A FUNCTIONAL COMPUTER ALGEBRA SYSTEM FOR POLYNOMIALS

BY

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A Thesis Submitted to the Graduate Faculty of

WAKE FOREST UNIVERSITY GRADUATE SCHOOL OF ARTS AND SCIENCES

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

Mathematics

April, 2023

Winston-Salem, North Carolina

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Acknowledgements

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Abstract

This paper explores the `polynomial-algorithms` package which is a computer algebra system written in Haskell using the functional programming paradigm. This package implements a Polynomial type for use in several algorithms including a recursive version of Buchberger's algorithm for finding a reduced Gröbner basis for a polynomial ideal. We will contrast this package with imperative approaches using more traditional languages.

Chapter 1: Introduction

The theory of polynomials is a well-studied field of mathematics, rich with results which provide tools for mathematicians in various areas. Polynomial functions are among the simplest continuous functions to work with analytically or algebraically. Computationally, they are unquestionably the most important class of functions. It is then imperative to have a robust collection of software that can assist the mathematicians who study them. Many tools currently exist for this purpose, from dedicated high-level languages like Macaulay2 and Singular to libraries for general purpose languages like SageMath and SimPy. These software collections are referred to as *computer algebra systems* or CAS's and while most have capabilities beyond polynomial manipulation, it is this feature that is the focus here.

The implementation of these tools varies greatly. The Macaulay2 engine is written in C++[3] while SageMath is built with Python[8], which in turn often calls C. Their respective interfaces often use a mixture of procedural, object-oriented, functional, and declarative design. This certainly makes sense in the case of SageMath as Python itself hybridizes these approaches. If a CAS is a package for a general purpose language like Python, it should obey the design principles of that language. For dedicated CAS's like Macaulay2 though, this design choice bears consideration. There is a tendency in modern languages to incorporate features which favor a variety of paradigms. The popularity of hybrid languages like Python and JavaScript has encouraged most newer languages like Rust and Go to follow in their footsteps. There are good reasons for this, however there are also benefits to sticking to a single paradigm.

It is true that these paradigms are more a philosophical way of approaching soft-

ware design than a well-defined feature of a language and given that, no language can be said to be strictly single paradigm. It is clear however that certain languages lend themselves to being more compatible with one paradigm than others. For example, Java is generally seen as a paragon of object-oriented programming. Every module contains exactly one class (with potentially many subclasses). Even the drivers are classes and must be instantiated before any program can be executed. One can use Java in a functional way, but doing so defeats the purpose of using Java. This is in contrast to a language like Python where it is difficult to escape the use of objects and classes, but higher order functions like folds and maps are used frequently, as are list comprehensions and other hallmarks of functional programming. At the same time, most programmers' first experience with Python is in writing plain procedural scripts. The freedom offered by such a language enables one to chose the most appropriate approach to the current task, but in doing so, it takes away the cohesiveness and predictability of a more *pure* language like Java.

Similarly to the way Java is considered a paragon of the object-oriented school, Haskell is often one of the first two languages people think of when one mentions functional programming, with the other of course being Lisp. The functional style offers many advantages over a procedural or object-oriented approach as evidenced in how much modern languages have borrowed from Haskell. Rust's type system strongly resembles that of Haskell[6] and modern JavaScript libraries like React leverage functional programming more than any other paradigm.

Haskell and the functional programming style are particularly well-suited to mathematics. The notion of a 'function' in procedural programming is a bit of a misnomer. Mathematically, a function f is a domain X, a codomain Y, and some subset of $X \times Y$ called the graph of f. What is the domain of the `print()` function in Python? What is its codomain? What is its graph? These inconsistencies vanish in Haskell. A

Haskell function *is* a mathematical function. For this reason, mathematicians often regard the functional paradigm as the natural choice for mathematical software. What better way to determine the injectivity of a function than with a function whose injectivity can be determined?!

It is in this spirit that the `polynomial-algorithms` package was written. Taking advantage of the benefits offered by Haskell (as well as a few language extensions), this package offers a Polynomial type that is itself built on Monomial and Coefficient types. Several algorithms, most notably a modified version of Buchberger's algorithm to find a reduced Gröbner basis for a list of polynomials, are featured in this software. In addition, its scalability and modularity should provide mathematicians with an invaluable tool when analyzing polynomials.

Chapter 2: Mathematical Background

One thing that sets functional programming apart from imperative paradigms is its reliance on mathematics. Writing algorithms or any sufficiently complex code will always require some familiarity with mathematics, and the best and most efficient code is often written by programmers competent in mathematical reasoning. However, this quality is even more important when working in a language like Haskell. This fact may be seen just by reading documentation. A good technical writer will always write to their audience. If one is writing a UDP server in Java, it may be assumed that anyone maintaining that server will be familiar with terms like *port*, *buffer*, or *packet*, so they may appear frequently in documentation. If one is writing an operating system in C, terms like *fork* and *process ID* are ubiquitous enough to appear in documentation without explanation. However, one is unlikely to encounter mentions of algebraic ring theory in the documentation for such projects. When using Haskell, this is quite common[1].

It is therefore imperative (pun intended) for a prospective Haskeller[1] to have a working knowledge of several areas of mathematics not usually necessary for writing programs in a procedural or object-oriented style. This is doubly important when using Haskell to write a CAS. The fact that the recommended backgrounds for the problem domain and the solution domain coincide is further evidence that Haskell is indeed a good fit for such a task.

As any software engineer will tell you, having a firm understanding of mathematical logic can be helpful in writing and debugging code. Boolean algebra is essential for the standard control flow mechanisms used in all languages. De Morgan's laws

[1]The term *Haskeller* refers to a regular user of Haskell.

and other properties of first order logic can assist in cleaning up messy functions. It is well-known that a familiarity with combinatorics and graph theory can be helpful for those writing abstract data types or analyzing networks, and linear algebra and multivariate calculus are essential for working with 3D graphics. However, abstract algebra and category theory are rarely covered in the standard curriculum for a computer science degree.

When using Haskell, a working knowledge of algebraic group theory and ring theory, while not strictly necessary, can provide useful insight into why certain typeclasses and functions behave the way they do. When writing a CAS, this recommendation is upgraded to a strict requirement. Even more central in the design of Haskell and other functionally minded languages is the presence of the ever-intimidating category theory. Even by theoretical mathematicians, this subject is often referred to as *abstract nonsense*[5]. Having a reputation as as being among the most abstruse of mathematical topics, the mere fact that something as concrete and useful as writing software could employ such a concept is itself remarkable.

Using such a mathematically inclined language to implement a CAS is clearly an endeavour that requires a thorough understanding of the mathematics at play. Of particular importance is the theory of polynomials.

2.1 Polynomials

Polynomials in a single variable with real coefficients are a familiar object of study, not only to professional mathematicians, but to anyone who has taken a high school math class. They are easy to understand and work with. Finding their roots, taking their derivatives, and analyzing their graphs are among the easier tasks a mathematician will attempt. We even estimate arbitrary analytic functions as polynomials via Taylor's theorem. Many of the algorithms that form the backbone of low-level

software are based on this idea. It should come as no surprise then that generalizing these familiar objects is a popular practice. Using fields other than $\mathbb{R}$ is a natural first step. In fact, when using an algebraically closed field like $\mathbb{C}$, polynomials behave even more nicely than they did over $\mathbb{R}$ as we don't have to wonder how many roots a degree n polynomial may have; the answer is always n (up to multiplicity).

The next natural generalization is to allow for multiple (though still finitely many) variables. This generalization does introduce some complexity. For one thing, the idea of ordering the terms, which was taken for granted in the single variable case, now becomes a nontrivial discussion. The tools from multivariate calculus can be helpful in understanding the analytic nature of multivariate polynomials, while an introductory course in abstract algebra or algebraic geometry often addresses the more algebraic concerns by introducing term orders, algorithms, and the relationship between the ideals containing these objects and the affine varieties they generate.

2.1.1 Monomials

There are two competing ways to view a polynomial. The first is as a function from some ring into itself. It is this view that is usually encountered first, and for most analytic purposes, this is sufficient. The other way is as a formal linear combination of indeterminates. This is the view that we prefer in this discussion. These indeterminates comprise a free monoid[2], and (at least for now) we may assume the monoidal operation to be commutative. We use the term *monomial* when referring to elements of a free monoid in the context of polynomials.

In this view, the phrase *polynomial in n variables* really means a linear combination of monomials drawn from a free monoid with n distinct generators. Each generator is called a variable. When dealing with polynomials in a single variable, we

[2]Recall that a monoid is a set with an associative binary operation and an identity element. Being free means there are no unforced relationships between elements. For example, $x \neq y^n$ for any n.

usually use the symbol x to refer to the generator of this monoid. So the polynomial f defined as

$$f(x) = x^2 - 8x + 15$$

is a formal linear combination of the monomials x^2, x, and 1 taken from the free monoid $\langle x \rangle$ with weights 1,−8, and 15. In the case of a three variable polynomial, the monoid used is generated by three elements. Here, there are two competing notational conventions. The first is to denote the generators as x, y, and z. The second is to denote the generators as x_1, x_2, and x_3. The former is often more readable, but the latter more easily generalizes to n variables.

This viewpoint of polynomials as a formal combination of monomials rather than a function may be summed up in the following few definitions courtesy of Cox, Little and O'Shea[2]:

Definition 1. A **monomial** in $x_1, \ldots, x_n$ is a product of the form

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. The **total degree** of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$.

We can simplify the notation for monomials as follows: let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers. Then we set

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

When $\alpha = (0, \ldots, 0)$, note that $x^\alpha = 1$. We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the total degree of the monomial x^α.

Definition 2. A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in a field k is a finite linear combination (with coefficients in k) of monomials. We will write a polynomial

f in the form

$$f = \sum_{\alpha} a_\alpha x^\alpha, \quad a_\alpha \in k,$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in k is denoted $k[x_1, \ldots, x_n]$.

Definition 3. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in $k[x_1, \ldots, x_n]$.

(i) We call a_α the **coefficient** of the monomial x^α.

(ii) If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a **term** of f.

(iii) The **total degree** of $f \neq 0$, denoted $\deg(f)$, is the maximum $|\alpha|$ such that the coefficient a_α is nonzero. The total degree of the zero polynomial is undefined.

One observation is that the set $k[x_1, \ldots, x_n]$ forms a ring under the standard polynomial addition and multiplication operations. In the case where $n = 1$, the ring $k[x]$ contains the familiar polynomials in a single variable. An important question that often arises when working with polynomials in more than one variable is how to define the leading term. Given a nonzero polynomial $f \in k[x]$, let

$$f = c_0 x^m + c_1 x^{m-1} + \cdots + c_m,$$

where $c_i \in k$ and $c_0 \neq 0$ (thus $\deg(f) = m$). Then we say that $c_0 x^m$ is the **leading term** of f and write $\text{LT}(f) = c_0 x^m$. The leading term of a polynomial is a surprisingly essential characteristic. For example, when executing single variable polynomial long division, the first step is to make sure the polynomial is expressed with its leading term first, and at each subsequent step, that must remain the case; but what about when there are multiple variables? Given the polynomial $g \in k[x, y, z]$ defined as

$$g = xy^2z^3 + x^5 + x^3y^2z,$$

what is the leading term of g? We will have to be careful about how we define the leading term in this case, as there is not just one obvious way.

This leads us to a discussion of *monomial orderings*. There are uncountably many ways to order the monomials in a free monoid, but the majority of them will not be compatible with polynomial multiplication in the way we would like. The orderings that we may use must satisfy a few special properties, espoused in the following definition[2]:

Definition 4. A **monomial ordering** on $k[x_1, \ldots, x_n]$ is a relation $>$ on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying:

(i) $>$ is a total ordering[3].

(ii) If $x^\alpha > x^\beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $x^\alpha x^\gamma > x^\beta x^\gamma$.

(iii) $>$ is a well-ordering.

It turns out that the third condition above is equivalent to two other statements that are easier to work with.

Theorem 5. *Let X be a commutative free monoid and suppose the first two conditions in the definition above are satisfied. The the following are equivalent:*

1. *$>$ is a well-ordering on X.*
2. *Every strictly decreasing sequence in X eventually terminates.*
3. *$x^\alpha > 0$ for all $\alpha \in \mathbb{Z}^n_{\geq 0}$.*

For a proof of this theorem, see Cox, Little and O'Shea[2]. This allows us to show that certain algorithms terminate by showing that some term strictly decreases at each step of the algorithm.

[3]Recall $>$ is a total ordering if for all α, β exactly one of $x^\alpha > x^\beta$, $x^\alpha = x^\beta$, or $x^\beta > x^\alpha$ is true.

Now that we have a well-defined way to order monomials, we are ready to define a few more terms.

Definition 6. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[x_1, \ldots, x_n]$ and let $>$ be a monomial order.

(i) The **multidegree** of f is

$$\text{multideg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0)$$

(the maximum is taken with respect to $>$).

(ii) The **leading coefficient** of f is

$$\text{LC}(f) = a_{\text{multideg}(f)} \in k.$$

(iii) The **leading monomial** of f is

$$\text{LM}(f) = x^{\text{multideg}(f)}.$$

(iv) The **leading term** of f is

$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

There are many monomial orderings that are used frequently in research, however we will limit our attention to three specific examples. The first such example is perhaps the most intuitive.

Definition 7 (Lexicographic Order). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be in $\mathbb{Z}^n_{\geq 0}$. We say $x^\alpha >_{Lex} x^\beta$ if the leftmost nonzero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive.

The Lexicographic order is the order used in a dictionary. We begin with an ordering of the variables. The greater monomial is the one with the larger exponent in the

first variable. If those exponents are the same, we instead calculate the Lexicographic order without the highest variable. In our previous example, $g = xy^2z^3+x^5+x^3y^2z$, we see that the lead term under the Lexicographic order where $x > y > z$ is $\text{LT}(g) = x^5$. By convention, when denoting variables with subscripts, we take $x_1 > x_2 > x_3 > \dots$.

Our next monomial ordering is built out of the Lexicographic order.

Definition 8 (Graded Lex Order). Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $x^\alpha >_{GLex} x^\beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and $x^\alpha >_{Lex} x^\beta$.

This means that the Graded Lex order first orders by total degree, but breaks ties with the Lexicographic order. In our running example using the Graded Lex order, we see that $\text{LT}(g) = x^3y^2z$.

Our third and final monomial ordering is less intuitive than the previous two.

Definition 9 (Graded Reverse Lex Order). Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $x^\alpha >_{GRevLex} x^\beta$ if $|\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

Here, we are still ordering first by total degree, but we break ties in a manner that is in a way the reverse of Lexicographic order. Using the Graded Reverse Lex order, the lead term of g is xy^2z^3. As unintuitive as the Graded Reverse Lex order may seem, it turns out that this ordering is often the most efficient in many algorithms, including some we will explore here.

2.1.2 Ideals

In general, when R is a commutative ring, a subring I (not necessarily with identity) is an ideal if $ra \in I$ for all $r \in R$ and $a \in I$. For the polynomial ring $k[x_1, \dots, x_n]$ this means that an ideal is a nonempty subset closed under subtraction which absorbs polynomials by multiplication. For example, let $S \subseteq k[x]$ be the set of polynomials with no constant or linear terms, so that $0 \in S$ and for any nonzero $f \in S$, the

lowest degree term of f has degree at least two. Then S is easily seen to be an ideal of $k[x]$. In fact, S is what is known as a *principle ideal* because S is *generated* by a single polynomial, namely the polynomial x^2. This means that any polynomial in S may be formed by multiplying x^2 by some polynomial in $k[x]$. More generally, an ideal $I \in k[x_1, \ldots, x_n]$ is said to be generated by a set B of polynomials if every $f \in I$ may be expressed as $f = b_1 f_1 + \cdots + b_t f_t$ for some $b_1, \ldots, b_t \in B$ and some $f_1, \ldots, f_t \in k[x_1, \ldots, x_n]$. In this case, we write $I = \langle B \rangle$. In our example above, we would write $S = \langle x^2 \rangle$. It turns out that *every* ideal in $k[x]$ is generated by a single polynomial, making this ring quite important in algebra and number theory. When dealing with multiple variables, we have no such luck. There is no single polynomial that generates the ideal $\langle x, y \rangle \in k[x, y, z]$. However, it is true that every polynomial ideal is generated by *some* set since an ideal will generate itself. What is more impressive and not quite as obvious is the fact that every polynomial ideal is generated by some *finite* set. This result was first proved by David Hilbert in 1890 and is known as *Hilbert's Basis Theorem*[4].

Combining our notion of the leading term of a polynomial with ideals in a polynomial ring leads us to the following definition.

Definition 10. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal other than $\{0\}$, and fix a monomial ordering on $k[x_1, \ldots, x_n]$. Then:

(i) We denote by $\text{LT}(I)$ the set of leading terms of nonzero elements of I. Thus,

$$\text{LT}(I) = \{cx^\alpha \mid \text{ there exists } f \in I \setminus \{0\} \text{ with } \text{LT}(f) = cx^\alpha\}.$$

(ii) We denote by $\langle \text{LT}(I) \rangle$ the ideal generated by the elements of $\text{LT}(I)$.

If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal then by Hilbert's Basis Theorem, there is some finite set of polynomials $\{b_1, \ldots, b_t\} \subseteq k[x_1, \ldots, x_n]$ such that $I = \langle b_1, \ldots, b_t \rangle$. Since

$b_1, \ldots, b_t \in I$, it is clear that $\langle \text{LT}(b_1), \ldots, \text{LT}(b_t) \rangle \subseteq \langle \text{LT}(I) \rangle$. However, this containment may be proper. Only in a very special case do we achieve equality of these two sets.

2.1.3 Gröbner bases

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2.2 Categories

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Chapter 3: Functional Programming

Among the key features of a language like Haskell is the guarantee of functional purity. A function in Haskell does exactly one thing. It returns a value. That's it. It cannot alter the state of the program. It cannot cause anything to happen other than returning that value. This makes a Haskell program far more predictable than any program written outside of a purely functional paradigm. This is particularly useful when writing multithreaded code. The inability for a thread to alter any resource that another thread is using simplifies the task tremendously.

In this way, software written in Haskell, or indeed any purely functional language, resembles mathematics quite closely. As we have previously mentioned, a mathematical function is merely a domain set, a codomain set, and a graph. When thinking of a function as *doing something*, the only thing it *does* is transform a domain element into its associated codomain element. However, the very notion of a function doing anything is a bit of an abuse of language. Given sets X and Y, the existence of a function $f : X \to Y$ gives us a way to refer to certain elements of Y as $f(x)$ for some $x \in X$. It doesn't really preform an action. This is in contrast to a function in the computer programming sense. A function `foo` in the Python language certainly does something. When `foo` is called, it preforms its task and then it returns if and when it finishes. This is really the most fundamental difference in functional programming and imperative programming.

Recall the difference between a declarative sentence and an imperative sentence. The sentence "John has twelve eggs." is a declarative sentence while the sentence "Give John twelve eggs." is an imperative sentence. The foundations of mathematics are built on the use of statements. A statement is a declarative sentence with a

well-defined (or unambiguous) truth value[7]. Theorems are statements. Definitions are statements. A proof is a collection is statements. Mathematicians deal predominantly with statements. If we encounter a question, we first rephrase it as a statement before we attempt to verify its veracity. However, imperative programmers are more accustomed to dealing with imperative sentences (indeed, it is easy to forget that this is where the term *imperative programming* comes from.) An instruction is an imperative sentence. A traditional algorithm is a sequence of imperative sentences. `print("Hello world")` is an imperative sentence. It could be said that imperative programming is as much founded on the unambiguous imperative sentence as mathematics is founded on the unambiguous declarative sentence.

The procedural and object oriented design philosophies are both considered imperative styles. Functional programming is often considered a declarative style. What this means is that a line of Java or C code is usually meant to be read as an imperative sentence, while a line of Haskell code is meant to be read as a declarative sentence. This allows Haskell syntax to resemble the syntax of mathematics much more closely than other languages. For example, say we want to construct a list containing the first ten perfect squares. In C, this task amounts to a series of instructions:

```
int squares[10];
for (int i = 0; i < 10; i++)
    squares[i] = i*i;
```

In Haskell on the other hand, instead of specifying instructions, we merely declare the existence of such a list, much like we would in mathematics:

```
squares = [x^2 | x <- [0..9]]
```

In this way, the discussion of functions having side effects becomes moot as functions don't *do* anything. They are merely maps identifying domain elements with codomain elements, just as they are in mathematics.

When thinking about functions in this declarative manner, it becomes apparent

that the only real difference between functions and non-functions is that a function needs an additional element to be fully evaluated. If `fn` is a Haskell function that maps `X` to `Y` and `x` is a term of type `X`, then `fn` has type `X -> Y` and `fn x` has type `Y`. In Haskell, we write

```
fn :: X -> Y
fn x = y
```

This syntax should be quite familiar to mathematicians. Indeed, it is the traditional syntax of mathematics that inspired the design of Haskell syntax.

Hidden in this syntax is one of Haskell's principle features: Currying. Say we want a function that takes two integers as input and returns their sum. In C, this would look something like the following.

```
int addTwo(int x, int y) {
    return x + y;
}
```

In Haskell however, there is no such thing as a function that takes two inputs. Just like mathematics, a function has a single domain, so every function only has one input variable. Now of course, we may define a function out of a product set like $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. In Haskell this is accomplished with tuples.

```
addTwo :: (Int,Int) -> Int
addTwo (x,y) = x + y
```

However, there is another, more natural way to accomplish this in mathematics. Instead of using products, simply define a function from $\mathbb{Z}$ into the set of functions $\mathbb{Z} \to \mathbb{Z}$. This is the more common way to implement such a function in Haskell.

```
addTwo :: Int -> Int -> Int
addTwo x y = x + y
```

The strength of this approach is the ability to partially evaluate functions. If we want a function $\mathbb{Z} \to \mathbb{Z}$ that returns the sum of a number and five, we write `addTwo 5`. So `addTwo` is a function that takes an integer and returns a function, `addTwo 5` is a function that takes an integer and returns an integer. We say that `addTwo 5` is a

partially evaluated function. An expression like `addTwo 5 8` that is fully evaluated is called a *term*.

Functional programming is not without its challenges. Since every expression is a declarative statement, the concept of iterative control must be handled differently. In C, if we wish to perform a task until a certain condition is met, it is common to use a loop to achieve this.

```
int collatz(int n) {
    while (n != 1) {
        if (isEven n) {
            n = n / 2;
        } else {
            n = 3 * n + 1;
        }
    }
    return n;
}
```

However, "While some condition holds, perform some action" is an imperative sentence, not a declarative one, so it is not allowed in Haskell. How then do we handle iteration? The answer is with recursion[1].

```
collatz :: Int -> Int
collatz n
    | n == 1    = 1
    | isEven n  = collatz (n `div` 2)
    | otherwise = collatz (3 * n + 1)
```

In many cases, a recursive solution is cleaner and more readable than a loop, but even the most ardent Haskeller will admit that at times, this *feature* is a limitation. One challenge that arises is translating existing algorithms, which often use loops and other imperative mechanisms, into functional ones. For an example of this, see Chapter 4 of this document.

[1] *Recursion.* Definition: See recursion.

3.1 Types

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3.2 Kinds

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Morbi justo. Aenean nec dolor. In hac habitasse platea dictumst. Proin nonummy porttitor velit. Sed sit amet leo nec metus rhoncus varius. Cras ante. Vestibulum commodo sem tincidunt massa. Nam justo. Aenean luctus, felis et condimentum lacinia, lectus enim pulvinar purus, non porta velit nisl sed eros. Suspendisse consequat. Mauris a dui et tortor mattis pretium. Sed nulla metus, volutpat id, aliquam eget, ullamcorper ut, ipsum. Morbi eu nunc. Praesent pretium. Duis aliquam pulvinar ligula. Ut blandit egestas justo. Quisque posuere metus viverra pede.

Vivamus sodales elementum neque. Vivamus dignissim accumsan neque. Sed at enim. Vestibulum nonummy interdum purus. Mauris ornare velit id nibh pretium ultricies. Fusce tempor pellentesque odio. Vivamus augue purus, laoreet in, scelerisque vel, commodo id, wisi. Duis enim. Nulla interdum, nunc eu semper eleifend, enim dolor pretium elit, ut commodo ligula nisl a est. Vivamus ante. Nulla leo massa, posuere nec, volutpat vitae, rhoncus eu, magna.

Quisque facilisis auctor sapien. Pellentesque gravida hendrerit lectus. Mauris rutrum sodales sapien. Fusce hendrerit sem vel lorem. Integer pellentesque massa vel augue. Integer elit tortor, feugiat quis, sagittis et, ornare non, lacus. Vestibulum posuere pellentesque eros. Quisque venenatis ipsum dictum nulla. Aliquam quis quam non metus eleifend interdum. Nam eget sapien ac mauris malesuada adipiscing. Etiam eleifend neque sed quam. Nulla facilisi. Proin a ligula. Sed id dui eu nibh egestas tincidunt. Suspendisse arcu.

Maecenas dui. Aliquam volutpat auctor lorem. Cras placerat est vitae lectus. Curabitur massa lectus, rutrum euismod, dignissim ut, dapibus a, odio. Ut eros erat, vulputate ut, interdum non, porta eu, erat. Cras fermentum, felis in porta congue,

velit leo facilisis odio, vitae consectetuer lorem quam vitae orci. Sed ultrices, pede eu placerat auctor, ante ligula rutrum tellus, vel posuere nibh lacus nec nibh. Maecenas laoreet dolor at enim. Donec molestie dolor nec metus. Vestibulum libero. Sed quis erat. Sed tristique. Duis pede leo, fermentum quis, consectetuer eget, vulputate sit amet, erat.

Donec vitae velit. Suspendisse porta fermentum mauris. Ut vel nunc non mauris pharetra varius. Duis consequat libero quis urna. Maecenas at ante. Vivamus varius, wisi sed egestas tristique, odio wisi luctus nulla, lobortis dictum dolor ligula in lacus. Vivamus aliquam, urna sed interdum porttitor, metus orci interdum odio, sit amet euismod lectus felis et leo. Praesent ac wisi. Nam suscipit vestibulum sem. Praesent eu ipsum vitae pede cursus venenatis. Duis sed odio. Vestibulum eleifend. Nulla ut massa. Proin rutrum mattis sapien. Curabitur dictum gravida ante.

Phasellus placerat vulputate quam. Maecenas at tellus. Pellentesque neque diam, dignissim ac, venenatis vitae, consequat ut, lacus. Nam nibh. Vestibulum fringilla arcu mollis arcu. Sed et turpis. Donec sem tellus, volutpat et, varius eu, commodo sed, lectus. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Quisque enim arcu, suscipit nec, tempus at, imperdiet vel, metus. Morbi volutpat purus at erat. Donec dignissim, sem id semper tempus, nibh massa eleifend turpis, sed pellentesque wisi purus sed libero. Nullam lobortis tortor vel risus. Pellentesque consequat nulla eu tellus. Donec velit. Aliquam fermentum, wisi ac rhoncus iaculis, tellus nunc malesuada orci, quis volutpat dui magna id mi. Nunc vel ante. Duis vitae lacus. Cras nec ipsum.

Morbi nunc. Aliquam consectetuer varius nulla. Phasellus eros. Cras dapibus porttitor risus. Maecenas ultrices mi sed diam. Praesent gravida velit at elit vehicula porttitor. Phasellus nisl mi, sagittis ac, pulvinar id, gravida sit amet, erat. Vestibulum est. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Curabitur id sem

elementum leo rutrum hendrerit. Ut at mi. Donec tincidunt faucibus massa. Sed turpis quam, sollicitudin a, hendrerit eget, pretium ut, nisl. Duis hendrerit ligula. Nunc pulvinar congue urna.

Nunc velit. Nullam elit sapien, eleifend eu, commodo nec, semper sit amet, elit. Nulla lectus risus, condimentum ut, laoreet eget, viverra nec, odio. Proin lobortis. Curabitur dictum arcu vel wisi. Cras id nulla venenatis tortor congue ultrices. Pellentesque eget pede. Sed eleifend sagittis elit. Nam sed tellus sit amet lectus ullamcorper tristique. Mauris enim sem, tristique eu, accumsan at, scelerisque vulputate, neque. Quisque lacus. Donec et ipsum sit amet elit nonummy aliquet. Sed viverra nisl at sem. Nam diam. Mauris ut dolor. Curabitur ornare tortor cursus velit.

3.3 The Polynomial Type

Morbi tincidunt posuere arcu. Cras venenatis est vitae dolor. Vivamus scelerisque semper mi. Donec ipsum arcu, consequat scelerisque, viverra id, dictum at, metus. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut pede sem, tempus ut, porttitor bibendum, molestie eu, elit. Suspendisse potenti. Sed id lectus sit amet purus faucibus vehicula. Praesent sed sem non dui pharetra interdum. Nam viverra ultrices magna.

Aenean laoreet aliquam orci. Nunc interdum elementum urna. Quisque erat. Nullam tempor neque. Maecenas velit nibh, scelerisque a, consequat ut, viverra in, enim. Duis magna. Donec odio neque, tristique et, tincidunt eu, rhoncus ac, nunc. Mauris malesuada malesuada elit. Etiam lacus mauris, pretium vel, blandit in, ultricies id, libero. Phasellus bibendum erat ut diam. In congue imperdiet lectus.

Aenean scelerisque. Fusce pretium porttitor lorem. In hac habitasse platea dictumst. Nulla sit amet nisl at sapien egestas pretium. Nunc non tellus. Vivamus aliquet. Nam adipiscing euismod dolor. Aliquam erat volutpat. Nulla ut ipsum.

Quisque tincidunt auctor augue. Nunc imperdiet ipsum eget elit. Aliquam quam leo, consectetuer non, ornare sit amet, tristique quis, felis. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque interdum quam sit amet mi. Pellentesque mauris dui, dictum a, adipiscing ac, fermentum sit amet, lorem.

Ut quis wisi. Praesent quis massa. Vivamus egestas risus eget lacus. Nunc tincidunt, risus quis bibendum facilisis, lorem purus rutrum neque, nec porta tortor urna quis orci. Aenean aliquet, libero semper volutpat luctus, pede erat lacinia augue, quis rutrum sem ipsum sit amet pede. Vestibulum aliquet, nibh sed iaculis sagittis, odio dolor blandit augue, eget mollis urna tellus id tellus. Aenean aliquet aliquam nunc. Nulla ultricies justo eget orci. Phasellus tristique fermentum leo. Sed massa metus, sagittis ut, semper ut, pharetra vel, erat. Aliquam quam turpis, egestas vel, elementum in, egestas sit amet, lorem. Duis convallis, wisi sit amet mollis molestie, libero mauris porta dui, vitae aliquam arcu turpis ac sem. Aliquam aliquet dapibus metus.

Vivamus commodo eros eleifend dui. Vestibulum in leo eu erat tristique mattis. Cras at elit. Cras pellentesque. Nullam id lacus sit amet libero aliquet hendrerit. Proin placerat, mi non elementum laoreet, eros elit tincidunt magna, a rhoncus sem arcu id odio. Nulla eget leo a leo egestas facilisis. Curabitur quis velit. Phasellus aliquam, tortor nec ornare rhoncus, purus urna posuere velit, et commodo risus tellus quis tellus. Vivamus leo turpis, tempus sit amet, tristique vitae, laoreet quis, odio. Proin scelerisque bibendum ipsum. Etiam nisl. Praesent vel dolor. Pellentesque vel magna. Curabitur urna. Vivamus congue urna in velit. Etiam ullamcorper elementum dui. Praesent non urna. Sed placerat quam non mi. Pellentesque diam magna, ultricies eget, ultrices placerat, adipiscing rutrum, sem.

Morbi sem. Nulla facilisi. Vestibulum ante ipsum primis in faucibus orci luctus et

ultrices posuere cubilia Curae; Nulla facilisi. Morbi sagittis ultrices libero. Praesent eu ligula sed sapien auctor sagittis. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Donec vel nunc. Nunc fermentum, lacus id aliquam porta, dui tortor euismod eros, vel molestie ipsum purus eu lacus. Vivamus pede arcu, euismod ac, tempus id, pretium et, lacus. Curabitur sodales dapibus urna. Nunc eu sapien. Donec eget nunc a pede dictum pretium. Proin mauris. Vivamus luctus libero vel nibh.

Fusce tristique risus id wisi. Integer molestie massa id sem. Vestibulum vel dolor. Pellentesque vel urna vel risus ultricies elementum. Quisque sapien urna, blandit nec, iaculis ac, viverra in, odio. In hac habitasse platea dictumst. Morbi neque lacus, convallis vitae, commodo ac, fermentum eu, velit. Sed in orci. In fringilla turpis non arcu. Donec in ante. Phasellus tempor feugiat velit. Aenean varius massa non turpis. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae;

Aliquam tortor. Morbi ipsum massa, imperdiet non, consectetuer vel, feugiat vel, lorem. Quisque eget lorem nec elit malesuada vestibulum. Quisque sollicitudin ipsum vel sem. Nulla enim. Proin nonummy felis vitae felis. Nullam pellentesque. Duis rutrum feugiat felis. Mauris vel pede sed libero tincidunt mollis. Phasellus sed urna rhoncus diam euismod bibendum. Phasellus sed nisl. Integer condimentum justo id orci iaculis varius. Quisque et lacus. Phasellus elementum, justo at dignissim auctor, wisi odio lobortis arcu, sed sollicitudin felis felis eu neque. Praesent at lacus.

Vivamus sit amet pede. Duis interdum, nunc eget rutrum dignissim, nisl diam luctus leo, et tincidunt velit nisl id tellus. In lorem tellus, aliquet vitae, porta in, aliquet sed, lectus. Phasellus sodales. Ut varius scelerisque erat. In vel nibh eu eros imperdiet rutrum. Donec ac odio nec neque vulputate suscipit. Nam nec magna. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Nullam porta, odio et sagittis iaculis, wisi neque fringilla sapien, vel commodo

lorem lorem id elit. Ut sem lectus, scelerisque eget, placerat et, tincidunt scelerisque, ligula. Pellentesque non orci.

Etiam vel ipsum. Morbi facilisis vestibulum nisl. Praesent cursus laoreet felis. Integer adipiscing pretium orci. Nulla facilisi. Quisque posuere bibendum purus. Nulla quam mauris, cursus eget, convallis ac, molestie non, enim. Aliquam congue. Quisque sagittis nonummy sapien. Proin molestie sem vitae urna. Maecenas lorem. Vivamus viverra consequat enim.

Chapter 4: Algorithms

Nunc sed pede. Praesent vitae lectus. Praesent neque justo, vehicula eget, interdum id, facilisis et, nibh. Phasellus at purus et libero lacinia dictum. Fusce aliquet. Nulla eu ante placerat leo semper dictum. Mauris metus. Curabitur lobortis. Curabitur sollicitudin hendrerit nunc. Donec ultrices lacus id ipsum.

Donec a nibh ut elit vestibulum tristique. Integer at pede. Cras volutpat varius magna. Phasellus eu wisi. Praesent risus justo, lobortis eget, scelerisque ac, aliquet in, dolor. Proin id leo. Nunc iaculis, mi vitae accumsan commodo, neque sem lacinia nulla, quis vestibulum justo sem in eros. Quisque sed massa. Morbi lectus ipsum, vulputate a, mollis ut, accumsan placerat, tellus. Nullam in wisi. Vivamus eu ligula a nunc accumsan congue. Suspendisse ac libero. Aliquam erat volutpat. Donec augue. Nunc venenatis fringilla nibh. Fusce accumsan pulvinar justo. Nullam semper, dui ut dignissim auctor, orci libero fringilla massa, blandit pulvinar pede tortor id magna. Nunc adipiscing justo sed velit tincidunt fermentum.

Integer placerat. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Sed in massa. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Phasellus tempus aliquam risus. Aliquam rutrum purus at metus. Donec posuere odio at erat. Nam non nibh. Phasellus ligula. Quisque venenatis lectus in augue. Sed vestibulum dapibus neque.

Mauris tempus eros at nulla. Sed quis dui dignissim mauris pretium tincidunt. Mauris ac purus. Phasellus ac libero. Etiam dapibus iaculis nunc. In lectus wisi, elementum eu, sollicitudin nec, imperdiet quis, dui. Nulla viverra neque ac libero. Mauris urna leo, adipiscing eu, ultrices non, blandit eu, dui. Maecenas dui neque, suscipit sit amet, rutrum a, laoreet in, eros. Ut eu nibh. Fusce nec erat tempus urna

fringilla tempus. Curabitur id enim. Sed ante. Cras sodales enim sit amet wisi. Nunc fermentum consequat quam.

Ut auctor, augue porta dignissim vestibulum, arcu diam lobortis velit, vel scelerisque risus augue sagittis risus. Maecenas eu justo. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris congue ligula eget tortor. Nullam laoreet urna sed enim. Donec eget eros ut eros volutpat convallis. Praesent turpis. Integer mauris diam, elementum quis, egestas ac, rutrum vel, orci. Nulla facilisi. Quisque adipiscing, nulla vitae elementum porta, sem urna volutpat leo, sed porta enim risus sed massa. Integer ac enim quis diam sodales luctus. Ut eget eros a ligula commodo ultricies. Donec eu urna viverra dolor hendrerit feugiat. Aliquam ac orci vel eros congue pharetra. Quisque rhoncus, justo eu volutpat faucibus, augue leo posuere lacus, a rhoncus purus pede vel est. Proin ultrices enim.

Aenean tincidunt laoreet dui. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Integer ipsum lectus, fermentum ac, malesuada in, eleifend ut, lorem. Vivamus ipsum turpis, elementum vel, hendrerit ut, semper at, metus. Vivamus sapien tortor, eleifend id, dapibus in, egestas et, pede. Pellentesque faucibus. Praesent lorem neque, dignissim in, facilisis nec, hendrerit vel, odio. Nam at diam ac neque aliquet viverra. Morbi dapibus ligula sagittis magna. In lobortis. Donec aliquet ultricies libero. Nunc dictum vulputate purus. Morbi varius. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In tempor. Phasellus commodo porttitor magna. Curabitur vehicula odio vel dolor.

Praesent facilisis, augue a adipiscing venenatis, libero risus molestie odio, pulvinar consectetuer felis erat ac mauris. Nam vestibulum rhoncus quam. Sed velit urna, pharetra eu, eleifend eu, viverra at, wisi. Maecenas ultrices nibh at turpis. Aenean quam. Nulla ipsum. Aliquam posuere luctus erat. Curabitur magna felis, lacinia et, tristique id, ultrices ut, mauris. Suspendisse feugiat. Cras eleifend wisi vitae tortor.

Phasellus leo purus, mattis sit amet, auctor in, rutrum in, magna. In hac habitasse platea dictumst. Phasellus imperdiet metus in sem. Vestibulum ac enim non sem ultricies sagittis. Sed vel diam.

Integer vel enim sed turpis adipiscing bibendum. Vestibulum pede dolor, laoreet nec, posuere in, nonummy in, sem. Donec imperdiet sapien placerat erat. Donec viverra. Aliquam eros. Nunc consequat massa id leo. Sed ullamcorper, lorem in sodales dapibus, risus metus sagittis lorem, non porttitor purus odio nec odio. Sed tincidunt posuere elit. Quisque eu enim. Donec libero risus, feugiat ac, dapibus eget, posuere a, felis. Quisque vel lectus ut metus tincidunt eleifend. Duis ut pede. Duis velit erat, venenatis vitae, vulputate a, pharetra sit amet, est. Etiam fringilla faucibus augue.

Aenean velit sem, viverra eu, tempus id, rutrum id, mi. Nullam nec nibh. Proin ullamcorper, dolor in cursus tristique, eros augue tempor nibh, at gravida diam wisi at purus. Donec mattis ullamcorper tellus. Phasellus vel nulla. Praesent interdum, eros in sodales sollicitudin, nunc nulla pulvinar justo, a euismod eros sem nec nibh. Nullam sagittis dapibus lectus. Nullam eget ipsum eu tortor lobortis sodales. Etiam purus leo, pretium nec, feugiat non, ullamcorper vel, nibh. Sed vel elit et quam accumsan facilisis. Nunc leo. Suspendisse faucibus lacus.

Pellentesque interdum sapien sed nulla. Proin tincidunt. Aliquam volutpat est vel massa. Sed dolor lacus, imperdiet non, ornare non, commodo eu, neque. Integer pretium semper justo. Proin risus. Nullam id quam. Nam neque. Duis vitae wisi ullamcorper diam congue ultricies. Quisque ligula. Mauris vehicula.

4.1 The Division Algorithm

Curabitur nunc magna, posuere eget, venenatis eu, vehicula ac, velit. Aenean ornare, massa a accumsan pulvinar, quam lorem laoreet purus, eu sodales magna risus

molestie lorem. Nunc erat velit, hendrerit quis, malesuada ut, aliquam vitae, wisi. Sed posuere. Suspendisse ipsum arcu, scelerisque nec, aliquam eu, molestie tincidunt, justo. Phasellus iaculis. Sed posuere lorem non ipsum. Pellentesque dapibus. Suspendisse quam libero, laoreet a, tincidunt eget, consequat at, est. Nullam ut lectus non enim consequat facilisis. Mauris leo. Quisque pede ligula, auctor vel, pellentesque vel, posuere id, turpis. Cras ipsum sem, cursus et, facilisis ut, tempus euismod, quam. Suspendisse tristique dolor eu orci. Mauris mattis. Aenean semper. Vivamus tortor magna, facilisis id, varius mattis, hendrerit in, justo. Integer purus.

Vivamus adipiscing. Curabitur imperdiet tempus turpis. Vivamus sapien dolor, congue venenatis, euismod eget, porta rhoncus, magna. Proin condimentum pretium enim. Fusce fringilla, libero et venenatis facilisis, eros enim cursus arcu, vitae facilisis odio augue vitae orci. Aliquam varius nibh ut odio. Sed condimentum condimentum nunc. Pellentesque eget massa. Pellentesque quis mauris. Donec ut ligula ac pede pulvinar lobortis. Pellentesque euismod. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent elit. Ut laoreet ornare est. Phasellus gravida vulputate nulla. Donec sit amet arcu ut sem tempor malesuada. Praesent hendrerit augue in urna. Proin enim ante, ornare vel, consequat ut, blandit in, justo. Donec felis elit, dignissim sed, sagittis ut, ullamcorper a, nulla. Aenean pharetra vulputate odio.

Quisque enim. Proin velit neque, tristique eu, eleifend eget, vestibulum nec, lacus. Vivamus odio. Duis odio urna, vehicula in, elementum aliquam, aliquet laoreet, tellus. Sed velit. Sed vel mi ac elit aliquet interdum. Etiam sapien neque, convallis et, aliquet vel, auctor non, arcu. Aliquam suscipit aliquam lectus. Proin tincidunt magna sed wisi. Integer blandit lacus ut lorem. Sed luctus justo sed enim.

Morbi malesuada hendrerit dui. Nunc mauris leo, dapibus sit amet, vestibulum et, commodo id, est. Pellentesque purus. Pellentesque tristique, nunc ac pulvinar

adipiscing, justo eros consequat lectus, sit amet posuere lectus neque vel augue. Cras consectetuer libero ac eros. Ut eget massa. Fusce sit amet enim eleifend sem dictum auctor. In eget risus luctus wisi convallis pulvinar. Vivamus sapien risus, tempor in, viverra in, aliquet pellentesque, eros. Aliquam euismod libero a sem.

Nunc velit augue, scelerisque dignissim, lobortis et, aliquam in, risus. In eu eros. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Curabitur vulputate elit viverra augue. Mauris fringilla, tortor sit amet malesuada mollis, sapien mi dapibus odio, ac imperdiet ligula enim eget nisl. Quisque vitae pede a pede aliquet suscipit. Phasellus tellus pede, viverra vestibulum, gravida id, laoreet in, justo. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Integer commodo luctus lectus. Mauris justo. Duis varius eros. Sed quam. Cras lacus eros, rutrum eget, varius quis, convallis iaculis, velit. Mauris imperdiet, metus at tristique venenatis, purus neque pellentesque mauris, a ultrices elit lacus nec tortor. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent malesuada. Nam lacus lectus, auctor sit amet, malesuada vel, elementum eget, metus. Duis neque pede, facilisis eget, egestas elementum, nonummy id, neque.

Proin non sem. Donec nec erat. Proin libero. Aliquam viverra arcu. Donec vitae purus. Donec felis mi, semper id, scelerisque porta, sollicitudin sed, turpis. Nulla in urna. Integer varius wisi non elit. Etiam nec sem. Mauris consequat, risus nec congue condimentum, ligula ligula suscipit urna, vitae porta odio erat quis sapien. Proin luctus leo id erat. Etiam massa metus, accumsan pellentesque, sagittis sit amet, venenatis nec, mauris. Praesent urna eros, ornare nec, vulputate eget, cursus sed, justo. Phasellus nec lorem. Nullam ligula ligula, mollis sit amet, faucibus vel, eleifend ac, dui. Aliquam erat volutpat.

Fusce vehicula, tortor et gravida porttitor, metus nibh congue lorem, ut tempus

purus mauris a pede. Integer tincidunt orci sit amet turpis. Aenean a metus. Aliquam vestibulum lobortis felis. Donec gravida. Sed sed urna. Mauris et orci. Integer ultrices feugiat ligula. Sed dignissim nibh a massa. Donec orci dui, tempor sed, tincidunt nonummy, viverra sit amet, turpis. Quisque lobortis. Proin venenatis tortor nec wisi. Vestibulum placerat. In hac habitasse platea dictumst. Aliquam porta mi quis risus. Donec sagittis luctus diam. Nam ipsum elit, imperdiet vitae, faucibus nec, fringilla eget, leo. Etiam quis dolor in sapien porttitor imperdiet.

Cras pretium. Nulla malesuada ipsum ut libero. Suspendisse gravida hendrerit tellus. Maecenas quis lacus. Morbi fringilla. Vestibulum odio turpis, tempor vitae, scelerisque a, dictum non, massa. Praesent erat felis, porta sit amet, condimentum sit amet, placerat et, turpis. Praesent placerat lacus a enim. Vestibulum non eros. Ut congue. Donec tristique varius tortor. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Nam dictum dictum urna.

Phasellus vestibulum orci vel mauris. Fusce quam leo, adipiscing ac, pulvinar eget, molestie sit amet, erat. Sed diam. Suspendisse eros leo, tempus eget, dapibus sit amet, tempus eu, arcu. Vestibulum wisi metus, dapibus vel, luctus sit amet, condimentum quis, leo. Suspendisse molestie. Duis in ante. Ut sodales sem sit amet mauris. Suspendisse ornare pretium orci. Fusce tristique enim eget mi. Vestibulum eros elit, gravida ac, pharetra sed, lobortis in, massa. Proin at dolor. Duis accumsan accumsan pede. Nullam blandit elit in magna lacinia hendrerit. Ut nonummy luctus eros. Fusce eget tortor.

Ut sit amet magna. Cras a ligula eu urna dignissim viverra. Nullam tempor leo porta ipsum. Praesent purus. Nullam consequat. Mauris dictum sagittis dui. Vestibulum sollicitudin consectetuer wisi. In sit amet diam. Nullam malesuada pharetra risus. Proin lacus arcu, eleifend sed, vehicula at, congue sit amet, sem. Sed sagittis pede a nisl. Sed tincidunt odio a pede. Sed dui. Nam eu enim. Aliquam sagittis lacus

eget libero. Pellentesque diam sem, sagittis molestie, tristique et, fermentum ornare, nibh. Nulla et tellus non felis imperdiet mattis. Aliquam erat volutpat.

4.2 Buchberger's Algorithm

Vestibulum sodales ipsum id augue. Integer ipsum pede, convallis sit amet, tristique vitae, tempor ut, nunc. Nam non ligula non lorem convallis hendrerit. Maecenas hendrerit. Sed magna odio, aliquam imperdiet, porta ac, aliquet eget, mi. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Vestibulum nisl sem, dignissim vel, euismod quis, egestas ut, orci. Nunc vitae risus vel metus euismod laoreet. Cras sit amet neque a turpis lobortis auctor. Sed aliquam sem ac elit. Cras velit lectus, facilisis id, dictum sed, porta rutrum, nisl. Nam hendrerit ipsum sed augue. Nullam scelerisque hendrerit wisi. Vivamus egestas arcu sed purus. Ut ornare lectus sed eros. Suspendisse potenti. Mauris sollicitudin pede vel velit. In hac habitasse platea dictumst.

Suspendisse erat mauris, nonummy eget, pretium eget, consequat vel, justo. Pellentesque consectetuer erat sed lacus. Nullam egestas nulla ac dui. Donec cursus rhoncus ipsum. Nunc et sem eu magna egestas malesuada. Vivamus dictum massa at dolor. Morbi est nulla, faucibus ac, posuere in, interdum ut, sapien. Proin consectetuer pretium urna. Donec sit amet nibh nec purus dignissim mattis. Phasellus vehicula elit at lacus. Nulla facilisi. Cras ut arcu. Sed consectetuer. Integer tristique elit quis felis consectetuer eleifend. Cras et lectus.

Ut congue malesuada justo. Curabitur congue, felis at hendrerit faucibus, mauris lacus porttitor pede, nec aliquam turpis diam feugiat arcu. Nullam rhoncus ipsum at risus. Vestibulum a dolor sed dolor fermentum vulputate. Sed nec ipsum dapibus urna bibendum lobortis. Vestibulum elit. Nam ligula arcu, volutpat eget, lacinia eu, lobortis ac, urna. Nam mollis ultrices nulla. Cras vulputate. Suspendisse at risus

at metus pulvinar malesuada. Nullam lacus. Aliquam tempus magna. Aliquam ut purus. Proin tellus.

Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Donec scelerisque metus. Maecenas non mi ut metus porta hendrerit. Nunc semper. Cras quis wisi ut lorem posuere tristique. Nunc vestibulum scelerisque nulla. Suspendisse pharetra sollicitudin ante. Praesent at augue sit amet ante interdum porta. Nunc bibendum augue luctus diam. Etiam nec sem. Sed eros turpis, facilisis nec, vehicula vitae, aliquam sed, nulla. Curabitur justo leo, vestibulum eget, tristique ut, tempus at, nisl.

Nulla venenatis lorem id arcu. Morbi cursus urna a ipsum. Donec porttitor. Integer eleifend, est non mattis malesuada, mi nulla convallis mi, et auctor lectus sapien ut purus. Aliquam nulla augue, pharetra sit amet, faucibus semper, molestie vel, nibh. Pellentesque vestibulum magna et mi. Sed fringilla dolor vel tellus. Nunc libero nunc, venenatis eget, convallis hendrerit, iaculis elementum, mi. Nullam aliquam, felis et accumsan vehicula, magna justo vehicula diam, eu condimentum nisl felis et nunc. Quisque volutpat mauris a velit. Pellentesque massa. Integer at lorem. Nam metus erat, lacinia id, convallis ut, pulvinar non, wisi. Cras iaculis mauris ut neque. Cras sodales, sem vitae imperdiet consequat, pede purus sollicitudin urna, ac aliquam metus orci in leo. Ut molestie ultrices mauris. Vivamus vitae sem. Aliquam erat volutpat. Praesent commodo, nisl ac dapibus aliquet, tortor orci sodales lorem, non ornare nulla lorem quis nisl.

Sed at sem vitae purus ultrices vestibulum. Vestibulum tincidunt lacus et ligula. Pellentesque vitae elit. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Duis ornare, erat eget laoreet vulputate, lacus ipsum suscipit turpis, et bibendum nisl orci non lectus. Vestibulum nec risus nec libero fermentum fringilla. Morbi non velit in magna gravida hendrerit. Pellentesque quis

lectus. Vestibulum eleifend lobortis leo. Vestibulum non augue. Vivamus dictum tempor dui. Maecenas at ligula id felis congue porttitor. Nulla leo magna, egestas quis, vulputate sit amet, viverra id, velit.

Ut lectus lectus, ultricies sit amet, semper eget, laoreet non, ante. Proin at massa quis nunc rhoncus mattis. Aliquam lorem. Curabitur pharetra dui at neque. Aliquam eu tellus. Aenean tempus, felis vitae vulputate iaculis, est dolor faucibus urna, in viverra wisi neque non risus. Fusce vel dolor nec sapien pretium nonummy. Integer faucibus massa ac nulla ornare venenatis. Nulla quis sapien. Sed tortor. Phasellus eget mi. Cras nunc. Cras a enim.

Quisque nisl. In dignissim dapibus massa. Aenean sem magna, scelerisque nec, ullamcorper quis, porttitor ut, lectus. Fusce dignissim facilisis tortor. Vivamus gravida felis sit amet nunc. Nam pulvinar odio vel enim. Pellentesque sit amet est. Vivamus pulvinar leo non sapien. Aliquam erat volutpat. Ut elementum auctor metus. Mauris vestibulum neque vitae eros. Pellentesque aliquam quam. Donec venenatis tristique purus. In nisl. Nulla velit libero, fermentum at, porta a, feugiat vitae, urna. Etiam aliquet ornare ipsum. Proin non dolor. Aenean nunc ligula, venenatis suscipit, porttitor sit amet, mattis suscipit, magna. Vivamus egestas viverra est. Morbi at risus sed sapien sodales pretium.

Morbi congue congue metus. Aenean sed purus. Nam pede magna, tristique nec, porta id, sollicitudin quis, sapien. Vestibulum blandit. Suspendisse ut augue ac nibh ullamcorper posuere. Integer euismod, neque at eleifend fringilla, augue elit ornare dolor, vel tincidunt purus est id lacus. Vivamus lorem dui, commodo quis, scelerisque eu, tincidunt non, magna. Cras sodales. Quisque vestibulum pulvinar diam. Phasellus tincidunt, leo vitae tristique facilisis, ipsum wisi interdum sem, dapibus semper nulla velit vel lectus. Cras dapibus mauris et augue. Quisque cursus nulla in libero. Suspendisse et lorem sit amet mauris malesuada mollis. Nullam id

justo. Maecenas venenatis. Donec lacus arcu, egestas ac, fermentum consectetuer, tempus eu, metus. Proin sodales, sem in pretium fermentum, arcu sapien commodo mauris, venenatis consequat augue urna in wisi. Quisque sapien nunc, varius eget, condimentum quis, lacinia in, est. Fusce facilisis. Praesent nec ipsum.

Suspendisse a dolor. Nam erat eros, congue eget, sagittis a, lacinia in, pede. Maecenas in elit. Proin molestie varius nibh. Vivamus tristique purus sed augue. Proin egestas semper tortor. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Vestibulum orci enim, sagittis ornare, eleifend ut, mattis at, ligula. Nulla molestie convallis arcu. Ut eros tellus, condimentum at, sodales in, ultrices vel, nulla.

4.3 Efficiency

Duis magna ante, bibendum eget, eleifend eget, suscipit sed, neque. Vestibulum in mi sed massa cursus cursus. Pellentesque pulvinar mollis neque. Fusce ut enim vitae mauris malesuada tincidunt. Vivamus a neque. Mauris pulvinar, sapien id condimentum dictum, quam arcu rhoncus dui, id tempor lacus justo et justo. Proin sit amet orci eu diam eleifend blandit. Nunc erat massa, luctus ac, fermentum lacinia, tincidunt ultrices, sapien. Praesent sed orci vitae dolor sollicitudin adipiscing. Cras a neque. Ut risus dui, interdum at, placerat id, tristique eu, enim. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Etiam adipiscing eros vestibulum dolor. Pellentesque aliquam, diam eget eleifend posuere, augue eros porttitor lectus, ac dignissim dui metus nec felis. Quisque lacinia. Vestibulum tellus. Suspendisse nec wisi. Aenean ac felis. Aliquam ultrices metus et nulla.

Praesent sed est non nibh tempus venenatis. Praesent rhoncus. Curabitur sagittis est sit amet neque. Sed commodo malesuada lectus. Phasellus enim tellus, tempor

ut, tristique eu, aliquam eu, quam. Aenean quis quam quis wisi gravida vehicula. Pellentesque a massa a leo pretium rhoncus. Suspendisse ultrices. Donec lacinia malesuada massa. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Donec pretium ornare mauris. Phasellus auctor erat eget enim. Integer scelerisque, felis eu consequat fringilla, lorem wisi ultricies velit, id vehicula purus nulla eget odio. Nullam mattis, diam a rutrum fermentum, odio sapien tristique quam, id mollis tellus quam in odio. Mauris eu sapien. Donec aliquam lorem sit amet lorem pharetra lobortis.

Donec ac velit. Sed convallis vestibulum sapien. Vivamus tempor lacus sed lacus. Nunc ut lorem. Ut et tortor. Nullam varius wisi at diam. Etiam ultricies, dolor sit amet fermentum vulputate, neque libero vestibulum orci, vitae fringilla neque arcu aliquet ante. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Quisque venenatis lobortis augue. Sed tempor, tellus iaculis pellentesque pharetra, pede dui malesuada mauris, vel ultrices urna mauris ac nibh. Etiam nibh odio, ultricies vehicula, vestibulum vitae, feugiat eleifend, felis. Vivamus pulvinar. Aliquam erat volutpat. Nulla egestas venenatis metus. Nam feugiat nunc quis elit egestas sagittis. Sed vitae felis. In libero arcu, rhoncus in, commodo eget, auctor in, enim. Vivamus suscipit est. Nulla dapibus, magna vel aliquet egestas, massa massa hendrerit lacus, ac rutrum tellus tellus sit amet felis. Cras viverra.

Suspendisse eu nunc. Aliquam dignissim urna sit amet mauris. Cras commodo, urna ut porttitor venenatis, arcu metus sodales risus, vitae gravida sapien ligula in est. Donec vulputate sollicitudin wisi. Donec vehicula, est id interdum ornare, nibh tellus consectetuer justo, a ultrices felis erat at lectus. In est massa, malesuada non, suscipit at, ullamcorper eu, elit. Nam nulla lacus, bibendum sit amet, sagittis sed, tempor eget, libero. Praesent ligula. Suspendisse nulla. Etiam diam. Nulla ante diam, vestibulum et, aliquet ac, imperdiet vitae, urna. Fusce tincidunt lacus vel

elit. Maecenas dictum, tortor non euismod bibendum, pede nibh pretium tellus, at dignissim leo eros eget pede. Nulla venenatis eleifend eros. Aenean ut odio dignissim augue rutrum faucibus. Fusce posuere, tellus eget viverra mattis, erat tellus porta mi, at facilisis sem nibh non urna. Phasellus quis turpis quis mauris suscipit vulputate. Sed interdum lacus non velit. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae.

Vivamus vehicula leo a justo. Quisque nec augue. Morbi mauris wisi, aliquet vitae, dignissim eget, sollicitudin molestie, ligula. In dictum enim sit amet risus. Curabitur vitae velit eu diam rhoncus hendrerit. Vivamus ut elit. Praesent mattis ipsum quis turpis. Curabitur rhoncus neque eu dui. Etiam vitae magna. Nam ullamcorper. Praesent interdum bibendum magna. Quisque auctor aliquam dolor. Morbi eu lorem et est porttitor fermentum. Nunc egestas arcu at tortor varius viverra. Fusce eu nulla ut nulla interdum consectetuer. Vestibulum gravida. Morbi mattis libero sed est.

Nam quis enim. Quisque ornare dui a tortor. Fusce consequat lacus pellentesque metus. Duis euismod. Duis non quam. Maecenas vitae dolor in ipsum auctor vehicula. Vivamus nec nibh eget wisi varius pulvinar. Cras a lacus. Etiam et massa. Donec in nisl sit amet dui imperdiet vestibulum. Duis porttitor nibh id eros.

Mauris consectetuer, wisi eu lobortis scelerisque, urna nibh feugiat quam, id congue eros justo eget orci. Ut tellus. Maecenas mattis sapien sed eros. Aliquam quis lectus. Donec nec massa ac turpis semper cursus. Etiam consectetuer ante vel odio. Aliquam tincidunt felis non dolor. Cras id augue ut nisl pretium placerat. Phasellus sapien sapien, pharetra sed, aliquam nec, suscipit a, nibh. Suspendisse risus. Nulla ut mi eget tellus sollicitudin euismod. Vestibulum malesuada malesuada dui. Ut at est ac dui aliquam sagittis. Aliquam erat volutpat.

Curabitur ullamcorper est in mauris. Praesent ac massa. Quisque enim odio, lobortis nec, mattis ut, luctus et, mauris. Mauris eu risus. Cum sociis natoque

penatibus et magnis dis parturient montes, nascetur ridiculus mus. Duis eu ligula. Nulla vehicula leo tincidunt erat. Maecenas et nunc. Sed ut sapien. Vestibulum in est. Vestibulum rhoncus.

Donec metus metus, condimentum eu, accumsan nec, vulputate non, purus. Vestibulum ullamcorper vehicula sapien. Mauris risus odio, hendrerit ac, congue ac, ullamcorper at, odio. Aenean leo justo, commodo vitae, placerat blandit, malesuada vel, sem. Donec sit amet ante eget mauris adipiscing sollicitudin. Curabitur posuere sem et leo. Nulla ultricies mauris. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Fusce sollicitudin augue vel tellus. Vivamus mauris eros, pharetra vel, lacinia pretium, egestas a, nibh. Morbi a ligula.

Donec vitae turpis. Suspendisse porttitor. Mauris aliquam purus vitae tellus. Morbi metus diam, tempus ac, cursus ut, ultricies quis, nulla. Praesent nec justo. In lobortis. Donec nec lectus a neque laoreet rhoncus. Quisque in risus nec wisi lacinia ullamcorper. In placerat. Proin facilisis sollicitudin libero. Integer eget neque et pede placerat aliquet. Aliquam purus nulla, pulvinar ut, facilisis quis, sodales sed, magna. Curabitur nulla lectus, rutrum id, bibendum ut, sagittis eget, diam. Sed porta dolor eget est. Integer hendrerit orci. In hac habitasse platea dictumst.

Chapter 5: Conclusion

Ut facilisis. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Sed pellentesque, turpis sit amet aliquet porta, risus odio venenatis felis, at porta tellus lacus vitae nisl. Donec augue. Quisque consequat, pede laoreet pellentesque posuere, urna sapien tempor justo, eu aliquam tortor nunc id mauris. Fusce pretium, purus facilisis consequat mattis, ligula leo pretium mauris, ac suscipit augue sapien sit amet ipsum. Praesent et ligula eget tortor dapibus blandit. Duis rutrum felis eget dolor. Vestibulum quis elit. Integer dignissim, velit at scelerisque congue, ipsum nulla dignissim dolor, lacinia scelerisque neque erat a mi. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Quisque ipsum lectus, euismod et, lacinia eu, iaculis eu, pede. Etiam justo quam, cursus ut, vulputate vel, feugiat ut, eros. Fusce eleifend mollis ipsum.

Nulla facilisi. Nunc nec elit. Integer ornare convallis tortor. Proin ac diam. In est sapien, laoreet euismod, mattis a, tincidunt at, risus. Vivamus risus. Vestibulum aliquam, urna aliquam porttitor accumsan, nulla tortor ullamcorper elit, ut consequat augue purus sit amet libero. Vivamus nisl lacus, commodo vel, dignissim ut, vestibulum id, pede. Curabitur malesuada hendrerit libero. Mauris quis dolor in tellus varius posuere. Sed vulputate elit at wisi. Fusce vitae neque. Nulla consectetuer, nunc ac eleifend laoreet, mi nulla commodo wisi, vel faucibus ligula lectus ut arcu. Vivamus hendrerit.

Sed varius, nulla vitae tincidunt lobortis, nibh ipsum sollicitudin libero, et commodo tellus massa in neque. Nulla facilisi. Aenean nec lectus. Aliquam fermentum. Duis ut magna et augue interdum gravida. Morbi elit. Fusce malesuada tempus ipsum. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Mauris iaculis enim non metus. Nullam dui magna, congue et, suscipit

sed, aliquam vel, turpis. Quisque ultricies.

Suspendisse feugiat sapien laoreet ante. Integer fringilla, erat eget adipiscing ultrices, nibh dui sollicitudin nunc, in lobortis arcu odio vitae erat. Fusce bibendum ultricies lacus. Mauris eleifend ligula a ante. Etiam faucibus cursus pede. Mauris enim eros, malesuada eu, mattis sit amet, blandit in, nulla. Fusce sit amet purus id mi posuere tincidunt. Mauris sit amet quam vitae quam semper accumsan. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Nam a justo at quam accumsan euismod. Duis tincidunt tristique risus. Ut vel nibh vel libero varius malesuada. In hac habitasse platea dictumst. Morbi sagittis mattis lorem. Pellentesque metus tellus, rutrum vitae, malesuada et, pharetra accumsan, ante. Quisque ac metus ac nisl gravida pellentesque. Sed dapibus feugiat sapien. Vestibulum nec nunc eget sem aliquam lobortis. Suspendisse aliquam quam quis metus.

Suspendisse in odio. In elit diam, cursus vitae, venenatis in, molestie in, leo. Cras ornare. Nulla libero. Phasellus feugiat mattis libero. Sed vehicula aliquam ligula. Nullam lacinia, felis vel dignissim sodales, enim lectus lobortis diam, quis nonummy mauris odio auctor tortor. Integer in dui nec lacus bibendum ultrices. Etiam odio elit, aliquam et, porttitor id, interdum cursus, elit. Nulla eleifend tempor mauris. In vel arcu quis pede laoreet vulputate.

Morbi pharetra magna a lorem. Cras sapien. Duis porttitor vehicula urna. Phasellus iaculis, mi vitae varius consequat, purus nibh sollicitudin mauris, quis aliquam felis dolor vel elit. Quisque neque mi, bibendum non, tristique convallis, congue eu, quam. Etiam vel felis. Quisque ac ligula at orci pulvinar rutrum. Donec mi eros, sagittis eu, consectetuer sed, sagittis sed, lorem. Nunc sed eros. Nullam pellentesque ante quis lectus. Vivamus lacinia, sapien vel fermentum placerat, purus nisl aliquet odio, et porta wisi dui nec nunc. Fusce porta cursus libero.

Quisque eu mi a augue mollis posuere. Donec tincidunt, lorem at vestibulum

pulvinar, felis purus nonummy urna, at accumsan purus dui nec leo. Praesent tortor turpis, vehicula in, aliquet ut, dignissim ac, leo. Curabitur sagittis mi id eros. In magna. Sed vitae elit facilisis elit semper sollicitudin. Curabitur convallis tempor nulla. Nullam non turpis a pede sagittis ultrices. Etiam vulputate pede in ligula. Sed a ante id metus pellentesque suscipit. Sed adipiscing justo vitae sapien. Nunc posuere, pede ullamcorper gravida egestas, justo libero tincidunt arcu, vitae pellentesque arcu leo ut mauris. Pellentesque auctor mauris sit amet elit luctus fringilla. Cras sed wisi. Morbi luctus enim vitae tellus. Vivamus venenatis sodales libero.

In hac habitasse platea dictumst. Suspendisse potenti. Nulla pretium sem sit amet nisl. Nulla facilisi. Sed aliquam, turpis sed hendrerit gravida, nunc metus aliquam urna, eget pharetra nibh urna nec lectus. Duis in nisl a nisl commodo facilisis. Nunc placerat risus sed leo. Duis pellentesque porta libero. Praesent et enim. Aenean ullamcorper, ante sit amet fermentum mollis, ligula metus laoreet magna, accumsan accumsan nibh wisi at wisi. Nam tincidunt tempor neque. Maecenas dolor. Donec interdum nisl. Aliquam quam libero, interdum quis, volutpat sed, semper ut, eros. Pellentesque sodales auctor quam. Nullam suscipit massa nec elit. Nullam vulputate.

Aliquam a nulla. Suspendisse suscipit. Etiam lectus ante, interdum sit amet, euismod venenatis, condimentum eu, urna. Etiam at turpis. Cras quis ligula. Cras varius, sapien non pellentesque bibendum, mauris wisi sodales sem, ac commodo mauris neque non felis. Sed sollicitudin tincidunt arcu. Nullam vel lectus sit amet magna tincidunt tempor. Phasellus a ante. Donec et diam.

Proin sit amet augue. Praesent lacus. Donec a leo. Ut turpis ante, condimentum sed, sagittis a, blandit sit amet, enim. Integer sed elit. In ultricies blandit libero. Proin molestie erat dignissim nulla convallis ultrices. Aliquam in magna. Etiam sollicitudin, eros a sagittis pellentesque, lacus odio volutpat elit, vel tincidunt felis dui vitae lorem. Etiam leo. Nulla et justo.

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https://www.linkedin.com/in/tommymeek123/
https://github.com/tommymeek123

EDUCATION

- **Master of Science, Mathematics** | GPA: 3.88 — Winston-Salem, NC
  Wake Forest University — Expected Graduation: May 2023
- **Bachelor of Science, Mathematics, Computer Science** | GPA: 3.92 — Cullowhee, NC
  Western Carolina University — Graduated: May 2021
- **Associate of Science** | GPA: 4.00 — Sylva, NC
  Southwestern Community College — Graduated: December 2018

PROFESSIONAL EXPERIENCE

- **Teaching Assistant** | Wake Forest University — August 2021 - Present
  - Responsible for leading weekly study sessions to ensure student success and teaching undergraduate courses in the absence of the professor.
  - Tutored Wake Forest undergraduate students in a one-on-one setting and in groups as large as 20.
  - Graded up to 50 undergraduate student assignments per week while maintaining my own coursework.
- **Math/Computer Science Tutor** | Western Carolina University — August 2019 - July 2021
  - Voted *Most Valuable Tutor* by my coworkers and peers.
  - Tutored Western Carolina undergraduate students in a one-on-one setting and in groups as large as 15.
  - Responsible for mentoring up to 30 walk-in clients per day for all math and computer science courses.
  - Assisted students in computer science lab courses by answering questions and verifying the correctness of their Python or Java lab assignments.
- **Table Games Dealer** | Harrah's Cherokee Casino — September 2009 - July 2019
  - Proficiently dealt casino games including craps and blackjack at a high pace.
  - Provided service and entertainment to up to hundreds of guests per day.
  - Ensured game security by being mindful of all active players and other nearby guests.

TECHNICAL SKILLS

- **Proficient with**: Python, Java, JavaScript, Haskell, MATLAB, Git, LaTeX
- **Familiar with**: C, C++, MIPS, Rust, HTML, CSS, SQL, R, Bash, PowerShell, Vim, Macaulay2

RESEARCH

- Master's thesis: "A Functional Computer Algebra System for Polynomials" (in progress)
- "An Axiomatic and Contextual Review of the Armitage and Doll Model of Carcinogenesis"
  Published in *Spora: A Journal of Biomathematics* Volume 8 (2022)
- "Algebraic Properties of a Hypergraph Lifting Map"
  Published in *Integers* Volume 21 (2021)
- "Avoiding Monochromatic Sub-paths in Uniform Hypergraph Paths and Cycles"
  Available at https://arxiv.org/abs/2003.00035

PRESENTATIONS

UNCG Regional Mathematics and Statistics Conference (Virtual)
Avoiding Blue Edges in 3-uniform Hyperpaths
University of North Carolina Greensboro | Greensboro, NC | November 14, 2020

WCU SURP Symposium (Virtual)
Algebraic Properties of a Hypergraph Lifting Map
Western Carolina University | Cullowhee, NC | October 16, 2020

SCC Lunch and Learn
Canceled due to COVID-19
Undergraduate Research at Western Carolina University
Southwestern Community College | Sylva, NC | March 20, 2020

Mathematical Association of America Southeastern Section Meeting
Canceled due to COVID-19
Avoiding Blue Edges in 3-uniform Hyperpaths
High Point University | High Point, NC | March 13, 2020

Joint Mathematics Meetings Undergraduate Student Poster Session
Monochromatic Subhypergraphs in Stochastic Processes on Hypergraphs
Colorado Convention Center | Denver, CO | January 17, 2020

AWARDS AND ACHIEVEMENTS

- 2020-2021 WCU Senior Mathematics Award
- 2020-2021 WCU Advanced Computer Science Award
- 2020-2021 WCU Dean's Outstanding Scholar in Mathematics
- Score of 10 on the 2020 William Lowell Putnam Mathematical Competition

GRANTS AND SCHOLARSHIPS

- 2022 Wake Forest summer thesis grant $3,000.00
- 2020-2021 L.E.A.R.N. Scholarship $2,000.00
- 2020-2021 Breitenbach Scholarship $1,381.00
- 2020-2021 George A. Milton Scholarship $1,213.00
- 2020 SURP research grant $5,300.00
- 2019-2020 CURM research grant $3,000.00