SICP Exercise 1.13

Real number Fibonacci sequence

Let $\tilde{f}(n) = \phi^n/\sqrt{5}$. Let $R(x)$ be a rounding function. First, let's check the base cases.

$$R(\tilde{f}(0)) = R\left(\frac{1}{\sqrt{5}}\right) = 0, \tag{1}$$

$$R(\tilde{f}(1)) = R\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right) = 1. \tag{2}$$

Now we can test whether the recurrence relation holds.

$$\tilde{f}(n) = \tilde{f}(n-1)\tilde{f}(n-2), \tag{3}$$

$$\frac{(1+\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}}, \tag{4}$$

$$(1+\sqrt{5})^2 = 2(1+\sqrt{5}) + 4, \tag{5}$$

$$6 + 2\sqrt{5} = 6 + 2\sqrt{5}. \tag{6}$$

Thus the recurrence relation holds for the real number recurrence relation.

Integer Fibonacci sequence

Introducing the rounding correction:

$$\hat{f}(n) = \psi^n = \frac{(1-\sqrt{5})^n}{2^n\sqrt{5}}, \tag{7}$$

in combination with the real number Fibonacci term

$$F(n) = \tilde{f}(n) - \hat{f}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}, \tag{8}$$

we can check the base cases

$$F(0) = \frac{1-1}{\sqrt{5}} = 0, \tag{9}$$

$$F(1) = \frac{1+\sqrt{5} - (1-\sqrt{5})}{2\sqrt{5}} = 1. \tag{10}$$

Again, test the recurrence relation holds.

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}} \tag{11}$$