School of Mechatronic Systems Engineering
Simon Fraser University
MSE483/782 Midterm Exam I

February 1, 2019 (Duration: 1.5 hours)
**Please read the following before signing your name**

- The exam is closed-book. A 2-page formula sheet is permitted but must not contain any solved problems. The formula sheet has to be returned with the questions.
- Questions have equal weights.
  **Please clearly specify any assumptions you make and write legibly. You may lose marks if your work is not clear.**

Name: Student I.D. Number:

1) The electromechanical dynamics of a DC motor is given by

$$LJ\frac{d^3\theta}{dt^3} + (Lb + RJ)\frac{d^2\theta}{dt^2} + (Rb + k_m^2)\frac{d\theta}{dt} = k_m v - R\tau_L$$

where $\theta$ is the motor angular displacement, $L$ is the motor winding inductance, $J$ is the rotor inertia, $b$ is the coefficient of viscous friction, $R$ is the motor winding resistance, $k_m$ is the motor constant, $v$ is the input voltage applied to the motor, and $\tau_L$ is the load torque. Obtain a state space representation of the motor dynamics.

Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = \ddot{\theta} \Rightarrow \dot{x}_1 = x_2$ ; $\dot{x}_2 = x_3$

$$\dot{x}_3 = \dddot{\theta} = \frac{1}{LJ}\left(k_m v - R\tau_L - (Lb+RJ)\ddot{\theta} - (Rb+k_m^2)\dot{\theta}\right)$$

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$$= \frac{1}{LJ}\left(k_m v - R\tau_L - (Lb+RJ)x_3 - (R_b+k_m^2)x_2\right)$$

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -\frac{R_b+k_m^2}{LJ} & -\frac{Lb+RJ}{LJ}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{k_m}{LJ} & \frac{-R}{LJ}\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}v\\ \tau_L\end{bmatrix}}_{u}$$

define as

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2) For the following system, obtain an approximate output $y$ in response to the input $u = 0.01u_s(t)$, where $u_s(t)$ is the unit step function. Assume that the system is resting at its equilibrium point before the step input is applied.

$$\begin{aligned} \dot{x}_1 &= (1+sin(x_2))x_1 + x_1 sin(x_2) + u \\ \dot{x}_2 &= -x_1 cos(x_1) + x_2 sin(x_1)) - u \\ y &= sin(x_1) \end{aligned}$$

$\bar{x}_1 = 0,\ \bar{x}_2 = 0,\ \bar{u} = 0$ is an equilibrium point since it satisfies the equation

Linearized system : about $\bar{x}_1 = 0$, $\bar{x}_2 = 0$, $\bar{u} = 0$ $\longrightarrow$

$$\dot{\tilde{x}}_1 = \tilde{x}_1 + \tilde{u}$$
$$\dot{\tilde{x}}_2 = -\tilde{x}_1 - \tilde{u}$$
$$\tilde{y} = \tilde{x}_1$$

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Note $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{u} = u$

$$\dot{\tilde{x}} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tilde{u}$$

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$$\tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}$$

$$y(s) = C(sI-A)^{-1}B\,U(s) \longrightarrow Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{0.01}{s}$$

$$\begin{bmatrix} s-1 & 0 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2-s} \begin{bmatrix} s & -1 \\ 0 & s-1 \end{bmatrix}^T = \frac{1}{s(s-1)} \begin{bmatrix} s & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\therefore Y(s) = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ -1 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{s(s-1)} \frac{0.01}{s} = \frac{0.01 \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ -1-s+1 \end{bmatrix}}{s^2(s-1)} = \frac{\cancel{s} \times 0.01}{s^{\cancel{2}}(s-1)}$$

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$$\frac{.01}{s(s-1)} = Y(s) = \frac{a}{s} + \frac{b}{s-1} \longrightarrow a = -0.01,\ b = 0.01$$

$$\therefore Y(s) = \frac{-0.01}{s} + \frac{0.01}{s-1} \longrightarrow y(t) = 0.01(-1+e^t)\,u_s(t)$$

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of the following system

its

3) Determine the matrix exponential and hence ~~the~~ zero input response ~~of state vector $x(t) = [x_1\ x_2]^T$~~ for ~~the following system~~ when the initial conditions are $x(0) = [2\ 3]^T$ of the system

$$\dot{x}_1 = -2x_1 + u$$
$$\dot{x}_2 = x_1 - x_2$$

$$\dot{x} = \underbrace{\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$e^{At} \longleftrightarrow (sI-A)^{-1} \rightsquigarrow (sI-A) = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+1 & 1 \\ 0 & s+2 \end{bmatrix}^T = \frac{1}{(s+2)(s+1)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} \end{bmatrix}$$

$$\therefore\ e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

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$$X(t) = e^{At} X(0) + \cancel{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau} \to 0$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ 2e^{-t} - 2e^{-2t} + 3e^{-t} \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{bmatrix} \longleftarrow$$ zero-input response

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4) Transform the following system into the diagonal form representation

$$\dot{x} = \begin{bmatrix}-2 & 1\\ 2 & -3\end{bmatrix} \underset{\downarrow\, x}{} + \begin{bmatrix}0\\ 1\end{bmatrix} u$$

$$\text{eig}(A) \longrightarrow \det(\lambda I - A) = 0 \longrightarrow \begin{vmatrix}\lambda+2 & -1\\ -2 & \lambda+3\end{vmatrix} = 0$$

$$\lambda^2 + 5\lambda + 6 - 2 = 0 \longrightarrow \lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda+1)(\lambda+4) = 0 \longrightarrow \lambda_1 = -1,\ \lambda_2 = -4 \qquad 5$$

$$Av = \lambda v \longrightarrow \begin{cases} \begin{bmatrix}-2 & 1\\ 2 & -3\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = -1\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} \longrightarrow \begin{cases}-2v_{11} + v_{12} = -v_{11}\\ 2v_{11} - 3v_{12} = -v_{12}\end{cases} \\[2ex] \begin{bmatrix}-2 & 1\\ 2 & -3\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = -4\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} \longrightarrow \begin{cases}-2v_{21} + v_{22} = -4v_2\\ 2v_{21} - 3v_{22} = -4v_{2i}\end{cases}\end{cases}$$

$$\longrightarrow \begin{cases} v_1 : \begin{cases} v_{11} = v_{12}\\ 2v_{11} = 2v_{12}\end{cases} \longrightarrow v_1 = \begin{bmatrix}1\\ 1\end{bmatrix} \quad 5 \\[2ex] v_2 : \begin{cases} v_{22} = -2v_{21}\\ -v_{22} = 2v_{21}\end{cases} \longrightarrow v_2 = \begin{bmatrix}1\\ -2\end{bmatrix} \quad 5\end{cases} \Longrightarrow T = \begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix} \begin{matrix}\\ \\ \downarrow \quad \downarrow\\ v_1 \quad v_2\end{matrix}$$

$$T^{-1} = \frac{1}{-2-1}\begin{bmatrix}-2 & -1\\ -1 & 1\end{bmatrix}^T = \frac{-1}{3}\begin{bmatrix}-2 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}2/3 & 1/3\\ 1/3 & -1/3\end{bmatrix}$$

$$x = Tz \Longrightarrow T\dot{z} = ATz + Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\dot{z} = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} z + \begin{bmatrix}2/3 & 1/3\\ 1/3 & -1/3\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} u$$

diagonalized form

$$\dot{z} = \begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix} z + \begin{bmatrix}1/3\\ -1/3\end{bmatrix} u \qquad 10$$