CS258: Information Theory

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Recap: fundamental information quantities

- Definition of KL-divergence, entropy, mutual information
- Information diagram: relationship of entropy, mutual information, etc.
- Some fundamental properties: non-negative, convexity/concativy, inequalities
- How to solve problems via information quantities

Lecture 3: Error Correcting

- Repetition code
- Hamming code
- Decomposability of entropy

Noisy world

Noisy world

Noisy world

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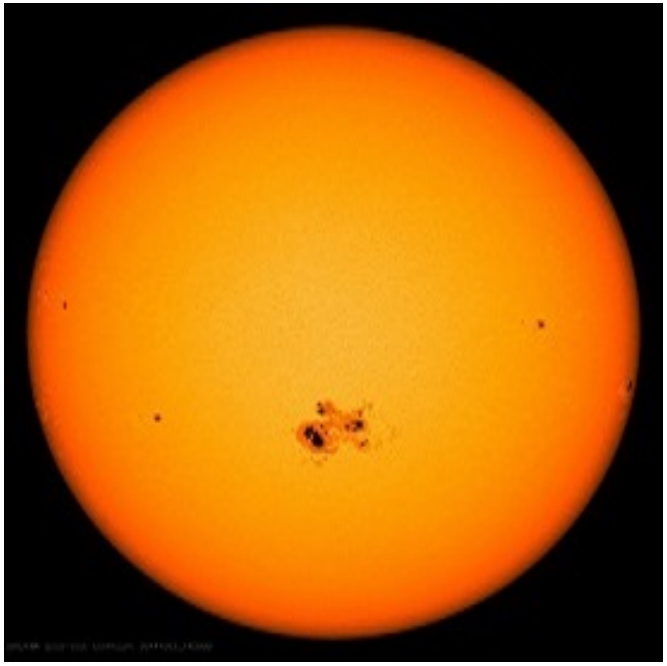

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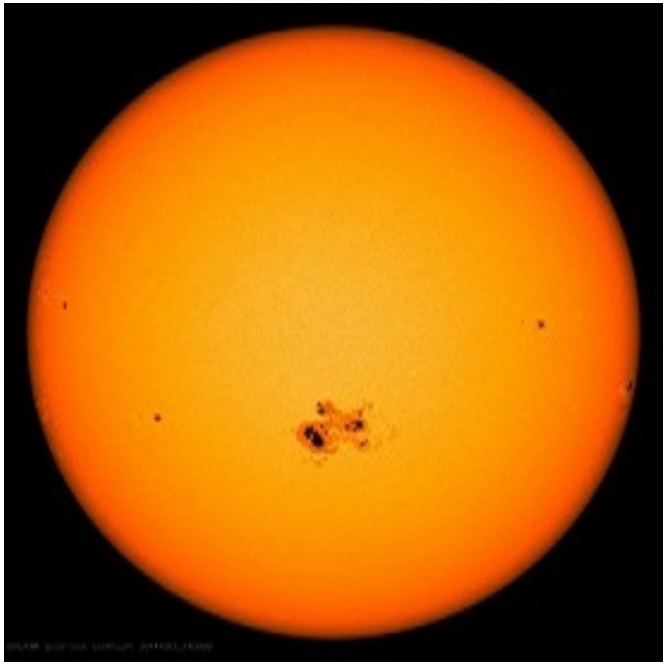
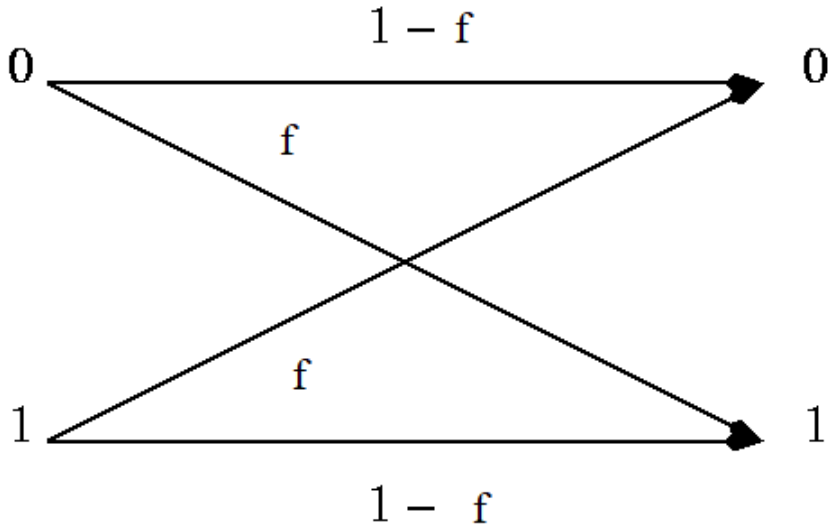


We need to correct them!

Mathematical model: binary symmetric channel



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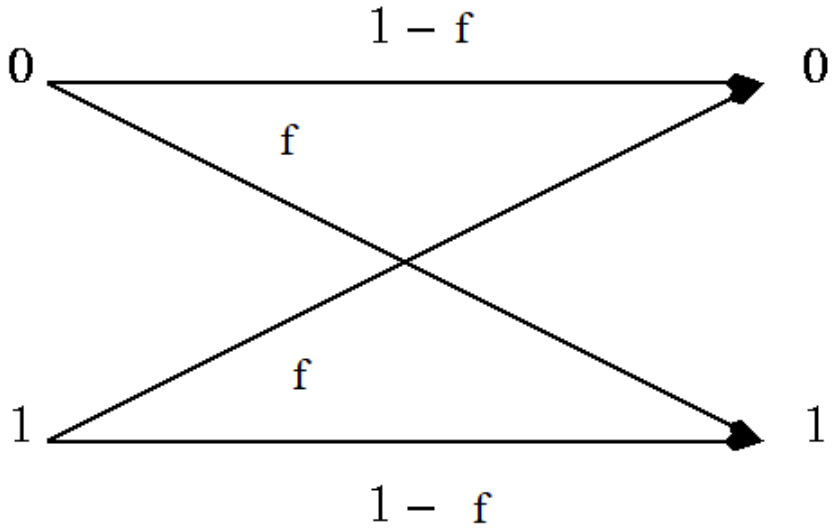

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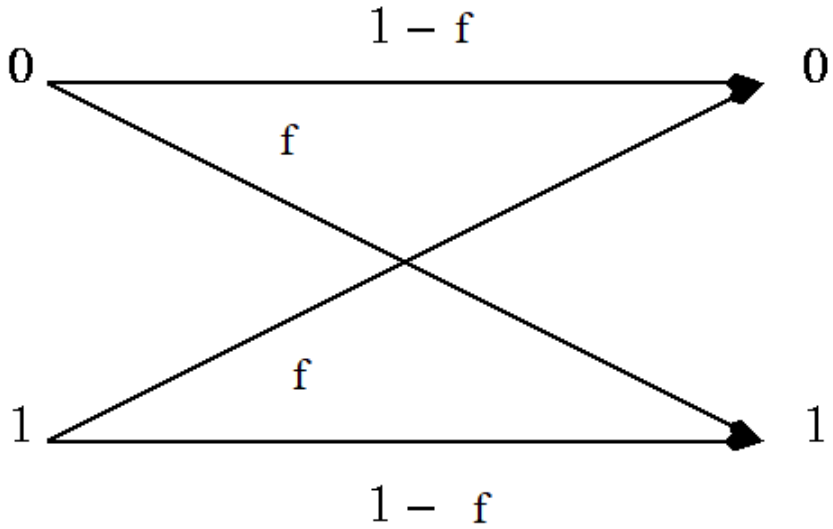

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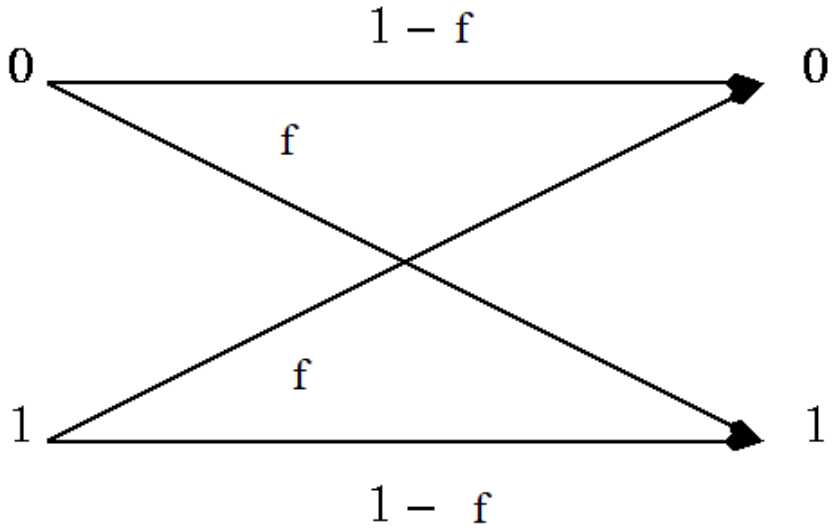

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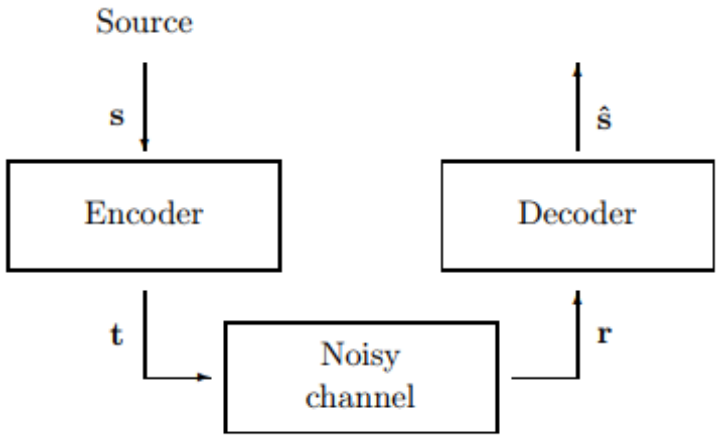
Question: how to transmit a message with very low error probability (e.g. 10^{-15})?

A system solution



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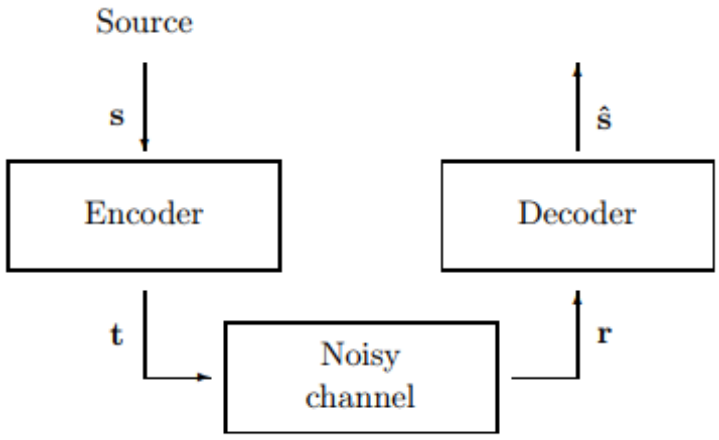

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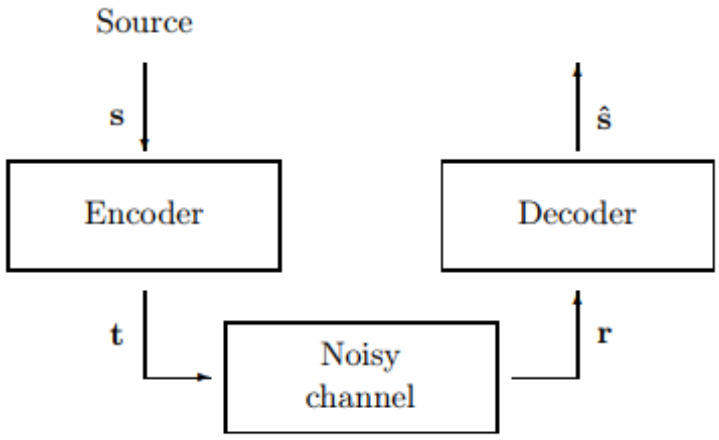

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Information theory: What is the best error-correcting performance we could achieve?
Coding theory: The creation of practical encoding and decoding systems.

Repetition codes

R_k: Repeat each bit k times; e.g., $R_3 :=$

s	t
0	000
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Repeat every bit of the message a prearranged number of times

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Encoding is obvious. How to decoding? Majority vote? Is it optimal?

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Optimal decoding

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- [2.] Assume $f < 0.5$, the winning hypothesis is the one with the most 'votes'.

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$$P(\mathbf{r}|s) = P(\mathbf{r}|t(s)) = \prod_{n=1}^{N} P(r_n|t_n(s))$$

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Each factor $\frac{P(r_n|t_n(1))}{P(r_n|t_n(0))}$ equals $\frac{1-f}{f}$ if $r_n = 1$ and $\frac{f}{1-f}$ if $r_n = 0$. Thus majority vote is optimal if $f < 0.5$ □

Majority vote decoder

An example transmission using R_3

s	0	0	1	0	1	1	0
t	000	000	111	000	111	111	000
n	000	001	000	000	101	000	000
r	000	001	111	000	010	111	000
$\hat{s}$	0	0	1	0	0	1	0

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Proper redundancy is needed to reliable communication. Tradeoff exists between rate of information and error probability.

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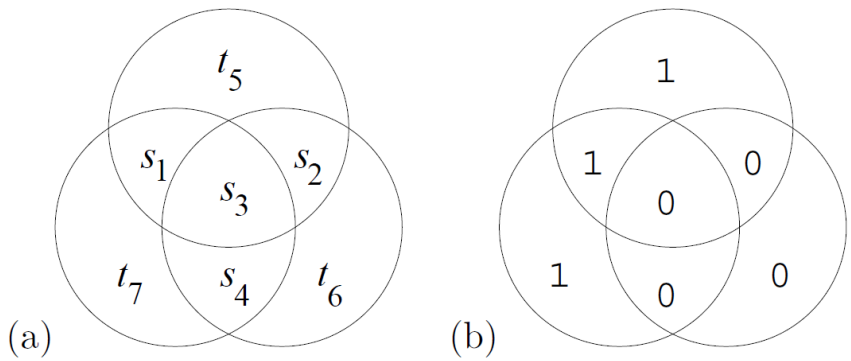

Encoder of (7,4) Hamming code:

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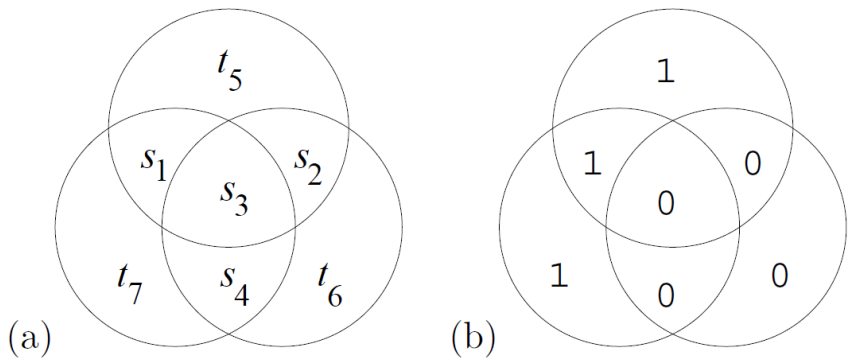

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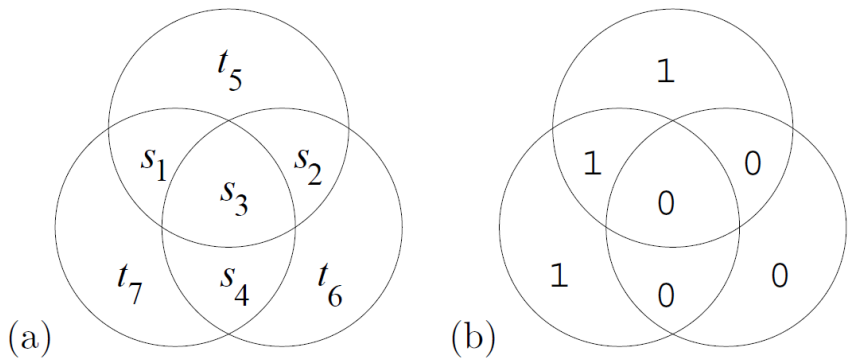

Encoder of (7,4) Hamming code:

- $t_1 t_2 t_3 t_4$ are set equal to $s_1 s_2 s_3 s_4$
- The parity-check bits $t_5 t_6 t_7$ are set so that the parity within each circle is even

Codewords of (7,4) Hamming code

s	t	s	t	s	t	s	t
0000	0000000	0100	0100110	1000	1000101	1100	1100011
0001	0001011	0101	0101101	1001	1001110	1101	1101000
0010	0010111	0110	0110001	1010	1010010	1110	1110100
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- Higher information rate
- More complicated encoder

Matrix form of Hamming (7,4) code

$$t = sG,$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Decoding the (7,4) Hamming code

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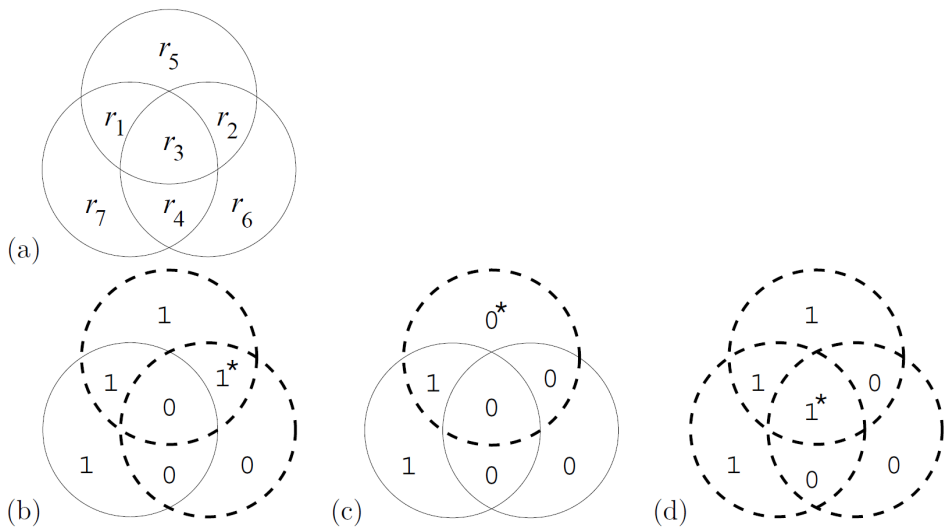
Not efficient!

Syndrome decoding for the Hamming code

The pattern of violations of the parity checks is called the syndrome, and can be written as a binary vector (In Fig. b, the syndrome is $z = (1, 1, 0)$). (Syndrome: Happy (parity 0) and unhappy (parity 1))

Syndrome decoding

Find a unique bit that lies inside all the 'unhappy' circles and outside all the 'happy' circles. Flip this bit for correction.



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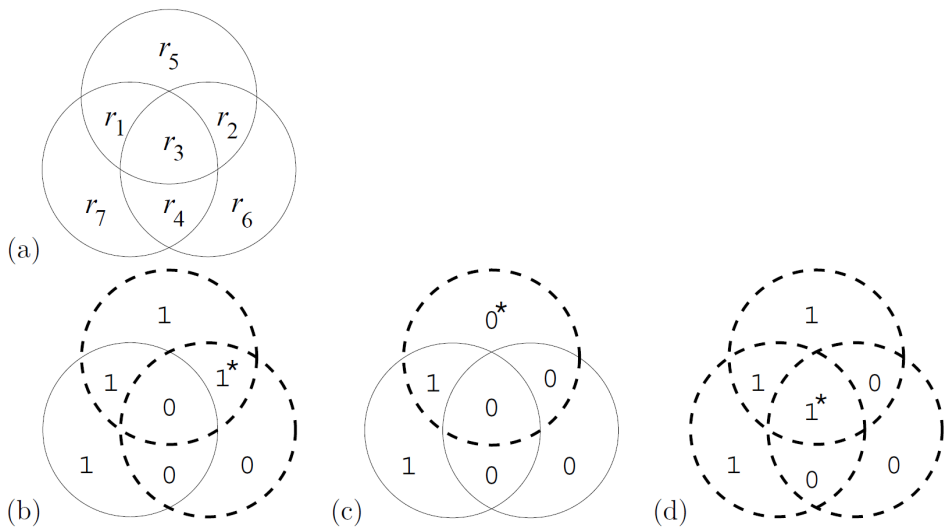

Syndrome z	000	001	010	011	100	101	110	111
Unflip this bit	none	r_7	r_6	r_4	r_5	r_1	r_2	r_3

Decoding: matrix form of (7, 4) Hamming code

Denote

$$\mathbf{t} = \mathbf{sG}, \mathbf{G}^T = \begin{pmatrix} \mathbf{I_4} \\ \mathbf{P} \end{pmatrix},$$

where $\mathbf{I_4}$ is the identity matrix, then the syndrome vector $\mathbf{z} = \mathbf{Hr}$, where the parity-check matrix $\mathbf{H}$ is given by $\mathbf{H} = [-\mathbf{P}\ \mathbf{I_3}]$. Since $-1 \equiv 1 \pmod 2$, $\mathbf{H} = [\mathbf{P}\ \mathbf{I_3}]$.

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G and H

All the codewords $\mathbf{t} = \mathbf{G^t s}$ of the code satisfy $\mathbf{Ht} = \mathbf{0}$, i.e., $\mathbf{HG^t} = \mathbf{0}$
In general, $\mathbf{G} = \left[\mathbf{I}_k | \mathbf{P}\right], \mathbf{H} = \left[-\mathbf{P}^\top | \mathbf{I}_{n-k}\right]$.

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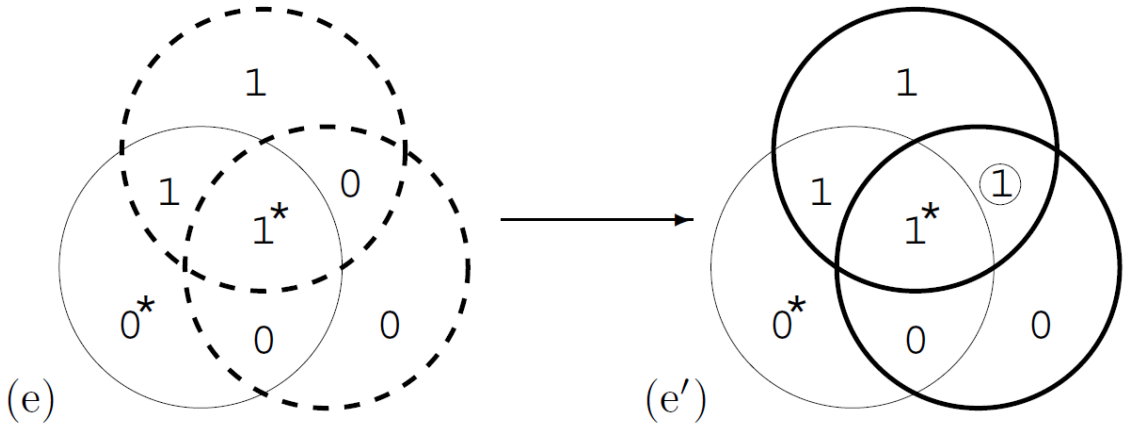
Since the received vector $\mathbf{r} = \mathbf{G^t s} + \mathbf{n}$, the syndrome-decoding problem is to find the most probable noise vector n satisfying the equation

$$\mathbf{Hn} = \mathbf{z}$$

A decoding algorithm that solves this problem is called a *maximum-likelihood decoder*.

More than two bits are flipped



$$1000101 \rightarrow 1010100$$

When two bits, r_3 and r_7, are received flipped. The syndrome, 110, makes us suspect the single bit r_2; so our optimal decoding algorithm flips this bit. If we use the optimal decoding algorithm, any two-bit error pattern will lead to a decoded seven-bit vector that contains three errors.

$$1000101 \rightarrow 1010100 \rightarrow 1110100$$

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There is a tradeoff between p_b and R. The feasible region of (p_b, R) is solved by C. E. Shannon in 1948.

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Exercise

1.3, 1.6, 2.28, 2.29

Decomposability of the entropy

The entropy of any probability distribution $p = \{p_1, p_2, ..., p_I\}$ is that

$$H(p) = H(p_1, 1 - p_1) + (1 - p_1)H(p_2/(1 - p_1), ..., p_I/(1 - p_1))$$

Generalizing further,

$$\begin{aligned} H(p) = & H(p_1 + ... + p_m, p_{m+1} + ... + p_I) \\ & + (p_1 + ... + p_m)H(p_1/(p_1 + ... + p_m), ..., p_m/(p_1 + ... + p_m)) \\ & + (p_{m+1} + ... + p_I)H(p_{m+1}/(p_{m+1} + ... + p_I), ..., p_I/(p_{m+1} + ... + p_I)) \end{aligned}$$

An unbiased coin is flipped until one head is thrown. What is the entropy of the random variable $x \in \{1, 2, 3, ...\}$, the number of flips?

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$$H(X) = H_2(f) + (1 - f)H(X)$$