Chapter 0 Foundations of Mathematical Reasoning and Expression

Section 1 The Real Number System

Number, numeral, digit, decimal, fraction, and ordinal are all words we use when describing the quantity, order, or amount of something. Why do we have so many words for basically the same thing? What's the difference between a number and a numeral? There are also different families of numbers. What are they all about? The formal rules of mathematics have been developed so that people can communicate with each other. An architect's drawings contain lines and measurements that the builder can understand and translate that into the practical details of materials and labor required for the building project. Here are some informal definitions of commonly used words which we often take for granted.

> Definition: A ***number*** is a measure of something's size. It refers to the actual value of a thing. "There were 10 ducklings in number." If the word ***number*** can be replaced with the word ***count***, ***value***, or ***size*** in a sentence, then the correct word is being used.

> Definition: A ***digit*** is one of the unique characters that are used in the writing of a number. Hindu-Arabic numerals are base 10 which means 10 unique characters: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

> Definition: A ***numeral*** is the written form of a number. It refers to the written expression as a series of digits, and sometimes includes a decimal point as well as commas (to aid in reading). "What is the *value* of 7 in the numeral 472.15?" Answer: 7x10 or 70

> Definition: An ***ordinal*** is the expression used to describe the order or placement of a thing. Here is a partial list of ordinals: first, second, third, fourth, fifth, sixth.

Everything in modern mathematics has a basis in our decimal numeral system. We will start with the simplest numbers: the whole numbers. The word 'decimal' comes from a Latin root word meaning "a tenth". Our number system is what we call a positional number system. This means that numbers have digits (represented by 0 through 9) and each digit has value that is a multiple of a power of ten. Nine is the largest value of a power of ten. Anything past that goes to the next power of 10.

> Example 1: The number 452 is really $4x100 + 5x10 + 2x1$ where $100 = 10^2$, $10 = 10^1$, and $1 = 10^0$. Note: any number raised to the 0 power = 1.

The position of each digit in a numeral matters because we need to understand that 452 is not the same number as 524, or 245. The set, or collection, of numbers that represent the numbers we use to count things (not including 0) is what we call natural numbers. Then by including the number 0 we refer to this as the whole numbers. By including all of the negatives of these, we have the integers. Each of these sets of numbers has a fancy Roman-style letter for its name.

Example 2: The set of counting numbers can be seen as a list inside {}:
{1,2,3,4,5,6,7,8,9, … ∞} and is denoted $\mathbb{N}$

These are the whole numbers:
{0, 1, 2, 3, 4, 5, … ∞} is denoted $\mathbb{W}$ (not as common)

These are the integers:
{ $-\infty$… , -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, … ∞} is denoted $\mathbb{Z}$

These are the rational numbers:
{ numbers = a ratio of two whole numbers } is denoted $\mathbb{Q}$. Why $\mathbb{Q}$? It comes from the word “quotient” which is the result of a division of two numbers

These are the real numbers (informal definition):
{ all numbers that can be *approximated* by a rational number } is denoted $\mathbb{R}$. We sometimes will write $\mathbb{R} = (-\infty, \infty)$ meaning numbers that extend infinitely far in both positive and negative directions.

Do they belong? If a number is a member of one of these sets, is it also in some of the others as well? How do they overlap? Similar in concept to support levels in a crowd-funding platform, with each new level all of the rewards of the previous levels are granted as well. Membership gets more exclusive as the requirements are more restrictive. Here is a way to visualize belonging to these sets of numbers. Think: inheritance. The lower levels inherit all the properties of the levels above them.

Base level: $\mathbb{R}$
Level 1:
$\mathbb{Q}$
Level 2:
$\mathbb{Z}$
Level 3:
$\mathbb{W}$
Level 4:
$\mathbb{N}$

Conversation Starter: Why do you think there are so many vocabulary words for numbers?

Conversation Starter: Can you think of number systems that are not decimal (base 10) systems? If so, what are some ways other number systems are different? the same?

Try It: Using this word bank: **whole, integer, rational, real** ($\mathbb{W}, \mathbb{Z}, \mathbb{Q}, or\ \mathbb{R}$), write an 'is ___ but is not ___' sentence about each number shown that reads as follows:
Example: e The number e is **real** but is not **rational**. Try to use each word at least twice!

2.015 -3 $\frac{5}{3}$ 0 6001 π $\sqrt{2}$ -$\sqrt{4}$

Section 2 What's The Big Deal About Fractions?

In September of 1978 the rock band Styx released its eighth studio album. The album was cleverly named Pieces of Eight. The title track was about the human pursuit of wealth and the consequences that often come with the relentlessness of it. What exactly is a "piece of eight" and why mention it here?

For about 360 years in the western world, when Europeans were expanding to North and South America, one of the most widely used currencies was the Spanish "Reale de ocho", which was minted from silver and imprinted as official currency backed by the government of Spain. One "cob" or piece of silver was worth 8 reales = 1 peso and depended on weight and composition rather than shape. The value was in the silver content (93%). It doesn't take much of an imagination to see how this led to fraud globally, and scandal. Silver is a soft enough metal that these cobs could be clipped or shaved, then could be 'hammered' or combined with a common metal like tin to restore the weight. The reduced cobs could contain as little as 65-75% of the original silver. To combat fraud, eventually the world saw rounded coins which were harder to clip (no corners!); these replaced the irregular-shaped cobs. (citation needed)
https://www.youtube.com/watch?v=5BebknA0PeU

For sake of this discussion and simplicity, let's assume that an early 1700s traveler could purchase a week's stay at an inn for one of these silver coins, but the traveler only wanted to stay one night. All the traveler has for currency are these silver coins (pesos). Well, it turns out the traveler is in luck. The town has a blacksmith who happens to have shears strong enough to cut one of these coins in half. Since this is an acceptable practice, the traveler visits the blacksmith and asks for the coin to be cut into 8 "bits". Why eight? Well think about the math involved, as well as the physical constraints. Here is an object about the size of the lid of a pickle jar that needs to be divided up into smaller pieces. The simplest thing to do is to cut it in half. Cut something in half and now there are 2 half-moon-shaped pieces. Cut each of those in half and there are now 4 wedge-shaped pieces. Now cut each of those in half. There are 8 roughly equal-sized pieces. In the early 1700s and into the 1800s each of these slim wedge-shaped pieces was called a "bit" which were worth 1 reale. Sound familiar? It should. Fast forward to the 20th century and a bit is an eighth of a byte, the binary code that computers are based on.

Returning to the inn, the traveler is able to pay 2 bits for the room for one night and a meal. In mathematical terms the traveler paid $\frac{2}{8}$ peso which is also $\frac{1}{4}$ peso. The fraction is the word mathematicians came up with when describing pieces of a whole.

At first, all fractions were what we call ***unit fractions***, in that the numerator was always 1. In a ledger from the 14th century, one might see written */4* which literally means $\frac{1}{4}$. The slash was used interchangeably with a horizontal fraction bar. The choice of how to express a fraction was often due to limitations of printing and typography. Historically fractions were used before the printing press, which meant making copies by hand – using ink or a stylus (like a chisel) in leather. (citation needed) https://mathshistory.st-andrews.ac.uk/Miller/mathsym/fractions/

Conversation starter: Can any fraction be decomposed as a sum and/or difference of unit fractions? Why not try it with the fraction $\frac{3}{8}$ without resorting to the obvious $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$?

In geometry, the ratio of the circumference of a circle to its diameter is always the same, regardless of the size of the circle. This ratio is one of the most famous ***irrational*** (real but not rational) numbers in mathematics. Since it can't be written as an exact decimal, or as a repeating decimal, the number instead has been given a name – π which is a letter of the Greek alphabet pronounced "pie"). π can be approximated using the fraction $\frac{22}{7}$. Sometimes the decimal approximation 3.14 is used. A grassroots effort began in 1988, then officially designated by the U.S. Congress in 2009 to mark March 14th as "Pi Day" celebrating the famous number.

Conversation starter: $\frac{22}{7}=3\frac{1}{7}$. Can you think of any reason this combination of numerals and digits might appeal to some people? Is there a cultural connection we can make?

Here is another application of the fraction in our everyday life. This is a tale as old as time, or at least as old as humans were tracking the number of days in a solar year. The calendar year really only matters because of one thing: agriculture. Well two things: agriculture and finance. Talk to an astrophysicist or astronomer and they will tell you that the solar calendar year is decreasing. It's happening very slowly, because of the changing gravitational forces of the solar system in which the Earth is one planet among eight. When we say 'one year' talking to a physicist we are referring to the 'tropical year', the length of time in days, hours, minutes, and seconds from one vernal equinox to the next. Right now, the length of a year is (2023) 365 days 5 hours 48 minutes 46 seconds. (citation needed) https://www.britannica.com/science/solar-year

This explains the need for a 'leap year' but it does not exactly explain why we include an extra day every four years. Until you do the math, that is. Also, we don't always insert an extra day every 4 years. There are some exceptions. But we are getting ahead of ourselves. At the time of

Julius Ceasar, calculating the length of a solar year to be about 365.25 days was a pretty big deal. It's worth noting that 6 hours out of 24 hours in a day is exactly 0.25 day. Leap year calculation was based on the length of a year = 365 days and 6 hours. Not bad.

$$6\ hours = 0.25\ day = \frac{1}{4} day$$

The fraction ¼ informs the decision to add a day every 4 years. The denominator measures how many years to wait before adding an extra day.

But the length of a solar year is not 365 days and 6 hours. The difference between 6 hours and 5 hours + 48 minutes may not seem like a big deal. It's not in the short term. But historically as time went on, adjusting by too many leap days began to be noticed around the 16th century. To put it in perspective, after 2 thousand years, our seasons would be off by nearly two weeks. Instead of 20th June being the longest day of the year in the northern hemisphere, the longest daylight day would be more like 7th June (Day 158 in a non-leap year by Julian calendar measurements).

Pope Gregory XIII, a 16th century head of the Roman church, instituted a new calculation for leap years that was based on a more accurate measurement of the length of a solar year. This calculation is in use today. The Gregorian year is 365 days + 5 hours + 49 minutes + 12 seconds.

Conversation starter: Length of a solar year $\cong 365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400}$. What is that in decimal form? How do these fractions determine leap year calculations?

Section 3 From Fractions and Decimals to Percents

Our journey into numbers takes us to a unit of measure that we call the percent. More than just a symbol next to a number, the % even has its own key along with the number 5 on computer keyboards. The symbol itself looks like *0 divided by 0*, where the slash means to divide. Division of a nonzero number by smaller and smaller numbers, approaching 0, yields larger and larger results. When we try to divide 0 by 0 the result is something we call *indeterminate*. This is one of the oddities in mathematics, and something we can experiment with as well. The purpose here is to explore what is meant by indeterminate. In words, we mean to say *can not be determined without further analysis*. Let's see why that is.

Weird things can happen when numbers get close to 0. Some of the more interesting areas of research involve objects or phenomena that measure infinitesimally small. In biology, much of medicine occurs at the microscopic and cellular level. In chemistry, some of the most destructive forces on earth involve the smallest atom – the hydrogen atom. In physics a whole area of study involves what happens to things moving at high speed but whose mass is virtually indetectable.

Since we can't actually divide 0 by 0, let's choose a number that is close to 0 and see what happens in 3 situations:

Start with $x = 0.01$ Calculate these 3 expressions using 0.01 in place of x and increasing the number of decimal positions, making x smaller and smaller each time:

	$2 \cdot x \div x$	$x^2 \div x$	$3 \cdot x \div (x^2)$
$x = 0.01$			
$x = 0.001$			
$x = 0.0001$			
$x = 0.00001$			

Conversation starter: What do you notice? What do you wonder? Compare and contrast the results you see in the table above. Each of the 3 calculations should give us insight about 0 divided by 0. Summarize your findings and what can you infer?

Clearly % is not 0 divided by 0. Instead, the symbol can be better explained in terms of other symbols that have been used for the same thing. We can deduce how the symbol % came to be. In the box below, every symbol is another version of % seen in writings from about the 14th century onward. The *p* is the abbreviation for the Latin word *per* and *c* represents *cento*.

p/c $/100$ pc^o $per\ \frac{0}{0}$ %

The fact is: % is a unit of measure that forces us to make a correction when performing calculations with other numbers. It's really that simple.

In practical terms, suppose a customer at a restaurant wants to leave a 15% tip on a food order that totaled $32 without sales tax. 15% literally means 15 cents for every dollar spent. We don't need a calculator for this.

- Take 10% of $32 first, by moving the decimal point one place to the left. 10% of $32 is $3.20.
- Another 5% is half of that. $1.60 is half of $3.20.
- Add them up. $3.20 + $1.60 is $4.80.
- If the customer is in a hurry, they can simply leave a $5 bill. Keep the change.

If someone had their phone, and forgot how to use the special % key, the tip could be calculated by multiplying 32 x 0.15. Converting 15% to the standard unit allows it to be multiplied by $32

We won't refer to the standard unit as "decimal form" because what if there is already a decimal point in the %? This can lead to confusion. In other words, suppose an individual wants to know how much interest they will be charged on their next credit card bill, when their balance is $431.87. Finance charge is calculated by taking the annual percentage rate (APR) and dividing it by 12 (because there are 12 months in a year). Suppose the APR = 17.99%.

How much finance charge will there be on this month's bill? The APR is given as a % even though there is also a decimal point in the number. It still needs to be converted, or we have a calculator that will do the conversion for us.

Solution: Take the APR (%) and divide by 12, and then divide that by 100. We could also save time by dividing by 1200. Do it both ways and see:

$$17.99 \div 12 \cong 1.49917 \ Then \ 1.49917 \div 100 = 0.0149917$$

Alternatively:

$$17.99 \div 1200 \cong 0.0149917$$

Pro tip: When rounding decimal numbers, either don't round until the very end OR make sure to include as many decimal spaces as possible until the final calculation is made.

Now multiply by the balance: $431.87 \times 0.0149917 = 6.4745 \cong \$6.47$

If the credit card holder wants to see their balance go down the payment will need to be greater than $6.47, and they will need to avoid using this card on any more purchases unless they are prepared to make a larger payment every month.

Section 4 How Big or Small; High or Low; Near or Far

Let's start with how we measure things in mathematics. If we want to know how big or how small something is, we quantify it. This means using the **decimal system** to assign a numeric value, and the **real number system** to understand how to place numbers in an ordered fashion. The problem with mathematicians is that we sometimes forget that context matters. Numbers mean different things when we assign units to them, or when we compare them to other numbers.

For example, let's start with a number; it will be **negative**: -68. Now we ask a mathematical question: How big is -68? Or perhaps it is not big at all. Perhaps it is very small. How big is anything really? Why does it matter? Honestly these questions are meaningless without context. But in math we often must solve problems that do not seem to have any real purpose because we are simply 'doing the math'. We are left to wonder what the purpose might be. What could -68 represent, then? It might be 68 feet below sea level. It might be 68 degrees below freezing on a Celsius scale. It might be a weight loss of 68 pounds. It might be a payment of $68.00. It might be a drop of 68 feet. It could be 68 miles south. 68 degrees rotation (pivot) in a negative direction.

Conversation starter: Name two other uses and meanings for the number – 68 that are not mentioned here.

All of these involve a size of 68 and a direction or change that is opposite the positive direction or change:

Description	**Context**	**Interpretation**
68 feet below sea level	depths range from sea level (0) to 1 mile below sea level (-5,280 ft)	given the context this depth is relatively shallow (low)
68 degrees below zero	temperatures vary from -70 to 120 degrees F	low temperature compared with other temperatures in the range
weight loss of 68 pounds	Blue whales weigh around 400,000 pounds	small as a portion of the weight of a blue whale
payment of $68.00	what was paid to the babysitter for 2.5 hours	big compared with typical pay for 2.5 hours babysitting
drop of 68 feet	plane flying at an altitude of 36,000 feet	very small compared with overall altitude
68 miles south	works anywhere from 5 miles away to 80 miles away from home each day	big (far) compared with other distances traveled each day for work
rotate 68 degrees down	scope can be pivoted between 85 degrees upward and 85 degrees downward	pretty steep angle compared with the range that the scope can be tilted down or up

Measuring the **difference** between two numbers is another way we decide how big something is, or how small something is, how close or how far. One would also suppose that calculating the difference between two numbers tells how far apart they are. Yet a lot of what we do in math depends on numbers always being positive or always being negative. We will see this come up again and again throughout this book. One basic premise is that **distance calculations** must result in a **positive number**. In math we deal with unknowns a lot. These unknowns are symbolized by letters that represent numbers, sometimes called **variables**, other times called **parameters** or **constants**.

When we want to use math to describe a pattern or show how we calculate something based on a formula, we use variables and other symbols for the calculations which are representative of that pattern. Recognizing patterns is about making something repeatable. Can the same calculation be applied to different numbers to achieve the same, or at least similar and predictable, result? That's the essence of mathematics.

When working with unknown quantities, especially measurements of length and distance, we need to make sure that length is a positive number. Otherwise, we might overestimate or underestimate something. If we want to guarantee that a number is positive, because it is unknown, then we must do one of these two things:

Definition: A number which we will label as a is given. We do not know if a is a positive or negative value (that part is hidden). We define the magnitude, or positive value (as in distance from 0) of a as follows:

$$\boldsymbol{magnitude}\ of\ a = |a|$$

$$|a| = \sqrt{a^2}$$

The first of these is called the "absolute value" of a number written as $|a|$. When we see these two vertical lines on either side of a quantity, we read it out loud using the description "***absolute value of***".

The second calculation is used more often in algebra and then calculus, especially when calculating distance or length of something. When we read out loud the expression $\sqrt{a^2}$ we say the words "***the square root of a-squared***". Here are some examples. It is VERY important not to treat absolute value the same way as grouping symbols like ( ) or [ ] .

When we see | | in an expression, we must treat it as an operator that measures something about the quantity inside. In this case, the quantity inside the | | must be determined first, and then decide its absolute value (or magnitude).

$$|-12| = 12$$

$$|5 \cdot 3| = |15| = 15$$

$$|-6 \cdot -1| = |6| = 6$$

$$|-6| \div (-2) = 6 \div (-2) = -3$$

Did you notice that every time we encounter | | we first *reconcile what is inside* | |? Before any other ***external*** calculations can be made, we must ***activate*** the ***operator*** | |.

By squaring, then taking the square root we can show that – 5 and + 5 both have the same magnitude:

$$(-5)^2 = (-5) \cdot (-5) = 25$$

and also $5^2 = 5 \cdot 5 = 25$

Taking the square root: $\sqrt{25} = 5$

Without using | | we found both – 5 and + 5 have a magnitude of 5 (distance from 0) This may all seem way to complicated of a process just to make a number positive. It's probably one of the reasons students lose interest in mathematics. Why does something so simple need to look so complex? The honest answer is that we needed to develop a common language – one that can be used to describe complex situations – so that people doing a math problem in China, or India, Australia, or Nigeria could understand one another.

Absolute value operator and square root expressions are universally adopted symbols all over the world. That's something special. How many other disciplines can make that claim?

Conversation starter: Develop your own universal way to describe distance of an object from a reference point (like 0). Remember that whatever you come up with, distance always needs to be a positive number. Your algorithm will be tested and must work for any positive or negative number.

Section 5: Relations and the Rectangular Coordinate System

For this investigation, we will need a standard card deck of 52 cards with 4 suits (♣♦♥♠) and 13 cards in each suit. Shuffle the deck and place it face down on the table. Before we begin, we will assign numbers based on suit (color) and whether the card is a number card or face card. We will also be visualizing the results of our experiment on a graph. The graph will include a horizontal axis, numbered from $-10\ to\ 10$ and a vertical axis also numbered from $-10\ to\ 10$. The two axes cross at 0, forming a $+$ shape. At the center is the point (0,0) which we call the *origin*.

Hearts ♥ and diamonds ♦ will be considered negative numbers (as with debits and credits on a ledger). Clubs ♣ and spades ♠ will be positive numbers. The Ace will be 1, the numbered cards will be valued by their number, and all face cards will be 0.

The experiment involves choosing two cards in succession. The first card will be “x” and the second card will be “y”. Our objective is to see if any patterns emerge. After the first two cards are drawn and recorded (tabulated), put them aside and choose two new cards. Remember, the first card is always “x” and the second card is always “y”.

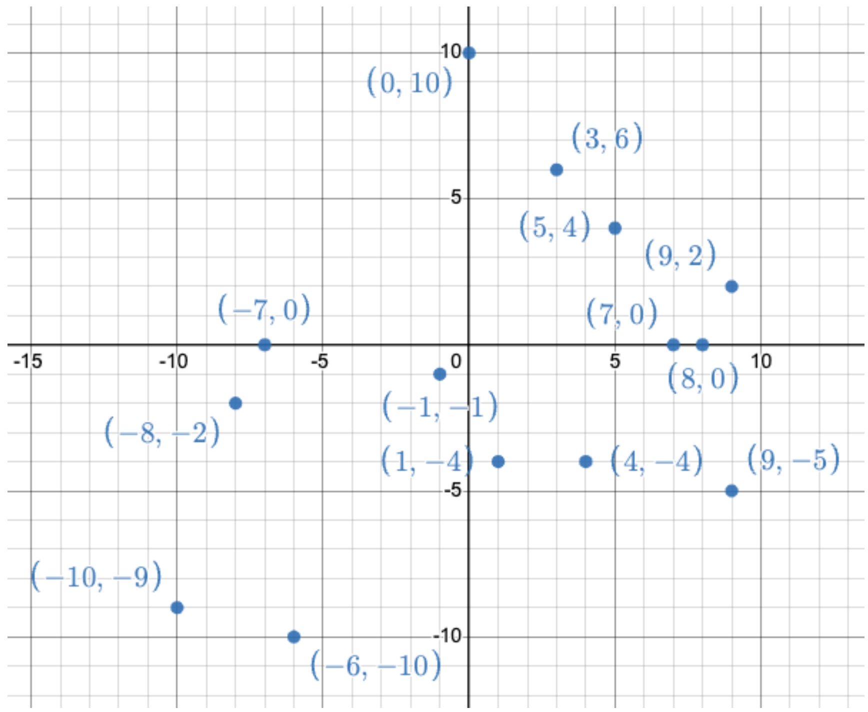
Example: First card is 8 ♣ and second card is Q ♣. x = 8 and y = 0 for the first ordered pair (8, 0). Next, two more cards were drawn. A ♣ and 4 ♦. x = 1, y = -4 for the second ordered pair (1, -4). Repeat this process until you have 15 pairs. 30 cards will have been drawn from the deck without replacement. (That’s important).

Your results will differ each time. Here is an example of one possible outcome.

x	y
8	0
1	-4
9	-5
-1	-1
3	6
-8	-2
-7	0
-6	-10
5	4
0	10
7	0
-7	0
4	-4
9	2
-10	-9

Looking at this table, no discernible pattern emerges. In fact, there may not be a pattern at all. Something interesting occurred. The ordered pair (-7,0) appears twice. Perhaps this is not unusual since there are 12 face cards in the deck. The number 0 is bound to occur a few times. But to appear next to a red 7, of which there are only two… now that seems a bit unusual!

The rectangular coordinate system, with horizontal and vertical axes, provides us with a means of visualizing our observations. We can use an online (free) graphing tool, plot the points by hand, or even use a spreadsheet and make a scatter plot.



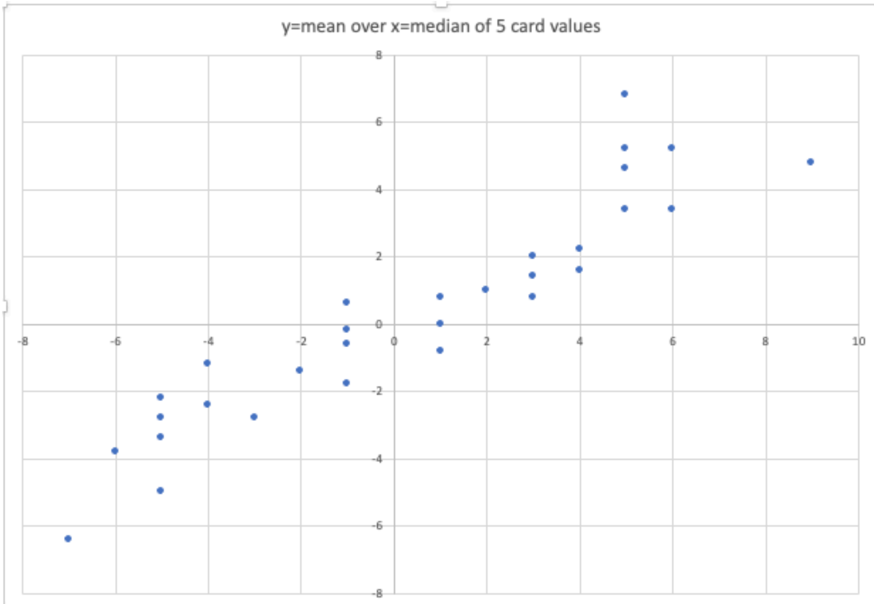
What we do with this is try to decide if there is a pattern. Does the value of 'x' at all seem to dictate, or impact, the value of 'y'? From this graph alone, perhaps it is a bit inconclusive. But there does not appear to be a relationship.

Let's do something different. This time, remove all the face cards (J, Q, K). Red cards will still be negative and black cards will still be positive. But now we will draw 5 cards and place them on the table. Let x be the median (middle) number, and y will be the average.

Example: 5 cards are drawn from a shuffled deck. 5♦, 10♥, A♠,9♥, and 2♥. As numbers, these are -5, -10, +1, -9, and -2. Arranging them in order we have: -10, -9, -5, -2, +1. The middle number is -5 = x. The average is (-10 + -9 + -5+ -2 + 1) ÷ 5 which is also -5 = y.
Our first ordered pair is (-5, -5). Note: The average can have a decimal.

Keep going…

Draw 5 more cards and let x = median, y = average (mean). After drawing 20 cards, return all cards to the deck, shuffle, and repeat the experiment until you have 30 ordered pairs (!). Yes, it's a lot. Have fun with it.



	x	y
1	-5	-5
2	5	6.8
3	-1	-0.2
4	5	3.4
5	4	1.6
6	-5	-2.2
7	1	0.8
8	6	3.4
9	-7	-6.4
10	1	0
11	3	2
12	6	5.2
13	2	1
14	-5	-2.8
15	1	-0.8
16	-1	-0.6
17	-1	0.6
18	3	0.8
19	-6	-3.8
20	-1	-1.8
21	4	2.2
22	5	5.2
23	-4	-2.4
24	9	4.8
25	-2	-1.4
26	-5	-3.4
27	3	1.4
28	-3	-2.8
29	5	4.6
30	-4	-1.2

The x and y columns can be put into a spreadsheet and make a scatter plot of the data as seen here:

We begin to see the value of making a graph, or plot, of the data. Relationships between two quantities can be more obvious once we see the visual representation. This is especially true with large amounts of data. Had we not performed this experiment 30 times, and instead only 10 times, we may not have seen a clear pattern.

What seems clear here is that the farther away (from 0) is the middle, or center, of the 5 numbers, the same thing will also be true about the mean, or average. In statistics median and mean are both measures of center. But we are getting ahead of ourselves.

Conversation starter: Suppose we jumbled up our y values so that they no longer pair up with the x's. Does a scatter plot still make sense? What in fact *would* still make sense?

Your turn: Why not come up with your own experiment involving an 'x' and a 'y'. It can be anything that can be measured using numbers. Plot the data and see if a relationship can be shown.

For more reading and/or fun:

https://www.beaglelearning.com/blog/ambiguity-and-nuance-in-math/

https://www.mathsisfun.com/numbers/numbers-numerals-digits.html

https://www.skillsyouneed.com/num/percentages.html

https://shadycharacters.co.uk/2015/03/percent-sign/

Online calculator resources:

Online graphing tool: https://www.desmos.com/calculator

Decimal time to H:M:S https://www.calculatorsoup.com/calculators/time/decimal-to-time-calculator.php