993335168	1.486891
8	3.465	40.11544012	0.974681401	1.472864
9	3.415	40.70278184	0.946755021	1.451611
10	3.515	39.54480797	1.003013688	1.494118
Average	3.4448	40.35067348	0.963350276	1.464278

Taking ratio of average fall time and comparing with ratio of masses, suggested that the helicopter follows linear model same as mentioned in paper.

However, the total C_D in the paper is less than that obtained in our experiments, which can be mainly due to change in dimensions.

The differences in results can also be attributed to the different paper chosen for performing experiments, stiffer card paper in our experiments versus the copy machine paper used for experiments in the paper.

Conclusions

- The weight of the thicker paper requires more paper clips to achieve stability, hence it is better to calibrate using 3 pins rather than 2 pins and validate for the rest.
    - Evidenced by the tests with 3 paper clips, which exhibited more instability on an average test than what is seen with the thicker paper
- Terminal velocities increased by about 4 in/s with each additional paper clip attached to the helicopter
- Comparing ratio of masses with ratio of fall time for helicopter 1 suggested that the helicopter follows quadratic model. However, calibration and validation agreed more for the linear model. Hence just comparing ratio of masses and fall time is NOT sufficient to determine which model the helicopter follows.
- When Helicopter 2 and 3 were calibrated using 3 pins using quadratic dependence, they validated perfectly, i.e. CDF obtained from experimental values of helicopters with 4 and 5 pins was completely within 95% CI bounds of P-box. However there was a large area metric and disagreement of experimental CDF and P-box when helicopters were assumed to follow linear model.

MATLAB Codes

For 95% CI P-box Calibration

```
clc;close all, clear all;
figure
subplot(2,2,[1 2])
%Cd values for helicopter 3 with 2 paper clips
CDs=[0.91308361,0.753552091,0.718910702,1.434742169,0.801008529,0.630518557,0
.538005672,0.508798227,,0.874910158,0.789008571,0.779473812];
std_ =std(CDs);
mean_=mean(CDs);
Nsu_=1000; %total Cds to generate
post__=normrnd(mean_,std_,Nsu_,1); %posterior distribution
histfit(post__)
xlabel('x')
ylabel('f(x)')
title('Posterior distribution')
cddat=ones(11,1000) %matrix to store all the mu and sigma values;

%% P-box from the posterior of mean

figure(1)
subplot(2,2,3)
for i=1:Nsu_
    Ns=10 %each set will have Ns values of Cd
    cd=normrnd(post__(i),std_,Ns,1);% using the same std as exp.
    cdsort=sort(cd);
    [Fim,xim]=ecdf(cdsort);
    cddat(:,i)=xim';%add the xim cdf values as a column in cddat
    hold on, stairs(xim,Fim,'b');
end
% 95% CI for the p-box
for i=1:11
    temp=sort(cddat(i,:));
    cilow(i)=temp(26)
    cihigh(i)=temp(974);
end

%cilow=cddat(:,[26])'; %lower bound, pick the 26th column

%cihigh=cddat(:,[974])';%upper bound, pick the 974th column



%% Plot the experimental CDF
yobs=CDs;
ysort=sort(yobs);
[Fi,xi]=ecdf(ysort);%get the analytical CDF
hold on, stairs(xi,Fi,'r');
xlabel('C_D value');
ylabel('CDF value');
title('Pbox');

%% Plot the 95% confidence interval CDF
```

```
subplot(2,2,4)
[Fi,xi] = ecdf(ysort);
hold on, stairs(xi,Fi,'r');
[rx,ry]=stairs(xi,Fi);
hold on, stairs(cilow,Fim,'g');
[glx,gly]=stairs(cilow,Fim);
hold on, stairs(cihigh,Fim,'g');
[ghx,ghy]=stairs(cihigh,Fim);
xlabel('C_D value');
ylabel('CDF value');
title('P-box with 95% CI');


%% Area metric

cdrange=linspace(min(rx),max(rx),100);

% >> [x, index] = unique(x);
% >> yi = interp1(x, y(index), xi);

[glx,index]=unique(glx);
cil = interp1(glx,gly(index),cdrange);
[ghx,index]=unique(ghx);
ciu = interp1(ghx,ghy(index),cdrange);
[rx,index]=unique(rx);
r_exp= interp1(rx,ry(index),cdrange);
sum=0;
for i=1:100
    if r_exp(i)>cil(i)
        sum=sum+r_exp(i)-cil(i);
    end
    if r_exp(i)<ciu(i)
        sum=sum+ciu(i)-r_exp(i);
    end
end
sum=sum/100
```

For Validation of Helicopter against calibrated P-box

```
clc;close all, clear all;



H1P1=[1.147227231
1.025756737
1.147227231
1.08564254
1.162888855
1.100879442
1.100879442
1.040568917
1.162888855
1.131671788
```

```
];
k_=length(H1P1);
pd=fitdist(H1P1,'Normal')

% mu =   1.20351   [1.1419, 1.25284]
% sigma = 0.0734226   [0.0513016, 0.128852]


mu=linspace(1.074,1.146,30);
x_=length(mu);
sig=linspace(0.032,0.09,30);
y_=length(sig);
[X,Y]=meshgrid(mu,sig);

L=meshgrid(mu,sig);
%%For Non-informative Prior
for i=1:x_
    for j=1:y_
        L(i,j)=1;
    end
end

%Likelihood Function
for i=1:x_
    for j=1:y_
        for k_=1:k_
            L(i,j)=L(i,j)*normpdf(H1P1(k_),mu(i),sig(j));
        end
    end
end

subplot(3,2,[1 2])
contour(X,Y,L);
xlabel('mu C_D');
ylabel('sig C_D');
title('Contour Plot')


%% Generate a p-box from the joint posterior distribution

%helicopter 1 with 2 paper clips
Nsu=100;  %generate 100 CDF sets
% select mu from the joint distribution
mu1_1=mu
nmu=length(mu1_1);

%select std from the joint distribution
sd1_1 =sig
nsd=length(sd1_1)

%Initialize a 11*100 matrix of ones that will store all the xCdF values
cddat = ones(11,900);
count=0;
```

```
%% P-box from the joint distribution


subplot(3,2,3)
for i=1:nmu
    for j=1:nsd
        count=count+1;
        Ns=10; %Each mu will have 10 CDs corresponding to different std
        cd=random('normal',mu1_1(i),sd1_1(j),Ns,1);
        cdsort=sort(cd); % sort them in ascending order to plot
        [Fim,xim]=ecdf(cdsort);
        cddat(:,count)=xim';%add the xim cdf values as a column in cddat
        hold on, stairs(xim,Fim,'b');
    end
end
% 95% CI for the p-box
for i=1:11
    temp=sort(cddat(i,:));
    cilow(i)=temp(24)
    cihigh(i)=temp(876);
end

%cilow=cddat(:,[26]); %lower bound, pick the 26th column
%cihigh=cddat(:,[974]);%upper bound, pick the 974th column

%% Plot the experimental CDF 1 calibration
yobs=H1P1;
ysort=sort(yobs);
[Fi,xi]=ecdf(ysort);%get the analytical CDF
hold on, stairs(xi,Fi,'r');
xlabel('C_D value');
ylabel('CDF value');
title('Pbox');

%% Plot the 95% confidence interval CDF

subplot(3,2,4)
[Fi,xi] = ecdf(ysort);
hold on, stairs(xi,Fi,'r');
[rx,ry]=stairs(xi,Fi);
hold on, stairs(cilow,Fim,'g');
[glx,gly]=stairs(cilow,Fim);
hold on, stairs(cihigh,Fim,'g');
[ghx,ghy]=stairs(cihigh,Fim);
xlabel('C_D value');
ylabel('CDF value');
title('P-box - Calibration - 3 Pins');


%% Area metric

cdrange=linspace(0,1,100);

% >> [x, index] = unique(x);
```

```
% >> yi = interp1(x, y(index), xi);

[gly,index]=unique(gly);
cil = interp1(gly,glx(index),cdrange);
[ghy,index]=unique(ghy);
ciu = interp1(ghy,ghx(index),cdrange);
[ry,index]=unique(ry);
r_exp= interp1(ry,rx(index),cdrange);
sum=0;
for i=1:100
    if r_exp(i)<cil(i)
        sum=sum-r_exp(i)+cil(i);
    end
    if r_exp(i)>ciu(i)
        sum=sum-ciu(i)+r_exp(i);
    end
end
sum=sum/100;
sum_c=sum;


%% Plot the experimental CDF 1 calibration  ---V1
yobs=[1.193593758
1.193593758
1.045064412
1.210705995
1.077218367
1.029170128
1.093478037
1.142987819
1.262773478
1.298094109
];
ysort=sort(yobs);
[Fi,xi]=ecdf(ysort);%get the analytical CDF
% hold on, stairs(xi,Fi,'r');
% xlabel('C_D value');
% ylabel('CDF value');
% title('Pbox');

%% Plot the 95% confidence interval CDF

subplot(3,2,5)
[Fi,xi] = ecdf(ysort);
hold on, stairs(xi,Fi,'r');
[rx,ry]=stairs(xi,Fi);
hold on, stairs(cilow,Fim,'g');
[glx,gly]=stairs(cilow,Fim);
hold on, stairs(cihigh,Fim,'g');
[ghx,ghy]=stairs(cihigh,Fim);
xlabel('C_D value');
ylabel('CDF value');
title('Validation - 4 pins');
```

```
%% Area metric

cdrange=linspace(0,1,100);

% >> [x, index] = unique(x);
% >> yi = interp1(x, y(index), xi);

[gly,index]=unique(gly);
cil = interp1(gly,glx(index),cdrange);
[ghy,index]=unique(ghy);
ciu = interp1(ghy,ghx(index),cdrange);
[ry,index]=unique(ry);
r_exp= interp1(ry,rx(index),cdrange);
sum=0;
for i=1:100
    if r_exp(i)<cil(i)
        sum=sum-r_exp(i)+cil(i);
    end
    if r_exp(i)>ciu(i)
        sum=sum-ciu(i)+r_exp(i);
    end
end
sum=sum/100;
sum_v_1=sum;


%% Plot the experimental CDF 1 calibration  ---V2
yobs=[1.161115016
1.252149182
1.039438407
1.289524718
1.270768245
1.215323285
1.252149182
1.289524718
1.12566322
1.233667529
];
ysort=sort(yobs);
[Fi,xi]=ecdf(ysort);%get the analytical CDF
% hold on, stairs(xi,Fi,'r');
% xlabel('C_D value');
% ylabel('CDF value');
% title('Pbox');

%% Plot the 95% confidence interval CDF

subplot(3,2,6)
[Fi,xi] = ecdf(ysort);
hold on, stairs(xi,Fi,'r');
[rx,ry]=stairs(xi,Fi);
hold on, stairs(cilow,Fim,'g');
[glx,gly]=stairs(cilow,Fim);
hold on, stairs(cihigh,Fim,'g');
[ghx,ghy]=stairs(cihigh,Fim);
```

```
xlabel('C_D value');
ylabel('CDF value');
title('Validation - 5 pins');


%% Area metric

cdrange=linspace(0,1,100);

% >> [x, index] = unique(x);
% >> yi = interp1(x, y(index), xi);

[gly,index]=unique(gly);
cil = interp1(gly,glx(index),cdrange);
[ghy,index]=unique(ghy);
ciu = interp1(ghy,ghx(index),cdrange);
[ry,index]=unique(ry);
r_exp= interp1(ry,rx(index),cdrange);
sum=0;
for i=1:100
    if r_exp(i)<cil(i)
        sum=sum-r_exp(i)+cil(i);
    end
    if r_exp(i)>ciu(i)
        sum=sum-ciu(i)+r_exp(i);
    end
end
sum=sum/100;
sum_v_2=sum;


sum_c
sum_v_1
sum_v_2
```

References:

1. Park, C., Choi, J., & Haftka, R. T. (2016). Teaching a Verification and Validation Course Using Simulations and Experiments With Paper Helicopters. *Journal of Verification, Validation and Uncertainty Quantification, 1*(3), 031002. doi:10.1115/1.4033889
2. Oberkamph, W., & Roy, C. (n.d.). *VERIFICATION AND VALIDATION IN SCIENTIFIC COMPUTING*. New York: Cambridge University Press, .
3. Measures of Variability. (n.d.). Retrieved April 10, 2017, from http://onlinestatbook.com/2/summarizing_distributions/variability.html

Solved example - Bayesian module

Difficulty Level: Medium

Problem: (The problem is more of information and little analysis)

Inspired by nature, several models are made to replicate same phenomenon. However, they must be assessed for accuracy by comparison with experimental data, which in statistical terms is called Validation. Whichever model validates more accurately is considered better or according to William of Occam, when you have 2 competing theories giving exactly same predictions, the simpler one is better. Validation is performed considering few guidelines,

- A validation experiment should try to emphasize the inherent **synergism** that is attainable between computational and experimental approaches.
- **Independence** must be maintained in obtaining the computational and experimental data.

Now suppose your class is performing a helicopter project. Students are divided into three groups, validating two different models,

Quadratic model: Which assumes quadratic dependence of drag force on the velocity
Linear model: Which assumes quadratic dependence of drag force on the velocity

To maintain **independence** one group 1 performs experiments and provides the drag coefficient for both models. Group 2, given a joint posterior distribution, samples and creates a Probability box(P-box) for further validation. You are in the 3rd group, responsible for maintaining **synergism** between group 1 and 2 by making a **joint posterior distribution** given the following values of C_D

Assume that the drag coefficient from tests follow a normal distribution with the mean and standard deviation of the true C_D. Start with a non-informative prior for the mean and standard deviation.

Test	1	2	3	4	5	6	7	8	9	10	Avg.
CD_quadratic	1.308	1.259	1.025	1.210	1.226	1.131	1.259	1.131	1.210	0.981	1.172
CD_linear	1.382	1.355	1.223	1.329	1.338	1.285	1.355	1.285	1.329	1.197	1.308

- Additional Reading Assignment : Read about P-box

Solution:

Joint Posterior Distribution:

The posterior probability is the probability of the parameter θ given random variable X. It is denoted as $p(\theta|X)$. The likelihood, on the other hand is the probability that we get the random variable X, given θ. It is denoted by $p(X|\theta)$. The two are related by the familiar Bayes rule:

$$p(\theta|x)=p(x|\theta)p(\theta)/p(x)$$

where x stands for the observations. Here $p(\theta)$ is called the prior and in general, the posterior probability = likelihood*prior

We assume a uniform prior for C_D as our non-informative prior. In such case, the likelihood becomes the joint probability density. To compute the likelihood we calculate the proability distribution value at each point for each C_D value and take their product.

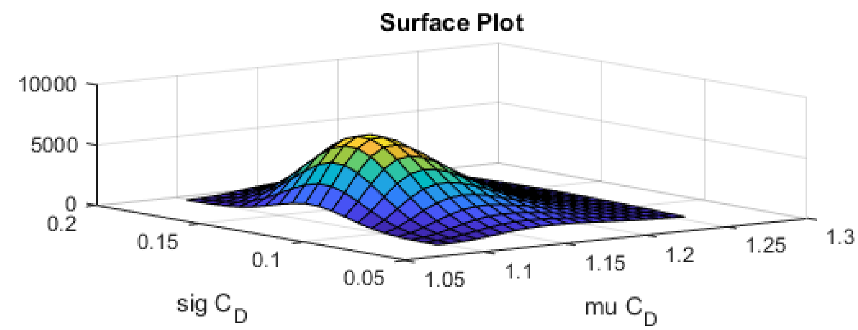
From our data, we have a bunch of $Mean_{CD}$ and σ_{CD}. To get the joint posterior distribution of C_D and σ_{test}, we look for the cases where their joint occurance is maximum.

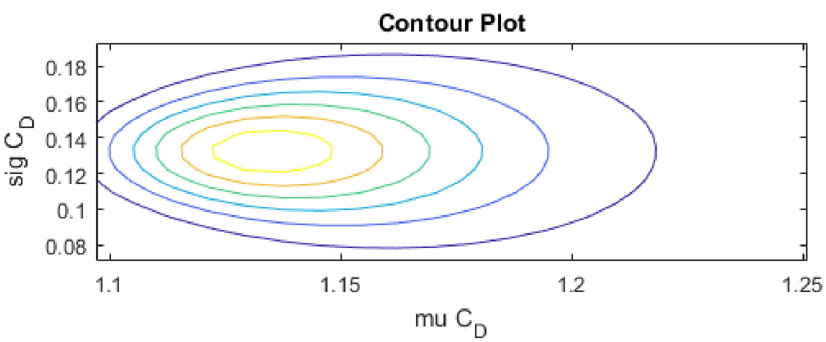
The joint posterior distribution can be represented as a surface plot or a contour plot.

Given Normal distribution of C_D,

Mean and sigma for C_D following quadratic speed dependence are,

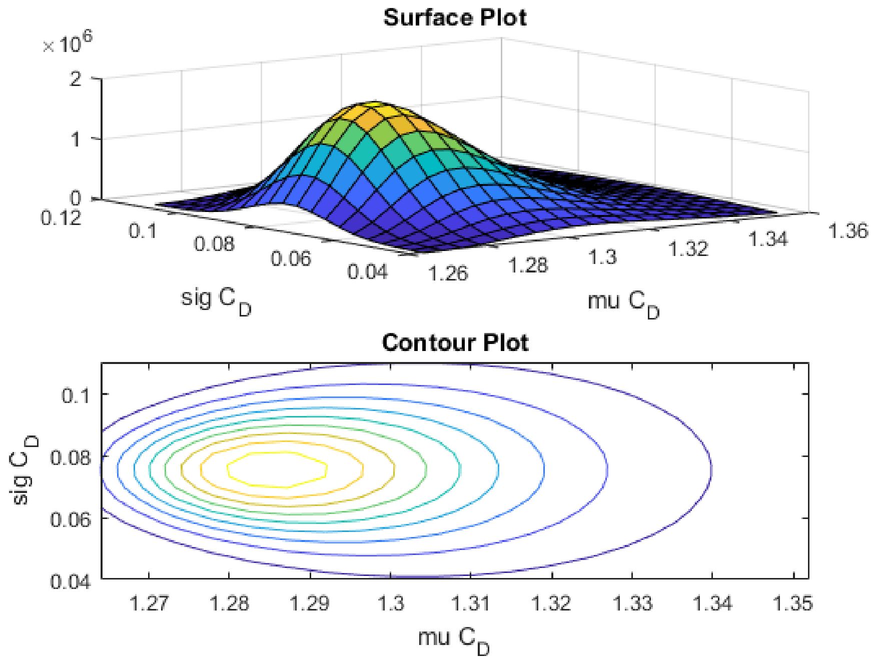
mu = 1.20351 [1.15419, 1.25284]
sigma = 0.0734226 [0.0513016, 0.128852]





Mean and sigma for C_D following linear speed dependence are,

mu =   1.30842   [1.26558, 1.35127]
sigma = 0.0598913   [0.0411953, 0.109338]



Matlab Code for obtaining the mean, sigma and plotting the joint posterior distribution is given below:

```
CD=[1.382385119,1.355970117,1.223895105,1.329555114,1.338360115,1.28553011
1,1.355970117,1.285530111,1.329555114,1.197480103];
];
k_=length(CD);
pd=fitdist(CD,'Normal')

mu=linspace(1.264,1.352,20);
x_=length(mu);
sig=linspace(0.040,0.11,20);
y_=length(sig);
[X,Y]=meshgrid(mu,sig);

L=meshgrid(mu,sig);
%%For Non-informative Prior
for i=1:x_
    for j=1:y_
        L(i,j)=1;
    end
end

%Likelihood Function
```

```
for i=1:x_
    for j=1:y_
        for k_=1:k_
            L(i,j)=L(i,j)*normpdf(CD(k_),mu(i),sig(j));
        end
    end
end

figure
subplot(2,1,1)
surf(X,Y,L);
xlabel('mu C_D');
ylabel('sig C_D');
title('Surface Plot')

subplot(2,1,2)
contour(X,Y,L);
xlabel('mu C_D');
ylabel('sig C_D');
title('Contour Plot')
```