Section 10.2: Average Variable Cost and Revenues

In the previous lesson we explored what insights can be gained when we know both a function and its derivative function. We asked and answered questions about memory and recall using the two functions. That example set us up to explore the Total Cost function, how it is written as a sum of Fixed Cost and Variable Cost. Then we introduced two new concepts related to cost: Unit Cost and Marginal Cost, which was used to estimate the Cost of producing one more item than the current production level.

Our goals in this lesson are:

1) Determine what fixed costs and variable costs are from the cost function.
2) Calculate the Average Variable Cost (Unit Cost) function.
3) Compare Average Variable Costs at different production levels and interpret the results.
4) Write the Revenue function given an equation for price (demand).
5) Calculate the Marginal Revenue function.
6) Compare Marginal Revenue at different production levels.

We need one more definition before continuing our investigation of cost to produce, given $\boldsymbol{x}$ is the **quantity** in production.

Definition: Given $C(x) = Fixed\ Cost + Variable\ Cost$ and $Variable\ Cost = VC(x)$ for producing x units of a product, **Average Variable Cost** is the average cost per item on the interval $[0, x]$.

Average Variable Cost on the interval *(0, x]* =

$$\frac{\Delta y}{\Delta x} = \frac{C(x)-C(0)}{x-0} = \frac{(Fixed+Variable) - Fixed}{x} = \frac{Variable\ Cost}{x}$$

which we are calling $AVC(x)$.

$$AVC(x) = \frac{VC(x)}{x}$$

Compare that to $UnitC(x) = \frac{C(x)}{x}$

We will be concerned with questions involving average variable cost and average revenue in a business setting. In our previous lesson we were introduced to a function that we called "Unit Cost". We need to distinguish between these two concepts. First and foremost, we always want to think of "average" as something that is **calculated over an interval**. The interval ***(0,x]*** is used to define "Average Variable Cost". The key difference is that the Unit Cost includes the fixed cost when calculating cost per item. Average Variable Cost subtracts out the fixed cost, as seen in the definition.

> **Example:** Given the cost function for production of *cell phone cases* is $C(x) = 750 + 0.01x^2$, find a) Fixed cost, b) Variable Cost, c) Unit cost, d) Average Variable Cost. Find e) the marginal cost $C'(x)$ using one of the toolkit function formulas.
> **Part 1:** Calculate all 6 function values when x = 150.
> **Part 2:** Give a sentence or two describing what each of these calculations is telling us about cost when production is 150.

Fixed cost is the part of C(x) that is constant.

$$Fixed\ cost = 750$$

Variable cost is the part of C(x) that contains x.

$$VC(x) = 0.01x^2$$

Unit cost is C(x) divided by x.

$$UnitC(x) = \frac{750 + 0.01x^2}{x} = (0.01x^2 + 750) \div x$$

Average variable cost is VC(x) divided by x. Note that $x^2 \div x = x$.

$$AVC(x) = \frac{0.01x^2}{x} = 0.01x$$

Finally, we will write the marginal cost function.

Marginal cost $C'(x)$ uses the toolkit combo $[C(x) = x^2, C'(x) = 2x]$. We multiply in front by 0.01. (The derivative of 750 is zero.)

$$\boldsymbol{C'(x) = 0.01 \cdot 2 \cdot x = 0.02x}$$

Part 1: When x = 150, $C(150) = 750 + 0.01 \cdot 150^2 = 975$

$$Fixed\ cost = 750 \text{ (does not depend on x),}$$
$$VC(150) = 0.01 \cdot 150^2 = 225$$
$$UnitC(150) = (225 + 750) \div 150 = 6.50$$
$$AVC(150) = 0.01 \cdot 150 = 1.50$$
$$C'(150) = 0.02 \cdot 150 = 3.00$$

Part 2: Write a sentence or two telling what each of these calculations says about cost when production is at 150.

The total cost to produce 150 cell phone cases is $975

The fixed cost of production, regardless of how many cell phone cases are produced, is $750.

The variable cost for producing 150 cell phone cases is $225, which it the total variable cost for all 150 in production.

The unit cost for producing 150 cell phone cases is $6.50, which means at the production level of 150 it costs the company $6.50 per case, which includes the fixed cost (overhead and basic operations).

The average variable cost for producing 150 cell phone cases is $1.50. At the production level of 150 without considering fixed cost, it costs the company on average $1.50 per case.

The marginal cost at the production level of 150 cell phone cases is $3.00. This means it would cost the company an additional $3.00 to make one more cell phone case and increase production to 151.

Conversation Starter: When is unit cost preferred over average variable cost when looking at numbers? Same question but reverse it. When should average variable cost matter more than unit cost?

Try this: For $C(x) = 750 + 0.01x^2$, find the total cost, unit cost, average variable cost, and marginal cost when x = 200, 250, 300, and 350. We already calculated for x = 150. Put your answers into a grid or table. What do these numbers tell us? Complete this sentence: Marginal cost is ________ as much as average variable cost.

If we knew that our information is applicable for production up to 420 (for example), we could enter the equations into a spreadsheet and calculate for all multiples of 10 or 20 starting with x = 0 all the way up to 420. If multiples of 10, there will be 21 additional rows. Use the Fill…Down feature to calculate these formulas all the way down.

	A	B	C	D	E	F	G
1	x	Fixed Cost	Variable Cost	Total Cost	Unit Cost	Average Variable Cost	Marginal Cost
2	0	750	=0.01*A2^2	=B2+C2	These do not apply when x = 0		
3	20	750	=0.01*A3^2	=B3+C3	=D3/A3	=C3/A3	=0.02*A3
4	40	750					
5	60	750					
…	…	…	…	…	…	…	…
23	420	750					

Now we will talk about revenue. We have already learned that revenue is a product of price with quantity. Price is determined by the demand function, and x is quantity. So we will start there.

In our cell phone case manufacturing company, the demand equation is given by $price = f(x) = 15 - 0.018x$. What is the revenue function, $R(x)$?

$$R(x) = x \cdot (15 - 0.018x)$$

Since $R(0) = 0$, we have an interesting situation when it comes to Unit Revenue and Average Variable Revenue. The Unit Revenue is the price per unit. That price is dependent on the amount in production, which we assume to be equal to the amount in demand. If production = demand, then we have a perfectly competitive situation. This concept can get more complicated, but for the purposes of our investigation we will assume that production = demand.

> **Definition:** When production (supply) = demand, then the average revenue per unit sold is equal to the price.
>
> $$AR(x) = price(x)$$

We also can derive meaning from the idea of marginal revenue. In the same way that we can discover how cost would increase if production increased by a small amount, we could also discover how revenue would increase if production/sales increased by a small amount.

> **Definition:** Given total revenue, $R(x)$, **marginal revenue** for increase in production by a small amount from x to x + h is given by:
>
> $$\frac{\Delta R}{\Delta x} = \frac{R(x+h)-R(x)}{h} = [R(x+h) - R(x)] \div h.$$
>
> When $R(x)$ is given by an equation that has a derivative, **marginal revenue** is $R'(x) = \frac{dR}{dx}$.

In general, $price = b - mx$, where the slope is $-m$ and $b = price\ when\ x\ is\ 0$.

The choice of $-m$ is to signify there will always be a negative sign with the x term. This means our revenue equation will always be in the form of

$$R(x) = x \cdot (b - mx)$$

We do this so that we can have a formula for R'(x) based solely on the values of $b\ and\ m$.

Definition: The pair of functions $R(x) and\ R'(x)$ will typically be as seen here, when $price = b - mx$.

$$R(x) = x \cdot (b - m \cdot x)$$

$$R'(x) = b - 2 \cdot m \cdot x$$

In our cell phone case manufacturing business, $b = 15$ and $m = 0.018$.

$$R(x) = x \cdot (15 - 0.018x) \text{ and}$$

$$R'(x) = 15 - 2 \cdot 0.018x = 15 - 0.036x$$

We want to make sure the coefficient of x is doubled.

Example: For $R(x)$ as given above, calculate average revenue and marginal revenue when x = 20, 30, and 60. Then write a sentence or two explaining these results.

Solution: When asked to repeat the same calculation for more than one input, it is always a good idea to make a table. It's easier to visualize the results that way.

x	Average Revenue	Marginal Revenue
	15-0.018x	15-0.036x
20	14.64	14.28
30	14.46	13.92
60	13.92	12.84

Here we see that the average revenue per item sold is the same as the price. But just because the average revenue goes down, does not mean that profits will also go down. The marginal revenue shows us what the revenue would be if we produced and sold one additional item at the given production level (x). It makes sense that these numbers would also decrease as x increases because if production is increasing, so is demand… which means the price is going down as will our marginal revenue also go down.

Conversation Starter: Why is marginal revenue decreasing at a faster rate than average revenue?

Hint: The answer has to do with the fact that marginal revenue seeks to increase production and increase demand at the same time, which means lowering prices. Average revenue is about what has happened up to the current level of production/demand.

Practice Exercises

1) The cost to produce backpacks is given by the equation
$C(x) = 22 + 2.7x - 0.01x^2$

a) What is the fixed cost?
b) What is the unit cost function?
c) What is the variable cost function?
d) Find the Average Variable Cost function.
e) Use the answer in d) to find the average variable cost for x = 10; x = 25; x = 50; x = 75
f) What do the numbers mean?

2) The demand for backpacks is given by: $price\ (p) = 12 - 0.11x$

a) Write the revenue function R(x)
b) What units are the "y" variable for R(x)?
c) Write the average revenue function.
d) Write the marginal revenue function.
e) Use the answer in c) and d) to find average revenue and marginal revenue when x = 10; x = 25; x = 50; x = 75
f) Explain these results.