Chapter 10 Applications of the Derivative
Section 10.1: Cost, Unit Cost, and Marginal Cost

Now that we have established the existence of a function and its derivative (which is also as a function) we will explore two scenarios in which the derivative helps us answer questions about a phenomenon.

Example 1: Our first function pair concerns memory. Specifically, it deals with how much content a person tends to recall after several days. The variable "x" again is a measure of time, this time in days. The variable "y" is a measure of the portion of material that is recalled at that time. The y variable can be interpreted as a percentage since the portion at the beginning, at x = 0, is 100. It should also be noted that this equation (formula) has no proven experimental basis in science, except that it is within the realm of what is possible. We use it for illustration purposes only. The decline of recall is represented, and we will capture the fact that recall is not linear. It starts with a function, a model for the relationship between time (x) and how much material is recalled (y).

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"Recall" letter blocks on a table

Our "recall function" model is:

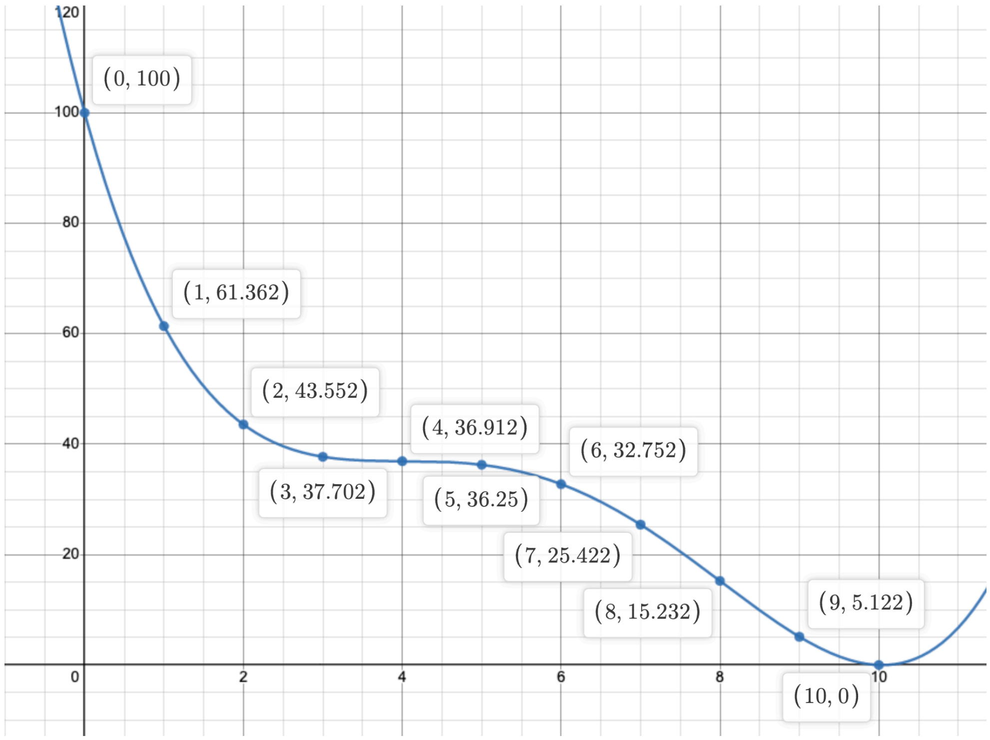
$$y = f(x) = 0.082x^4 - 1.97x^3 + 15.75x^2 - 52.5x + 100$$

and is valid for x between 0 and 10. The domain is [0, 10]
The derivative function is

$$\frac{dy}{dx} = f'(x) = 0.328x^3 - 5.91x^2 + 31.5x - 52.5$$

The domain of the derivative can't include the endpoints. We only have rate of change for x in $0 < x < 10$. Up to 9.999… but not 10.

Here is a graph of f(x): (After 10 days, or x = 10, the function no longer applies) We include a table of values for clarity and support of multiple representations of data. We use " > 0 " when we want to get as close to 0 as possible, and we use " < 10 " when we want to get as close to 10 as possible.



Let's complete a table of (x,y) values that represent the relationship between *amount of recall over x days*. That is f(x). Let's also complete a table of derivative (slope) values which will show *how fast or slow the rate of decline in memory is over x days*.

x	y=f(x)
0	**100**
1	**61.4**
2	**43.5**
3	**37.7**
4	**36.9**
5	**36.25**
6	**32.75**
7	**25.4**
8	**15.2**
9	**5.1**
10	**0**

*Two places where the derivative is 0 are approximated

x	$\frac{dy}{dx} = f'(x)$
> 0	**-52.5**
1	**-26.6**
2	**-10.5**
3	**-2.3**
4	**0***
5	**-1.75**
6	**-5.4**
7	**-9.1**
8	**-10.8**
9	**-8.6**
< 10	**0***

But what do all these numbers mean? What insights can be gained?

Some questions we can ask to help understand what is "mathematically" significant in this situation involve ***interpreting*** what we see on the graph and the numbers in these two tables.

1) Where does the ***greatest rate of decline*** in recall occur?

This question asks about "rate of decline" which is measured by the derivative. Looking at the slopes at the various points, and taking their absolute values, we would conclude that the greatest rate of decline occurs at or very near x = 0. The slope of – 52.5 is the steepest and gives us the "greatest rate of decline".

2) What are the ***units*** for $\frac{dy}{dx}$?

The units for $\frac{dy}{dx}$ are the units of y ***per*** unit of x. In this situation that means we are measuring change in "amount of recall per day".

3) Why is the derivative (slope) negative? Why is it never positive?

The derivative is negative any time the y values are decreasing. We can only assume that memory continues to fade over time, which is why the derivative is never positive.

4) What does it mean when the derivative is zero?

The derivative (slope) is zero at an instant when no change is happening. In this situation the rate of decline slowed down to zero and then started to speed up again. In math terms, we call this a "saddle point". What might be more descriptive is to call it a brief, momentary pause or levelling of the y variable before it started to decline again.

5) At what point does the amount of recall get cut in half? (50%)

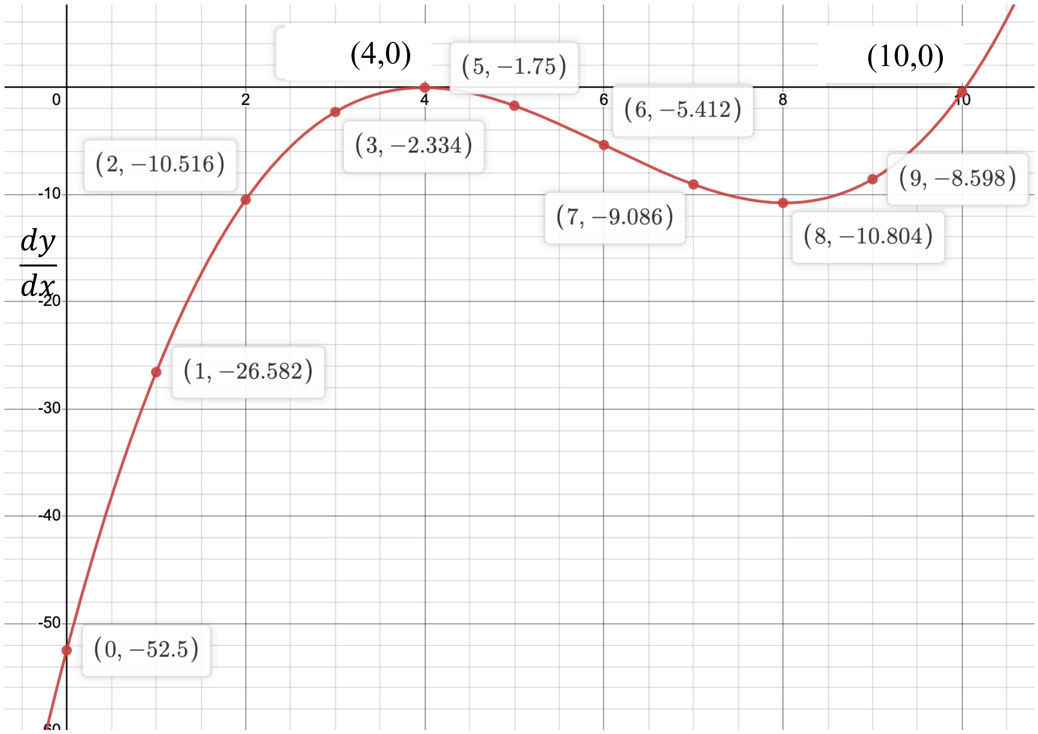
We don't know the exact time, but it happens when y = 50. That would place it between days 1 and 2. We can approximate the time to be 1.5 days.

6) Why does the graph stop at x = 10?

The amount of recall reaches zero. In other words, whatever content was memorized is all gone. Without further study, or something else entering the situation, day 10 is when this model ceases to be useful.

Conversation Starter: Think of one or two additional questions to answer about the situation that can be answered from the y = f(x) equation and one or two that can be answered from the $\frac{dy}{dx} = f'(x)$ equation.

Here is a graph of the derivative function:



The derivative function is quite distinct from the recall function. That's because now the coordinates of each point are not (x, y) but $\left(x, \frac{dy}{dx}\right)$. Both equations are polynomials, but the derivative is one degree less than the degree of the original function. Also worth noting is that the derivative graph is completely in the 4th quadrant, where the first coordinate is positive, and the second coordinate is negative. This occurs because the slope at any point on the original function is negative. We also see the two places when the derivative becomes zero.

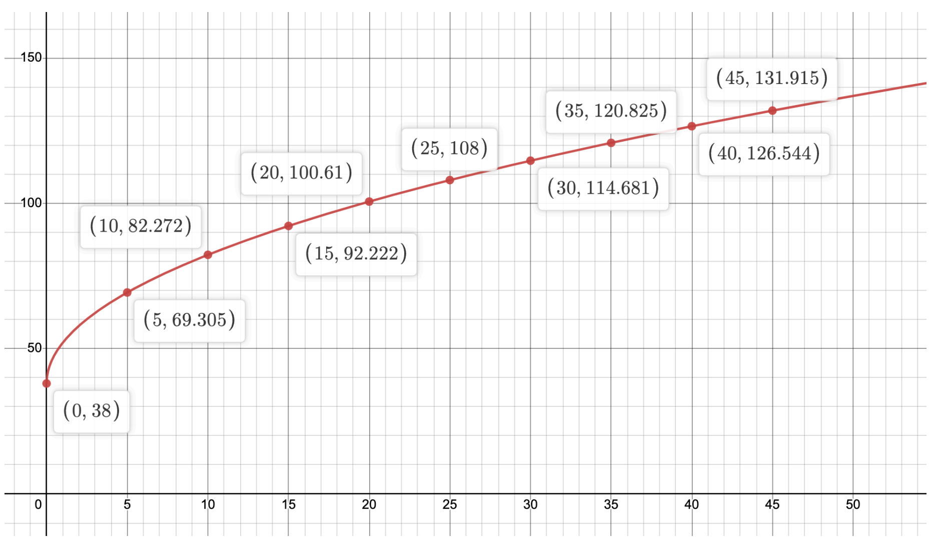
Example 2: The second example in this lesson has to do with cost to produce or manufacture a product. To make it interesting we will be manufacturing ceramic teapots. The cost function in business is often written using the variable "q" which represents the quantity of items in production. The reason for this is that quantity is tied to demand. But for purposes of clarity and mathematical consistency, we will continue to use the "x" variable as the input, or independent variable in our equation/model.

Let “x” be the quantity of teapots produced each week and y = C(x) is the overall (total) cost to produce “x” teapots. We know from a previous chapter that $C(x) = Fixed\ Cost + Variable\ Cost$. Keep in mind that our cost function will have a “fixed cost” amount and the rest is the “variable cost”. Let the fixed cost be \$38 and the variable cost is $14\sqrt{x}$. Let’s write a formula for the cost:

The Total Cost function is $C(x) = 38 + 14\sqrt{x}$. Variable Cost is $VC(x) = 14\sqrt{x}$.

Before we look at the graph of this, what do we expect to see (is it increasing or decreasing?) Does it ever reach a maximum? or a minimum? The more we produce, the more it is likely to cost to produce the items, therefore it is our expectation that the graph of C(x) will be increasing. The minimum is likely to occur when we are producing nothing (x = 0) and the maximum is only affected by our materials, equipment, and available labor. In theory, if we had unlimited supplies, equipment, and personnel there would be no maximum.

Let’s see if our expectation is verified by the graph.
Our total cost function is a variation of one of our toolkit functions, $y = \sqrt{x}$. Compare the graph of that function to our total cost graph. The shapes are the same, with only a few minor differences!



Try this: Make a table of values for C(x) choosing x to be multiples of 5 from 0 to 45.

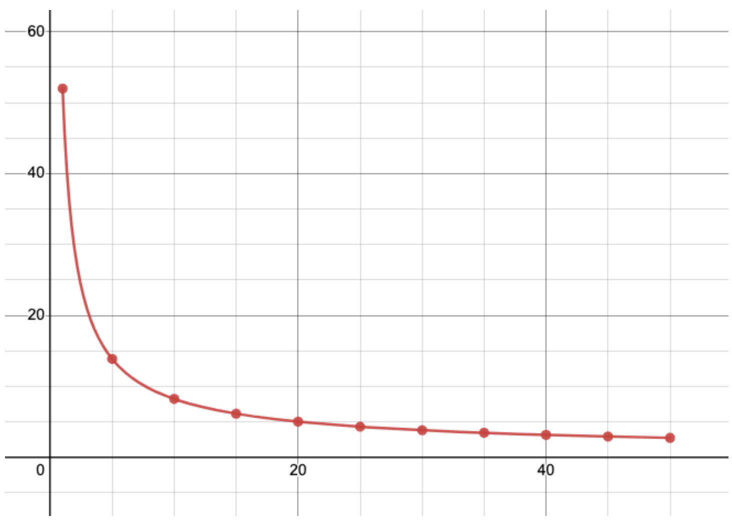
There are in fact two more functions that we will be using in this example. The first is the "unit cost" or cost per teapot. Unit cost is a measure of how *efficient* our production is. Do we expect the **cost per item** to increase or decrease the more items we make?

> **Definition:** The **unit cost function**, which we will designate $UnitC(x)$ is found by dividing the total cost to make x items by x:
>
> $$UnitC(x) = C(x) \div x$$

In this example then we have $UnitC(x) = \frac{38+14\sqrt{x}}{x}$. Notice that x can't = 0.

Before we see the graph of this unit cost, what do we expect to see? Is it increasing? decreasing? Should the ***cost per teapot*** to make 10 teapots be more or less than the ***cost per teapot*** to make 20? Are there any clues in the information provided so far that might give us insight? In general, the answer here can vary greatly. Materials and supplies may be cheaper in bulk when making more of something. Then again, more production may mean more labor requirements. Our model (equation) will tell us the situation.

First let's generate a table of values starting with x = 1 and then use multiples of 5 up to 50.



x	UnitC(x)
1	52
5	13.86
10	8.23
15	6.15
20	5.03
25	4.32
30	3.82
35	3.45
40	3.16
45	2.93
50	2.74

Now can we answer the question about cost per teapot? It would be unrealistic to believe that the unit cost will continue to go down. There must be a limit. In fact, there is! We have additional information. Models like this are generally only applicable ***for specific values of x***. **It turns out that we do not have the capacity to make more than 45 teapots.** This means that our unit cost has both a ***minimum and a maximum***. The most it costs to make a single teapot is $52.00 (when we make only one). The least it costs to make each teapot is $2.93, when we are making all 45 of them (full production).

In our teapot factory, as the total cost goes up, the unit cost goes down. This tells us that overall, the efficiency of our manufacturing process is such that making more teapots (at least up to a certain point) saves money in the labor and materials required per teapot.

Let's tackle some questions we could ask about this situation (before we talk about the derivative). Each one of these questions is answered from the cost and unit cost functions since *none* of them involve a ***rate of change***.

1) Our operating budget is $125.00. If we wanted to minimize unit cost and stay within budget, what would our production be? What would be the unit cost?
    a. Start with the total cost graph. We want a production amount where the total cost is $125. We can use the formula to help as well. When x = 38, $C(38) = 38 + 14\sqrt{38} \approx \$124.30$. If we increase production to 39, we have $C(39) = 38 + 14\sqrt{38} \approx \$125.43$ which is over budget. Our production would be 38.
    b. Using the Unit cost function, we can calculate the unit cost when x = 38. $UnitC(38) = (38 + 14\sqrt{38}) \div 38 \approx \$3.27$ per teapot to produce.

2) At what production level is the total cost between $120 and $121?
    a. Answer: From the graph, we see the point (35, 120.825). Since 120.825 is between $120 and $121 our production amount is 35.

3) At what production level (approximately) is the unit cost $7.00? $4.00? $3.00?

a. For a unit cost of $7.00, the production is between 10 and 15. From the graph, we estimate x = 13.
b. The point (27,4) is a good estimate for the $4.00 unit cost. The production level would be about 27 teapots.
c. The point (44,3) is a good estimate for the $3.00 unit cost. The production level would be about 44 teapots.

How would we verify these answers? Answer: Calculate them using the equation.

In a production setting, we also have a quantity called the "marginal cost" function. This is the derivative of the cost function, and it measures the ***cost to produce one more item*** if currently producing "x" items.

Definition: Given the total cost C(x) to produce x items, the **marginal cost** is the amount it costs to increase production by a **small amount** from x to x + h. It is given by the average rate of change formula:

$$\frac{\Delta C}{\Delta x} = \frac{C(x+h)-C(x)}{h} = [C(x+h) - C(x)] \div h.$$

When $C(x)$ is given by an equation that has a derivative, **marginal cost** is $C'(x) = \frac{dC}{dx}$.

In this example, $C(x) = 38 + 14\sqrt{x}$ which means that $C'(x) = \frac{7}{\sqrt{x}} = 7 \div \sqrt{x}$. Let's make a table using x values 1, 2, and then multiples of 5:

x	*C'(x)*
1	$7.00
2	4.95
5	3.13
10	2.21
15	1.81
20	1.57
25	1.40
30	1.28
35	1.18
40	1.11
45	-----

We will now use the marginal cost to answer some questions about ***increasing*** production:

a) When x = 20, how much would it cost to produce one more teapot?
   a. Answer: $C'(20) = 7 \div \sqrt{20} \approx \$1.57$. Notice this is not the same as the unit cost when making 20 teapots!
b) How many teapots are being made if the cost to produce one more is $1.18?

a. In other words, what is x when C'(x) = 1.18? Looking at our table, we can say x = 35

Practice Exercises:

1) Let the recall function be given by $y = \frac{100}{x^2+1}$. Compare this with the toolkit function $y = \frac{1}{x^2+1}$. Name two ways they are basically "the same". Name two ways they are different.

2) Given that $y = \frac{100}{x^2+1}$ its derivative is $\frac{dy}{dx} = \frac{-200x}{(x^2+1)^2} = -200x \div (x^2+1)^2$

   what is the slope at x = 3? Interpret this number in the "recall" function context.

3) The total cost to produce "x" items is given by $C(x) = -0.2x^2 + 20x + 120$ and the marginal cost is $C'(x) = -0.4x + 20$; we also know that the maximum number of items that can be made is 50.
   a. What is the fixed cost?
   b. What is the variable cost equation?
   c. What is the unit cost equation?
   d. What is the unit cost if the current production amount is 30?
   e. If the current production amount is 30, what would it cost to make one more?
   f. At what production amount would you be if it cost $2.00 to make just one more?