School of Mechatronic Systems Engineering
Simon Fraser University
MSE483/782 Midterm Exam

5 questions
$5 \times 20 = 100$

February 24, 2020 (Duration: 2 hours)
Please read the following before signing your name

- The exam is closed-book. A 2-page formula sheet is permitted but must not contain any solved problems. The formula sheet will be checked during the exam and can be taken out.
- Questions have equal weights.
- **Please clearly specify any assumptions you make and write legibly. You may lose marks if your work is not clear.**

Name: Solution Student I.D. Number:

1) A vehicular suspension system can be modelled as a mass suspended by a *leaf spring* with a cubic nonlinearity and a nonlinear damper. The dynamics are given by

$$m\ddot{x} = -k_1 x - k_2 x^3 - d(1 - x^2)\dot{x}$$

where x is the suspension's displacement from equilibrium, k_1, k_2 are coefficients of the leaf spring, and d is the damping coefficient. Using Jacobian linearization, obtain a linear state space model of the system when the output of interest is $\dot{x}$.

Let $x_1 = x$, $x_2 = \dot{x} \implies \dot{x}_1 = x_2$

$$\dot{x}_2 = \ddot{x} = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x^3 - \frac{d}{m}(1 - x_1^2)x_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 - \frac{d}{m}(1 - x_1^2)x_2 \end{bmatrix} = f(x)$$ 10

Linearizing about $x_1 = 0$, $x_2 = 0$ (equilibrium point, $\dot{x}_1, \dot{x}_2 = 0$), we have

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} - \frac{d}{m}(-2x_1)x_2 - \frac{3k_2}{m}x_1^2 & -\frac{d}{m}(1 - x_1^2) \end{bmatrix}_{x_1, x_2 = 0}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & -\frac{d}{m} \end{bmatrix}$$

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$$\implies \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & -\frac{d}{m} \end{bmatrix}}_{A} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \quad y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

2) A harmonic oscillator can be described by the following state equations

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \;\; 1]x\end{aligned} \qquad (1)$$

Find the state transition matrix and output due to a unit step input.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow e^{At} = ? \longrightarrow$ state transition matrix

$(sI - A)^{-1} = ? \longrightarrow \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \dfrac{1}{s^2+1} \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{T}$

$(sI-A)^{-1} = \dfrac{1}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \Rightarrow$

$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \leftarrow 10$

$Y(s) = c\,(sI-A)^{-1} b\, U(s) = [0 \;\; 1] \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s}$

$= [0 \;\; 1] \begin{bmatrix} \frac{1}{s^2+1} \\ \frac{s}{s^2+1} \end{bmatrix} \frac{1}{s} = \dfrac{\cancel{s}}{s^2+1} \dfrac{1}{\cancel{s}} = \dfrac{1}{s^2+1}$

$\therefore\; y(t) = \sin t \leftarrow 10$

3 4) An n-dimensional state space system is given by $\dot{x} = Ax + Bu$, $y = Cx + Du$, where x is the state vector, u is the input vector, and y is the output vector. Show that the transfer function of the system from input to output is not changed by a linear non-singular transformation of the state.

$x = Tz$ wher T is nonsingular

Transfer function of the original system is

$$H(s) = C(sI - A)^{-1} B + D$$

For system in z, we have

$$T\dot{z} = ATz + Bu \qquad \Longrightarrow \qquad \begin{cases} \dot{z} = \underbrace{T^{-1}AT}_{\hat{A}} z + \underbrace{T^{-1}B}_{\hat{B}} u \\ y = \underbrace{CT}_{\hat{C}} z + Du \end{cases}$$

$$y = CTz + Du$$

Transfer function for new system

$$G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D = CT(sI - T^{-1}AT)^{-1} T^{-1}B + D$$

$$G(s) = C\left[T(sI - T^{-1}AT)T^{-1}\right]^{-1} B + D$$

$$= C\left[sTT^{-1} - TT^{-1}ATT^{-1}\right]^{-1} B + D$$

$$\therefore G(s) = C(sI - A)^{-1} B + D$$

$$\Longrightarrow G(s) = H(s)$$

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3) Transform the following system into the diagonal form representation and comment on controllability of the system

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B} u$$

Eigenvalues/eigenvectors :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Longrightarrow \lambda I - A = \begin{bmatrix} s-1 & 0 & -1 \\ 0 & s-1 & 0 \\ 0 & 0 & s-2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \Longrightarrow \lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \lambda_1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Longrightarrow \begin{cases} v_{11} + v_{13} = v_{11} \\ v_{12} = v_{12} \\ 2v_{13} = v_{13} \rightarrow v_{13} = 0 \end{cases} \Longrightarrow v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ 0 \end{bmatrix}$$

$\lambda_1, \lambda_2 = 1$ $\rightarrow$ $\therefore \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\leftarrow$ v_{11}, v_{12} any values $\rightarrow$ 2 independent vectors can be chosen

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \lambda_3 \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 2v_{31} \\ 2v_{32} \\ 2v_{33} \end{bmatrix}$$

$$\rightarrow \begin{cases} v_{31} + v_{33} = 2v_{31} \longrightarrow v_{33} = v_{31} \\ v_{32} = 2v_{32} \longrightarrow v_{32} = 0 \\ 2v_{33} = 2v_{33} \longrightarrow \text{any value} \end{cases} \Longrightarrow v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

5 $\quad \lambda_3 = 2$

$$\Longrightarrow T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow x = Tz \Longrightarrow \dot{x} = T\dot{z}$$

$$Ax + Bu = T\dot{z} \Longrightarrow ATz + Bu = T\dot{z} \Longrightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$T^{-1} = \frac{1}{\det(T)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T = \frac{1}{1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow T^{-1}B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow \dot{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \rightsquigarrow$$ System uncontrollable!

10 $\quad$ Not controllable $\leftarrow$ 2 modes $\leftarrow$

4 5) The A matrix in the state space system $\dot{x} = Ax + bu$, $y = Cx$, is given by

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Verify if a column vector b can be found such that the system is controllable?

$P = [b \;\; Ab] \longleftarrow$ Controllability matrix

$$= \begin{bmatrix} b_1 & 2b_1 + b_2 \\ b_2 & 2b_2 \end{bmatrix}$$

$$\det(P) = 2b_1 b_2 - b_2(2b_1 + b_2)$$

$$= \cancel{2b_1 b_2} - \cancel{2b_1 b_2} - b_2^2 \neq 0$$

As long as $b_2 \neq 0$ the system is controllable

Thus $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\neq 0$ $\forall\, b_1, b_2$ $(b_2 \neq 0)$

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