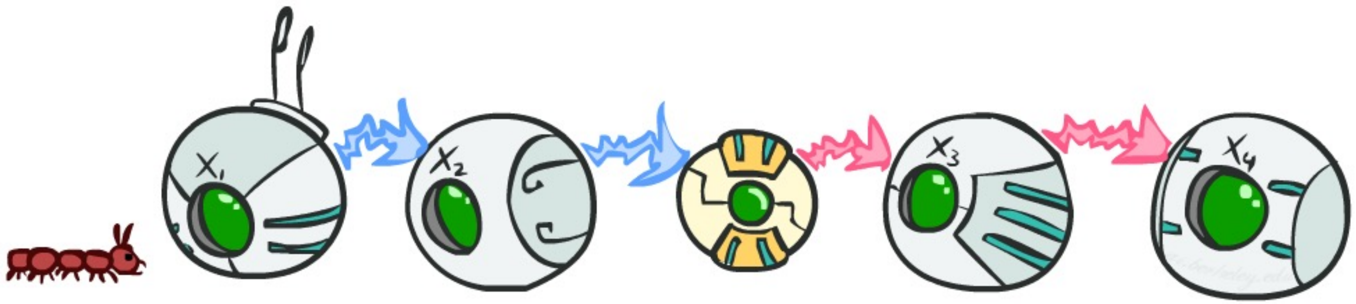
Artificial Intelligence

Markov Models

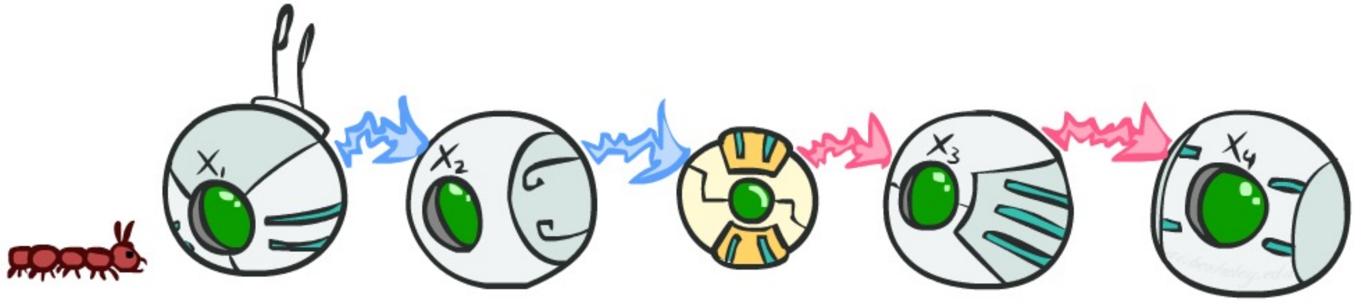


[These slides were created by Dan Klein, Pieter Abbeel, and Sergey Levine. http://ai.berkeley.edu.]

Uncertainty and Time

- Often, we want to reason about a ***sequence*** of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time into our models

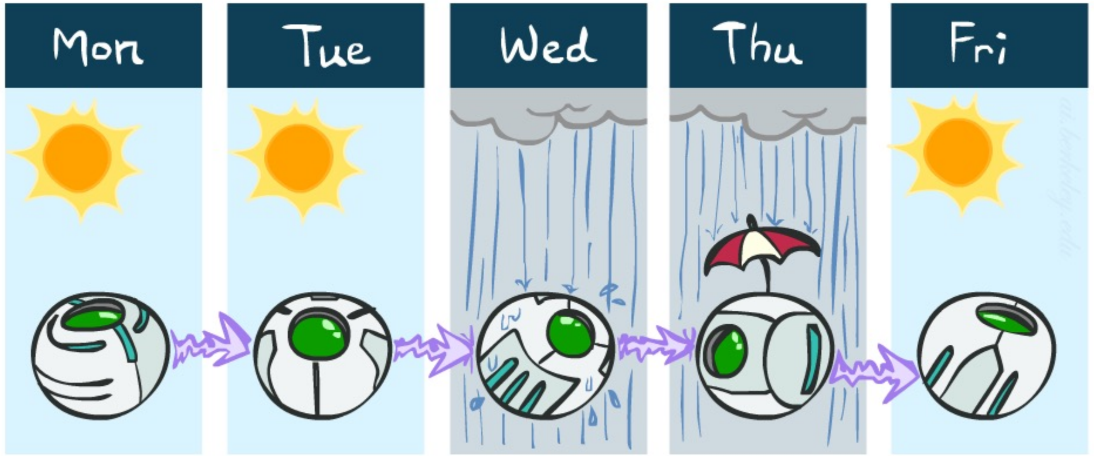
Markov Models



- Basic conditional independence:
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

Example Markov Chain: Weather

- States: X = {rain, sun}

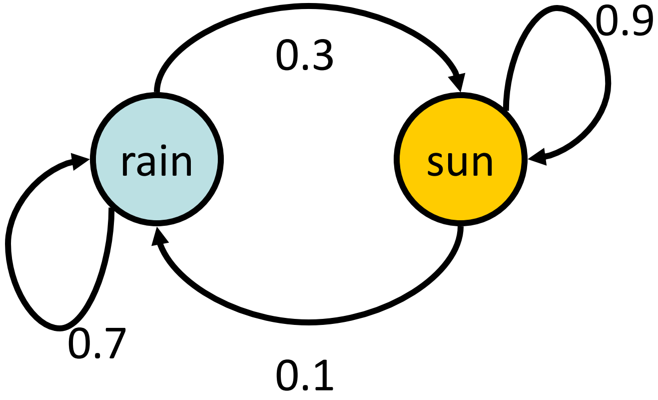


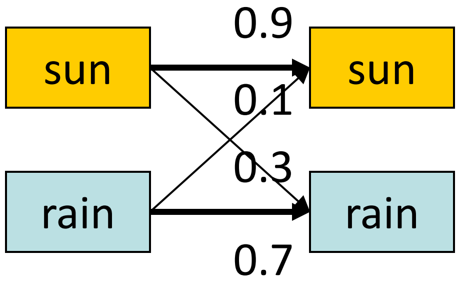
- Initial distribution: 0.5 sun

- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Two new ways of representing the same CPT

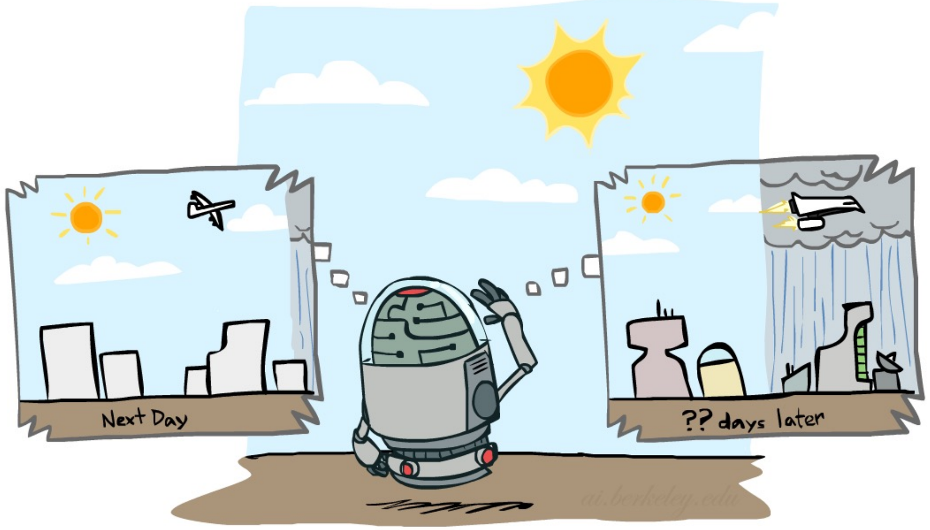




Forward Algorithm

- Question: What's P(X) on some day t?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$



$$\color{red}{P(x_1)} = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

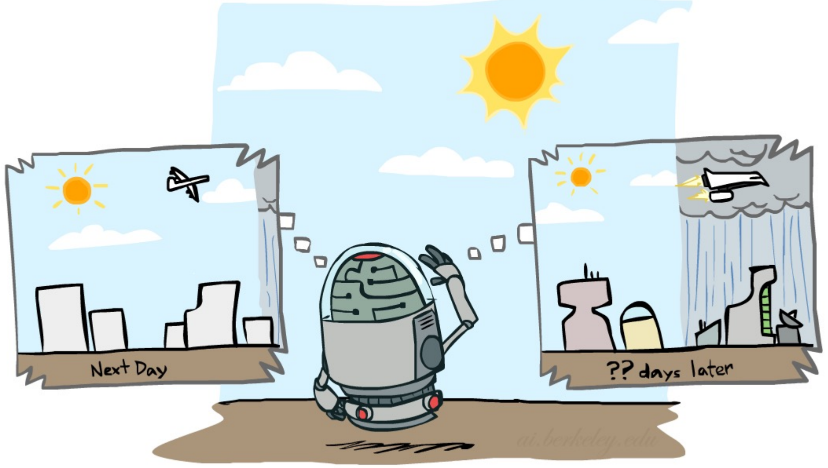
$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Stationary Distributions

- Question: What's P(X) at time t = infinity?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$



$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

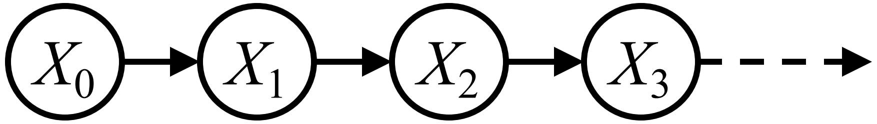
Also: $P_\infty(sun) + P_\infty(rain) = 1$

$$P_\infty(sun) = 3/4$$
$$P_\infty(rain) = 1/4$$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Markov Models (aka Markov chain/process)

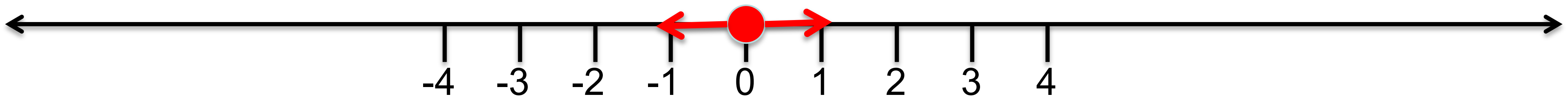
- Value of X at a given time is called the ***state*** (usually discrete, finite)



$P(X_0)$ $P(X_t \mid X_{t-1})$

- The ***transition model*** $P(X_t \mid X_{t-1})$ specifies how the state evolves over time
- ***Stationarity*** assumption: transition probabilities are the same at all times
- ***Markov*** assumption: "future is independent of the past given the present"
  - X_{t+1} is independent of $X_0,\ldots, X_{t-1}$ given X_t
  - This is a ***first-order*** Markov model (a kth-order model allows dependencies on k earlier steps)
- Joint distribution $P(X_0,\ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t-1})$

Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial probability: starts at 0
- Transition model: $P(X_t = k \mid X_{t-1} = k\pm1) = 0.5$
- Applications: particle motion in crystals, stock prices, gambling, genetics, etc.
- Questions:
  - How far does it get as a function of t?
    - Expected distance is $O(\sqrt{t})$
  - Does it get back to 0 or can it go off for ever and not come back?
    - In 1D and 2D, returns w.p. 1; in 3D, returns w.p. 0.34053733

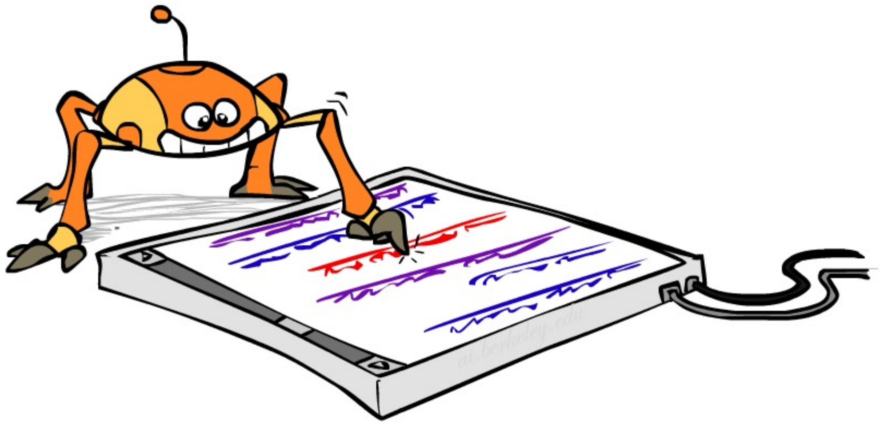
Example: n-gram models

We call ourselves *Homo sapiens*—man the wise—because our **intelligence** is so important to us.
For thousands of years, we have tried to understand *how we think*; that is, how a mere handful of matter can perceive, understand, predict, and manipulate a world far larger and more complicated than itself. ….

- State: word at position *t* in text (can also build letter n-grams)
- Transition model (probabilities come from empirical frequencies):
  - Unigram (zero-order): $P(Word_t = i)$
    - "logical are as are confusion a may right tries agent goal the was . . ."
  - Bigram (first-order): $P(Word_t = i \mid Word_{t-1} = j)$
    - "systems are very similar computational approach would be represented . . ."
  - Trigram (second-order): $P(Word_t = i \mid Word_{t-1} = j, Word_{t-2} = k)$
    - "planning and scheduling are integrated the success of naive bayes model is . . ."
- Applications: text classification, spam detection, author identification, language classification, speech recognition

Example: Web browsing

- State: URL visited at step *t*
- Transition model:
  - With probability *p*, choose an outgoing link at random
  - With probability (1-*p*), choose an arbitrary new page
- Question: What is the ***stationary distribution*** over pages?
  - I.e., if the process runs forever, what fraction of time does it spend in any given page?
- Application: Google page rank



Back to Weather Prediction

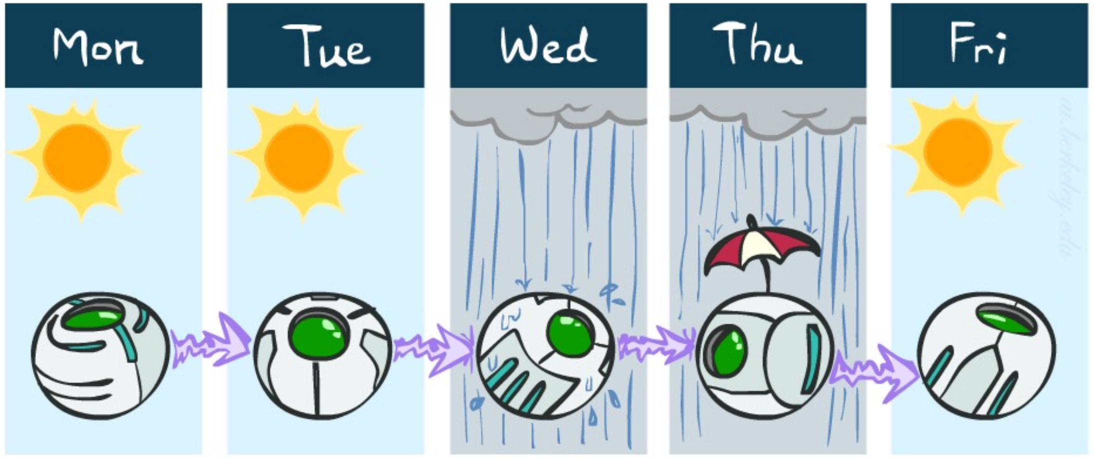
- States {rain, sun}

- Initial distribution $P(X_0)$

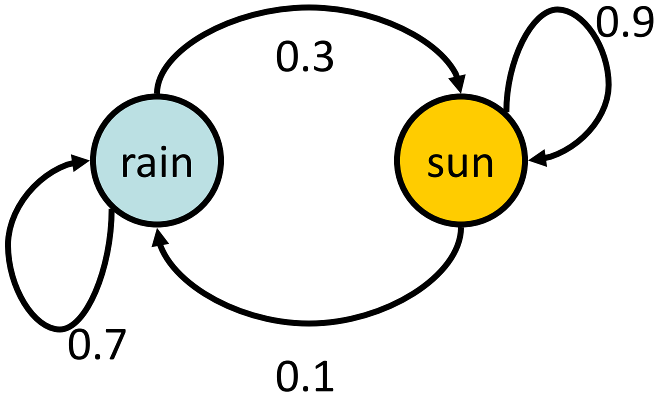
$P(X_0)$	
sun	rain
0.5	0.5

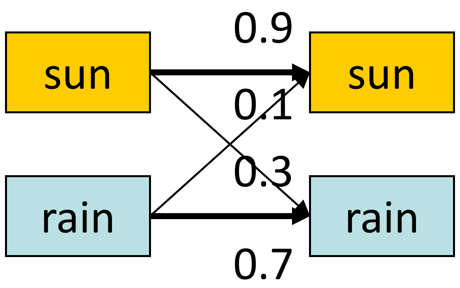
- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Two new ways of representing the same CPT

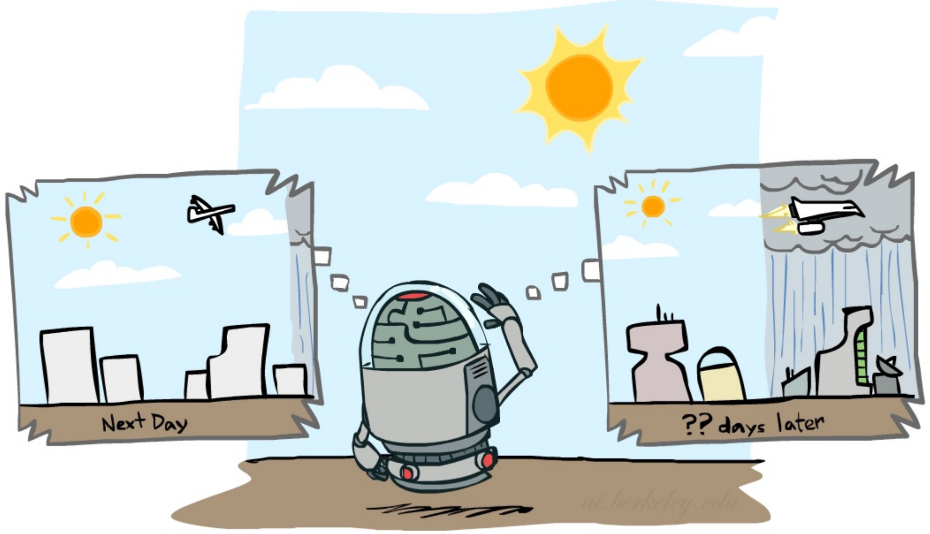




Weather prediction

- Time 0: <0.5,0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

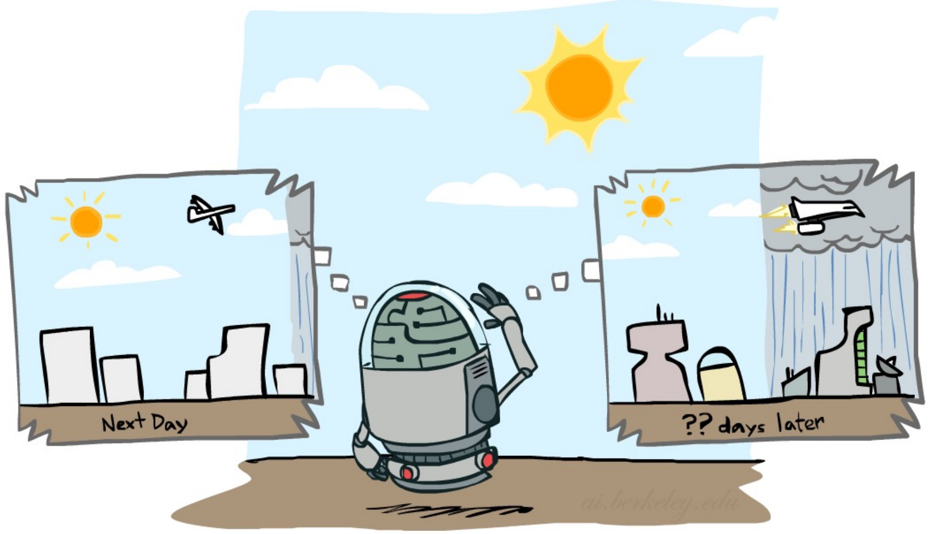


- What is the weather like at time 1?
  - $P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$
  - $= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$
  - $= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

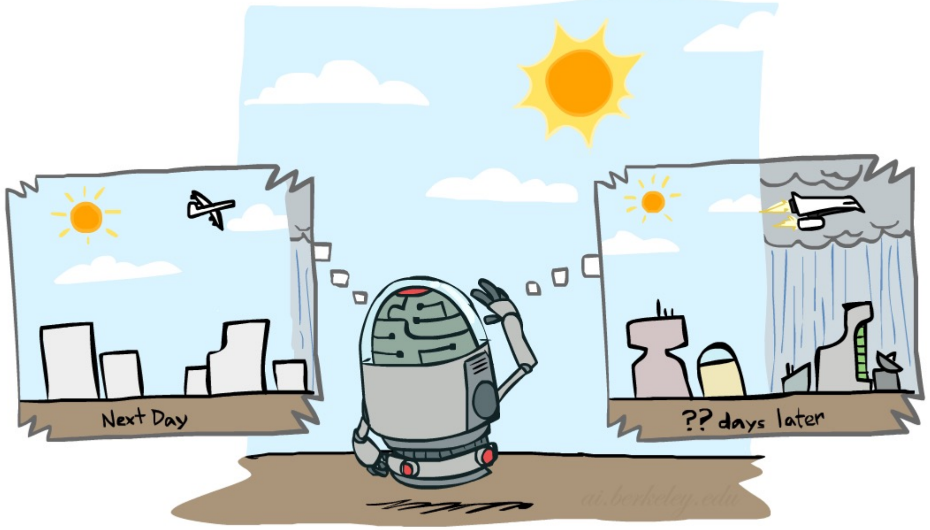


- What is the weather like at time 2?
  - $P(X_2) = \sum_{x_1} P(X_2, X_1 = x_1)$
  - $= \sum_{x_1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$
  - = 0.6<0.9,0.1> + 0.4<0.3,0.7> = <0.66,0.34>

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 3?
  - $P(X_3) = \sum_{x_2} P(X_3, X_2{=}x_2)$
  - $= \sum_{x_2} P(X_2{=}x_2)\, P(X_3 \mid X_2{=}x_2)$
  - $= 0.66\langle 0.9, 0.1\rangle + 0.34\langle 0.3, 0.7\rangle = \langle 0.696, 0.304\rangle$

Forward algorithm (simple form)

Probability from previous iteration

Transition model

- What is the state at time t?
  - $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$
  - $= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\, P(X_t \mid X_{t-1}=x_{t-1})$
- Iterate this update starting at t=0

And the same thing in linear algebra

- What is the weather like at time 2?
  - $P(X_2) = 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle = \langle 0.66,0.34\rangle$
- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- I.e., multiply by T^T, transpose of transition matrix

Stationary Distributions

- The limiting distribution is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^\mathrm{T} P_\infty$
- Solving for P_∞ in the example:

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$$0.9p + 0.3(1-p) = p$$

$$p = 0.75$$

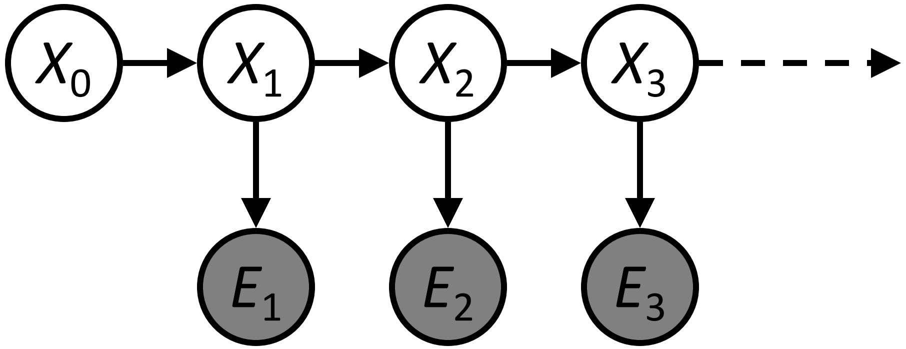
Stationary distribution is <0.75,0.25> ***regardless of starting distribution***

Hidden Markov Models

Hidden Markov Models

- Usually the true state is not observed directly

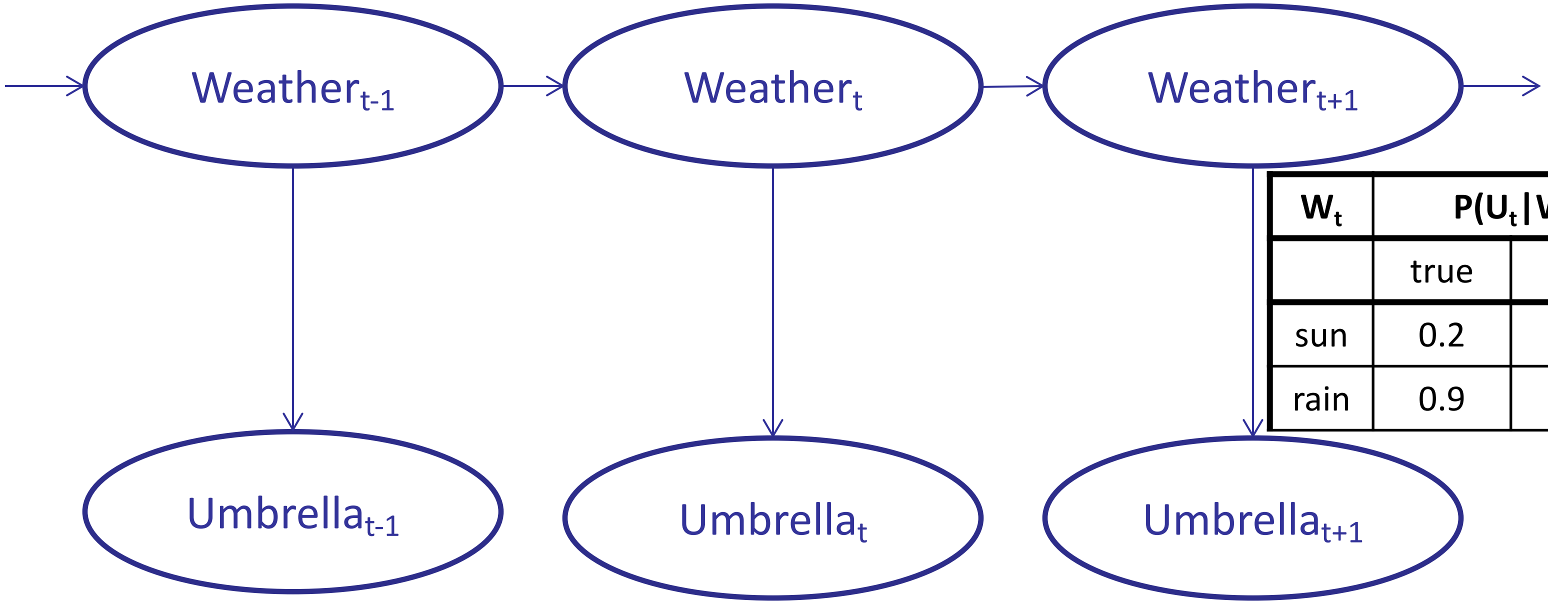
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe evidence E at each time step
  - X_t is a single discrete variable; E_t may be continuous and may consist of several variables



Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t | X_{t-1})$
  - Sensor model: $P(E_t | X_t)$

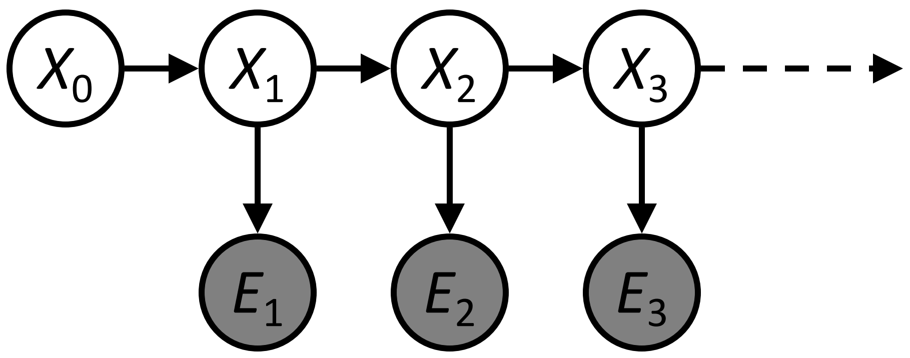
W_{t-1}	$P(W_t \| W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



W_t	$P(U_t \| W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as probability model

- Joint distribution for Markov model: $P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:
  $P(X_0,X_1,\ldots, X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1}) P(E_t \mid X_t)$
- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?



Useful notation:

$X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Filtering / Monitoring

- Filtering, or monitoring, or state estimation, is the task of maintaining the distribution $\boldsymbol{f}_{1:t} = P(X_t|e_{1:t})$ over time
- We start with $\boldsymbol{f}_0$ in an initial setting, usually uniform
- Filtering is a fundamental task in engineering and science
- The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program; core ideas used by Gauss for planetary observations

Filtering algorithm

- Aim: devise a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$
- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t, e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t)$

Apply Bayes' rule

Apply conditional independence

Condition on X_t

Apply conditional independence

Filtering algorithm

- $P(X_{t+1} | e_{1:t+1}) = \alpha \, P(e_{t+1} | X_{t+1}) \, P(X_{t+1} | e_{1:t})$
- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \, \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \, \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) \, P(X_{t+1} | x_t)}_{\text{Predict}}$
- $\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$
- Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states
- Time and space costs are ***constant***, independent of t
- $O(|X|^2)$ is infeasible for models with many state variables
- We get to invent really cool approximate filtering algorithms

And the same thing in linear algebra

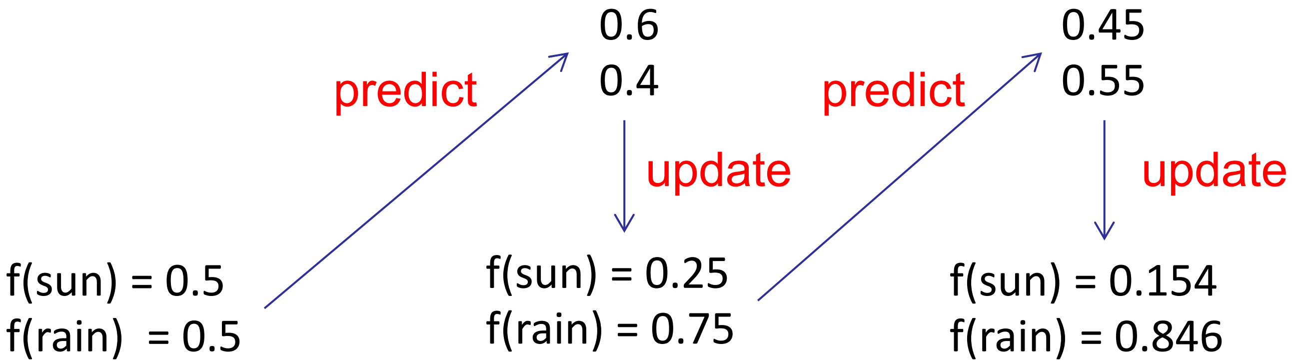
- Transition matrix T, observation matrix O_t
  - Observation matrix has state likelihoods for E_t along diagonal
  - E.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\boldsymbol{f}_{1:t+1} = \alpha\, O_{t+1} T^{\mathrm{T}} \boldsymbol{f}_{1:t}$

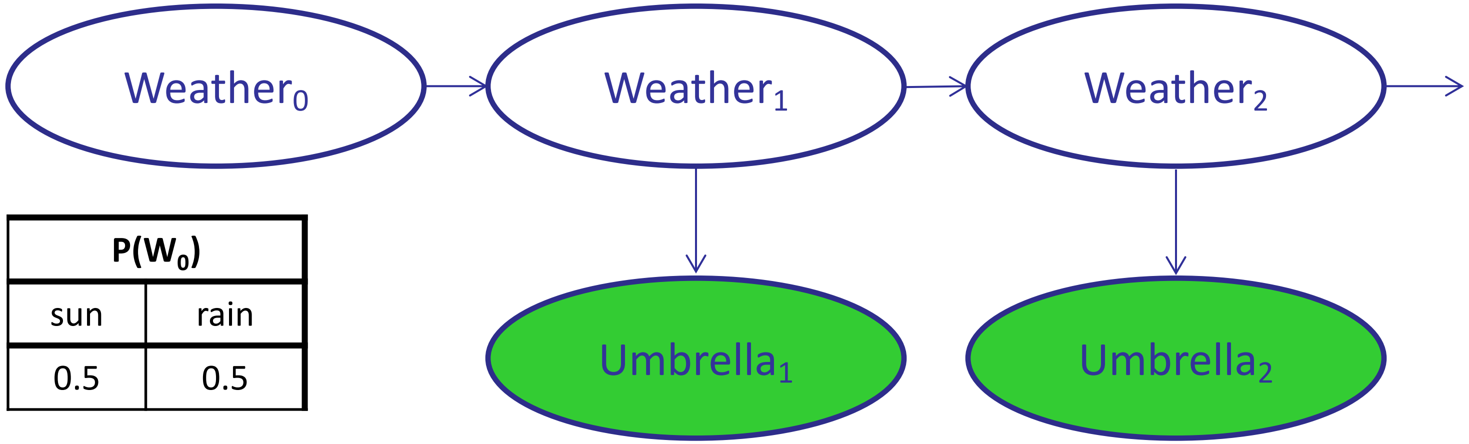
X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Weather HMM

$\alpha\, P(e_{t+1}|X_{t+1})\, P(X_{t+1}|\, e_{1:t})$





P(W$_0$)	
sun	rain
0.5	0.5

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

https://youtu.be/mwn8xhgNpFY