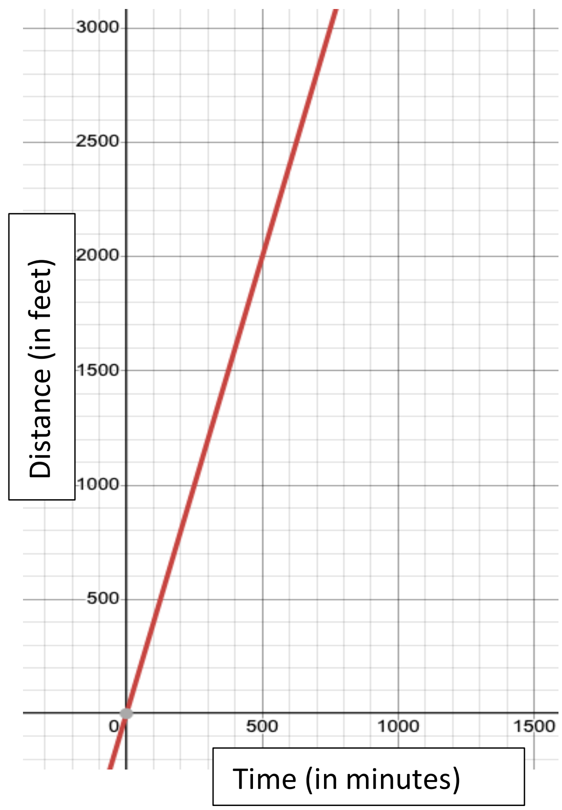
Chapter 9 The Derivative as Rate of Change

Section 1: Rate of Change on an Interval

In this chapter we will explore the meaning of the phrase 'change of y with respect to x'. How one quantity changes with respect to the other is at the heart of calculus. This is an informal introduction to one-variable calculus of the derivative. Our goal is to define what we mean by 'the derivative', how it relates to slope, and apply what we learn to business concepts. We will develop the concepts starting with what we know and build on that until we can apply what we've learned to problems involving time value of money.



It all starts with the idea of slope as rate of change when we have a linear relationship between two quantities. This was studied in Chapter 2.

Consider the fable 'The Tortoise and the Hare' as a basis for understanding the relationships between time, distance, and speed. Speed is the measure of the **rate of change** of distance over time, after all. In chapter 2 we learned that when the relationship between two quantities is linear, there is a constant **slope** (rate of change). In the fable, the straight-line relationship between distance and time most closely describes the movement of the tortoise, so we will start there. The graph shows us the steady increase in distance (measured in feet) over time (measured in minutes).

On the graph at the right, the point (0,0) is on the graph because at the beginning of the race time = 0 and distance = 0. Another point that can be identified rather easily is the point (500, 2000).

Try this: Using the graph provided, answer the following questions about the tortoise.

1) Locate the point (500,2000) on the graph and make a statement about what the 500 minutes, 2000 feet means.

2) Calculate the speed of the tortoise from two points on the graph. (0,0) is a handy one. Give your answer in feet per minute.

Hint: $slope = \frac{\Delta y}{\Delta x} = \frac{rise}{run} = \frac{vertical\ change}{horizontal\ change}$.

3) Determine how long (in minutes, and then in hours/minutes) it takes for the tortoise to finish the race. (How many feet is ½ mile?)

Make use of the graph, a table of values, or an equation… or all three… to understand the race from the tortoise's point of view.

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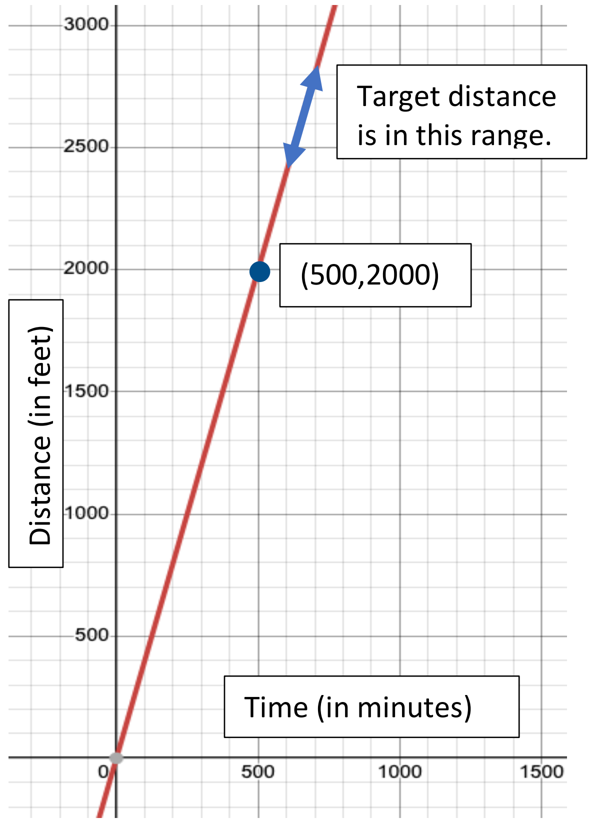
(image of a tortoise – cartoonish)

Solutions:

1) Locate that point on the graph. (see graph) What it means: In 500 minutes, the tortoise has moved 2000 feet toward the finish line.

2) Calculate the speed in feet per minute that the tortoise moves.

Speed of the tortoise = $\frac{2000-0}{500-0} =$
$2000 \div 500 = 4.00$
$feet/minute$



3) How long to finish?

Since 5,280 feet = 1 mile, we take ½ of that. So ***½ mile = 2,640 feet***.

To find out how long it takes for the tortoise to travel that distance, we first see that it must be greater than 500 minutes because at that time the tortoise will have traveled 2,000 feet. The tortoise still has to go another 640 feet. Let's start there and use the equation for distance as a function of time (now that we know the slope). We now know $y = 4 \cdot x$, so we will make a table. From the graph, we can also ***estimate*** that the distance of 2,640 will fall *somewhere between 600 and 700 minutes*.

x	y	$\boldsymbol{y = 4 \cdot x}$
600	2400	$600 \cdot 4$
620	2480	$620 \cdot 4$
640	2560	$640 \cdot 4$
660	**2640**	$660 \cdot 4$
680	2720	$680 \cdot 4$
700	2800	$700 \cdot 4$

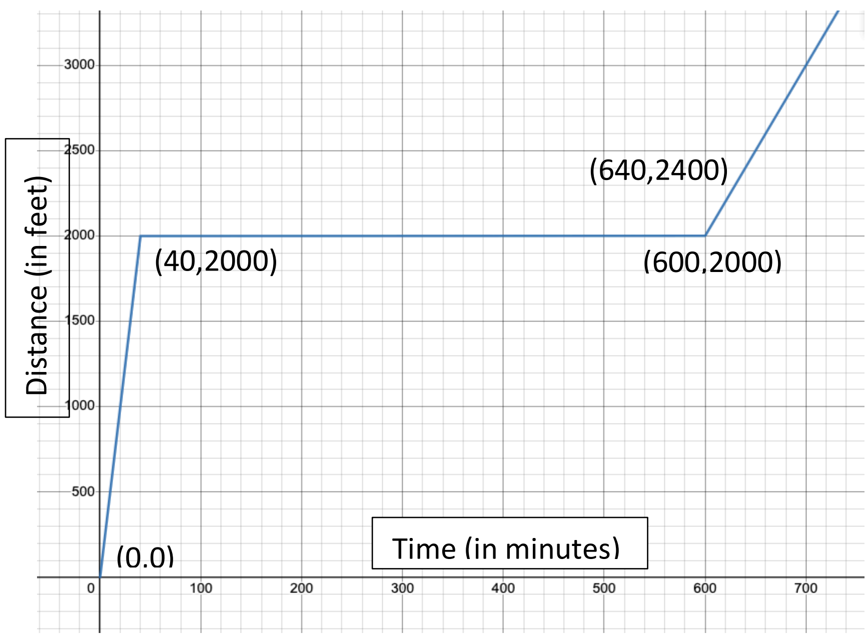
If needed, we can narrow down the values of x if the target distance is in between two numbers on our table. The solution is there. When x = 660, y = 2640, our target distance. 660 minutes converted to hours is $660 \div 60$ which gives us 11 hours! What a marathon!

There is a pattern in the table!
Whenever x goes up by **20**, y goes up by **80**. Why?

> **Definition:** The ratio for slope $\frac{\Delta y}{\Delta x} = m$ of a linear function can be rewritten as a formula for **change in y**, if we know the change in x.
>
> $$\Delta y = m \cdot \Delta x$$

From the table, $\Delta x = 20$ and $m = 4$. Using the slope formula in solving for change in y we have $\Delta y = 20 \cdot 4 = 80$.
The values of y increase by 80 each time x increases by 20!

The hare, on the other hand, took off like a sprinter. But then something happened. In the story, we know that the hare stopped to take a nap, eat, nap some more, eat some more. How is this behavior reflected on the graph of the hare's distance over time?



This graph tells a story of a sprinter taking off, then stopping for a long time, then resuming the race at a reduced speed. The steeper the line, the greater the speed. Horizontal line means no progress (no speed). The hare stopped at 2,000 feet with only 640 feet to go. Meanwhile, the tortoise passed the napping hare at 500 minutes. The hare did not start trying to catch the tortoise until 600 minutes into the race. Will the hare be able to catch up in time?

Try this:

1) What was the time and distance at which point the hare stopped to relax (feeling overconfident)?
2) How fast had the hare been moving until that moment?
3) At what time did the hare realize that the tortoise might win, and started to race again?
4) After all the napping and eating, the hare could not run quite as fast. What was the speed on that final part of the race?
5) Who won? How long was it before the hare crossed the finish line?
6) Summarize the hare's race in the following statement: For the first ____ minutes the hare ran at a constant speed of ____ feet per minute. Then, for ____ minutes the hare made no progress (speed = 0). The hare then ran at a speed of ____ feet per minute to finish in ____ minutes, which is _____ hours and _____ minutes.

Answering these questions will give us a complete mathematical picture of what occurred in the race.

Solutions:

1) What was the time and distance at which point the hare stopped to relax (feeling overconfident)?

   $(40{,}2000)$ *which means* 40 *minutes, at* $2{,}000$ *feet*

2) How fast had the hare been moving until that moment?

$(2000 - 0) \div (40 - 0) = 50$ feet per minute

3) At what time did the hare decide to join the race again? At 600 minutes into the race, or 10 hours. The amount of time that the hare wasted was 600 – 40 = 560 minutes

4) After all the napping and eating, the hare could not run quite as fast. What was the speed on that final leg of the race? We need to calculate the slope between the points (600,2000) and (640,2400). Slope = speed: $\frac{\Delta y}{\Delta x} = \frac{2400-2000}{640-600} = (2400 - 2000) \div (640 - 600) = \mathbf{10}$ feet per minute.

5) Who won? How long was it before the hare crossed the finish line?

At the 660-minute mark, the hare managed to narrow the gap but did not catch up. 2600 feet was 40 feet behind the tortoise, who was at the finish line.

x	y	$\boldsymbol{\Delta y = 10 \cdot \Delta x}$
600	2000	
620	2200	$2000 + 10 \cdot 20 = 2000 + 200$
640	2400	$2200 + 200$
660	**2600**	$2400 + 200$
680	2800	$2600 + 200$
700	-	-

The tortoise won.

To get the time for the hare crossing the finish line, we start at the point (660,2600) and since we need 40 more feet distance, we will solve for Δx

a) use $\Delta y = 40 = 10 \cdot \Delta x$
b) which means $40 = 10 \cdot 4$
c) $\Delta x = 4$
d) $Add\ 4\ to\ the\ time$ for a race time of 664 minutes, or 11 hours + 4 minutes

6) Summarize the hare's race in the following statement: For the first **40** minutes the hare ran at a constant speed of **50** feet per minute. Then, for **560** minutes the hare made no progress (speed = 0). The hare then ran at a speed of **10** feet per minute to finish in **664** minutes which was **11** hours and **4** minutes.

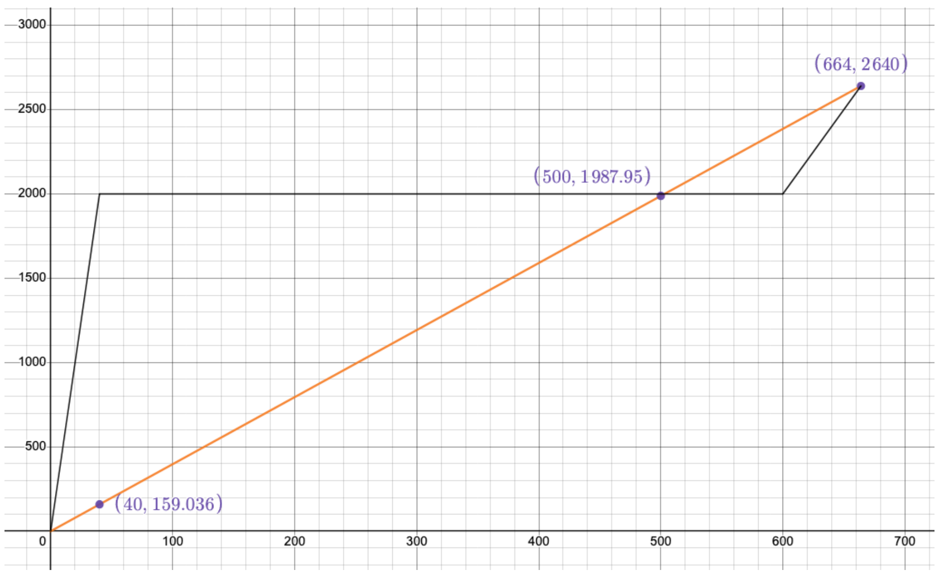
Since the hare was inconsistent, going from 50 feet per minute to 0 feet per minute, and then finishing at 10 feet per minute, we might be interested to know what the hare's ***AVERAGE*** rate of speed was throughout the entire *664 minutes* of the race.

By making a straight line from (0,0) to the finish (664,2640) we can see how the hare's race would have looked at an average rate of speed the whole way.

The average speed of the hare was:

$$slope = \frac{\Delta y}{\Delta x} = 2640 \div 664 \approx 3.9759$$

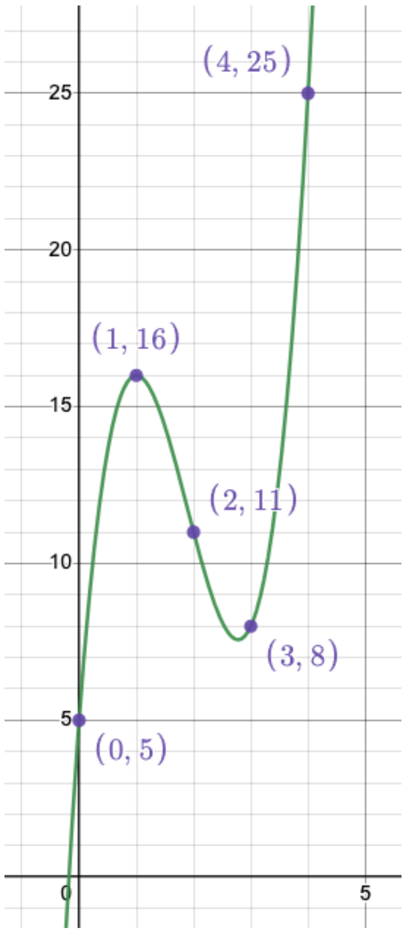
Notice that the average speed of the hare was a ***hair*** less than the tortoise's speed of 4 feet per minute. This graph shows the progress had the hare run the entire race at 3.9759 feet per minute.



Useful results of this investigation:

1) If you know a couple of points on a line and the slope, you can get more points on the line by increasing x by a certain amount (Δx) and then increase or decrease y by multiplying the slope times the change in x: $\Delta y = m \cdot \Delta x$. (If slope is negative, then y would decrease)
2) When the change of one quantity with respect to another varies over an interval, we can use a straight line to find the AVERAGE rate of change over that interval.

Here we have a graph that shows typical ups and downs we might see in places like the stock market, job growth, prices of a certain commodity, etc. Fluctuations happen over time. When we don't have a linear relationship between two quantities, we need a different definition for slope *at a point*.



Later in this chapter we will be learning how to use information called the ***derivative***, which is a measure of slope on a curve at any point. The derivative is ***instantaneous rate of change***, like the current speed on a car's speedometer shows your speed at a moment in time. When you are driving, your actual speed changes based on traffic, speed limit, and road conditions.

We will define instantaneous rate of change by exploring **AVERAGE** rate of change over a small interval, letting the intervals of x get smaller and smaller until we begin to see a pattern. In chapter 1 we learned about average rate of change of any function, linear or not, over an interval.

On this graph, what is the AVERAGE rate of change between **x = 0 to x = 4**? What information do we need? Once we have all the numbers we need, what do we do with those numbers?

The first thing we need is to recognize that "**x=0 to x=4**" is an ***interval*** on the x axis. Average rate of change occurs between two given x values.
What we need is to identify the two points (x,y) for the formula. These two points are $(0,5)\ and\ (4,25)$.
Using a straight-edge instrument helps with that because the curve is very steep.
Now we calculate the slope between the two points (0,5) and (4,25).

$$Average\ rate\ of\ change = \frac{\Delta y}{\Delta x} = \frac{25-5}{4-0} = \frac{20}{4} = 5$$

The conclusion then is to say, "**the average rate of change between x = 0 to x = 4 is 5**".

This means that ***on average***, y increases by 5 whenever x increases by 1. This is what market analysts mean when they say that fluctuations in the market average out over time. What happens minute by minute, or hour by hour is not as important as what happens over a specific time interval on average.

Try this: Make a graph of the curve, and on the same graph show the straight line from (0,5) to (4,25) that models the average rate of change.

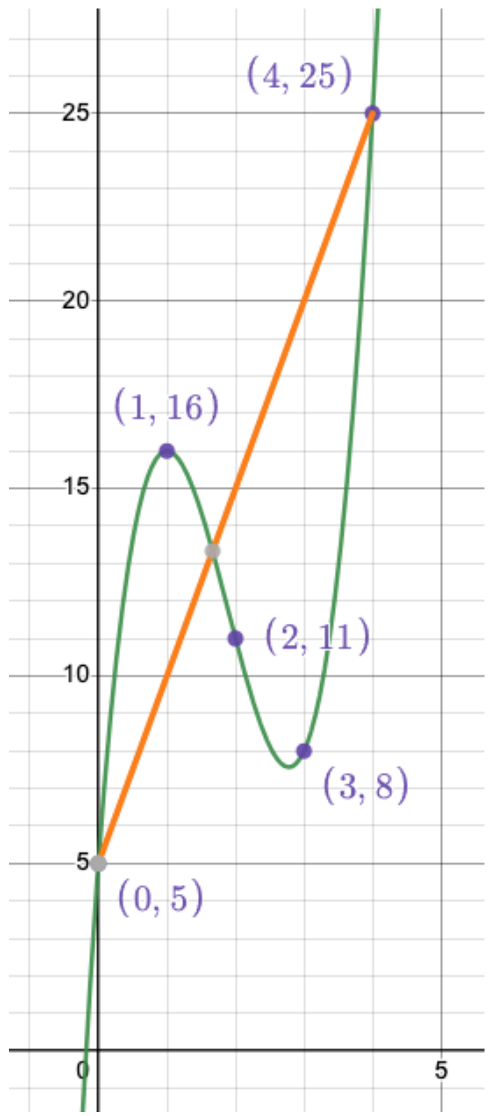
Solution: (see graph)

The straight line from (0, 5) to (4, 25) has a slope of 5. Here is a follow-up question to consider. Suppose we keep the orange line anchored at (0, 5) but move the other end to another point on the curve.

1) What is the average rate of change on the curve for x in [0,1], ie from (0,5) to (1, 16)?
2) What is the average rate of change on the curve for x in [0, 2], ie from (0, 5) to (2, 11)?

What if we move both ends of the straight line?

3) What is the average rate of change on the curve for x in [1, 3], ie from (1, 16) to (3, 8)?



Answers:

1) 11
2) 3
3) – 4

More Problems To Try

1) Average rate of change given a table of values

2) Average rate of change given an equation

3) Average rate of change when $\Delta x = 0.5$

Section 2: Toolkit Functions and Tangent Line to a Curve

For the purposes of our investigations, there are some useful equations that describe different kinds of relationships between two quantities, which we designate x and y. In previous chapters, we have learned about five of these: linear, quadratic, polynomial, exponential and logarithmic. As we go forward with different applications, it will be necessary to introduce a few more, and to name them. Here are the toolkit functions that we will be using throughout the remainder of this course. Keep in mind that new functions can be created by adding, subtracting, multiplying, or dividing any of these by any other. There are infinite possibilities, especially when we consider different coefficients.

Toolkit Function Name	Formula
Constant	$f(x) = 1$
Identity	$f(x) = x$
Power – Square	$f(x) = x^2$
Power – Cubic	$f(x) = x^3$
Rational - Reciprocal	$f(x) = \frac{1}{x}$
Rational – Pi Function	$f(x) = \frac{1}{x^2 + 1}$
Square Root	$f(x) = \sqrt{x}$
Exponential	$f(x) = e^x$
Natural Logarithm	$f(x) = \ln x$

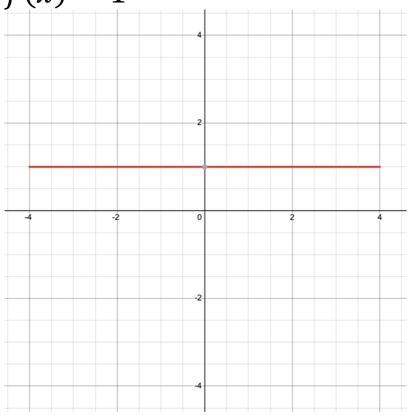
For three of these functions, there are some values of x for which they are undefined. Also, most of our applications will be restricted to the first quadrant (that is, both x and y will be positive numbers). For that reason, we will need to be clear about the domain (the values of x for which our model is needed).

Proceed with caution with these functions:

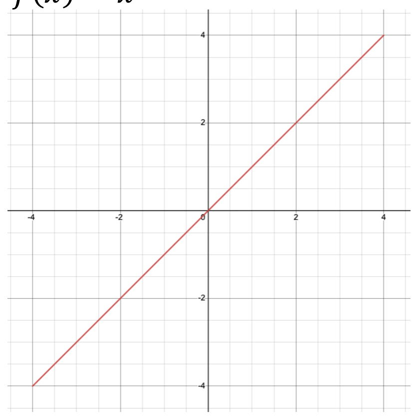
Toolkit Function	Values of x to be assumed unless stated otherwise
$f(x) = \frac{1}{x}$	$x > 0$ (x is positive and not zero) ***or*** $x < 0$ (x is negative and not zero)
$f(x) = \sqrt{x}$	$x \geq 0$ (x is positive and also zero)
$f(x) = \ln x$	$x \geq 1$ (x is 1 or larger)

Graphs of Toolkit Functions

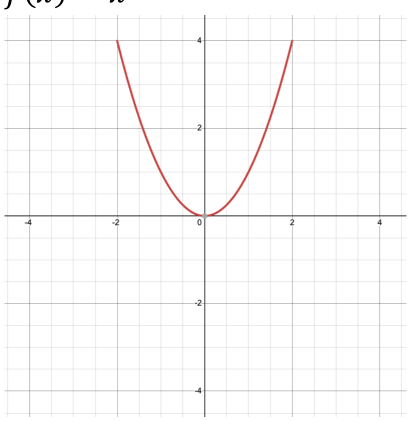
$f(x) = 1$



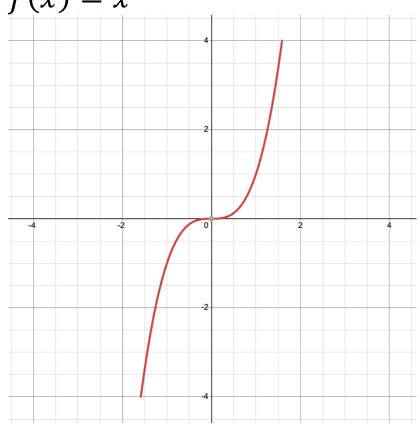
$f(x) = x$



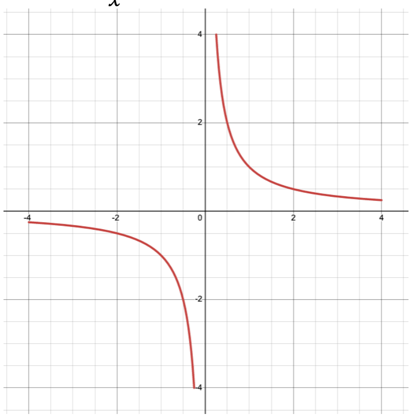
$f(x) = x^2$



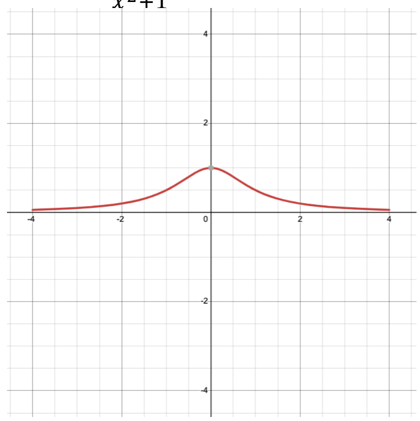
$f(x) = x^3$



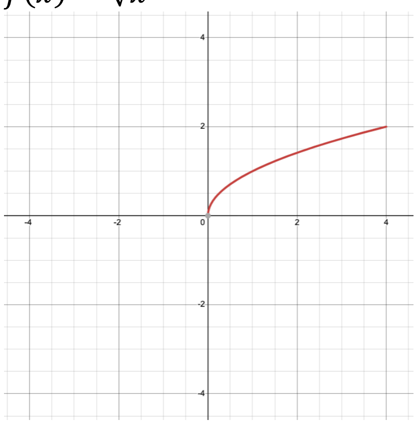
$f(x) = \frac{1}{x}$



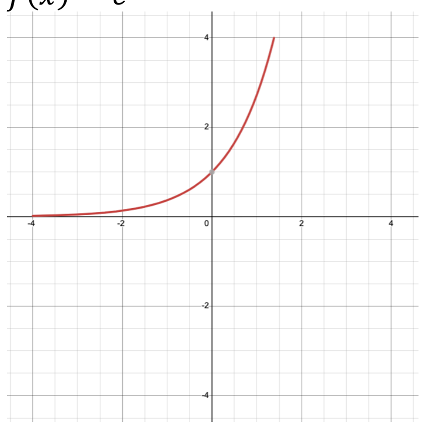
$f(x) = \frac{1}{x^2+1}$



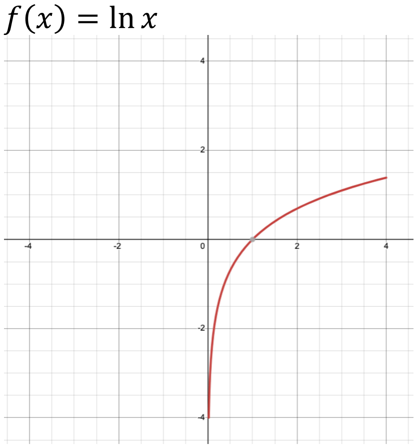
$f(x) = \sqrt{x}$



$f(x) = e^x$



$f(x) = \ln x$



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(image of a toolbox)

Try this: Make your own graph of each of the 9 toolkit functions. Use graph paper (preferred). Make a second, smaller graph that can be cut out and pasted to an index card. On one side of the index card, put the graph. On the other side, put the equation. Use these to drill yourself until you can answer all of them with no errors. Make sure you can do both – identify the equation from the graph; identify the graph from the equation.

If this were a traditional Calculus course, we would be concerning ourselves with function operations, composition of functions, limits and continuity. But all of our models will be based on data; we will assume that the functions are well-behaved and are basic variations on the toolkit functions.

Let's consider motion again, but this time we have a projectile. A person throws a baseball up in the air. The height of the ball is dependent on three things: the force of gravity (which is acceleration), the speed at which the ball is thrown, and the initial height of the ball above the ground. Also, the ***units of measure matter*** in this situation because we will use the force of gravity as $-9.81\ meters/sec^2$. So height is measured in meters and speed is measured in meters/second.

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(a baseball in motion)

Here is what we know: The speed of the ball when thrown is **27.2** m/s and the height above the ground is **0.5** meter. Let the variable "x" be time in seconds. Then by Newton's Second Law of Motion

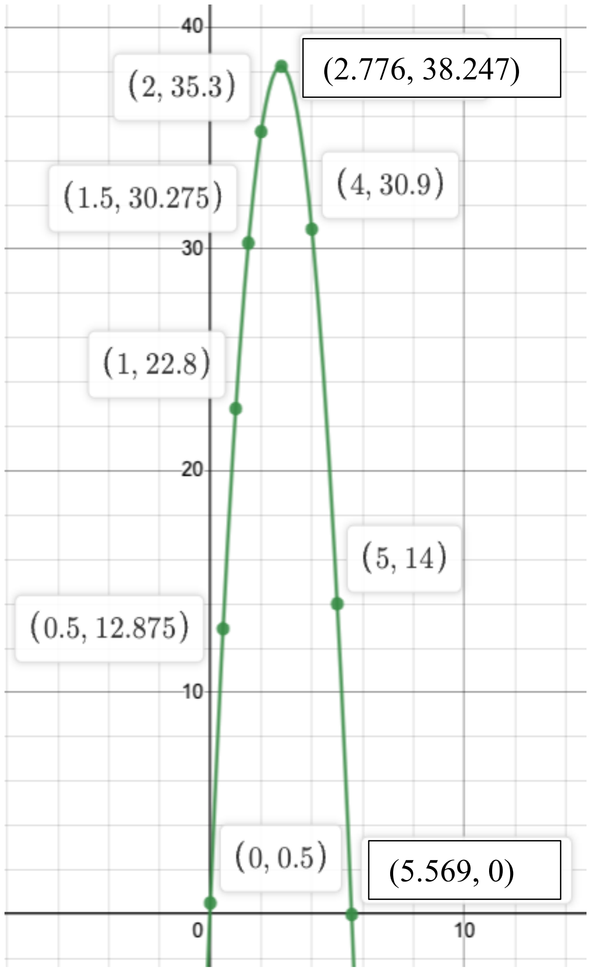
$$h(x) = \mathbf{0.5} + \mathbf{27.2}x - 4.9x^2$$

[-4.9 is half gravity].

We can use technology to graph that function. We will start with a table of values. It doesn't make sense to choose negative numbers for x. We are stopping at 5 because when x = 6 we get a negative number. The ball can't go below ground. Ground level is y = 0. These y values are measured in meters, so we have a general idea that the ball is going up, then coming back down.

x	y
0	0.5
1	22.8
2	35.3
3	38
4	30.9
5	14

The relationship between height in meters and time in seconds can be seen on this graph. We can label points that come from our table of values, and points that appear to be important to the situation.



The type of function is a quadratic function and belongs to the family of "square" functions.
Given the equation, the graph, and the known positions of the ball that are labeled here, what can we say about this situation?

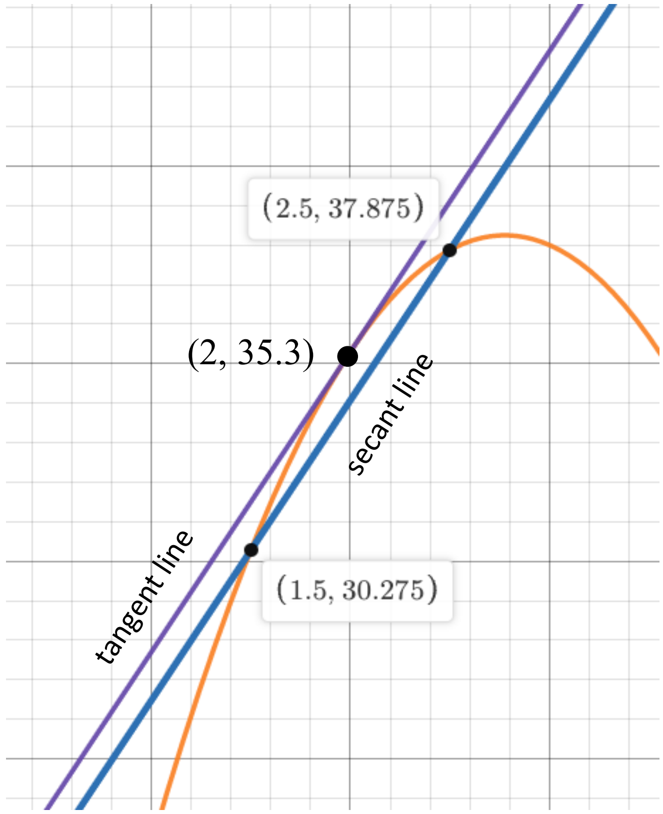
1) We can say that the ball reached a maximum height of 38.25 meters above the ground.
2) We can say that the ball reached its maximum height at 2.8 seconds after being thrown.
3) We can say that it took a total of 5.6 seconds to fall back down to the ground.

What we can't say (yet) is what the ball's speed was at any of these points.

Suppose we wanted to know exactly ***how fast the ball was going at the 2 second spot***. We'd need to be able to determine the slope of the curve at that point, which is difficult because this is not a straight line. Slope is sometimes negative. Speed will be the absolute value at these points.

Now is the time to introduce the *two lines* that help us determine what the slope is on a curve. These two lines need to be drawn at or near the point of interest.

Here we have a closeup of the graph of our projectile, concentrated on the part that is near x = 2.



The point (2, 35.3) is labeled. This is where we want to know the speed of the ball, which is the slope at that point.

The first straight-edge line we need is called the ***tangent line*** to the curve at a point. It is the line that passes through (2, 35.3) and *only touches the graph at that point*. (shown here in purple)

The second line is called the ***secant line***. There are lots of secant lines, but here we see one that appears to be parallel to the tangent line (which is important!)

Definition: Let a curve with equation $y = f(x)$ be given. The **tangent line** to the curve at a point (x_0, y_0) is the straight line with slope equal to the slope of the curve at the point. $y_0 = f(x_0)$ perhaps goes without saying.

We are going through a lot of effort just to find the speed of the baseball at 2 seconds. Part of the difficulty is with the tangent line. We only have one point. We don't know the slope because that is what we are solving for. The secant line will come to the rescue. Secant lines are useful because they intersect with the curve in ***TWO points***… which means we can calculate the slope! Notice the two values of x that were chosen have an x coordinate that happen to be ½ unit from 2. 2 + ½ = 2.5 and 2 – ½ is 1.5.

This is exactly what we want:
If the ***tangent line and the secant line are parallel***, then they have the same slope. We can calculate the slope of the secant line, and that will tell us what the slope of the tangent line is, which in turn tells us the speed of the ball at x = 2 seconds. Remember parallel means two lines are not the same line and will never intersect because they remain the same distance apart.

So far, we have accomplished quite a bit:

1) a closeup on our graph of the curve near the point (2, 35.3).
2) we have drawn to the best of our ability the **tangent line** to the curve that passes through our point
3) we have drawn to the best of our ability a **secant line** that passes through two nearby points on the curve and ***appears to be parallel*** to the tangent line!
4) We have identified the two points that the secant line passes through on the curve: (1.5, 30.275) and (2.5, 37.875)

Now we have enough information to calculate the speed of the baseball.

Slope of the secant line is:

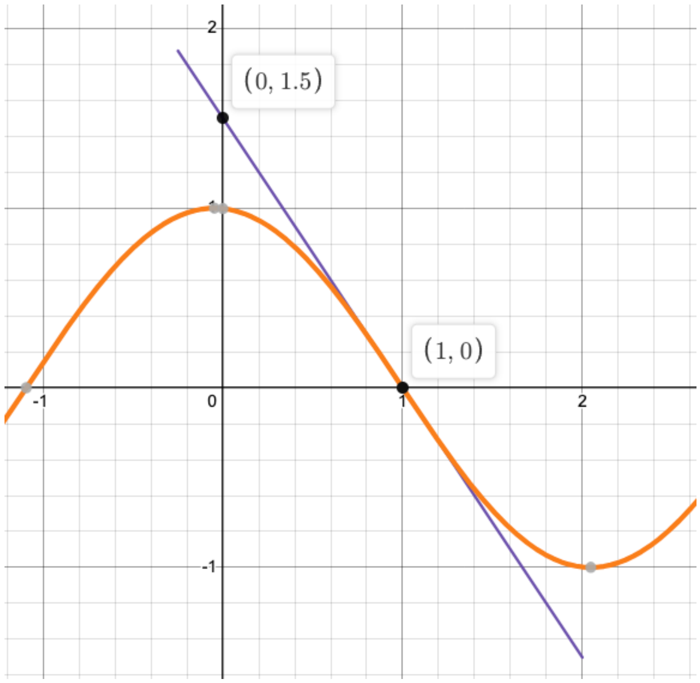
$$slope = \frac{\Delta y}{\Delta x} = \frac{(37.875 - 30.275)}{(2.5 - 1.5)} = \frac{7.6}{1} = 7.6 \div 1 = 7.6$$

This is the slope that matches the slope of the tangent line, which is the speed of the ball at x = 2.

The summary statement is that 2 seconds after being thrown, the ball is traveling at an approximate speed of 7.6 meters per second.

Since we are talking about estimating slope at a point on a curve, there is another way we can do this via the graph. Instead of finding a secant line that is parallel to the tangent line, sometimes we can extend the tangent line to one of the axes. Without technology, it can be difficult to pinpoint the exact spot where a line crosses with either the x or y axis.

This is why we have options. When one approach doesn’t work, try another. Also, we are looking for a reasonable estimate. Let’s see an example.



We have a curve but no equation. We only have the graph. The goal is to estimate the slope of the curve at the point (1, 0).

The tangent line to the curve passing through (1, 0) is drawn and extended so that we can see clearly that it crosses the y axis at the point (0, 1.5).

Since we have two points, there is no need for a secant line. In this example, finding a parallel secant line would be very difficult anyway.

The slope between the points (0, 1.5) and (1, 0) is $m = \frac{\Delta y}{\Delta x} = -1.5$

That result is our estimate of the slope of the curve at the point (1, 0).

Definition: The slope on a curve at a point is called the **derivative** at a point. Instead of using the symbol $\frac{\Delta y}{\Delta x}$ (which is slope between two points), we use the symbol $\frac{dy}{dx}$ at (x_0, y_0).

Finally, we can estimate slope at a point using a table. We select x values closer and closer to the desired x until we see a pattern emerging when we calculate the slopes of the secant lines. We can do this without graphing at all if we have the equation.

Example: For the function $f(x) = \sqrt{3x + 16}$, find the derivative at x = 3.

Setting up the table. We need three columns, one for our x's, the second for the y's, and the third for the slope between two points. Use headings as shown in grey. We have a choice of values for x. It was decided to start with 0 and creep closer to 3. We'll start with these and see if we can determine a pattern.

x	y = f(x)	Calculate the slope between (3, 5) and a **second point**
3	**5 =** $\sqrt{\mathbf{3 \cdot 3 + 16}}$	
Second point		
0		
1		
2		
2.5		

Next we will calculate all the y values from the function. Once we have the y values, calculate the slopes for each new point with the original (in this example, the original is (3, 5).

x	y = f(x)	Calculate the slope between (3, 5) and a **second point**
3	**5 =** $\sqrt{\mathbf{3 \cdot 3 + 16}}$	
Second point		
0	4 = $\sqrt{3 \cdot 0 + 16}$	$\frac{(5-4)}{(3-0)} = 0.333$
1	4.36	$\frac{(5-4.36)}{(3-1)} = 0.32$
2	4.69	$\frac{(5-4.69)}{(3-2)} = 0.31$
2.5	4.85	$\frac{(5-4.85)}{(3-2.5)} = 0.30$
We probably need one more, perhaps two more, rows because so far the slopes keep going down by 0.01. ***We need the numbers to get closer together.***		
2.8	4.94	$\frac{(5-4.94)}{(3-2.8)} = 0.30$
2.9	4.97	$\frac{(5-4.97)}{(3-2.9)} = 0.30$

When we calculated the slope of the last three secant lines from the table, accurate out to two decimal places, our slopes were identical. We were better off including at least one more calculation than was necessary because it allows us to check our work better.

Conclusion: The derivative of $f(x) = \sqrt{3x + 16}$ at x = 3 is about 0.30.

> **Try this:** For the function $f(x) = \sqrt{2x + 12}$, find the derivative at the point (2, 4). Use a table and select values of x that are: 0, 1, 1.5, 1.8, 1.9.

Solution:

x	y = f(x)	Calculate the slope between (2, 4) and a **second point**
2	**4 =** $\sqrt{\mathbf{2 \cdot 2 + 12}}$	
Second point		
0	3.464 = $\sqrt{2 \cdot 0 + 12}$	$\frac{(4 - 3.464)}{(2 - 0)} = 0.268$
1	3.74	$\frac{(4 - 3.74)}{(2 - 1)} = 0.26$
1.5	3.87	$\frac{(4 - 3.87)}{(2 - 1.5)} = 0.26$
1.8	3.95	$\frac{(4 - 3.95)}{(2 - 1.8)} = 0.25$
1.9	3.975	$\frac{(4 - 3.975)}{(2 - 1.9)} = 0.25$
1.99	3.9975	$\frac{(4 - 3.9975)}{(2 - 1.99)} = 0.25$

That last row was insurance. We were already confident that our answer would be 0.25, but now we are even more confident that we would have been had we stopped at 1.9. The inclusion of more decimal places became more necessary the closer to 2 we became.

Conclusion: The derivative of function $f(x) = \sqrt{2x + 12}$ at the point (2, 4) is about 0.25 or 1/4.

Practice Exercises:

Exercise 1: Create a "Table of Values" for each of the toolkit functions using at least 5 values of x. Then construct your own graph. Does your graph resemble the ones in this section? Make sure to select both positive and negative x values ***when appropriate*** in order to practice calculator operations with negative numbers. [When are negative values of x not appropriate?]

Exercise 2: Given a ball thrown vertically upward according to the formula

$$h(x) = \mathbf{0.5} + \mathbf{27.2}x - 4.9x^2$$

a) A line is ***tangent*** to the curve at x = 0.5 seconds. What is the y coordinate of the point where the tangent line and the curve intersect?
b) A line is ***secant*** to the curve and passes through (1, ____) and (0, _____). What are the y coordinates of the two points?
c) Find the slope of the secant line in b)
d) Use this number to estimate the speed of the ball at 0.5 seconds.
e) Is this a "reliable" estimate, given that the actual speed of the ball is 22.3 meters/second, 0.5 second after being thrown?