Section 10.3: Optimization with the Derivative

In these two chapters we have focused primarily on **five** relevant calculations on functions. Let's list them now, and note what we **expect**, what we **need**, and how it is **used**.

The first two apply to a **specified value**, or values of x and result in a number result.

1) *Average rate of change* of a function on an ***interval*** [x_1, x_2]. We also called this the *slope of the secant line* that passes through two points on the graph of the function.
    a. we **expect** to get a number for an answer, because both x_1 and x_2 are numbers.
    b. we **need** the x and y coordinates of the two points: $(x_1, y_1) and\ (x_2, y_2)$ either from the graph of the function or calculate the "y's" from the equation/formula.
    c. we **calculate** the slope using $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{change\ in\ y}{change\ in\ x}$.
    d. we **can use** this to estimate slope at a point on a curve if the secant line appears to be a good representation of the slope at a specific point on the curve.
2) *Instantaneous rate of change* of a function at a point. We call this the slope or derivative of the function at a (specified) point. It is the *slope of the tangent line* to the graph that passes through only one point (x_1, y_1).
    a. we **expect** to get a number for an answer, because we are given a specific value of x (the value of x is a number).
    b. we can **estimate** this from the graph by sketching the tangent line which passes through (x_1, y_1). Identify a second set of coordinates (another point) and calculate the slope – OR --
    c. we **need** the equation of the derivative for a more accurate answer.
    d. we **calculate** it from the equation of the derivative if known (plug in numbers): Evaluate it using x_1.

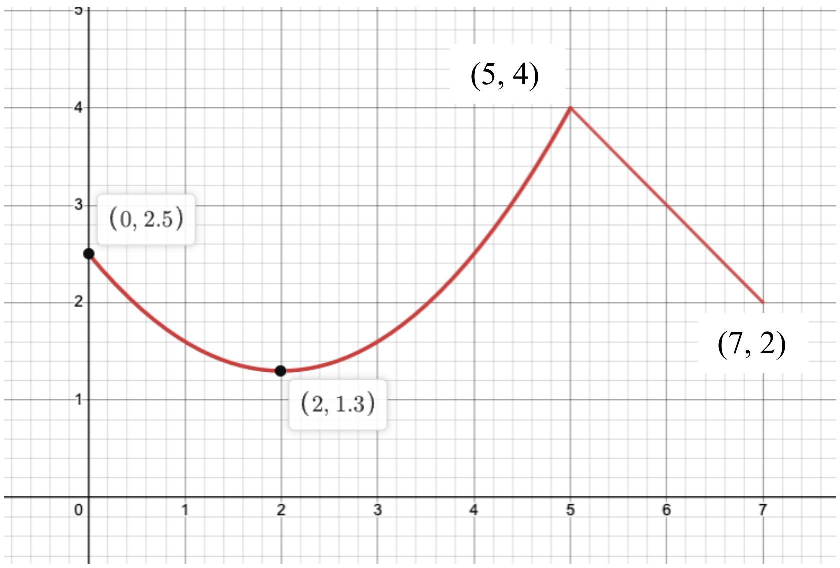
The last three involve the creation of a ***new function*** or formula using x as the variable, which then in turn can be used to calculate one of these: unit values, average over [0,x], marginal amount, or a rate of change, *at any value of x.*

3) *Unit function*, which in Economics measures $ per unit of production/sales.
    a. we **expect** to get a new function from the one that is given.
    b. we **need** the equation/formula of the quantity that we want to express as a unit quantity.
    c. we **write** the expression from a given function $f(x)$. The unit function is defined to be $Unit_{f(x)} = \frac{f(x)}{x} = f(x) \div x$.
    d. we **use** the new formula/function to calculate the unit cost/revenue/profit at a specified value of x.
4) *Average of a function* on **[0,x]** which itself is a new function of x.
    a. we **expect** to get a new function from the one that is given.
    b. we **need** the equation/formula of the quantity that is being averaged.
    c. we **simplify** this expression $\frac{f(x)-f(0)}{x}$. When $f(0)=0$ this becomes $\frac{f(x)}{x}$ and looks like the unit function.
    d. we **use** the new formula to **calculate** the average at a specified value of x.
5) *Derivative of a function*, which in Economics is called "*marginal*" equation, measuring the change in a cost, revenue, or profit with respect to production quantity. In a physical application it might be called "*velocity*" or "*speed*" at which a measurable unit is changing with respect to time.
    a. we **expect** to have the companion function, which we call "f-prime".
    b. we **need** to have a means of finding the derivative formula, or it is given along with the function that represents the quantity in question.
    c. we **use** the new formula to **calculate** derivatives at a specified value.

In Calculus we refer to the process of optimization when we are looking for either a minimum or maximum of a quantity over its domain or a portion thereof (on an interval of x). Any potential maximum or minimum is called an ***extremum***. The plural of that word is "***extrema***", from Latin. Suppose we are looking for the maximum and minimum value of a function f(x) on an interval $[a, b]$.

This means
$f(a)\ and\ f(b)\ both\ exist. The\ function\ exists\ everywhere\ in\ between\ as\ well.$

Consider this graph of a function on the interval [0,7].



The Extreme Value Theorem in Calculus *guarantees* that f(x) has both an absolute minimum and an absolute maximum somewhere between x=0 and x=7. We do need the function to be continuous (unbroken). Not only do we have a guarantee, but we know where they have to be. Extrema MUST occur at

a) one or both endpoints, or...
b) a place on the interior where the derivative does not exist, but the function is continuous (like a cusp), or...
c) a place where the derivative is zero.

From the graph, let's identify the endpoints. $(0, 2.5)\ and\ (7, 2)$

Name any point where the derivative does not exist. $(5, 4)$

Let's name any point where the derivative is zero. $(2, 1.3)$

Now the absolute maximum will be the one with the highest y value. Here it is the point $(5, 4)$. The absolute minimum will be the one with the lowest value. Here it is the point $(2, 1.3)$. In general, absolute min/max can occur at more than one point. Our concern is to identify only the ones we care about, on a specified closed interval of x.

In a business production setting, we have several functions that qualify for both optimization and for marginal analysis. Two of these we've already seen.

Marginal cost and marginal revenue functions have been introduced to calculate how revenue and cost would be impacted if production were to increase by a small amount.

Shutterstock Image ID
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Image of a blue teapot

In a previous example, we considered a business production setting in which teapots were made. The cost function was given as $C(x) = 38 + 14\sqrt{x}$ and its companion function (the derivative) is $C'(x) = \frac{7}{\sqrt{x}} = 7 \div \sqrt{x}$. Let's take another look at a table for $C'(x)$ using x values 1, 2, and then increment "x" by multiples of 5. Values of x do not go past 45.

We will now use the marginal cost to answer some questions about ***optimizing*** production:

x	***C'(x)***
1	$7.00
2	4.95
5	3.13
10	2.21
15	1.81
20	1.57
25	1.40
30	1.28
35	1.18
40	1.11
45	---

Is C'(x) = 0 anywhere? No.

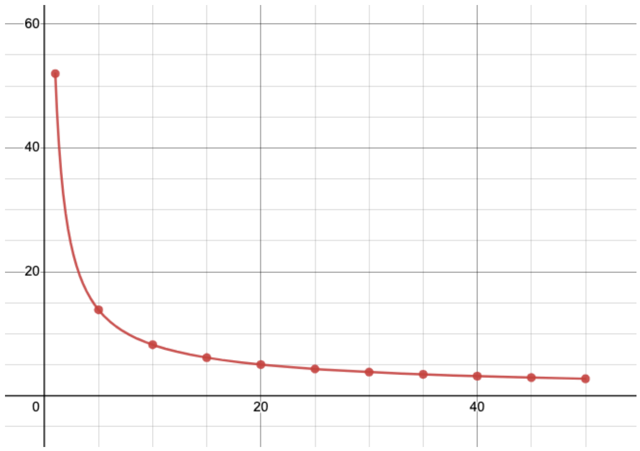
On the interval [0,45] not including the endpoints, is there a place where the derivative does not exist?

No because our formula works for any x value in this interval. In fact, the derivative is always positive, which supports the data that cost increases as production increases.

Where are the extrema for cost on x in [0, 45]?
The only extrema for cost occur at the endpoints.

What is the maximum cost? What is the minimum cost? The extrema for cost occur at the endpoints, with the minimum occurring at x = 0 when total cost = $38. The maximum occurs at x = 45 with total cost = $131.915

Optimizing for total cost may not be the goal anyway. Perhaps instead the goal is to optimize unit cost, or cost per unit. This should have the effect of increasing the gap between revenue and cost, which has to do with profit. Let's take another look at the graph of unit cost in the teapot factory.



We saw this graph before, and we see that the unit cost is decreasing on the interval [1, 45]. Unit cost does not make sense for anything less than x = 1. And the operation is most efficient when at full production of 45 teapots.

We will now look at revenue, and then introduce the profit function in our teapot manufacturing business.

Suppose that demand for our teapots is such that

$$price = 10.5 - 0.16x$$

What are $R(x)$ and $R'(x)$? From the price equation, b = 10.5 and m = 0.16.

$R(x) = x \cdot (10.5 - 0.16x)$

$R'(x) = 10.5 - 0.32x$

x	C(x)	C'(x)	R(x)	R'(x)
0	$38.00	---	0.00	---
1	$52.00	$7.00	10.34	10.18
2	$57.80	4.95	20.36	9.86
5	$69.305	3.13	48.50	8.90
10	$82.27	2.21	89.00	7.30
15	$92.22	1.81	121.50	5.70
20	$100.61	1.57	146.00	4.10
25	$108.00	1.40	162.50	2.50
30	$114.68	1.28	171.00	0.90
35	$120.825	1.18	171.50	-0.70
40	$126.54	1.11	164.00	-2.30
45	$131.915	---	148.50	---

Not far from the place where our unit cost is minimized we seem to have a maximum revenue in the vicinity of production x = 35. We see that in two ways. On the table the largest value of R(x) occurs at x = 35, *AND* R'(x) goes from a positive number to a negative one. Another theorem in Calculus called the Intermediate Value Theorem (IVT) guarantees that the derivative function will hit all the values between two values on our table. Why is this important? Because extrema occur where the derivative is zero (0). And *zero is between 0.90 and -0.70.* So, there is an x between 30 and 35 where R'(x) = 0... which makes that either a maximum or minimum revenue!

Let's state that another way.

Observation #1: On the table, we noticed that revenues increased up to $171.50, then started to go down again. This was our first clue that there may be a maximum right at x = 35, or at least nearby.

Observation #2: We know from the Extreme Value Theorem that if there is going to be a maximum or minimum, it will occur at one of the endpoints or at a place where the derivative is zero. It seems rather obvious that the smallest revenue occurs when x = 0. But the other endpoint has a revenue of $148.50, which is smaller than the revenues at x = 30 and x = 35.

Observation #3: If we were to graph the revenue function, we could see that the shape of the graph is a quadratic, and that the vertex occurs at x = 32.813. Recall that the vertex of a quadratic occurs where the derivative (slope) is zero.

Observation #4: We have further evidence of a place where the derivative is zero and a potential maximum. R'(x) went from a positive number to a negative number. We know from IVT Calculus that there must be an x where R' = 0.

Our manufacturing operation requires that we choose a whole number for production, so let's complete the table filling in all values of x between 30 and 35. We can see that x = 33, which is the nearest whole number to 32.8, is where we would maximize revenue.

We have just *optimized* revenue. We found the maximum.

x	*C(x)*	*C'(x)*	*R(x)*	*R'(x)*
10	$82.27	2.21	89.00	7.30
15	$92.22	1.81	121.50	5.70
20	$100.61	1.57	146.00	4.10
25	$108.00	1.40	162.50	2.50
30	$114.68	1.28	171.00	0.90
31	115.95	1.26	171.74	0.58
32	117.20	1.24	172.16	0.26
33	118.42	1.22	172.26	-0.06
34	119.63	1.20	172.04	-0.38
35	$120.825	1.18	171.50	-0.70
40	$126.54	1.11	164.00	-2.30
45	$131.915	---	148.50	---

But the goal of any business is to maximize profit. This brings us to the last two equations that we will need in this chapter.

Definition: Let profit = P(x) and the marginal profit is P'(x).

$$P(x) = R(x) - C(x)$$

$$P'(x) = R'(x) - C'(x)$$

Maximum profit occurs when $P'(x) = 0$ which means we want to find the value(s) of x where

$$R'(x) \cong C'(x)$$

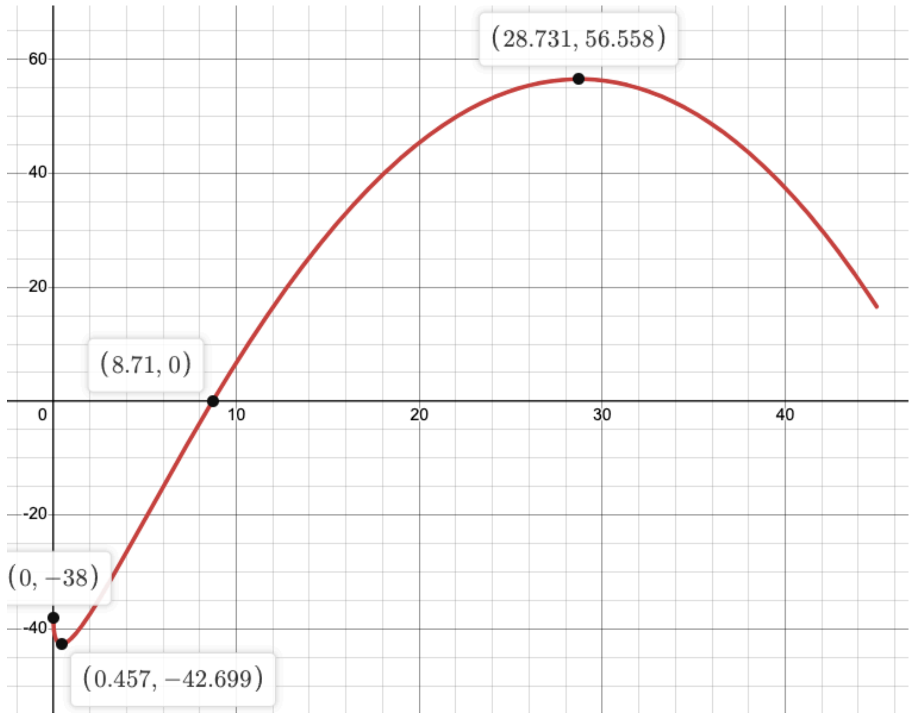
This gives us really two options for optimizing profit. Let's look at both. The first option is to work from the table. See if we can find a place where $R'(x) \cong C'(x)$

Let's not give up just because we don't see it right away, either. Using previously calculated values, the closest we get is at x = 30. We should include all values between 25 and 30. As close to zero as possible is what we are looking for.

x	*C'(x)*	*R'(x)*	*R' – C'*
10	2.21	7.30	5.09
15	1.81	5.70	3.89
20	1.57	4.10	2.53
25	1.40	2.50	1.10
26	1.37	2.18	0.81
27	1.35	1.86	0.51
28	1.32	1.54	0.22
29	1.30	1.22	-0.08
30	1.28	0.90	-0.38
31	1.26	0.58	-0.68
32	1.24	0.26	-0.98
33	1.22	-0.06	-1.28
34	1.20	-0.38	-1.58

Again, the IVT comes in handy. In the $R'(x) - C'(x)$ column the value goes from a positive number to a negative one between x = 28 and x = 29. Comparing the two values, *-0.08 is closest to zero*. We maximize profit when we are only producing and selling 29 teapots at the price of $price = 10.5 - 0.16 \cdot 29 = \$5.86$ to retailers.

Spreadsheets are one way to see everything everywhere all at once. But sometimes it can be too much. Using a tool of our choice, we can graph the profit function and locate the maximum on the graph.



Profit from production and sales of between 0 and 45 ceramic teapots is shown here. The precise maximum occurs when x = 28.731 and y = 56.558. When x = 29 the profit is 56.548.

Try this: Go back and gather all information we have concerning the "cell phone case" business. Using a spreadsheet (or table of values), make columns for ***x, price***, ***C(x)***, ***C'(x)***, ***R(x)***, ***R'(x)*** and then two more columns for ***P(x)*** and "***R'(x) – C'(x)***". There should be eight columns in all. Assume x in [0, 420]. Do not go past 420.

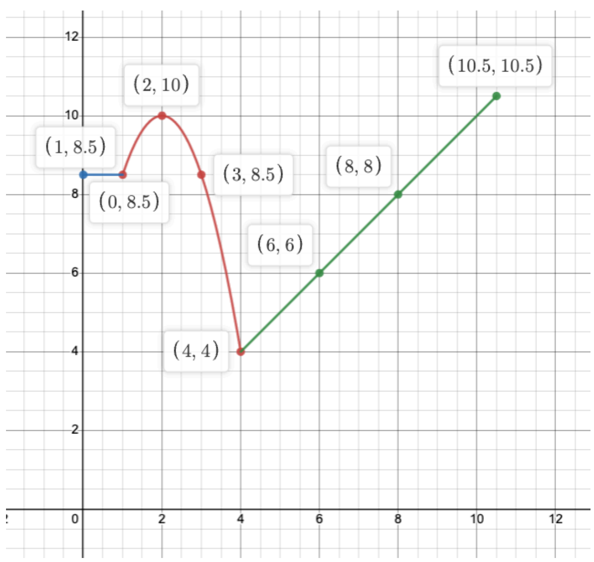
Find the maximum revenue and maximum profit. Write a sentence or two about what is required – in terms of production, sales, and price, to maximize profit.

Section 10.3 **Practice Exercises**

1) The cost to produce a certain video game is given by the equation $C(x) = 15000 + 3.1x - 0.008x^2 + 0.004x^3$ [x is in thousands of video games produced]

a) What is the fixed cost?
b) What is the variable cost to produce "x"?
c) Find the Average Variable Cost function.
d) Use the answer in c) to find the average variable cost for x = 50; x = 100; x = 150; x = 200

What do the numbers mean?



2) Consider the graph of a function f(x) Answer the following questions. Of the points that are labelled, can you identify the endpoints?

____

Name the place(s) where the derivative does not exist.

____

Is there a place where the derivative is zero?

____

Name an interval where the derivative (slope) is zero ________________

List all extrema:

____

Where is the absolute maximum? _________ Absolute minimum? _________

For the following questions, please use the same cost function as in question #1 and the price function $price = 500 - 1.2x$. Try to get the derivatives of cost and revenue on your own and see if you can do it. We are also considering the interval $0 \leq x \leq 225$. For the purposes of exercises, the derivatives are

$$C'(x) = 3.1 - 0.016x + 0.012x^2$$
$$R'(x) = 500 - 2.4x$$

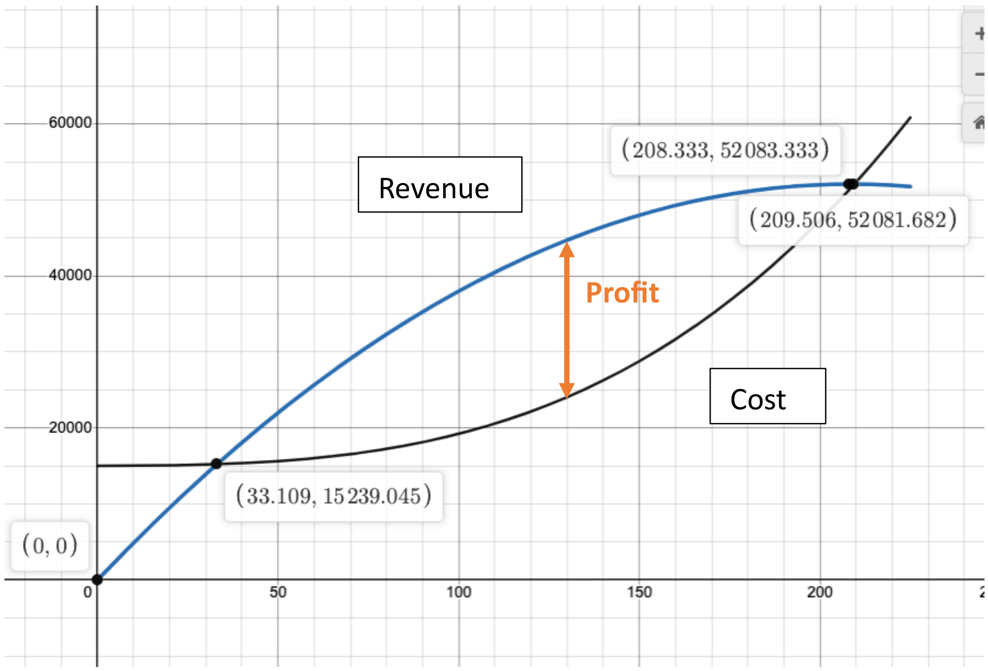
3) Where is the "Break Even" point? At what production level will revenue = cost?
4) On what interval of x do we see revenue exceeding cost?
5) Use any of the techniques we have used in this course (including graphing in Excel) to find the maximum cost for $0 \leq x \leq 225$

6) Use any of the techniques we have used in this course (including graphing in Excel) to find the maximum revenue for $0 \leq x \leq 225$

7) Write a formula for the profit function. Make a graph of profit for $0 \leq x \leq 225$ and make note that it could sometimes be negative. (Feel free to use Excel for the graph).

8) What is the maximum profit? Solve the equation ***C'(x)=R'(x)*** using any of the techniques that we have learned. Does this reconcile with the graph that we see of Cost and Revenue? Explain your reasoning.

(see graph on the next page)



Hint: Maximum profit occurs at the point (127.107, 194.942)