MSE 427/727 Midterm Solution

Q1

	Element 1	Element 2	Element 3
Nodes	1 – 3	2 – 3	3 – 4
L	25 ft	25 ft	10 ft
θ	53.13°	126.87°	90°
$\cos\theta$	0.6	–0.6	0
$\sin\theta$	0.8	0.8	1

$$[K]_1 = \frac{AE}{25}\left[\begin{array}{c|c} \lambda & -\lambda \\ \hline -\lambda & \lambda \end{array}\right], \quad \lambda = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

$$[K]_2 = \frac{AE}{25}\left[\begin{array}{c|c} \lambda & -\lambda \\ \hline -\lambda & \lambda \end{array}\right], \quad \lambda = \begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{bmatrix}$$

$$[K]_3 = \frac{AE}{10}\left[\begin{array}{c|c} \lambda & -\lambda \\ \hline -\lambda & \lambda \end{array}\right], \quad \lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Invoking boundary conditions, we only need the displacements of node $\underline{\underline{3}}$

$$[K] = \begin{bmatrix} \frac{AE}{25}(0.36+0.36) + \frac{AE}{10}(0) & \frac{AE}{25}(0.48-0.48) + \frac{AE}{10}(0) \\ \frac{AE}{25}(0.48-0.48) + \frac{AE}{10}(0) & \frac{AE}{25}(0.64+0.64) + \frac{AE}{10}(1) \end{bmatrix}$$

$$\Rightarrow [K] = AE\begin{bmatrix} \frac{0.72}{25} & 0 \\ 0 & \frac{1.28}{25} + \frac{1}{10} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} F_{3x} = 0 \\ F_{3y} = -10^{kip} \end{bmatrix} = AE\begin{bmatrix} \frac{0.72}{25} & 0 \\ 0 & \frac{1.28}{25} + \frac{1}{10} \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_3 = 0 \\ v_3 = -0.0088^{\ in} \end{cases}$$

ft

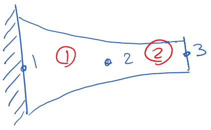
$v_3 = -0.00\ldots$

Forces in element with length of 10 ft

$$F = \sigma A = \frac{EA}{L}\begin{bmatrix} -C & -S & C & S \end{bmatrix}\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

$$= \frac{30\times10^{6}\,\frac{lb}{in^2} \times 3\,in^2}{(10\times12)\,in}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.0088\,in \\ 0 \\ 0 \end{Bmatrix} = 6600\,lb$$

Q2)



$$[K]_1 = [B]^T E [B] A_0 \int_0^{\frac{L}{2}} (1-x^2)\, dx = [B]^T E [B] A_0 \left(x - \frac{x^3}{3}\right)\Big|_0^{\frac{L}{2}}$$

$$= [B]^T E [B] A_0 \left(\frac{L}{2} - \frac{L^3}{24}\right)$$

$$= \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A_0 \left(\frac{L}{2} - \frac{L^3}{24}\right) = \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} E A_0 \left(\frac{L}{2} - \frac{L^3}{24}\right)$$

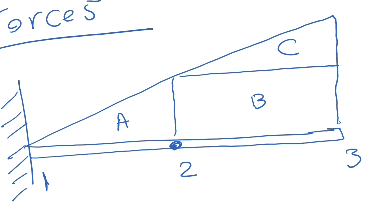
$$\Rightarrow [K]_1 = E A_0 \left(\frac{1}{2L} - \frac{L}{24}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K]_2 = [B]^T E [B] A_0 \int_{\frac{L}{2}}^{L} (1-x^2)\, dx = [B]^T E [B] A_0 \left(x - \frac{x^3}{3}\right)\Big|_{\frac{L}{2}}^{L}$$

$$= [B]^T E [B] A_0 \left(L - \frac{L^3}{3} - \frac{L}{2} + \frac{L^3}{24}\right) = [B]^T E [B] A_0 \left(\frac{L}{2} - \frac{7L^3}{24}\right)$$

$$= \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} E A_0 \left(\frac{L}{2} - \frac{7L^3}{24}\right) = E A_0 \left(\frac{1}{2L} - \frac{7L}{24}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Forces



$$A = \frac{1}{2} \times \frac{L}{2} \times C\frac{L}{2} = \frac{CL^2}{8}$$

$$f_{1x} = \frac{1}{3} \times \frac{CL^2}{8} = \frac{CL^2}{24}$$

$$f_{2x} = \frac{2}{3} \times \frac{CL^2}{8} = \frac{CL^2}{12}$$

$$B = \frac{L}{2} \times C\frac{L}{2} = \frac{CL^2}{4}$$

$$f_{2x} = \frac{1}{2} \times \frac{CL^2}{4} = \frac{CL^2}{8}$$

$$f_{3x} = \frac{CL^2}{8}$$

$$C = \frac{1}{2} \times \frac{L}{2} \times C\frac{L}{2} = \frac{CL^2}{8}$$

$$f_{2x} = \frac{1}{3} \times \frac{CL^2}{8} = \frac{CL^2}{24}$$

$$f_{3x} = \frac{2}{3} \times \frac{CL^2}{8} = \frac{CL^2}{12}$$

$$\Rightarrow \begin{bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \end{bmatrix} = \begin{bmatrix} R_x + \frac{CL^2}{24} \\ \frac{CL^2}{12} + \frac{CL^2}{8} + \frac{CL^2}{24} \\ \frac{CL^2}{8} + \frac{CL^2}{12} \end{bmatrix} = \begin{bmatrix} R_x + \frac{CL^2}{24} \\ \frac{CL^2}{4} \\ \frac{5}{24} CL^2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} (K_{ii})_1 & (K_{ij})_1 & \\ (K_{ji})_1 & (K_{jj})_1 + (K_{ii})_2 & (K_{ij})_2 \\ & (K_{ji})_2 & (K_{jj})_2 \end{bmatrix}$$

$$\left(\frac{1}{L} - \frac{7L}{\ldots}\right) \Big] \begin{bmatrix} u_2 \end{bmatrix} \quad \begin{bmatrix} CL^2 \end{bmatrix}$$

$$\Rightarrow EA_0 \begin{pmatrix} \left(\frac{1}{2L}-\frac{L}{24}\right)+\left(\frac{1}{2L}-\frac{7L}{24}\right) & -\left(\frac{1}{2L}-\frac{7L}{24}\right) \\ -\left(\frac{1}{2L}-\frac{7L}{24}\right) & \left(\frac{1}{2L}-\frac{7L}{24}\right) \end{pmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{cL^2}{4} \\ \frac{5}{24}cL^2 \end{bmatrix}$$

you get full mark, if you get to this point

Q3) $\int w(x)\, v(x)\, dx = m_1\phi_1 + m_2\phi_2 + f_{1y} v_1 + f_{2y} v_2$

$$w(x) = \begin{cases} -w & 0 < x < \frac{L}{2} \\ 0 & \frac{L}{2} < x < L \end{cases}$$

$$\int w(x)\, v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] w \int_0^{\frac{L}{2}} \left(\frac{x}{L} - 1\right) x^3\, dx$$

$$+ \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\phi_1 + \phi_2)\right] w \int_0^{\frac{L}{2}} \left(\frac{x}{L} - 1\right) x^2\, dx$$

$$+ \left[\phi_1 w\right] \int_0^{\frac{L}{2}} \left(\frac{x}{L} - 1\right) x\, dx + v_1 w \int_0^{\frac{L}{2}} \left(\frac{x}{L} - 1\right) dx$$

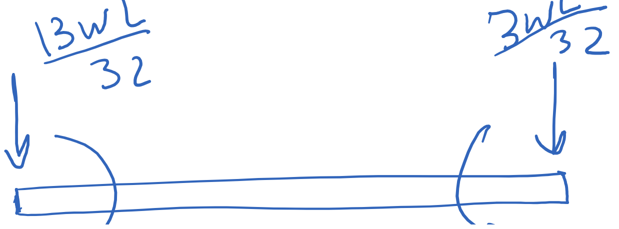
$$= m_1\phi_1 + m_2\phi_2 + f_{1y} v_1 + f_{2y} v_2$$

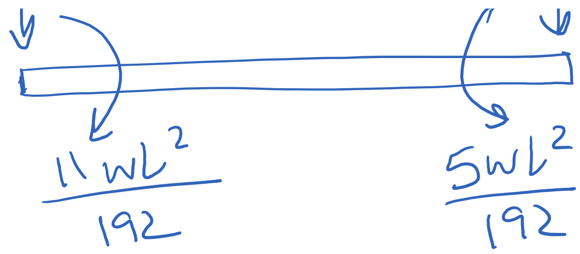
$\phi_1 = 1$ & the rest $= 0$ $\longrightarrow$ $m_1 = -\frac{11 w L^2}{192}$

$\phi_2 = 1$ & the rest $= 0$ $\longrightarrow$ $m_2 = \frac{5 w L^2}{192}$

$v_1 = 1$ & the rest $= 0$ $\longrightarrow$ $f_{1y} = -\frac{13 w L}{32}$

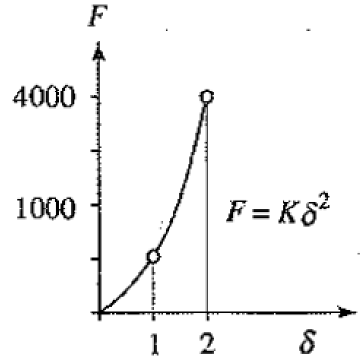
$v_2 = 1$ & the rest $= 0$ $\longrightarrow$ $f_{2y} = -\frac{3 w L}{32}$





For full mark, you should show the integration steps

$$U = \int_0^x (kx^2)\,dx$$



$$U = \frac{kx^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3}\,kx^3 - 500\,x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 500$$

$$0 = 1000\,x^2 - 500$$

$\Rightarrow \quad x = 0.707$ in. (equilibrium value of displacement)

$$\pi_{p\ \min} = \frac{1}{3}\,(1000)\,(0.707)^3 - 500\,(0.707)$$

$$\pi_{p\ \min} = -235.7 \text{ lb·in.}$$

Q5) $L_1 = \sqrt{(x_5-x_1)^2+(y_5-y_1)^2+(z_5-z_1)^2} = 108 \text{ in}$

$$C_x = \frac{x_5-x_1}{L_1} = \frac{0-(-72)}{108} \Rightarrow C_x = 0.667$$

$$C_y = \frac{y_5-y_1}{L_1} = \frac{0-(-36)}{108} \Rightarrow C_y = 0.333$$

$$C_z = \frac{z_5-z_1}{L_1} = \frac{72-0}{108} \Rightarrow C_z = 0.667$$

Element Stress

$$\sigma_1 = \frac{E}{L_1}\begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_5 \\ v_5 \\ w_5 \end{bmatrix}$$

$$= \frac{3\times10^6}{108}\begin{bmatrix} -0.667 & -0.333 & -0.667 & 0.667 & 0.333 & 0.667 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0014 \\ 0 \\ -0.00042 \end{bmatrix}$$

$$\Rightarrow \sigma_1 = -180 \text{ psi}$$