1 Approximability of Echo Chamber Problem

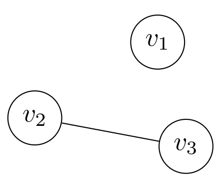
Theorem 1. *Echo Chamber Problem (ECP) has no $n^{1-\epsilon}$-approximation algorithm for any ϵ unless $\mathcal{P} = \mathcal{NP}$*

Proof. We show this by presenting a direct reduction from Maximum Independent Set (MIS), which is known having the mentioned hardness factor.

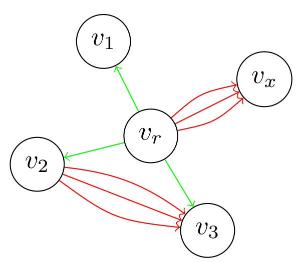
Let $G_1 = (V_1, E_1)$ be an undirected and unweighted graph and $\lambda \geq \frac{\alpha}{1-\alpha}$, $\lambda \in \mathbb{N}$ and $n_1 \coloneqq |V_1|$. We construct the *interaction* graph $G_2 = (V_2, E_2^+, E_2^-)$ as follows

- for each vertex $v_i \in V_1$ we add a vertex in G_2
- for each edge $e_{ij} \in E_1$ we add λn_1 negative edges between v_i and v_j
- add a vertex v_r and a positive edge between it and any other vertex that we already inserted in G_2
- add a vertex v_x and λn_1 negative edges between v_x and v_r

Furthermore, all the edges in G_2 are associated to the same content C and the same thread $T \in \mathcal{T}_C$. An illustration of the conversion can be found in Figure 1.



(a) G_1, undirected graph



(b) G_2, directed signed graph, for $\lambda = 1$

Figure 1: Example construction of the interaction graph G_2 from G_1, for $\alpha = \frac{1}{2}$

Claim 2. *Content C is controversial.*

Proof. Let m_2^- and m_2^+ be the number of negative and positive edges in G_2, respectively.

By construction $m_2^+ = n_1$ and $m_2^- \geq \lambda n_1$. Also, for $a, b, c \in \mathbb{R}^+$ it holds that $\frac{a+b}{a+b+c} \geq \frac{a}{a+c}$. Consequently

$$\eta(C) = \frac{m_2^-}{m_2^- + m_2^+} \geq \frac{\lambda n_1}{\lambda n_1 + n_1} = \frac{\lambda}{\lambda + 1} = \frac{\frac{\alpha}{1-\alpha}}{\frac{\alpha}{1-\alpha} + 1} \geq \alpha \tag{1}$$

□

So content C is controversial. This reduces the Echo Chamber Problem on G_2 to the maximization of

$$\xi(U) = \sum_{T \in S_C(U)} |T[U]| \tag{2}$$

Claim 3.

$$OPT(ECP) = OPT(MIS) \tag{3}$$

Proof. Let $I \subseteq V_1$ be an independent set of G_1 of size $|I| > 1$. Consider the associated solution in G_2 in which $U = I \cup \{v_r\}$. By construction it will contain $|I|$ positive edges, so T will not be controversial and also

$$OPT(ECP) \geq \xi(U) = |T[U]| = |I| \implies OPT(ECP) \geq OPT(MIS) \tag{4}$$

Now let $S \subseteq V_2$ be a solution of the Echo Chamber problem on G_2, and suppose $\xi(S) > 0$. It is easy to see that $v_r \in S$ and that $v_x \notin S$. Let $J \coloneqq S \backslash \{v_r\}$.

Suppose that 2 vertices v_i, $v_j \in J$ are linked in G_1. By construction there are at least λn_1 negative edges in $T[S]$, thus

$$\eta(T[S]) \geq \frac{\lambda n_1}{\lambda n_1 + |S - 1|} \geq \frac{\lambda n_1}{\lambda n_1 + n_1} = \frac{\lambda}{\lambda + 1} \geq \alpha \tag{5}$$

This means that $T[S]$ is controversial $\implies \xi(S) = 0 \implies$ *contradiction*. Consequently J contains vertices which are independent in G_1. Therefore $T[S]$ contains only positive edges; more specifically

$$\xi(S) = |T[S]| = |S| - 1 = |S \setminus \{v_r\}| = |J| \tag{6}$$

Thus

$$OPT(MIS) \geq |J| \implies OPT(MIS) \geq OPT(ECP) \tag{7}$$

So the optimal value of the constructed instance of Echo Chamber Problem exactly equals that of the Maximum Independent Set instance, so it has an hardness factor at least as large as that of MIS. □

This concludes the proof of Theorem 1. □

2 Approximability of Densest Echo Chamber Problem

Theorem 4. *Densest Echo Chamber Problem (DECP) has no $n^{1-\epsilon}$-approximation algorithm for any ϵ unless $\mathcal{P} = \mathcal{NP}$*

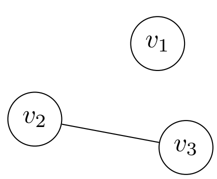
Proof. We again show this by presenting a direct reduction from Maximum Independent Set.

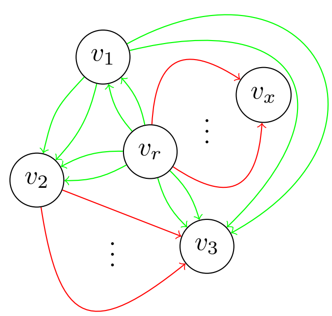
Let $G_1 = (V_1, E_1)$ be an undirected and unweighted graph and $\lambda \geq \frac{\alpha}{1-\alpha}$, $\lambda \in \mathbb{N}$. We construct the *interaction* graph $G_2 = (V_2, E_2^+, E_2^-)$ as follows

- for each vertex $v_i \in V_1$ we add a vertex in G_2
- for each edge $e_{ij} \in E_1$ we add $\lambda(n_1 + 1)^2$ negative edges between v_i and v_j
- for each edge $e_{ij} \in V_1 \times V_1, e_{ij} \notin E_1$ we add 2 positive edges between v_i and v_j
- add a vertex v_r and 2 positive edges between it and any other vertex that we already inserted in G_2
- add a vertex v_x and λn_1^2 negative edges between v_x and v_r

Furthermore, all the edges in G_2 are associated to the same content C and the same thread $T \in \mathcal{T}_C$. An illustration of the conversion can be found in Figure 2.

(a) G_1, undirected graph





(b) G_2, directed signed graph

Figure 2: Example construction of the interaction graph G_2 from G_1

Claim 5. *Content C is controversial.*

Proof. By construction $m_2^+ \leq n_1^2$ and $m_2^- \geq \lambda n_1^2$. Thus

$$\eta(C) = \frac{m_2^-}{m_2^- + m_2^+} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1^2} = \frac{\lambda}{\lambda + 1} = \frac{\frac{\alpha}{1-\alpha}}{\frac{\alpha}{1-\alpha} + 1} \geq \alpha \tag{8}$$

□

So content C is controversial. This reduces the Densest Echo Chamber Problem on G_2 to the maximization of

$$\psi(U) = \sum_{T \in S_C(U)} \frac{|T[U]|}{|U|} \tag{9}$$

Claim 6.

$$OPT(DECP) = OPT(MIS) \tag{10}$$

Proof. Let $I \subseteq V_1$ be an independent set of G_1 of size $n_I := |I| > 1$. Consider the associated solution in G_2 in which $U = I \cup \{v_r\}$.

By construction it will contain

- $2 \cdot n_I$ positive edges between v_r and $v_i \in I$
- $n_I(n_I - 1)$ positive edges between vertices $v_i \in I$

$|I|$ positive thus T will not be controversial and also

$$\psi(U) = \frac{|T[U]|}{|U|} = \frac{2n_I + \cdot n_I(n_I - 1)}{n_I + 1} = \frac{n_I^2 + n_I}{n_I + 1} = n_I \tag{11}$$

Consequently

$$OPT(ECP) \geq \psi(U) = |I| \implies OPT(DECP) \geq OPT(MIS) \tag{12}$$

Now let $S \subseteq V_2$ be a solution of the Densest Echo Chamber problem on G_2, and suppose $\psi(S) > 0$. It is easy to see that $v_r \in S$ and that $v_x \notin S$. Let $J := S \setminus \{v_r\}$.

Suppose that 2 vertices v_i, $v_j \in J$ are linked in G_1. By construction there are at least $\lambda(n_1 + 1)^2$ negative edges in $T[S]$, thus

$$\eta(T[S]) \geq \frac{\lambda(n_1 + 1)^2}{\lambda(n_1 + 1)^2 + n(n + 1)} \geq \frac{\lambda(n_1 + 1)^2}{\lambda(n_1 + 1)^2 + (n_1 + 1)^2} = \frac{\lambda}{\lambda + 1} \geq \alpha \tag{13}$$

This means that $T[S]$ is controversial $\implies \xi(S) = 0 \implies$ *contradiction*. Consequently J contains vertices which are independent in G_1. Therefore $T[S]$ contains only positive edges. As shown previously in Equation 11

$$\psi(S) = \frac{|T[S]|}{|S|} = |J| \tag{14}$$

Thus

$$OPT(MIS) \geq |J| \implies OPT(MIS) \geq OPT(ECP) \tag{15}$$

So the optimal value of the constructed instance of Densest Echo Chamber Problem exactly equals that of the Maximum Independent Set instance, so it has an hardness factor at least as large as that of MIS. $\square$

This concludes the proof of Theorem 4. $\square$