Section 9.3: The Derivative as a Function

In Lesson 2 we explored how to find the slope of a curve at a point using secant lines or by extending tangent lines to approximate the derivative. Let's summarize the different descriptions of this, and the vocabulary:

Start with y = f(x) as a function of x. Assume it is "well-behaved" on an interval so that it has a ***derivative (slope)*** at the point (a, b).
Each of these is the same thing:

1) The **slope** of f(x) at the point (a,b)
2) The **slope** of the tangent line that passes through (a,b)
3) The **instantaneous rate of change** of y with respect to x at the point (a,b)
4) The **derivative** of f(x) at the point (a,b)
5) The **slope** of a specific secant line, which is (on a graph) *parallel* to the tangent line at (a,b) and passes through two points on the graph of y = f(x).

We go from defining the derivative at a specific point to a definition of the derivative at ANY point, which means that the derivative can be understood as a function of x, as a companion function.

Start with **y = f(x)** as a function of x, and it **has a slope** at all points (x,y) in its domain, except any end points.

The ***derivative of f(x)*** is its companion function which we call $f'(x)$. In words, we say "***f-prime of x***". We can use the formula for the derivative to calculate the slope at any point.

We also say that $f'(x)$ is the "**rate of change of f with respect to x**".

Notation: $f'(x) = \frac{dy}{dx}$ [These are two ways of writing the same thing.]

Example 1: For $f(x) = x^2$ its derivative function is $f'(x) = 2x$. Consider the point (3, 9) which is on the graph of f(x). Then we can find the derivative (slope) at the point (3,9) using the formula for the derivative.

Solution: The $slope\ at\ (3,9) = f'(3) = 2 \cdot 3 = 6$

Try this: Find the derivative at the point (4, 16) for $f(x) = x^2$

Solution: The derivative at (4, 16) is $f'(4) = 2 \cdot 4 = 8$

Let's look at the derivative functions for all the toolkit functions. It should be noted that there are rules associated with finding derivatives of functions. These rules involve a level of mastery of advanced algebra that is not necessary when accomplishing the goals and objectives of the study of this book. Toward that end, consider the questions that follow this chart.

Toolkit Function Name	Formula	Companion Function (Derivative)
Constant	$f(x) = c$	$f'(x) = 0$
Identity	$f(x) = x$	$f'(x) = 1$
Power – Square	$f(x) = x^2$	$f'(x) = 2x$
Power – Cubic	$f(x) = x^3$	$f'(x) = 3x^2$
Rational - Reciprocal	$f(x) = \frac{1}{x}$	$f'(x) = -\frac{1}{x^2}$
Rational – Pi Function	$f(x) = \frac{1}{x^2+1}$	$f'(x) = \frac{-2x}{(x^2+1)^2}$
Square Root	$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$
Exponential	$f(x) = e^x$	$f'(x) = e^x$
Natural Logarithm	$f(x) = \ln x$	$f'(x) = \frac{1}{x}$

Conversation Starter: By looking at the graph of each toolkit function, can you predict whether the derivative should be positive or negative

a) to the right of x = 0

b) to the left of x = 0

Now look at the function equations for $f'(x)$. Is $f'(x)$ positive for x values that we expect? Is it negative for x values that we expect?

Interpretations of the Derivative:

What the derivative *measures* depends on the context, and the units of "x" and "y". Words associated with the derivative are as follows: velocity, speed, acceleration, marginal cost, marginal revenue, and marginal profit, inflow rate, outflow rate, evaporation rate, or rate of increase/decrease.

In business settings, the use of the word 'marginal' is used to describe how much of the y quantity is needed to increase x by ONE unit. We will see this in many examples involving cost, revenue, profit, and price. This is the focus of the next chapter.

Example 2: Suppose the number of pounds of beef that the market ***demands*** is given by the function $y = f(x) = \frac{1}{x}$. y is measured in *thousands* of pounds and x is price in \$. The companion function $f'(x) = -\frac{1}{x^2}$

a) In this context, what is $f'(x) = \frac{dy}{dx}$ a ***measure of***?

Answer: $\frac{dy}{dx}\ measures\ \frac{thousands\ of\ pounds}{dollar\ in\ price}$ which can be interpreted as "thousands of pounds (of beef) per dollar in price" .

In other words, $f'(x) = \frac{dy}{dx}$ measures how much demand ***changes*** for every increase of \$1 in price.

b) What is the demand for beef when price is *\$3.00*?

Answer: $f(3) = \frac{1}{3} =$ 0.333 thousand pounds

= 0.333 x 1,000 pounds

= 333 pounds

c) Find and interpret the *derivative of f(x) with respect to x* at $x = \$3.00$.

Answer: $f'(\$3.00) = f'(3) = -\frac{1}{3^2} = -\frac{1}{9} \approx -0.111$ which we interpret as "demand decreases by 0.111 thousand pounds if the price goes up from \$3 to \$4"

As before we can convert that number to pounds:

$-$ 0.111 thousand pounds per dollar
= $-$ 0.111 x 1,000 pounds per dollar
= $-$111 pounds per dollar

Try this: Answer parts b) and c) but when $x = \$5.00$

Solution: When x = \$5.00 the demand is 0.200 thousand pounds of beef, or 200 pounds. The derivative when x = \$5.00 is – 0.040 which is – 40 pounds of beef per dollar. We interpret that using a summary statement:

"When the price of beef is \$5.00 per pound, the demand is decreasing at the rate of – 40 pounds of beef per dollar of increase in the price."

Combinations of Toolkit Functions and Their Derivatives:

Linear combinations of functions and their derivatives behave the way we expect. When we add or subtract two functions, we add or subtract their derivative (companion) functions. When we have a coefficient with a function, we have a coefficient with the derivative function.

Things get a bit more complicated when the x variable is multiplied by a constant inside the function. In each of our toolkit functions, it is possible to rewrite the function so that the constant coefficient of x becomes a coefficient of y.

Example: Suppose $g(x) = \sqrt{3x}$. What is the derivative then?

Rewrite $g(x) = \sqrt{3} \cdot \sqrt{x}$ and since $\sqrt{3} \approx 1.732$,
$g(x) = 1.732 \cdot \sqrt{x}$ is acceptable.

Now we can establish derivatives of functions with a coefficient, and sums/differences of functions.

Definition: Suppose $f(x) = x^2$ and $g(x) = x$ are two toolkit functions. Suppose we have constant a not equal to 0.

If $h(x) = a \cdot f(x) = ax^2$.
Then $h'(x) = a \cdot f'(x) = a \cdot 2x = 2ax$.

In other words, coefficients get passed from the function to the derivative.

If $h(x) = f(x) + g(x)$ or if $h(x) = f(x) - g(x)$
In this example:
$h(x) = x^2 + x$ or $h(x) = x^2 - x$

Then $h'(x) = f'(x) + g'(x)$ or $h'(x) = f'(x) - g'(x)$
which means in this example:
$h'(x) = 2x + 1$ or $h'(x) = 2x - 1$

Continuing the previous example, find the derivative of $g(x) = 1.732\sqrt{x}$

Answer: The toolkit function here is $f(x) = \sqrt{x}$ and $f'(x) = \frac{1}{2\sqrt{x}}$.

$g'(x) = 1.732 \cdot \frac{1}{2\sqrt{x}}$ which simplifies to

$g'(x) = \frac{1.732}{2\sqrt{x}} = \frac{1.732}{2} \cdot \frac{1}{\sqrt{x}}$ and $1.732 \div 2 = 0.866$

so, we have $g'(x) = 0.866 \cdot \frac{1}{\sqrt{x}} = \frac{0.866}{\sqrt{x}}$

It isn't necessary to simplify. But when looking at an answer key, often the solution is simplified. It's nice to see where the 0.866 came from.