Hardness of Echo Chamber Problem

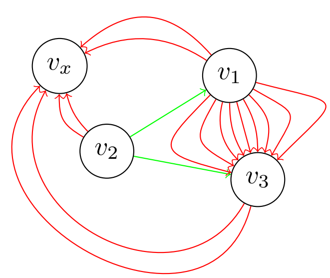
Theorem 1. *Solving exactly the Echo Chamber Problem is $\mathcal{NP}$-hard.*

Proof. We show this by presenting a direct reduction from Maximum Clique, which is well-known to have the mentioned hardness factor.

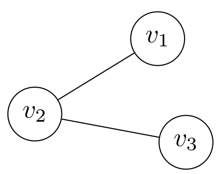
Let $G_1 = (V_1, E_1)$ be an undirected and unweighted graph and $\lambda \geq \frac{\alpha}{1-\alpha}$, $\lambda \in \mathbb{N}$. We construct the *interaction* graph $G_2 = (V_2, E_2^+, E_2^-)$ as follows

- for each vertex $v_i \in V_1$ we add a vertex in G_2
- for each edge $e_{ij} \in E_1$ add a positive edge between v_i and v_j
- for each edge $e_{ij} \in V_1 \times V_1, e_{ij} \notin E_1$ add λn_1^2 negative edges between v_i and v_j.
- add a vertex v_x and λn_1 negative edges between v_x and each other vertex v_i in G_2.

Furthermore, all the edges in G_2 are associated to the same content C and the same thread $T \in \mathcal{T}_C$. An illustration of the conversion can be found in Figure 1.



(a) G_1, undirected graph



(b) G_2, directed signed graph, for $\lambda = 1$

Figure 1: Example construction of the interaction graph G_2 from G_1, for $\alpha = \frac{1}{2}$

Claim 2. *Content C is controversial.*

Proof. Let m_2^+ be the number of positive edges in G_2.

By construction $m_2^+ = m_1$ and $m_2^- \geq \lambda n_1^2$ so

$$\eta(C) = \frac{m_2^-}{m_2^- + m_2^+} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + m_1} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1(n_1-1)/2} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1^2} = \tag{1}$$

$$= \frac{\lambda}{\lambda+1} = \frac{\frac{\alpha}{1-\alpha}}{\frac{\alpha}{1-\alpha}+1} \geq \alpha \tag{2}$$

□

This reduces the Echo Chamber Problem on G_2 to the maximization of

$$\xi(U) = \sum_{T \in S_C(U)} |T[U]| \tag{3}$$

Claim 3. *The solution of the Echo Chamber Problem for G_2 is a set of vertices $\{v_{ia}\}_{i\in I} \subseteq V_2$ which is a clique in G_1.*

Proof. Let $U := \{v_{ia}\}_{i\in I} \subseteq V_2$ be the solution of the Echo Chamber Problem on G_2.

We can assume $\xi(U) > 0$ (otherwise the proof is trivial)[1]. It is also easy to see that U does not contain v_x[2].

Now suppose that U does not induce a complete subgraph on G_1. This means that there is at least one missing edge $e_{ij} \in V_1 \times V_1$, $x_{ij} \notin E_1$ and consequently at least λn_1^2 negative edges in $T[U]$. Let $n_U := |U|$, then

$$\eta(T) \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_U(n_U-1)/2} \geq \frac{\lambda n_1^2}{\lambda n_1^2 + n_1^2} = \frac{\lambda}{\lambda+1} \geq \alpha \tag{4}$$

Therefore, thread T is controversial and does not contribute to the score $\implies \xi(U) = 0 \implies$ *contradiction.* □

Claim 4. *The solution of the Echo Chamber Problem for G_2 is a set of vertices U associated to a* maximum clique *of G_1.*

Proof. Suppose there is a set of vertices $\tilde{U} \neq U, |\tilde{U}| > |U|$ which is a clique for G_1. Then by construction it will contain only positive edge in G_2 and will be non-controversial. Also, being $|T[\tilde{U}]| > |T[U]| \implies \xi(\tilde{U}) > \xi(U) \implies$ *contradiction.* □

Claim 5. *The set of vertices defining a maximum clique on G_1 corresponds to a solution of the Echo Chamber Problem on G_2.*

[1] In this case any subset of V_2 maximizes the echo chamber score, and this would clearly violate Claim 3. For simplicity we can assume that in this case the algorithm returns a *singleton*.

[2] Similarly to the proof of Claim 2 it can be shown that if $v_x \in U$ then T becomes controversial

Proof. Let $U \subseteq V_1$ be a set of nodes defining a maximum clique on G_1 and $n_U = |U|$. By construction $T[U]$ will not be controversial and

$$\xi(U) = |T[U]| = n_U(n_U - 1)/2 \tag{5}$$

Now suppose $\exists \tilde{U} \subseteq V_2$ *s.t.* $\xi(\tilde{U}) > \xi(U)$. Due to Claim 3 U_2 induces a clique on G_1; consequently $T[\tilde{U}]$ has only positive edges and $\xi(\tilde{U}) = |T[\tilde{U}]| = n_{\tilde{U}}(n_{\tilde{U}} - 1)/2$.

And since $\xi(\tilde{U}) > \xi(U) \implies n_{\tilde{U}} > n_U \implies$ *contradiction*. ☐

This concludes the proof of Theorem 1. ☐