

## Hardness of Echo Chamber Problem

**Theorem 1.** *Solving exactly the Echo Chamber Problem is  $\mathcal{NP}$ -hard.*

*Proof.* We show this by presenting a direct reduction from Maximum Clique, which is well-known to have the mentioned hardness factor.

Let  $G_1 = (V_1, E_1)$  be an undirected and unweighted graph and  $\lambda \geq \frac{\alpha}{1-\alpha}$ ,  $\lambda \in \mathbb{N}$ . We construct the *interaction* graph  $G_2 = (V_2, E_2^+, E_2^-)$  as follows

- for each vertex  $v_i \in V_1$  we add a vertex in  $G_2$
- for each edge  $e_{ij} \in E_1$  add a positive edge between  $v_i$  and  $v_j$
- for each edge  $e_{ij} \in V_1 \times V_1, e_{ij} \notin E_1$  add  $\lambda n_1^2$  negative edges between  $v_i$  and  $v_j$ .
- add a vertex  $v_x$  and  $\lambda n_1$  negative edges between  $v_x$  and each other vertex  $v_i$  in  $G_2$ .

Furthermore, all the edges in  $G_2$  are associated to the same content  $C$  and the same thread  $T \in \mathcal{T}_C$ . An illustration of the conversion can be found in Figure 1.

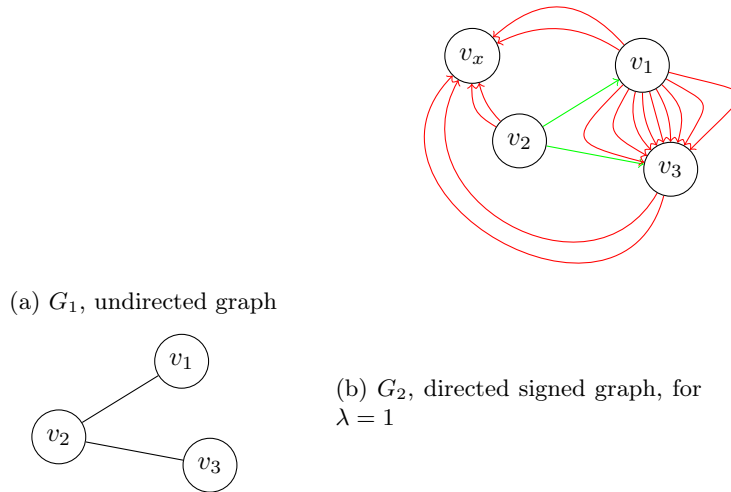


Figure 1: Example construction of the interaction graph  $G_2$  from  $G_1$ , for  $\alpha = \frac{1}{2}$

**Claim 2.** *Content  $C$  is controversial.*

*Proof.* Let  $m_2^+$  be the number of positive edges in  $G_2$ .

By construction  $m_2^+ = m_1$  and  $m_2^- \geq \lambda n_1^2$  so

$$\begin{aligned} \eta(C) &= \frac{m_2^-}{m_2^- + m_2^+} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + m_1} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1(n_1 - 1)} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1^2} = \quad (1) \\ &= \frac{\lambda}{\lambda + 1} = \frac{\frac{\alpha}{1-\alpha}}{\frac{\alpha}{1-\alpha} + 1} \geq \alpha \quad (2) \end{aligned}$$

□

This reduces the Echo Chamber Problem on  $G_2$  to the maximization of

$$\xi(U) = \sum_{T \in S_C(U)} |T[U]| \quad (3)$$

**Claim 3.** *The solution of the Echo Chamber Problem for  $G_2$  is a set of vertices  $\{v_{ia}\}_{i \in I} \subseteq V_2$  which is a clique in  $G_1$ .*

*Proof.* Let  $U := \{v_{ia}\}_{i \in I} \subseteq V_2$  be the solution of the Echo Chamber Problem on  $G_2$ .

We can assume  $\xi(U) > 0$  (otherwise the proof is trivial)<sup>1</sup>. It is also easy to see that  $U$  does not contain  $v_x$ <sup>2</sup>.

Now suppose that  $U$  does not induce a complete subgraph on  $G_1$ . This means that there is at least one missing edge  $e_{ij} \in V_1 \times V_1$ ,  $x_{ij} \notin E_1$  and consequently at least  $\lambda n_1^2$  negative edges in  $T[U]$ . Let  $n_U := |U|$ , then

$$\eta(T) \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_U(n_U - 1)} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1^2} = \frac{\lambda}{\lambda + 1} \geq \alpha \quad (4)$$

Therefore thread  $T$  is controversial and does not contribute to the score  $\Rightarrow \xi(U) = 0 \Rightarrow \text{contradiction}$ . □

**Claim 4.** *The solution of the Echo Chamber Score for  $G_2$  is a set of vertices  $U$  associated to the maximum clique of  $G_1$ .*

*Proof.* Suppose there is a set of vertices  $\tilde{U} \neq U, |\tilde{U}| > |U|$  which is a clique for  $G_1$ . Then by construction it will contain only positive edge in  $G_2$  and will be non-controversial. Also, being  $|T[\tilde{U}]| > |T[U]| \Rightarrow \xi(\tilde{U}) > \xi(U) \Rightarrow \text{contradiction}$ . □

This concludes the proof of Theorem 1. □

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<sup>1</sup>In this case any subset of  $V_2$  maximizes the echo chamber score, and this would clearly violate Claim 3. For simplicity we can assume that in this case the algorithm returns a *singleton*.

<sup>2</sup>Similarly to the proof of Claim 2 it can be shown that if  $v_x \in U$  then  $T$  becomes controversial