Math for Elementary Teachers: Part I

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1 Integers

We're getting pretty good at using positive numbers. Let's use some negative numbers!

1.1 History of Integers

We look at integers from a historical point of view.

You might have noticed that there is one common class of numbers that we haven't much discussed, yet: negative numbers. One of the choices that the author of your textbook has made is to discuss the various kinds of operations (addition, subtraction, multiplication, division) as a whole, including many of the types of numbers in the discussion. This helps to highlight the fact that multiplication, for instance, is the same thing no matter what kinds of numbers you use. Other textbooks take a different approach: looking at whole numbers and their operations, then fractions and their operations, and so on. We happen to think that our textbook has made the better choice (and you can agree or disagree), but we have deviated a little in this instance.

Now that you have noticed these facts, you might be wondering: why?

There are several reasons, but the ones we would like to discuss here are the pattern of history and the difficulty of finding a good representation.

These two reasons are very much related. If we look over how numbers as a concept were developed throughout history, we definitely find evidence of whole numbers by 3000BC in Ancient Egypt. Beginning around the same period of time in Ancient Babylon, people were starting to use a place-value system which included what we would call decimals, except the Ancient Babylonians used base 60 instead of base 10 as we do. The Ancient Egyptians also had notation for unit fractions, and elaborate methods of combining these unit fractions to express any other fraction they would like. If you're interested in this subject, you can find a nice timeline of very early mathematics at MacTutor[1] .

So, we have seen evidence of whole numbers, fractions, and even decimals from the beginning of mathematics. Negative numbers, however, are a different story. We see the first evidence of negative numbers appearing sometime between 200BC and 200AD in Chinese mathematics, and around the 7th century AD in Indian mathematics. Western European mathematicians didn't start really using negatives in their calculations until the 17th century AD, and didn't allow them as answers to problems or give them equal status amongst the positive numbers, decimals, and fractions until the 19th century!

To emphasize this point a little bit, here are some things that happened in history *before* most Western mathematicians were working with negative numbers.

- The Hindu-Arabic numerals[2] are developed in India and the Middle East around the 6th and 7th centuries. They are brought to Western Europe in the early 1200s.
- The bubonic plague hit Europe and Asia in the mid 1300s.
- Gutenberg invents the printing press around 1440.
- Christopher Columbus sails to the New World around 1492.
- Michelangelo begins painting the Sistine Chapel around 1508.
- Martin Luther publishes his 95 theses, and the Reformation begins around 1517.
- Galileo finds the Earth revolves around the sun around 1613.
- Descartes and Fermat invent coordinate geometry around 1637.
- Isaac Newton and Wilhelm Leibniz invent calculus around 1670.

[1] See MacTutor at http://www-history.mcs.st-and.ac.uk/Chronology/30000BC_500BC.html

[2] See Hindu-Arabic numerals at https://en.wikipedia.org/wiki/Arabic_numerals

(Of course, any such list leaves out an incredible number of important events! Our list here is focused on Western history, as these events are usually more familiar to people who have studied history in the United States. Also, in this time period, mathematics in Western Europe tended to be advancing more rapidly than in other parts of the world. You can find other world history timelines online if you are interested, such as World History AD Timeline[3] .)

Much of mathematicians' hesitation to treat negative numbers as actual numbers came from the idea that mathematics was a subject that should make sense in the real world, and model real-world phenomena. People didn't have a very good representation for negative numbers that made sense all the time. For example, let's consider a few problems.

Question 1 *For each of the problems below, write an expression that would solve the problem, i.e.* $3+4$.

(a) *Jaci has 8 apples, and Joseph has 12 apples. How many apples do the two children have together?* $\boxed{8+12}$

(b) *Jaci has* $\frac{1}{2}$ *of an apple, and Joseph has* $\frac{5}{7}$ *of an apple. How many apples do the two children have together?* $\boxed{\frac{1}{2}+\frac{5}{7}}$

(c) *Jaci has* -8 *apples, and Joseph has* -12 *apples. How many apples do the two children have together?* $\boxed{-8+-12}$

Now, you might say that negative numbers weren't too bad in that situation: after all, you could likely answer the question because you've recognized the structure of addition is the same in all three cases. But what does the problem actually *mean*? What does it mean to have -8 apples? Maybe you can reconcile this, as Indian mathematicians did in their earliest interpretations, to mean that Jaci owes someone 8 apples, and Joseph owes someone 12 apples, and perhaps this interpretation of negative numbers as debts makes sense in some cases.

Let's try another example.

Question 2 *For each of the problems below, write an expression that would solve the problem, i.e.* $3+4$.

(a) *Sasha has 8 bags, and each bag contains 12 sheets of stickers. How many sheets of stickers does Sasha have in total?* $\boxed{(8)(12)}$

(b) *Sasha has* $\frac{1}{2}$ *of a box of stickers, and the box originally contained* $\frac{5}{7}$ *of a sheet of stickers. How many sheets of stickers does Sasha have?* $\boxed{\frac{1}{2}\frac{5}{7}}$

(c) *Sasha has* -8 *bags, and each bag contains* -12 *sheets of stickers. How many sheets of stickers does Sasha have in total?* $\boxed{(-8)(-12)}$

Again, you could likely answer the third question, but the meaning here is likely even less clear. The question itself doesn't even seem to make sense – we have just asked what looks like the right question based on the patterns in the previous two parts. If we understand the objects in the groups to be debts, what does it mean to have a negative group? Why should the overall answer to this question be positive?

[3]See World History AD Timeline at `https://www.fincher.org/History/WorldAD.shtml`

We will look at several kinds of models for these problems in order to try to make some sense of negative numbers, and hopefully be able to write story problems that make sense. We expect this to be pretty challenging: after all, mathematicians struggled to make sense of these ideas for centuries!

1.2 Integers: Models

We discuss various models for integers.

Motivation

From the point of view of arithmetic, the whole numbers have had two serious deficiencies.

- We can add any two whole numbers and get another whole number, but the same isn't true for subtraction.
- We can multiply any two whole numbers and get another whole number, but the same isn't true for division.

We fix the second deficiency by introducing the set of fractions – we can now divide any whole number by any other whole number which is not zero and get a meaningful answer. In fact, the same is true for the entire (now expanded) set of integers and fractions together. To solve the first deficiency, we want to do something similar. The expansion of the whole number system to the system of integers, that is, introducing negative numbers to go along with positive numbers, is designed to resolve this problem.

From an application point of view, we need integers to be able to quantifiably describe situations in which we have what we will call an artificial zero.

Definition 1. *An **artificial zero** is a situation in which it makes sense and is useful for values of the quantities to be less than zero.*

Remember (or read in the section about the history of the integers) that finding such situations was difficult for mathematicians for many centuries! Many physical situations do not need integers because they have what we will call an absolute zero.

Definition 2. *An **absolute zero** is a situation in which it does not make sense and is not useful for values of the quantities to be less than zero.*

Explanation. *It makes no sense to say, "I have fewer than zero oranges", so this situation would have an absolute zero. "I am taking fewer than zero classes this semester" has an* (artificial zero / absolute zero) *, and "I have fewer than zero dollar bills in my wallet" has an* (artificial zero / absolute zero) *, while "It is fewer than zero degrees outside" has an* (artificial zero/ absolute zero) *.*

The terminology for absolute and artificial zeros helps us to distinguish these two types of situations, but we will not in general focus on identifying which situation is which or require you to use this terminology. After all, we want to work here with negatives as well as positives, so we will generally be in situations with an artificial zero!

In situations with an artificial zero, or a zero arbitrarily set, it makes sense to be talking about amounts that are less than zero. Here are some examples.

- *Temperature.* In the Fahrenheit and Celsius scales, zero is set at some arbitrary temperature (often for a reason), but it is possible for temperatures to be colder than that set zero. Notice that the Kelvin scale is different: zero is set at the point there is completely no heat, so a temperature less than zero Kelvin is not possible.
- *Finances.* In finances, a person is said to have zero financial worth when they don't have any money. Also, it makes sense for a person to have a negative financial worth in the sense of owing someone money. So, having a worth of -5 dollars means you need to somehow gain \$5 before you can say you are "even", with no debt and no profit. Notice that this can get complicated! If oranges are the currency,

one could say one has -5 oranges instead of -5 dollars even though it is impossible to physically have -5 oranges or -5 dollar bills. The idea of owing someone oranges or dollars makes this idea work.

- *Sports.* In football, we can take zero to be the line of scrimmage, or where the play begins. If a player who is running gets tackled behind this line, we say that the player has rushed for negative yardage.

  Example 1. *A running play starts at the 25 yard line and ends at the 19 yard line. In this case, the runner has gone backwards* $\underset{\text{given}}{\boxed{6}}$ *yards, so football commentators would say that the play went for a total of* $\underset{\text{given}}{\boxed{-6}}$ *yards.*

  In golf, "par" is defined as the number of shots experts think a golfer should need in order to be able to get the golf ball into the hole. We can take par as our zero in this situation. However, if the golfer takes fewer shots to get into the hole, we say the golfer is so many shots "under par", and if the golfer takes more shots to get into the hole, we say the golfer is so many shots "over par".

  Example 2. *A certain golf hole has a par of* 5*. If a golfer takes* 2 *shots to get into the hole, we say the golfer is* $\underset{\text{given}}{\boxed{3}}$ *shots under par, or at* $\underset{\text{given}}{\boxed{-3}}$ *with respect to par.*

  Notice that the actual number of shots is always a positive number, but par is playing the role of zero.

Representations of Integers

There are several main ways we represent integers, or several important tools we will discuss.

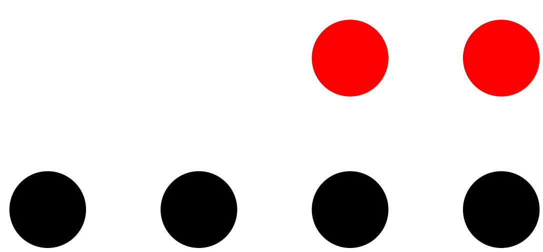
Stories. We use situations similar to those with an artificial zero described above. These stories are useful because we can start to picture what we would like to do with integers, why we would like to do these things, and most importantly we can use the stories to check whether or not our answers make sense.

Chips. We use one black chip to represent one positive unit and one red chip to represent one negative unit. If I have one black chip together with one red chip, I have the same amount in value as if I have no chips at all. We might say that one red chip cancels one black chip.

Example 3. *Sometimes these chips are used alongside a story. If Alisha has seven apples, but also owes Alex four apples, we could use the black chips to represent the apples she has. We would place* $\underset{\text{given}}{\boxed{7}}$ *black chips on the table, each with a value of* $\underset{\text{given}}{\boxed{1}}$ *apple. We could also use the red chips to represent the applese Alisha owes, and in this situation we would place* $\underset{\text{given}}{\boxed{4}}$ *red chips on the table, each with a value of* $\underset{\text{given}}{\boxed{-1}}$ *apples.*

One of the most important things to keep in mind while working with chips is that we can easily change the *representation* of our number without changing the *value* of our number. Let's investigate this phenomenon.

Question 3 *What is the total value of all of the chips in the picture below?*



The total value of the chips is $\underset{\text{given}}{\boxed{2}}$.

Question 4 *Suppose you would like to use chips to represent a total value of* 8*. Which of the following combinations of chips would give you this value?*

Select All Correct Answers:

(a) 8 *black chips* ✓

(b) 8 *red chips*

(c) 10 *black chips and* 2 *red chips* ✓

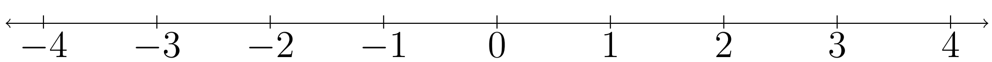
(d) 2 *black chips and* 10 *red chips*

(e) 24 *black chips and* 16 *red chips* ✓

(f) 3 *black chips and* 11 *red chips*

(g) 5 *black chips and* 3 *red chips*

Number lines. Previously, we marked zero and one on our number lines, and used the spacing between zero and one to mark all of the other positive numbers to the right of zero. With integers, we extend the number line to the left of zero as well, using the same spacing between zero and one.



Comparison of Integers

Like with the other number systems weve studied, we would like to discuss having two integer amounts and be able to tell which one is greater, lesser, or if the two amounts are equal. While we could just state a rule for this, as with all mathematics, if it is possible to come up with a rule from a sense-making point of view, it will be more understandable and useful.

Your first guess might be to try a one-to-one correspondence. However, with negative numbers we are sometimes dealing with sets of objects that don't physically exist, and so making a one-to-one correspondence is very difficult in this case. Also, when we work with models like the chips, our physical objects have different meanings, and so they cannot be placed in a one-to-one correspondence without extra care!

Example 4. *Imagine a situation in which Ava has* 4 *black chips, and Aleks has* 8 *red chips. The value of Ava's chips is* $\underset{\text{given}}{\boxed{4}}$ *, while the value of Aleks' chips is* $\underset{\text{given}}{\boxed{-8}}$*. If we line up their chips in a one-to-one correspondence, we see that* (Ava / Aleks) *has more chips, but* (Ava / Aleks) *has the greater value in chips.*

However, like in other number systems, we can use our story problems to look at real-life situations and ask, "Which situation is better?" or "Which number has more value in this situation?" For instance, we have things like the following.

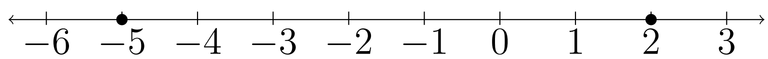
- Which financial worth is better? ("Better" here usually means more money.)

- Which team's total running yards is better? ("Better" here usually means more yards, with positive yards being better than negative yards.)
- Which temperature is better? (Here, "better" depends on the situation! We should probably ask instead, "Which temperature is hotter?")
- Which golf score is better? (Here, "better", with respect to par, would actually be the more negative number!)

Notice that in each case, one needs to define which attribute means greater and which means lesser.

We can also use a number line and take advantage of what we did with positive numbers, but now extending our ideas to negative numbers as well.

Example 5. *Consider the following number line, and the locations of* -5 *and* 2.



On the number line above, $\underset{\text{given}}{\boxed{-5}} < \underset{\text{given}}{\boxed{2}}$ *because* -5 *is to the* (left /right) *of* 2 *on the number line. However, of these two numbers,* $\underset{\text{given}}{\boxed{-5}}$ *is further from zero. If our home was located at* 0 *and we asked, "Which of* -5 *and* 2 *is further from home?" we might say that* $\underset{\text{given}}{\boxed{2}} < \underset{\text{given}}{\boxed{-5}}$.

Again, notice how complicated these ideas get! We should always keep in mind the context for our numbers.

Checks and Bills

We will end this section with two sets of conventions that we plan to use to help make sense of operations with integers. The first is a model that often helps us to write meaningful story problems, called the "checks and bills" model.

Definition 3. *In the* ***checks and bills model****, imagine that you are a small business owner, and that you live in a wonderful land where everyone always pays their bills. We will ask questions about the financial net worth (in dollars) of your business using the following.*

- *An addition sign means to "receive", generally in the mail.*
- *A subtraction sign means to "send", generally in the mail.*
- *A check will refer to a positive number.*
- *A bill will refer to a negative number.*

If we are using chips to represent the checks and bills in our stories, a black chip will mean \$1, and a red chip will mean \$−1.

Example 6. *Yesterday, you wrote a check for twelve dollars. What amount is represented by this check?* $\underset{\text{given}}{\boxed{12}}$

Today, you plan to write a bill for eighteen dollars. What amount is represented by this bill? $\underset{\text{given}}{\boxed{-18}}$

Yesterday, you sent the check you wrote in the mail. So, the twelve dollars should be (added /subtracted) *from your net worth.*

Operations and Number Lines

When we model our operations using number lines, we will use the following conventions.

- An addition sign means to face right (towards the positive numbers).
- A subtraction sign means to face left (towards the negative numbers).
- A positive number means to walk forwards.
- A negative number means to walk backwards.

The result that you see after following these procedures should answer the question, "Where on the number line are we now?" We can also often use our story situation, whether it is checks and bills or something else, to understand how to move on the number line. Some people think of this as asking the question, "Did we get good news or bad news?"

Example 7. *Receiving a bill in the mail is generally considered to be* (good news / bad news) *. If we received a bill for* \$24*, we should move towards the* (positive / negative) *numbers on the line. The number of steps we should move is* $\underbrace{\boxed{24}}_{\text{given}}$.

Receiving a check in the mail is generally considered to be (good news / bad news) *. If we received a check for* \$19*, we should move towards the* (positive / negative) *numbers on the line. The number of steps we should move is* $\underbrace{\boxed{19}}_{\text{given}}$.

1.3 Addition with Integers

We look at adding integers.

Now that we have thought about several ways to represent integers, as well as different models that will help us try to make sense of operations with integers, we are ready to dive in to talk about addition with integers. We will start by thinking again about addition with positive numbers, and try to use what we already know to build what we would like to know. We hope this is becoming one of your practices any time you encounter a new mathematical concept!

As we begin this section, you may want to consider re-reading the sections in your text about addition with whole numbers, since we plan to build from this beginning.

Checks and Bills

Let's begin with one of our most basic addition examples.

Example 8. *Johnny has 9 apples, and Suzy gives him 2 more apples. How many apples does Johnny have now? Write an expression using the addition sign which solves this problem.* $\boxed{9+2}$ given

If we change this example to a story about checks and bills, we might use the following instead.

Example 9. *Johnny opens his business for today with a net worth of \$9, and then he receives a check for \$2. What is Johnny's net worth now?*

Explanation. *Johnny begins with a net worth of \$9. From our checks and bills model, we know that receiving something means we should* (add / subtract) *its value from Johnny's total net worth. Since the object he receives is a check, the value should be* (positive / negative) . *Therefore, the expression we would write which solves this problem is* $\boxed{9+2}$ given.

Let's begin to include some negative numbers in our stories.

Question 5 *Johnny opens his business for today with a net worth of \$−8, and then he receives a check for \$4. What is Johnny's net worth now? As an expression involving the addition sign, Johnny's net worth is now* $\boxed{-8+4}$ given.

Question 6 *Johnny opens his business for today with a net worth of \$−22, and then he receives a bill for \$54. What is Johnny's net worth now? As an expression involving the addition sign, Johnny's net worth is now* $\boxed{-22+(-54)}$ given.

Since we are focusing our attention on addition, all of these story problems are about Johnny receiving something in the mail. His total net worth sometimes increases and sometimes decreases as a result of the value of what he receives in the mail.

There are several important things to notice with these examples. First, our story gives us a good idea whether Johnny's net worth at the end of the day should be positive or negative. If we begin with a positive net worth, and then add a positive number to it, we expect a positive net worth. If we begin with a negative net worth, and then add a negative number to it, we expect a negative net worth. If we start with a positive net worth, and then add a negative number to it, we could get a positive net worth or a negative net worth depending on the relative sizes of the numbers. Making these conclusions didn't require us to know any "rules" about negative numbers at all! Thinking about what the story tells us about the sign of the answer will be important when we start to ask questions about multiplication with negative numbers, so we would like to start practicing this skill now.

A second thing to notice is that we need to exercise care with the question we ask in our checks and bills stories. Writing a good question to go along with your story problem is for many people the most difficult part about writing story problems. Here are some examples of questions to avoid.

Example 10. *Johnny receives a check for \$18 and a bill for \$14. What is Johnny's net worth now?*

Explanation. *We don't actually have enough information to answer this question, because we don't know what Johnny's net worth was when the day began. If he began with a net worth of \$10, his new net worth is* $\underset{\text{given}}{\boxed{10 + 18 + (-14)}}$. *If he began the day with \$0, his new net worth is* $\underset{\text{given}}{\boxed{0 + 18 + (-14)}}$. *The second answer is likely what the writer of this question was aiming for, but we can't know for sure!*

Example 11. *Johnny starts the day with a net worth of \$7, and then receives a bill for \$5. How much did Johnny's net worth decrease?*

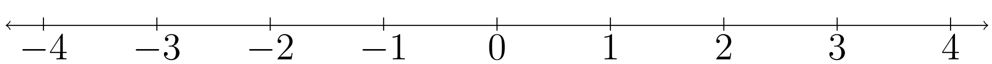
Explanation. *This story looks very similar to our first addition story at a glance, but if we read the question carefully, we see an important difference. We are asked how much Johnny's net worth* decreases, *so we do not need the information about his beginning net worth. Our bill was for \$5, so his net worth decreases by* $\underset{\text{given}}{\boxed{5}}$. *Notice that this answer is positive, even though our bill is a negative number, and that this story no longer models* $7 + (-5)$.

Any time you write a story problem, it's an excellent practice to go back and try to answer the question from an objective perspective. Is the question really asking what you intend? Is there any other way the question could be interpreted? Especially in class or on a homework assignment, ask someone else for their opinion!

Number Lines

Next, let's use a number line to solve some addition problems with integers. You can write a story problem to go along with each of these addition problems for extra practice.

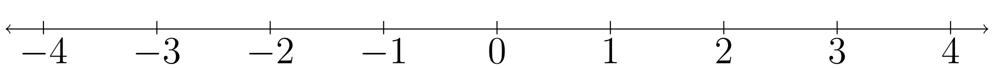
Example 12. *Imagine using a number line like the one below to solve the addition problem* $5 + 3$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{5}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{3}}$ *spaces* (forward / backward) , *since* 3 *is positive. Where on the number line are we now?*

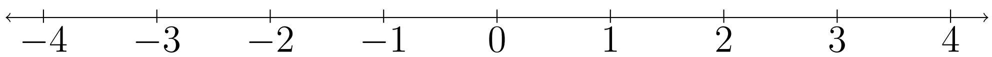
We are located at the tick labeled $\underset{\text{given}}{\boxed{8}}$.

Example 13. *Imagine using a number line like the one below to solve the addition problem* $-8 + 4$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{-8}}$*. Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{4}}$ *spaces* (forward / backward) *, since* 4 *is positive. Where on the number line are we now?* $\underset{\text{given}}{\boxed{-4}}$

Example 14. *Imagine using a number line like the one below to solve the addition problem* $(-22) + (-54)$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{-22}}$*. Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{54}}$ *spaces* (forward / backward) *, since* 54 *is negative. Where on the number line are we now?* $\underset{\text{given}}{\boxed{-76}}$

Again, notice that the direction of our movement on the number line gives us a sense as to whether the final answer should be positive or negative!

Patterns

Finally, we investigate addition of negative numbers via patterns.

Example 15. *Consider the sequence of addition problems.*

$$5 + 4 = \underset{\text{given}}{\boxed{9}}$$
$$5 + 3 = \underset{\text{given}}{\boxed{8}}$$
$$5 + 2 = \underset{\text{given}}{\boxed{7}}$$
$$5 + 1 = \underset{\text{given}}{\boxed{6}}$$
$$5 + 0 = \underset{\text{given}}{\boxed{5}}$$

As we move down the chart, moving one row down results in the final answer decreasing by $\underset{\text{given}}{\boxed{1}}$ *. So, if the pattern continues to hold, we expect the answer to* $5 + (-1)$ *to be* $\underset{\text{given}}{\boxed{4}}$ *, since it is one less than* 5.

Try your hand at recognizing patterns with some other addition problems.

Finally, notice that no matter how we approach the problems in this section, we are getting consistent answers. Whether we use a checks and bills story, a number line, or a pattern, we are always getting the same answer. This is not only comforting, it is necessary for addition as an operation!

1.4 Subtraction with Integers

We look at subtracting integers.

Let's look at subtraction with integers using the same three main ideas we used with addition: checks and bills, number lines, and investigating patterns. Remember that we are going to build from what we know about negative numbers, so you may want to re-read these sections in our text. Also, if you need to refresh your memory on checks and bills, chips, or number lines, you can go back to the section about models.

Checks and Bills

Let's begin with one of our most basic subtraction examples.

Example 16. *Johnny has 9 apples, and Suzy takes away 2 of his apples. How many apples does Johnny have now? Write an expression using the subtraction sign which solves this problem.* $\underbrace{\boxed{9-2}}_{\text{given}}$

If we change this example to a story about checks and bills, we might use the following instead.

Example 17. *Johnny opens his business for today with a net worth of \$9, and then he sends a check for \$2. What is Johnny's net worth now?*

Explanation. *Johnny begins with a net worth of \$9. From our checks and bills model, we know that sending something means we should* (add / subtract) *its value from Johnny's total net worth. Since the object he receives is a check, the value should be* (positive / negative) *. Therefore, the expression we would write which solves this problem is* $\underbrace{\boxed{9-2}}_{\text{given}}$.

Notice that Johnny's total net worth should decrease in this situation: if he sends a check to someone else, he should have less money overall!

Let's begin to include some negative numbers in our stories.

Question 7 *Johnny opens his business for today with a net worth of \$−8, and then he sends a check for \$4. What is Johnny's net worth now?*

As an expression involving the subtraction sign, Johnny's net worth is now $\underbrace{\boxed{-8-4}}_{\text{given}}$.

Again, after sending a check for \$4, Johnny's net worth should be less than it was at the beginning of the day.

Question 8 *Johnny opens his business for today with a net worth of \$−22, and then he sends a bill for \$54. What is Johnny's net worth now?*

As an expression involving the subtraction sign, Johnny's net worth is now $\underbrace{\boxed{-22-(-54)}}_{\text{given}}$.

In this question, notice that Johnny is now sending a bill. Since we assume this bill will be paid by the person to whom it was sent, Johnny's total net worth should be greater than it was at the beginning of the day.

You may have learned some rules about subtraction in the past. In particular, you might be familiar with what happens when we subtract a negative number. How can we make sense of what we already know, but in terms of checks and bills?

Example 18. *When we subtract a negative number, our checks and bills method says we are* (receiving / sending) *a* (check /bill) *. So, when writing such a story, we don't care what Johnny's net worth is at the beginning of the day. Let's call it \$A. Let's also say our bill is worth \$B. We want to compute* $A - (-B)$*. After the person to whom Johnny sent this bill pays, the result to Johnny will be that he has \$B more than \$A. So, we combine Johnny's original net worth \$A with the additional amount \$B to see that*

$$A - (-B) = \underbrace{\boxed{A + B}}_{\text{given}}.$$

We have a second way of thinking about subtraction problems which also lends itself nicely to stories about checks and bills. Again, we begin with an example about apples, and then transition this example to one about checks and bills.

Question 9 *Johnny had 12 apples, and then Suzy gave Johnny some apples. Johnny now has 19 apples. How many apples did Suzy give Johnny?*

As an expression using the subtraction sign, Johnny got $\underbrace{\boxed{19 - 12}}_{\text{given}}$ *apples from Suzy.*

In this example, we might have been thinking of the related addition problem

$$12 + ? = 19$$

which is equivalent to the subtraction problem

$$19 - 12.$$

Now, let's use our checks and bills context!

Question 10 *Johnny opens his business for today with a net worth of \$8 and then received something in the mail. After this, his net worth is now \$−3. What was the value of what Johnny received in the mail?*

As an expression using the subtraction sign, Johnny got $\underbrace{\boxed{-3 - 8}}_{\text{given}}$ *in the mail.*

Question 11 *Johnny opens his business for today with a net worth of \$−2 and then the mail arrives. Afterwards, Johnny's net worth is now \$−7. What was the value of what Johnny received in the mail?*

As an expression using the subtraction sign, Johnny got $\underbrace{\boxed{-7 - (-2)}}_{\text{given}}$ *in the mail.*

Again, notice how the checks and bills story helps us understand why subtracting a negative number is equivalent to adding the value of the bill. Don't hesitate to ask any questions you have about this concept!

With subtraction problems, notice that we are sometimes sending something in the mail, but sometimes receiving something as well! Be careful as you phrase your questions that you have the appropriate operation!

Along these lines, we should again be very careful when asking questions in our story problems. Here are some common pitfalls.

Example 19. *Johnny sends a check for \$18 and a bill for \$14. What is Johnny's net worth now?*

Explanation. *We don't actually have enough information to answer this question, because we don't know what Johnny's net worth was when the day began. If he began with a net worth of \$10, his new net worth is* $\underbrace{\boxed{10 - 18 - (-14)}}_{\text{given}}$*. If he began the day with \$0, his new net worth is* $\underbrace{\boxed{0 - 18 - (-14)}}_{\text{given}}$*. Neither of these expressions is the same as* $18 - 14$.

Example 20. *Johnny starts the day with a net worth of \$7, and then sends a bill for \$5. How much did Johnny's net worth change?*

Explanation. *This story looks very similar to some of our other stories, but if we read the question carefully, we see an important difference. We are asked how much Johnny's net worth* changes, *so we do not need the information about his beginning net worth. Our bill was for \$5, so his net worth changes by* $\underset{\text{given}}{\boxed{-5}}$. *This is not a story problem for* $7-(-5)$.

Example 21. *Johnny starts the day with a net worth of \$12, and then receives a bill in the mail. If Johnny now has \$9, what is the value of the bill Johnny received?*

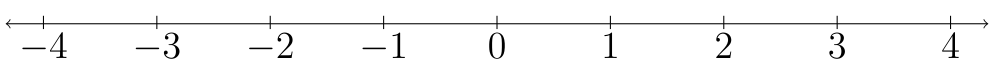
Explanation. *Again, this story may seem familiar at first glance. In this case, however, our story specified that what Johnny received in the mail was a bill. The value of this bill must have been \$*$\underset{\text{given}}{\boxed{3}}$ *, but notice that the answer to this question must be a positive number, even though* $9-12$ *is negative.*

Any time you write a story problem, it's an excellent practice to go back and try to answer the question from an objective perspective. Is the question really asking what you intend? Is there any other way the question could be interpreted? Especially in class or on a homework assignment, ask someone else for their opinion!

Number Lines

Next, let's use a number line to solve some subtraction problems with integers. You can write a story problem to go along with each of these expressions for extra practice.

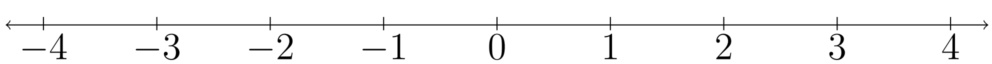
Example 22. *Imagine using a number line like the one below to solve the subtraction problem* $5-3$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{5}}$. *Since we are subtracting, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{3}}$ *spaces* (forward / backward) , *since* 3 *is positive. Where on the number line are we now?*

We are located at the tick labeled $\underset{\text{given}}{\boxed{2}}$.

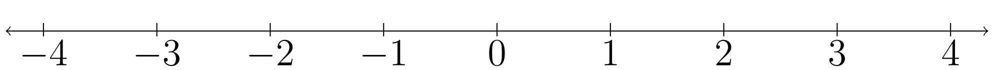
Example 23. *Imagine using a number line like the one below to solve the subtraction problem* $-8-4$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{-8}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{4}}$ *spaces* (forward / backward) , *since* 4 *is positive. Where on the number line are we now?*

We are located at the tick labeled $\underset{\text{given}}{\boxed{-12}}$.

Example 24. *Imagine using a number line like the one below to solve the subtraction problem* $(-22)-(-54)$.



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{-22}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{54}}$ *spaces* (forward / backward) , *since* 54 *is negative. Where on the number line are we now?*

We are located at the tick labeled $\underset{\text{given}}{\boxed{32}}$.

Why does subtracting a negative number give us the same result as adding that number? Using number lines, we can see that if we are subtracting, we are facing *left* while moving *backward*. The net result is the same as if we were facing right while moving forward. Try this out with some friends if you are skeptical.

Patterns

Finally, we investigate subtraction of negative numbers via patterns.

Example 25. *Consider the sequence of addition problems.*

$$5 - 4 = \underset{\text{given}}{\boxed{1}}$$

$$5 - 3 = \underset{\text{given}}{\boxed{2}}$$

$$5 - 2 = \underset{\text{given}}{\boxed{3}}$$

$$5 - 1 = \underset{\text{given}}{\boxed{4}}$$

$$5 - 0 = \underset{\text{given}}{\boxed{5}}$$

As we move down the chart, moving one row down results in the final answer increasing by $\underset{\text{given}}{\boxed{1}}$. *So, if the pattern continues to hold, we expect the answer to* $5 - (-1)$ *to be* $\underset{\text{given}}{\boxed{6}}$, *since it is one less than* 5. *We can also notice that the answer to* $5 - (-1)$ *is the same as the answer to* $5 + \underset{\text{given}}{\boxed{1}}$.

Try your hand at recognizing patterns with some other subtraction problems.

Finally, notice that no matter how we approach the problems in this section, we are getting consistent answers. Also, no matter how we look at subtraction of a negative number, we can see that it should be the same as addition. Subtraction as an operation remains the same, no matter how we model it.

1.5 Multiplication with Integers

We look at multiplying integers.

Now that we've conquered addition and subtraction, let's move on to multiplication. The first thing we should notice when comparing multiplication to addition and subtraction is that with multiplication, the wholes for each of our quantities are different, while with addition and subtraction they are all the same. For instance, we might have something like the following.

$$4 \text{ (apples)} + 9 \text{ (apples)} = 13 \text{ (apples)}$$
$$4 \text{ (baskets of apples)} \times 9 \text{ (apples per basket)} = 36 \text{ (apples)}$$

We need to continue to be careful in how we phrase our problems!

Checks and Bills

As we have done previously, let's begin by looking at a straightforward multiplication problem with whole numbers, and then slowly introduce integers. Remember to look back at our sections about multiplication in the text!

Question 12 *Johnny has 9 lunch bags, and each lunch bag has 2 apples in it. How many apples does Johnny have all together?*

As an expression involving the multiplication sign, Johnny has $\underbrace{\boxed{9 \times 2}}_{\text{given}}$ *apples.*

We can think of this problem as making 9 copies of the 2 apples per bag. As a story about checks and bills, then, we could think of making 9 copies of \$2.

Example 26. *Let's start to write our story problem for 9×2. Johnny is our usual small business owner. Starting with the second number, we know that these should be* (checks /bills) *, each worth \$2, since the 2 is positive.*

For the first number, we know that Johnny (receives /sends) 9 *of these checks, because the 9 is positive.*

But what should be our question? We could start by saying that Johnny's net worth is \$0, and ask what his net worth is now. The natural answer to this question would be

$$0 + (9 \times 2) = 9 \times 2,$$

but that extra zero can feel a little bit confusing.

We might instead ask, "How has Johnny's net worth changed?" In this case, no matter what he had to start with, he now has \$$9 \times 2$ *more* than he did previously. Let's proceed with this second formulation, but cautiously. We will need to make sure all of this still makes sense as we introduce negative numbers.

For our first question, let's change the second number to be negative.

Question 13 *Johnny receives 5 bills, each for \$10. After all bills are paid, how has Johnny's net worth changed?*

As an expression involving the multiplication sign, Johnny's net worth has changed by \$$\underbrace{\boxed{5 \times (-10)}}_{\text{given}}$.

Since Johnny is paying bills in this situation, it makes sense for his net worth to decrease, or for the change to be negative.

What if our first number is negative? In other words, what would it mean to have negative groups? It may feel a little bit forced, but we will use the convention that a negative group means we are *sending* that many copies of our check or bill.

Example 27. *Johnny opens his business for today with a net worth of \$0, and then sends 3 checks for \$8 each. What is Johnny's net worth now?*

Explanation. *Each of the checks is worth \$8, and there are 3 of them. This means that Johnny is sending out \$*$\underset{\text{given}}{\boxed{24}}$*. In other words, his total net worth should be \$*$\underset{\text{given}}{\boxed{-24}}$*. Writing this as an expression, we have*

$$\underset{\text{given}}{\boxed{0}} - \left(\underset{\text{given}}{\boxed{3}} \times \underset{\text{given}}{\boxed{8}}\right) = (-3) \times (8).$$

While beginning with a net worth of \$0 helps us to understand why negative groups might be represented by sending, this convention may still feel artificial. Instead, we could ask the following.

Question 14 *Johnny sends three checks, each for eight dollars. How much has Johnny's net worth changed?*

As an expression involving the multiplication sign, Johnny's net worth has changed by \$$\underset{\text{given}}{\boxed{(-3) \times 8}}$.

You might object that what we are doing here is actually calculating 3×8, and reasoning that the overall answer should be negative. That's okay! Remember that this concept is difficult, and we are doing the best we can.

Finally, let's look at what happens when we multiply two negatives together.

Question 15 *Johnny sends 22 bills, each for \$54. After all bills are paid, how much has Johnny's net worth changed?*

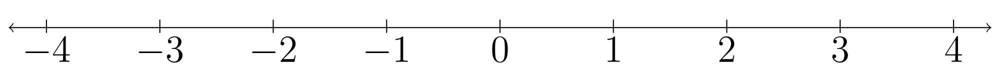
As an expression involving the multiplication sign, Johnny's net worth has changed by \$$\underset{\text{given}}{\boxed{(-22) \times (-54}}$.

In terms of the situation with checks and bills, should this answer be positive or negative? If Johnny is sending bills to other people, and then these other people pay their bills, this money comes back to Johnny. So, his net worth overall should increase, meaning the change should be positive. Notice that we didn't need to memorize any rules about negative numbers in order to come to this conclusion!

Number Lines

Next, let's use a number line to solve some multiplication problems with integers. You can write a story problem to go along with each of these expressions for extra practice.

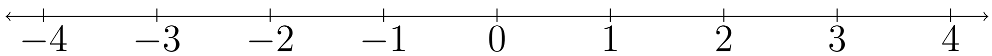
Example 28. *Imagine using a number line like the one below to solve the multiplication problem* $5 \times (-3)$.



Remember that multiplication is repeated addition. We want to add -3 *five times, or make five copies of* -3*. As with our story problems, the question or starting point requires care. With our stories, our starting* *net worth was* $\underset{\text{given}}{\boxed{0}}$ *, so with a number line we begin by standing at the tick marked with* $\underset{\text{given}}{\boxed{0}}$ *. Since we are adding, we face towards the* (right / left) *. Now, we can think of the steps as our groups, meaning the amount we move with each step should be* $\underset{\text{given}}{\boxed{3}}$ *ticks* (forward / backward) *, since* 3 *is positive. After taking* $\underset{\text{given}}{\boxed{5}}$ *such steps, where on the number line are we now?*

We are located at the tick labeled $\underset{\text{given}}{\boxed{-15}}$.

Example 29. *Imagine using a number line like the one below to solve the multiplication problem* $(-8)\times(-7)$*.*



We begin by standing on the number line at the tick marked with $\underset{\text{given}}{\boxed{0}}$ *. Since our groups are negative, we will face towards the* (right / left) *, and take* $\underset{\text{given}}{\boxed{8}}$ *total steps. Each step will move* $\underset{\text{given}}{\boxed{7}}$ *ticks* (forward / backward) *, since* 7 *is negative. Where on the number line are we now?*

We are located at the tick labeled $\underset{\text{given}}{\boxed{56}}$.

Notice again that if we multiply two negative numbers, we face backwards while moving backwards, for a net result of moving in the positive direction along the number line. Try this out with some friends if you are skeptical.

Patterns

Finally, we investigate subtraction of negative numbers via patterns.

Example 30. *Consider the sequence of addition problems.*

$$5 \times 4 = \underset{\text{given}}{\boxed{20}}$$

$$5 \times 3 = \underset{\text{given}}{\boxed{15}}$$

$$5 \times 2 = \underset{\text{given}}{\boxed{10}}$$

$$5 \times 1 = \underset{\text{given}}{\boxed{5}}$$

$$5 \times 0 = \underset{\text{given}}{\boxed{0}}$$

As we move down the chart, moving one row down results in the final answer decreasing by $\underset{\text{given}}{\boxed{5}}$ *. So, if the pattern continues to hold, we expect the answer to* $5 \times (-1)$ *to be* $\underset{\text{given}}{\boxed{-5}}$*, since it is five less than* 0*.*

If we are convinced by this pattern that a positive number times a negative number should be negative, we can use the commutative property to convince ourselves that a negative number times a positive number should also be negative. Then, we can use our patterns again to convince ourselves that a negative number times a negative number should be positive.

Example 31. *Consider the sequence of addition problems.*

$$-5 \times 4 = \underbrace{\boxed{-20}}_{\text{given}}$$

$$-5 \times 3 = \underbrace{\boxed{-15}}_{\text{given}}$$

$$-5 \times 2 = \underbrace{\boxed{-10}}_{\text{given}}$$

$$-5 \times 1 = \underbrace{\boxed{-5}}_{\text{given}}$$

$$-5 \times 0 = \underbrace{\boxed{0}}_{\text{given}}$$

As we move down the chart, moving one row down results in the final answer increasing by $\underbrace{\boxed{5}}_{\text{given}}$ *. So, if the pattern continues to hold, we expect the answer to* $-5 \times (-1)$ *to be* $\underbrace{\boxed{5}}_{\text{given}}$ *, since it is five more than* 0*.*

Extra Examples

Since multiplication with negatives is a complicated topic, here are two extra ways to think about this concept.

Example 32. *We mentioned very briefly when talking about patterns that the properties of addition and subtraction can help us to understand multiplication with negative numbers.*

In particular, we can model multiplication problems like $2 \times (-17)$ *without much trouble. In fact, most of our trouble came from attempting to understand what negative groups should mean. If we instead wanted to consider* $(-17) \times 2$*, we might quickly realize that the commutative property of multiplication gives us*

$$(-17) \times 2 = \underbrace{\boxed{2}}_{\text{given}} \times \underbrace{\boxed{-17}}_{\text{given}},$$

which we could model with a story, a number line, or chips.

Also, we might notice that since $-17 = (-1) \times 17$*, we could use the associative property of multiplication to write*

$$(-17) \times 2 = \left((-1) \times \underbrace{\boxed{17}}_{\text{given}} \right) \times 2 = (-1) \times \left(\underbrace{\boxed{17}}_{\text{given}} \times \underbrace{\boxed{2}}_{\text{given}} \right),$$

and we have plenty of practice computing 2×17*.*

This technique can also give us some help with things like $(-2)\times(-17)$.

$$\begin{aligned}
(-2)\times(-17) &= (-1)\left(\underset{\text{given}}{\boxed{2}}\right)\times(-1)\times\left(\underset{\text{given}}{\boxed{17}}\right)\\
&= (-1)\times\left(\underset{\text{given}}{\boxed{2}}\times\left((-1)\underset{\text{given}}{\boxed{17}}\right)\right)\\
&= (-1)\times\left(\left(\underset{\text{given}}{\boxed{2}}\times(-1)\right)\times\underset{\text{given}}{\boxed{17}}\right)\\
&= (-1)\times\left(\left((-1)\times\underset{\text{given}}{\boxed{2}}\right)\times\underset{\text{given}}{\boxed{17}}\right)\\
&= (-1)\times\left((-1)\times\left(\underset{\text{given}}{\boxed{2}}\times\underset{\text{given}}{\boxed{17}}\right)\right)\\
&= ((-1)\times(-1))\times\left(\underset{\text{given}}{\boxed{2}}\times\underset{\text{given}}{\boxed{17}}\right)
\end{aligned}$$

Again, as long as we understand what to do with $(-1)(-1)$, *we have plenty of practice computing* 2×17.

Example 33. *You might recall the equation*

$$\textit{Distance} = \textit{Time}\times\textit{Rate}.$$

Let's say that zero is our position at noon, and we have been traveling east on an east-west highway at a steady rate of 50 *miles per hour. Where were we at* 9*am?*

Notice that 9*am was three hours ago, and so this question is equivalent to asking for the solution to* $(-3)\times 50$. *Knowing that the answer is* $\underset{\text{given}}{\boxed{-150}}$, *we should interpret this result to say that at* 9*am, we were* $\underset{\text{given}}{\boxed{150}}$ *miles* (east / west) *of where we were at noon.*

Division

Now that we've had so much practice, we will generally leave you to think about division as the inverse of multiplication. We have discussed many ways to understand the sign of a multiplication problem with integers. Once we understand whether a result should be positive or negative, the problem is reduced to computing with positive numbers. What we mean by thinking of division as the inverse of multiplication is to turn your division problem back into a multiplication problem, and then use what you know about multiplication to understand the sign of the answer.

Example 34. *What is* $24\div(-6)$*?*

We can think of $24\div(-6)$ *as a multiplication problem by using*

$$(-6)\times ? = 24$$

or

$$?\times(-6) = 24.$$

While which problem we choose matters in our stories, the (associative / commutative / distributive) *property of multiplication gives us the same answer in either case. We also know from our work with multiplication that the sign of the answer must be* (positive / negative). *Since* $24\div 6 = \underset{\text{given}}{\boxed{4}}$, *we now know that* $24\div(-6)$ *should equal* $\underset{\text{given}}{\boxed{-4}}$.

As we have stated throughout, this is a complicated concept. You may need to work through the examples several times, as well as ask questions, before understanding completely. Don't worry, though – many of history's most notable mathematicians shared these struggles!