SICP Exercise 1.13

Real number Fibonacci sequence

Let $\tilde{f}(n) = \phi^n/\sqrt{5}$. Let $R(x)$ be a rounding function. First, let's check the base cases.

$$R(\tilde{f}(0)) = R\left(\frac{1}{\sqrt{5}}\right) = 0, \tag{1}$$

$$R(\tilde{f}(1)) = R\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right) = 1. \tag{2}$$

Now we can test whether the recurrence relation holds.

$$\tilde{f}(n) = \tilde{f}(n-1)\tilde{f}(n-2), \tag{3}$$

$$\frac{(1+\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}}, \tag{4}$$

$$(1+\sqrt{5})^2 = 2(1+\sqrt{5}) + 4, \tag{5}$$

$$6 + 2\sqrt{5} = 6 + 2\sqrt{5}. \tag{6}$$

Thus the recurrence relation holds for the real number recurrence relation.

Integer Fibonacci sequence

Introducing the rounding correction:

$$\hat{f}(n) = \psi^n = \frac{(1-\sqrt{5})^n}{2^n\sqrt{5}}, \tag{7}$$

in combination with the real number Fibonacci term

$$F(n) = \tilde{f}(n) - \hat{f}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}, \tag{8}$$

we can check the base cases

$$F(0) = \frac{1-1}{\sqrt{5}} = 0, \tag{9}$$

$$F(1) = \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = 1. \tag{10}$$

Again, test the recurrence relation holds.

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}} \tag{11}$$

$$(1+\sqrt{5})^n - (1-\sqrt{5})^n = 2\left[(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}\right] + 4\left[(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}\right], \tag{12}$$

which, after some algebra leads to

$$(1+\sqrt{5})^2)\frac{(1+\sqrt{5})^{n-2}}{(1-\sqrt{5})^{n-2}} = \frac{(1+\sqrt{5})^2(1+\sqrt{5})^{2n-4}}{(-4)^{n-2}}, \tag{13}$$

$$\left[(1+\sqrt{5})^2 - 2(1+\sqrt{5}) - 4\right]\frac{(1+\sqrt{5})^{2n-4}}{(-4)^{n-2}} = (1+\sqrt{5})^2 - 2(1+\sqrt{5}) - 4, \tag{14}$$

from which it can be seen that both sides are equal to 0.

Boundedness of correction term

As we can show that the correction term is always less than 1/2, we know that the nearest integer to a $\tilde{\phi}(n)$ term must be the corresponding Fibonacci number. Proof of boundedness is below.

$$4 < 5 < 9, \tag{15}$$

$$2 < \sqrt{5} < 3, \tag{16}$$

$$-2 > -\sqrt{5} > -3, \tag{17}$$

$$-1 > 1 - \sqrt{5} > -2, \tag{18}$$

$$-\frac{1}{2} > \frac{1-\sqrt{5}}{2} > -1, \tag{19}$$

$$0 > -\frac{-1}{2\sqrt{5}} > \left(\frac{1-\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} > -\frac{1}{\sqrt{5}} > -\frac{1}{2}, \tag{20}$$

or simplified we can see that

$$0 > \left(\frac{1-\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} > -\frac{1}{2}, \tag{21}$$

and as the term in the centre decreases in size as it is exponentiated, the inequality holds for all values of n.