CS258: Information Theory

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Recap: Error Correcting

- Channel model: BSC and its channel capacity
- Repetition code: encoder, decoder, information rate
- (7, 4) Hamming code: encoder, decoder, information rate

- Information content
- Shannon source coding theorem
- Typical set
- Fundamental tools

A mathematical brain-teaser

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Oddball Problem

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- You are also given a two-pan balance to use. In each use of the balance, there are three possible outcomes: equal, heavier, or lighter.

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Oddball Problem

- Given 12 balls, all equal in weight except for one that is either heavier or lighter.
- You are also given a two-pan balance to use. In each use of the balance, there are three possible outcomes: equal, heavier, or lighter.
- Design a strategy to determine which is the odd ball and whether it is heavier or lighter in as few uses of the balance as possible.

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(c) Once you have designed a strategy, draw a tree showing, for each of the possible outcomes of a weighing, what weighing you perform next. At each node in the tree, how much information have the outcomes so far given you, and how much information remains to be gained?

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(d) How much information is gained on the first step of the weighing problem if 6 balls are weighed against the other 6? How much is gained if 4 are weighed against 4 on the first step, leaving out 4 balls?

Ensemble and Information content

An ensemble X is a triple $(x; \mathcal{A}_X; \mathcal{P}_X)$, where the outcome x is the value of a random variable, which takes on one of a set of possible values, $\mathcal{A}_X = \{a_1, a_2, ..., a_i, ..., a_I\}$, having probabilities $\mathcal{P}_X = \{p_1, p_2, ..., p_I\}$, with $P(x = a_i) = p_i, p_i \geq 0$ and $\sum_{a_i \in \mathcal{A}_X} P(x = a_i) = 1$.

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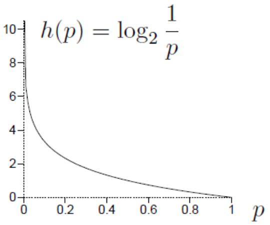

Guessing games

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1: is $x \geq 32$?

2: is $x \bmod 32 \geq 16$?

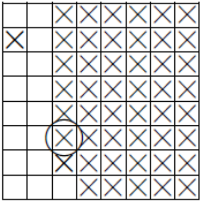
3: is $x \bmod 16 \geq 8$?

4: is $x \bmod 8 \geq 4$?

5: is $x \bmod 4 \geq 2$?

6: is $x \bmod 2 = 1$?

The game of submarine



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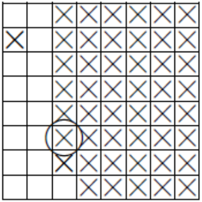

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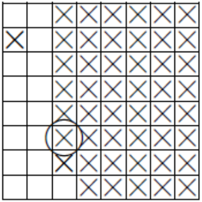

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If we hit the submarine when there are n squares left to choose from, then the total information gained is:

$$\log_2 \frac{64}{63} + \log_2 \frac{63}{62} + \ldots + \log_2 \frac{n+1}{n} + \log_2 \frac{n}{1} = log_2 64 = 6$$

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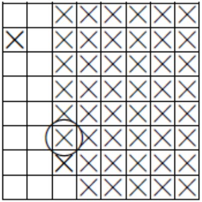

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Meaning of Shannon information content

Shannon information content measures the length of a binary file that encodes x.

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Denote by X^N the ensemble $(X_1, X_2, ..., X_N)$, where X_i's are independent identically distributed random variables. What is $H_\delta(X^N)$?

Shannon's source coding theorem

Theorem

Let X be an ensemble with entropy $H(X) = H$ bits. Given $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer N_0 such that for $N > N_0$,

$$|\frac{1}{N}H_\delta(X^N) - H| < \epsilon$$

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- Typicality: Law of large numbers
- Some useful fundamental inequalities

Typicality

By law of large numbers, the probability of a typical string $x \in \mathcal{A}_X^N$ is

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Typical set

$$T_{N\beta} := \{x \in \mathcal{A}_X^N : |\frac{1}{N} \log_2 \frac{1}{P(x)} - H| < \beta\}$$

'Asymptotic equipartition' principle (AEP). With N sufficiently large, the outcome $x = (x_1, x_2, ..., x_N)$ is almost certain to belong to a subset of $\mathcal{A}_X^N$ having only $2^{NH(X)}$ members, each having probability 'close to' $2^{-NH(X)}$.

Chebyshev's inequality 1

Let t be a non-negative real random variable, and let α be a positive real number. Then

$$P(t \geq \alpha) \leq \frac{\bar{t}}{\alpha},$$

where $\bar{t}$ is the mean of t.

$P(t \geq \alpha) = \sum_{t \geq \alpha} P(t)$. We multiply each term by $t/\alpha \geq 1$ and obtain: $P(t \geq \alpha) \leq \sum_{t \geq \alpha} P(t) t/\alpha$. We add the (non-negative) missing terms and obtain: $P(t \geq \alpha) \leq \sum_{t} P(t) t/\alpha = \bar{t}/\alpha$

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Chebyshev's inequality 2

Let x be a random variable, and let α be a positive real number. Then

$$P((x - \bar{x})^2 \geq \alpha) \leq \sigma_x^2/\alpha$$

Take $t = (x - \bar{x})^2$.

Take x to be the average of N independent random variables $h_1, h_2, \ldots, h_N$, having common mean $\bar{h}$ and common variance σ_h^2. Then

$$P((x - \bar{h})^2 \geq \alpha) \leq \sigma_h^2 / \alpha N$$

Take $\bar{x} = \bar{h}$ and $\sigma_x^2 = \sigma_h^2 / N$.

Proof of Source Coding Theorem

- $\frac{1}{N} H_\delta(X^N) < H + \epsilon$
- $\frac{1}{N} H_\delta(X^N) > H - \epsilon$

Reading and Exercise

Reading: Ch. 4 (MacKay)

Exercise

4.9, 4.10, 4.11, 4.12