Integers: Models and Operations

We discuss various models for integers as well as how to use these models to understand operations with integers.

Motivation

From the point of view of arithmetic, the whole numbers have had two serious deficiencies.

- We can add any two whole numbers and get another whole number, but the same isn't true for subtraction.
- We can multiply any two whole numbers and get another whole number, but the same isn't true for division.

We fix the second deficiency by introducing the set of fractions – we can now divide any whole number by any other whole number which is not zero and get a meaningful answer. In fact, the same is true for the entire (now expanded) set of integers and fractions together. To solve the first deficiency, we want to do something similar. The expansion of the whole number system to the system of integers, that is, introducing negative numbers to go along with positive numbers, is designed to resolve this problem.

From an application point of view, we need integers to be able to quantifiably describe situations in which we have what we will call an artificial zero.

Definition 1. *An **artificial zero** is a situation in which it makes sense and is useful for values of the quantities to be less than zero.*

Remember (or read in the section about the history of the integers) that finding such situations was difficult for mathematicians for many centuries! Many physical situations do not need integers because they have what we will call an absolute zero.

Definition 2. *An **absolute zero** is a situation in which it does not make sense and is not useful for values of the quantities to be less than zero.*

Explanation. *It makes no sense to say, "I have fewer than zero oranges", so this situation would have an absolute zero. "I am taking fewer than zero classes this semester" has an* (artificial zero / absolute zero) *, and "I have fewer than zero dollar bills in my wallet" has an* (artificial zero / absolute zero) *, while "It is fewer than zero degrees outside" has an* (artificial zero/ absolute zero) *.*

The terminology for absolute and artificial zeros helps us to distinguish these two types of situations, but we will not in general focus on identifying which situation is which or require you to use this terminology. After all, we want to work here with negatives as well as positives, so we will generally be in situations with an artificial zero!

In situations with an artificial zero, or a zero arbitrarily set, it makes sense to be talking about amounts that are less than zero. Here are some examples.

- *Temperature.* In the Fahrenheit and Celsius scales, zero is set at some arbitrary temperature (often for a reason), but it is possible for temperatures to be colder than that set zero. Notice that the Kelvin scale is different: zero is set at the point there is completely no heat, so a temperature less than zero Kelvin is not possible.

Learning outcomes:
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- *Finances.* Although having a financial worth of zero is often a bad situation, one can have less financial worth in the sense of owing someone money. So, having a worth of -5 dollars means you need to somehow gain \$5 before you can say you are "even", with no debt and no profit. Notice that this can get complicated! If oranges are the currency, one could say one has -5 oranges instead of -5 dollars even though it is impossible to physically have -5 oranges or -5 dollar bills. The idea of owing someone oranges or dollars makes this idea work.

- *Sports.* In football, the artificial zero is the line of scrimmage, or where the play begins. If a player who is running gets tackled behind this line, we say that the player has rushed for negative yardage.

  Example 1. *A running play starts at the 25 yard line and ends at the 19 yard line. In this case, the runner has gone backwards* $\underset{\text{given}}{\boxed{6}}$ *yards, so football commentators would say that the play went for a total of* $\underset{\text{given}}{\boxed{-6}}$ *yards.*

  In golf, "par" is the artificial zero which is defined as the number of shots experts think a golfer should need in order to be able to get the golf ball into the hole. However, if the golfer takes fewer shots to get into the hole, we say the golfer is so many shots "under par".

  Example 2. *A certain golf hole has a par of* 5*. If a golfer takes* 2 *shots to get into the hole, we say the golfer is* $\underset{\text{given}}{\boxed{3}}$ *shots under par, or at* $\underset{\text{given}}{\boxed{-3}}$ *with respect to par.*

  Notice that the actual number of shots is always a positive number, but par is playing the role of an artificial zero.

Representations of Integers

There are several main ways we represent integers, or several important tools we will discuss.

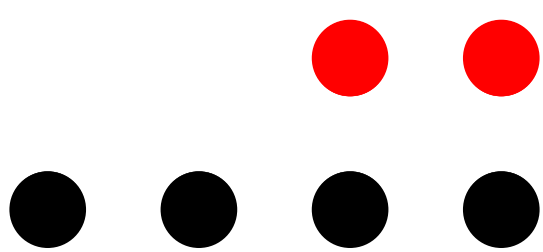
Stories. We use situations similar to those with an artificial zero described above. These stories are useful because we can start to picture what we would like to do with integers, why we would like to do these things, and most importantly we can use the stories to check whether or not our answers make sense.

Chips. We use one black chip to represent one positive unit and one red chip to represent one negative unit. If I have one black chip together with one red chip, I have the same amount in value as if I have no chips at all. We might say that one red chip cancels one black chip.

Example 3. *Sometimes these chips are used alongside a story, and we might say one black chip is* \$1 *while one red chip is* \$$\underset{\text{given}}{\boxed{-1}}$.

One of the most important things to keep in mind while working with chips is that we can easily change the *representation* of our number without changing the *value* of our number. Let's investigate this phenomenon.

Question 1 *What is the total value of all of the chips in the picture below?*



The total value of the chips is $\underset{\text{given}}{\boxed{2}}$.

Question 2 *Suppose you would like to use chips to represent a total value of* 8*. Which of the following combinations of chips would give you this value?*

Select All Correct Answers:

(a) 8 *black chips* ✓

(b) 8 *red chips*

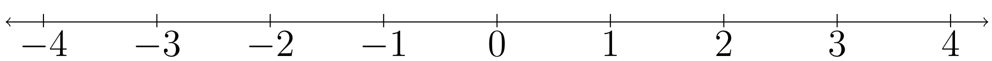
(c) 10 *black chips and* 2 *red chips* ✓

(d) 24 *black chips and* 16 *red chips* ✓

(e) 3 *black chips and* 11 *red chips*

(f) 5 *black chips and* 3 *red chips*

Number lines. Previously, we marked zero and one on our number lines, and considered how we could mark all of the other positive numbers to the right of zero. With integers, we extend the number line to the left of zero as well.



Comparison of Integers

Like with the other number systems weve studied, we would like to discuss having two integer amounts and be able to tell which one is greater, lesser, or if the two amounts are equal. While we could just state a rule for this, as with all mathematics, if it is possible to come up with a rule from a sense-making point of view, it will be more understandable and useful.

Your first guess might be to try a one-to-one correspondence. However, with negative numbers we are sometimes dealing with sets of objects that don't physically exist, and so making a one-to-one correspondence is very difficult in this case. Also, when we work with models like the chips, our physical objects have different meanings, and so they cannot be placed in a one-to-one correspondence without extra care!

Example 4. *Imagine a situation in which Ava has* 4 *black chips, and Aleks has* 8 *red chips. The value of Ava's chips is* $\underset{\text{given}}{\boxed{4}}$ *, while the value of Aleks' chips is* $\underset{\text{given}}{\boxed{-8}}$*. If we line up their chips in a one-to-one correspondence, we see that* (Ava / Aleks) *has more chips, but* (Ava / Aleks) *has the greater value in chips.*

However, like in other number systems, we can use our story problems to look at real-life situations and ask, "Which situation is better?" or "Which number has more value in this situation?" For instance, we have things like the following.

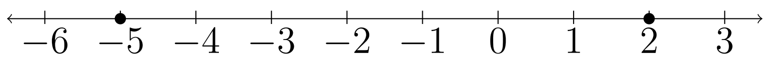
- Which financial worth is better? ("Better" here usually means more money.)
- Which team's total running yards is better? ("Better" here usually means more yards, with positive yards being better than negative yards.)

- Which temperature is better? (Here, “better” depends on the situation! We should probably ask instead, “Which temperature is hotter?”)
- Which golf score is better? (Here, “better”, with respect to par, would actually be the more negative number!)

Notice that in each case, one needs to define which attribute means greater and which means lesser.

We can also use a number line and take advantage of what we did with positive numbers, but now extending our ideas to negative numbers as well.

Example 5. *Consider the following number line, and the locations of* -5 *and* 2.



On the number line above, $\underset{\text{given}}{\boxed{-5}} < \underset{\text{given}}{\boxed{2}}$ *because* -5 *is to the* (left /right) *of* 2 *on the number line. However, of these two numbers,* $\underset{\text{given}}{\boxed{-5}}$ *is further from zero. If our home was located at* 0 *and we asked, “Which of* -5 *and* 2 *is further from home?” we might say that* $\underset{\text{given}}{\boxed{2}} < \underset{\text{given}}{\boxed{-5}}$.

Again, notice how complicated these ideas get! We should always keep in mind the context for our numbers.

Operations with Integers

As always, we will start with what we know and use our knowledge to develop what we don’t know. The meanings of our operations with negative numbers should be extensions of the meanings of our operations with whole numbers.

One context we often find useful when writing stories with negative numbers is called the “checks and bills” model.

Definition 3. *In the* ***checks and bills model****, imagine that you are a small business owner, and that you live in a wonderful land where everyone always pays their bills. We will ask questions about the financial net worth (in dollars) of your business using the following.*

- *An addition sign means to “receive”, generally in the mail.*
- *A subtraction sign means to “send”, generally in the mail.*
- *A check will refer to a positive number.*
- *A bill will refer to a negative number.*

If we are using chips to represent our checks and bills stories, a black chip will mean \$1, and a red chip will mean \$$-1$.

Example 6. *Yesterday, you wrote a check for twelve dollars. What amount is represented by this check?* $\underset{\text{given}}{\boxed{12}}$

Today, you plan to write a bill for eighteen dollars. What amount is represented by this bill? $\underset{\text{given}}{\boxed{-18}}$

Yesterday, you sent the check you wrote in the mail. So, the twelve dollars should be (added /subtracted) *from your net worth.*

When we work with number lines, we will use the following conventions.

- An addition sign means to face right (towards the positive numbers).
- A subtraction sign means to face left (towards the negative numbers).
- A positive number means to walk forwards.
- A negative number means to walk backwards.

The result that you see after following these procedures should answer the question, "Where on the number line are we now?" We can also often use our story situation, whether it is checks and bills or something else, to understand how to move on the number line. Some people think of this as asking the question, "Did we get good news or bad news?"

Example 7. *Receiving a bill in the mail is generally considered to be* (good news / bad news) *. If we received a bill for $24, we should move towards the* (positive / negative) *numbers on the line. The number of steps we should move is* $\boxed{24}$.
given

Receiving a check in the mail is generally considered to be (good news / bad news) *. If we received a check for $19, we should move towards the* (positive / negative) *numbers on the line. The number of steps we should move is* $\boxed{19}$.
given

Addition

Recall that when we think of addition as combining, we are taking two disjoint sets and joining them together into a new set. The sum of the two sets' amounts is the amount in the newly formed set. We also think of addition in a "join add-to" context, where we begin with one set and join a disjoint set to it. Let's recall our most basic addition example.

Example 8. *Johnny has 5 apples, and Suzy has 3 apples. How many apples do Johnny and Suzy have all together? Write an expression using the addition sign which solves this problem.* $\boxed{5+3}$
given

If we change this example to a story about checks and bills, we might use the following instead.

Example 9. *In yesterday's mail, Johnny received a check for $5 and another check for $3. What is the total value of Johnny's checks and bills from yesterday? Write an expression using the addition sign which solves this problem.* $\boxed{5+3}$
given

Now we can easily extend our story to one involving negative numbers.

Example 10. *In yesterday's mail, Johnny received a bill for $8 and a check for $4. What is the total value of Johnny's checks and bills from yesterday? Write an expression using the addition sign which solves this problem.* $\boxed{-8+4}$
given

Example 11. *In yesterday's mail, Johnny received a bill for $22 and another bill for $54. What is the total value of Johnny's checks and bills from yesterday? Write an expression using the addition sign which solves this problem.* $\boxed{-22+(-54)}$
given

There are several important things to notice with these examples. First, our story gives us a good idea whether the overall answer of the story should be positive or negative. In the case of two checks, we expect a positive number. In the case of two bills, we expect a negative number. In the case of a check and a bill, we would need to know the values of the objects. This observation sets the stage for some important questions we want to consider about multiplying with negative numbers. In particular, we would like to develop an intuition for why the product of two negative numbers is negative. You might pause for a few minutes here to see if you already have some ideas about this concept.

Second, we need to exercise care with the question we ask in our checks and bills stories. For instance, if we used the example of receiving two checks, one for \$5 and one for \$3, we could ask, "What is our net worth now?" In this case, we don't actually have enough information to answer the question. If we began the day with a net worth of \$84, our new net worth would be $\$(84 + 5 + 3)$. If instead we began the day in debt \$9, we are now in debt \$1. We can modify this question to add the information that we began the day with no profit and no debt, but then our net worth would be $\$(0 + 5 + 3)$. We now get the correct numerical value of \$8, since adding zero doesn't change our answer, but this is a different expression than $\$(5 + 3)$. Be on the lookout for similar pitfalls as you write your own story problems. You may find the "join add-to" model of addition easier to work with.

Example 12. *Johnny has a net worth of \$−4, and then he receives a check for \$14. What is Johnny's net worth now? Write an expression using the addition sign which solves this problem.* $\underset{\text{given}}{\boxed{-4 + 14}}$.

Next, let's use a number line to solve some addition problems with integers.

Question 3 *Imagine using a number line like the one below to solve the addition problem* $5 + 3$.

$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

We begin by standing on the number line at the tick marked with a $\underset{\text{given}}{\boxed{5}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{3}}$ *spaces* (forward / backward) , *since* 3 *is positive. Where on the number line are we now?* $\underset{\text{given}}{\boxed{8}}$

Question 4 *Imagine using a number line like the one below to solve the addition problem* $-8 + 4$.

$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

We begin by standing on the number line at the tick marked with a $\underset{\text{given}}{\boxed{-8}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{4}}$ *spaces* (forward / backward) , *since* 4 *is positive. Where on the number line are we now?* $\underset{\text{given}}{\boxed{-4}}$

Question 5 *Imagine using a number line like the one below to solve the addition problem* $(-22) + (-54)$.

$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

We begin by standing on the number line at the tick marked with a $\underset{\text{given}}{\boxed{-22}}$. *Since we are adding, we face towards the* (right / left) . *We will move* $\underset{\text{given}}{\boxed{54}}$ *spaces* (forward / backward) , *since* 54 *is negative. Where on the number line are we now?* $\underset{\text{given}}{\boxed{-76}}$

Again, notice that our movement on the number line gives us a sense as to whether the final answer should be positive or negative!

Finally, we investigate addition of negative numbers via patterns.

Example 13. *Consider the sequence of addition problems.*

$$5+4=\underbrace{\boxed{9}}_{\text{given}}$$

$$5+3=\underbrace{\boxed{8}}_{\text{given}}$$

$$5+2=\underbrace{\boxed{7}}_{\text{given}}$$

$$5+1=\underbrace{\boxed{6}}_{\text{given}}$$

$$5+0=\underbrace{\boxed{5}}_{\text{given}}$$

As we move down the chart, moving one row down results in the final answer decreasing by $\underbrace{\boxed{1}}_{\text{given}}$. *So, if the pattern continues to hold, we expect the answer to* $5+(-1)$ *to be* $\underbrace{\boxed{4}}_{\text{given}}$, *since it is one less than* 5.

Try your hand at recognizing patterns with some other addition problems.

Finally, notice that no matter how we approach the problems in this section, we are getting consistent answers. Whether we use a combining or join add-to addition structure, we get the same answer. Whether we use a checks and bills story, a number line, or a pattern, we are always getting the same answer. This is not only comforting, it is necessary for addition as an operation!