MSE 380 – Systems Modelling and Simulation

Midterm (Fall 2018)

Name: Solutions

Student Number: ____________________

Instructor: Flavio Firmani
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Duration: 1hr 50 min

There are 4 questions and 4 pages in this examination. The exam is worth 40 pts. Please ensure that you have all four pages before starting this exam.

Q1) Shown below is the set of differential equations that model a particular system. Develop the state equations of this system (i.e. do not write it in a matrix form). (4 pts)

$$a\ddot{y} + b\dot{y} - c\dot{x} + dx + fy - g = 0$$

$$h\ddot{y} - jy + k\ddot{x} + mx + n = 0$$

Define variables

$y_1 = y$ $\qquad$ $x_1 = x$

$y_2 = \dot{y}_1 = \dot{y}$ $\qquad$ $x_2 = \dot{x}_1 = \dot{x}$

$y_3 = \dot{y}_2 = \ddot{y}$

Sub variables in given eqs.

$$a\dot{y}_3 + by_2 - cx_2 + dx_1 + fy_1 - g = 0$$

$$hy_3 - jy_1 + k\dot{x}_2 + mx_1 + n = 0$$

State equations

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = y_3$$

$$\dot{y}_3 = \frac{1}{a}\left(-by_2 + cx_2 - dx_1 - fy_1 + g\right)$$

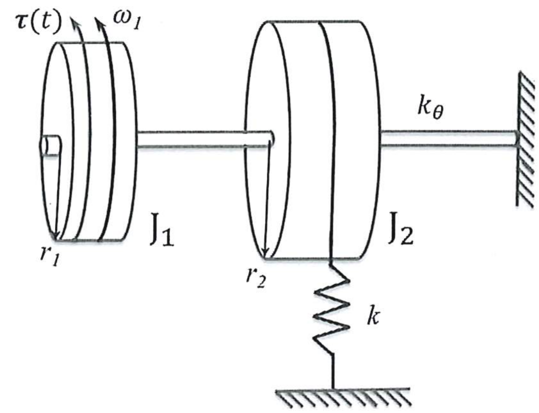
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{k}\left(-hy_3 + jy_1 - mx_1 - n\right)$$

Q2) Shown below is a mechanical system composed of two inertia elements connected with a rigid shaft The second inertia element is connected to the wall by a torsional spring and it is wrapped around by a cable (modeled as a linear spring) that is anchored to the floor. Let τ(t) be the input torque.

Both inertia elements form a rigid body

a) What is the order of the system? __3__ (1 pt)



b) Indicate the state variables using the "through" and "across" convention and their corresponding element (1 pts)

ω_1 - Inertia Elements

τ_θ - Torsional Spring

F - Linear Spring

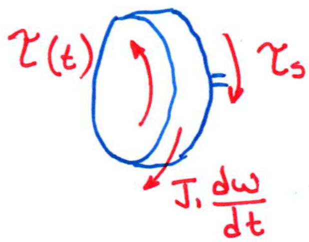
Compatibility eq.

$\omega = \omega_1 = \omega_2$

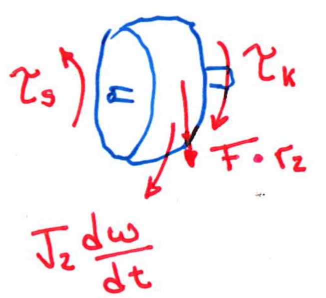
(same rigid body)

c) Develop the state space model of the form $\dot{x} = Ax + Bu$. (8 pts)

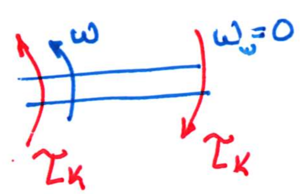
τ_s - torque through the shaft



$$J_1 \frac{d\omega}{dt} = \tau(t) - \tau_s \quad (1)$$



$$J_2 \frac{d\omega}{dt} = \tau_s - F \cdot r_2 - \tau_K \quad (2)$$



$$\frac{d\tau_K}{dt} = K_\theta \omega \quad (3)$$

$$\frac{dF}{dt} = K v = K(r_2 \cdot \omega) \quad (4)$$

τ_s is a redundant variable. Use eqs. 1 and 2 to eliminate it.

$$J_1 \frac{d\omega}{dt} = \tau(t) - \left(J_2 \frac{d\omega}{dt} + F \cdot r_2 + \tau_K \right)$$

$$\underbrace{(J_1 + J_2)}_{\text{Moments of inertia just add}} \frac{d\omega}{dt} = \tau(t) - F \cdot r_2 - \tau_K \quad (5)$$

Eqs. (3 - 5) are the state equations that lead to the following state model

$$\begin{bmatrix} \dot{\omega} \\ \dot{\tau}_K \\ \dot{F} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{J_1 + J_2} & -\frac{r_2}{J_1 + J_2} \\ K_\theta & 0 & 0 \\ K r_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \tau_K \\ F \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1 + J_2} \\ 0 \\ 0 \end{bmatrix} \tau(t)$$

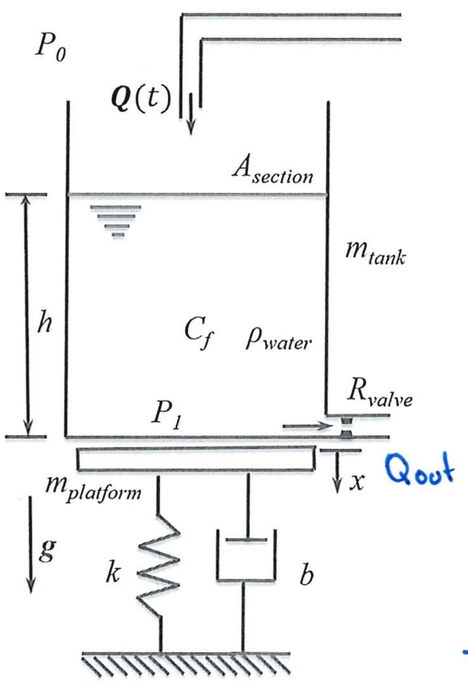
Q3) A scale is used to measure the mass of water over a period of time. The scale is modeled as a mass-spring-damper system. While a water line feeds the tank $Q(t)$, an open valve drains it.

a) Identify the elements of the system and for each element indicate whether it is a T-type, an A-type, or a D-type element. (1 pt)

b) What is the order of the system? 3 (1 pt)

c) Write the constitutive equations of all the elements. Consider the effect of gravity in your model. Do not consider water as a mass element. (6 pts)

d) Find the state model of the system $\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$ and the output equation $\boldsymbol{y} = \boldsymbol{C}x + \boldsymbol{D}u$, where the mass of water, the level of water and the displacement of the scale mass are the output variables (7 pts)



a) Water Tank (Fluid Capacitor) Type A

Outlet Valve (Fluid Resistance) Type D

Platform + Tank (Mass) Type A

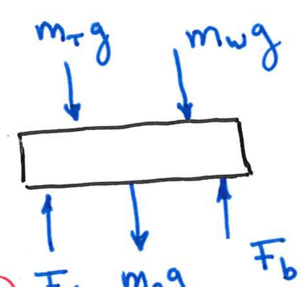
Spring Type T

Damper Type D

b) Since m_{plat} and m_{tank} are constant, we can consider them as a rigid body

There are 3 energy storage elements

c) Constitutive eqs.



$$C_f \frac{dP_{10}}{dt} = \sum Q = Q(t) - Q_{out} \quad (1)$$

$$Q_{out} = \frac{1}{R_v} P_{10} \quad (2)$$

$$(m_T + m_P)\frac{dv}{dt} = (m_T + m_P + m_w)g - F_K - F_b \quad (3)$$

$$\frac{dF}{dt} = kv \quad (4)$$

$$F_b = bv \quad (5)$$

d) Since $\rho = \frac{m_w}{V}$, then $m_w = \rho V = \rho A h$ (6)

From hydrostatic preassure $P_{10} = \rho g h$ $\quad h = \frac{P_{10}}{\rho g}$ (7)

Sub eq 7 in 6 $\quad m_w = \frac{A P_{10}}{g}$ (8)

State eqs.

$$\frac{dP_{10}}{dt} = \frac{1}{C_f}\left(Q(t) - \frac{1}{R_v} P_{10}\right)$$

$$\frac{dv}{dt} = g + \frac{AP_{10}}{m_T + m_p} - \frac{F_K}{m_T + m_p} - \frac{bv}{m_T + m_p}$$

$$\frac{dF}{dt} = kv$$

State model

$$\begin{bmatrix} \dot{P}_{10} \\ \dot{v} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_f R_v} & 0 & 0 \\ \frac{A}{m_T + m_p} & \frac{-b}{m_T + m_p} & -\frac{1}{m_T + m_p} \\ 0 & K & 0 \end{bmatrix} \begin{bmatrix} P_{10} \\ v \\ F \end{bmatrix} + \begin{bmatrix} \frac{1}{C_f} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q(t) \\ g \end{bmatrix}$$

Output

$$\begin{bmatrix} m_w \\ h \\ x \end{bmatrix} = \begin{bmatrix} \frac{A}{g} & 0 & 0 \\ \frac{1}{\rho g} & 0 & 0 \\ 0 & 0 & \frac{1}{K} \end{bmatrix} \begin{bmatrix} P_{10} \\ v \\ F \end{bmatrix}$$

$F = kx$

$x = \frac{1}{K} F$

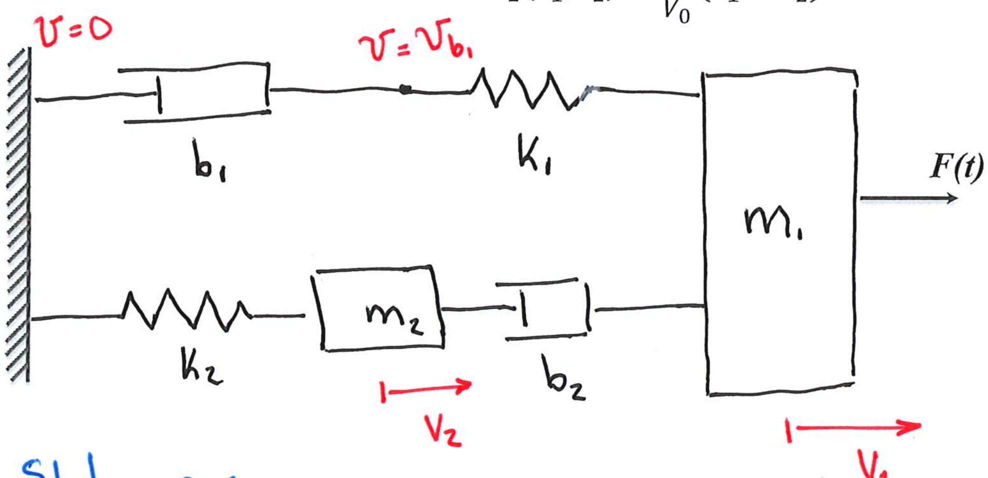
Q4) Shown below is the state space model of a mechanical system:

$$\begin{bmatrix} \dot{F}_1 \\ \dot{V}_1 \\ \dot{F}_2 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} -\frac{k_1}{b_1} & k_1 & 0 & 0 \\ -\frac{1}{m_1} & -\frac{b_2}{m_1} & 0 & \frac{b_2}{m_1} \\ 0 & 0 & 0 & k_2 \\ 0 & \frac{b_2}{m_2} & -\frac{1}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ V_1 \\ F_2 \\ V_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} F(t)$$

a) Draw the system based on the model. Label all the elements (e.g., k_1). (6 pts)

b) Assume that the operating point $[\bar{F}_1 \quad \bar{V}_1 \quad \bar{F}_2 \quad \bar{V}_2]$ has been found in relation to a steady input force $\bar{F}(t)$, i.e., do not calculate the operating point. Let b_2 be a function of V_1 and V_2 as shown below, where b_0 and V_0 are two parameters of the system. Linearize the system about the operating point. (5 pts)

$$b_2(V_1, V_2) = \frac{b_0}{V_0}(V_1 - V_2)$$

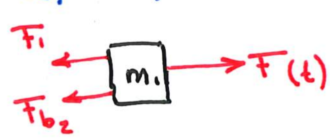


State eqs.

$$\frac{dF_1}{dt} = -\frac{k_1}{b_1}F_1 + k_1 V_1 = k_1\left(V_1 - \frac{F_1}{b_1}\right) = k_1(V_1 - V_{b_1})$$

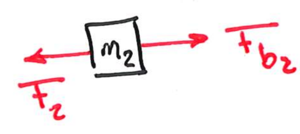
where $F_{b_1} = b_1 V_{b_1}$

$$m_1 \frac{dV_1}{dt} = F(t) - F_1 - b_2 V_1 + b_2 V_2 = F(t) - b_2(V_1 - V_2) - F_1$$



$$\frac{dF_2}{dt} = k_2 V_2$$

$$m_2 \frac{dV_2}{dt} = b_2 V_1 - F_2 - b_2 V_2 = b_2(V_1 - V_2) - F_2$$



The non-linear term appears in the second and forth eqs

$$f_2 = \dot{V}_1 = \frac{1}{m_1}\left(F(t) - \frac{b_0}{V_0}\left(V_1^2 - 2V_1V_2 + V_2^2\right) - F\right)$$

$$f_4 = \dot{V}_2 = \frac{1}{m_2}\left(\frac{b_0}{V_0}\left(V_1^2 - 2V_1V_2 + V_2^2\right) - F_2\right)$$

$$q = \begin{bmatrix} F_1 \\ V_1 \\ F_2 \\ V_2 \end{bmatrix} \qquad r = F(t)$$

$$\dot{\hat{q}} = \frac{\partial f(\bar{q}, \bar{r}, t)}{\partial q}\hat{q} + \frac{\partial f(\bar{q}, \bar{r}, t)}{\partial r}\hat{r}$$

$$\dot{\hat{V}}_1 = -\frac{1}{m_1}\hat{F}_1 - \frac{b_0}{m_1V_0}\left(2\bar{V}_1 - 2\bar{V}_2\right)\hat{V}_1 + 0\hat{F}_2 - \frac{b_0}{m_1V_0}\left(-2\bar{V}_1 + 2\bar{V}_2\right)\hat{V}_2 + \frac{1}{m_1}\hat{F}(t)$$

$$\dot{\hat{V}}_2 = 0\hat{F}_1 + \frac{b_0}{m_2V_0}\left(2\bar{V}_1 - 2V_2\right)\hat{V}_1 - \frac{1}{m_2}\hat{F}_2 + \frac{b_0}{m_2V_0}\left(-2\bar{V}_1 + 2\bar{V}_2\right)\hat{V}_2 + 0\hat{F}(t)$$

State model of the increments

$$\begin{bmatrix} \dot{\hat{F}}_1 \\ \dot{\hat{V}}_1 \\ \dot{\hat{F}}_2 \\ \dot{\hat{V}}_2 \end{bmatrix} = \begin{bmatrix} -\frac{K_1}{b_1} & K_1 & 0 & 0 \\ -\frac{1}{m_1} & -\frac{2b_0}{m_1V_0}\left(\bar{V}_1 - \bar{V}_2\right) & 0 & \frac{2b_0}{m_1V_0}\left(\bar{V}_1 - \bar{V}_2\right) \\ 0 & 0 & 0 & K_2 \\ 0 & \frac{2b_0}{m_2V_0}\left(\bar{V}_1 - \bar{V}_2\right) & -\frac{1}{m_2} & \frac{-2b_0}{m_2V_0}\left(\bar{V}_1 - \bar{V}_2\right) \end{bmatrix} \begin{bmatrix} \hat{F}_1 \\ \hat{V}_1 \\ \hat{F}_2 \\ \hat{V}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} \hat{F}(t)$$

Finally,

$$F_1(t) = \bar{F}_1 + \hat{F}_1(t)$$

$$V_1(t) = \bar{V}_1 + \hat{V}_1(t)$$

$$F_2(t) = \bar{F}_2 + \hat{F}_2(t)$$

$$V_2(t) = \bar{V}_2 + \hat{V}_2(t)$$