Integers: Models and Operations

We discuss various models for integers as well as how to use these models to understand operations with integers.

Motivation

From the point of view of arithmetic, the whole numbers have had two serious deficiencies.

- We can add any two whole numbers and get another whole number, but the same isn't true for subtraction.
- We can multiply any two whole numbers and get another whole number, but the same isn't true for division.

We fix the second deficiency by introducing the set of (fractions / even numbers / imaginary numbers) – we can now divide any whole number by any other whole number which is not $\underset{\text{given}}{\boxed{0}}$ and get a meaningful answer. In fact, the same is true for the entire (now expanded) set of integers and fractions together. To solve the first deficiency, we want to do something similar. The expansion of the whole number system to the system of integers, that is, introducing negative numbers to go along with positive numbers, is designed to resolve this problem.

From an application point of view, we need integers to be able to quantifiably describe situations in which we have what we will call an artificial zero.

Definition 1. *An **artificial zero** is a situation in which it makes sense and is useful for values of the quantities to be less than zero.*

Remember (or read in the section about the history of the integers) that finding such situations was difficult for mathematicians for many centuries! Many physical situations do not need integers because they have what we will call an absolute zero.

Definition 2. *An **absolute zero** is a situation in which it does not make sense and is not useful for values of the quantities to be less than zero.*

It makes no sense to say, "I have fewer than zero oranges", so this situation would have an absolute zero. "I am taking fewer than zero classes this semester" has an (artificial zero / absolute zero) , and "I have fewer than zero dollar bills in my wallet" has an (artificial zero / absolute zero) , while "It is fewer than zero degrees outside" has an (artificial zero/ absolute zero) .

In situations with an artificial zero, or a zero arbitrarily set, it makes sense to be talking about amounts that are less than zero. Here are some examples.

- *Temperature.* In the Fahrenheit and Celsius scales, zero is set at some arbitrary temperature (often for a reason), but it is possible for temperatures to be colder than that set zero. Notice that the Kelvin scale is different: zero is set at the point there is completely no heat, so a temperature less than zero Kelvin is not possible.

Learning outcomes:
Author(s):

- *Finances.* Although having a financial worth of zero is often a bad situation, one can have less financial worth in the sense of owing someone money. So, having a worth of -5 dollars means you need to somehow gain \$5 before you can say you are "even", with no debt and no profit. Notice that this can get complicated! If oranges are the currency, one could say one has -5 oranges instead of -5 dollars even though it is impossible to physically have -5 oranges or -5 dollar bills. The idea of owing someone oranges or dollars makes this idea work.

- *Sports.* In golf, "par" is the artificial zero which is defined as the number of shots (e.g., 5 shots) experts think a golfer should need in order to be able to get the golf ball into the hole. However, if the golfer takes fewer shots (e.g., 2) to get into the hole, we say the golfer is so many shots "under par".

  Example 1. *A certain golf hole has a par of* 5*. If a golfer takes* 2 *shots to get into the hole, we say the golfer is* $\underset{\text{given}}{\boxed{3}}$ *shots under par, or at* $\underset{\text{given}}{\boxed{-3}}$ *with respect to par.*

  Notice that the actual number of shots is always a positive number, but par is playing the role of an artificial zero. In football, the artificial zero is the line of scrimmage, or where the play begins. If a player who is running gets tackled behind this line, we say that the player has rushed for negative yardage.

  Example 2. *A running play starts at the 25 yard line and ends at the 19 yard line. In this case, the runner has gone backwards* $\underset{\text{given}}{\boxed{6}}$ *yards, so football commentators would say that the play went for a total of* $\underset{\text{given}}{\boxed{-6}}$ *yards.*

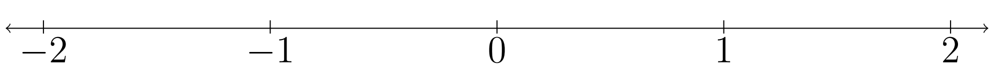
Representations of Integers

There are several main ways we represent integers, or several important tools we will discuss.

Stories. We use situations similar to those with an artificial zero described above. These stories are useful because we can start to picture what we would like to do with integers, why we would like to do these things, and most importantly we can use the stories to check whether or not our answers make sense.

Chips. We use one black chip to represent one positive unit and one red chip to represent one negative unit. If I have one black chip together with one red chip, I have the same amount in value as if I have no chips at all. We might say that one red chip cancels one black chip. Sometimes these chips are used alongside a story, and we might say one black chip is \$1 while one red chip is \$$\underset{\text{given}}{\boxed{-1}}$.

Number lines. Previously, we marked zero and one on our number lines, and considered how we could mark all of the other positive numbers to the right of zero. With integers, we extend the number line to the left of zero as well.



Comparison of Integers

Like with the other number systems weve studied, we would like to discuss having two integer amounts and be able to tell which one is greater, lesser, or if the two amounts are equal. While we could just state a rule for this, as with all mathematics, if it is possible to come up with a rule from a sense-making point of view, it will be more understandable and useful.

Your first guess might be to try a one-to-one correspondence. However, with negative numbers we are sometimes dealing with sets of objects that don't physically exist, and so making a one-to-one correspondence is very difficult in this case. Also, when we work with models like the chips, our physical objects have different meanings, and so they cannot be placed in a one-to-one correspondence without extra care!

Example 3. *Imagine a situation in which Ava has* 4 *black chips, and Aleks has* 8 *red chips. The value of Ava's chips is* $\underset{\text{given}}{\boxed{4}}$ *, while the value of Aleks' chips is* $\underset{\text{given}}{\boxed{-8}}$. *If we line up their chips in a one-to-one correspondence, we see that* (Ava / Aleks) *has more chips, but* (Ava / Aleks) *has the greater value in chips.*

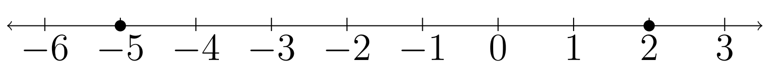
However, like in other number systems, we can use our story problems to look at real-life situations and ask, "Which situation is better?" or "Which number has more value in this situation?" For instance, we have things like the following.

- Which financial worth is better? ("Better" here usually means more money.)
- Which team's total running yards is better? ("Better" here usually means more yards, with positive yards being better than negative yards.)
- Which temperature is better? (Here, "better" depends on the situation! We should probably ask instead, "Which temperature is hotter?")
- Which golf score is better? (Here, "better", with respect to par, would actually be the more negative number!)

Notice that in each case, one needs to define which attribute means greater and which means lesser.

We can also use a number line and take advantage of what we did with positive numbers when saying that the number further to the (left / right) is the greater number.

Example 4. *Consider the following number line, and the locations of* -5 *and* 2.



On the number line above, $\underset{\text{given}}{\boxed{-5}} < \underset{\text{given}}{\boxed{2}}$ *because* -5 *is to the* (left / right) *of* 2 *on the number line. However, of these two numbers,* $\underset{\text{given}}{\boxed{-5}}$ *is further from zero. If our home was located at* 0 *and we asked, "Which of* -5 *and* 2 *is further from home?" we might say that* $\underset{\text{given}}{\boxed{2}} < \underset{\text{given}}{\boxed{-5}}$.

Again, notice how complicated these ideas get! We need to always keep in mind the context for our numbers.