School of Mechatronic Systems Engineering
Simon Fraser University
MSE483/782 Midterm Exam II

March 8, 2019 (Duration: 1.5 hours)
Please read the following before signing your name

- **The exam is closed-book. A 2-page formula sheet is permitted but must not contain any solved problems. The formula sheet may be taken out after being checked during/after the exam.**
- **Questions have equal weights.**
- **Please clearly specify any assumptions you make and write legibly. You may lose marks if your work is not clear.**

Name: Student I.D. Number:

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1) Consider the following system

$$\begin{aligned}\dot{x}_1 &= 2x_1 + u\\ \dot{x}_2 &= 3x_2 + u\end{aligned} \quad (1)$$

with initial conditions $x_1(0) = 1,\ x_2(0) = 1$. Indicate if the above initial state can be steered to zero in 1 second and obtain an appropriate input for doing so.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow P = [B \ \ AB] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$\det(P) = 3 - 2 = 1 \neq 0$

$\therefore$ Controllable! 16

$$u = -B^T e^{A^T(t_0 - t)} W^{-1}(\underset{0}{t_0}, \underset{1}{t_f})\, x_0$$

$$\text{where } W(\underset{0}{t_0}, \underset{1}{t_f}) = \int_0^1 e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)}\, d\tau$$

$$= \int_0^1 e^{-A\tau} BB^T e^{-A^T\tau}\, d\tau$$

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$$e^{-A^T\tau} = e^{-A\tau} = \begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{-3\tau} \end{bmatrix} \longrightarrow B^T e^{-A^T\tau} = [1 \ \ 1]\begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{-3\tau} \end{bmatrix} = [e^{-2\tau} \ \ e^{-3\tau}]$$

$$\therefore W(0,1) = \int_0^1 \begin{bmatrix} e^{-2\tau} \\ e^{-3\tau} \end{bmatrix} [e^{-2\tau} \ \ e^{-3\tau}]\, d\tau = \int_0^1 \begin{bmatrix} e^{-4\tau} & e^{-5\tau} \\ e^{-5\tau} & e^{-6\tau} \end{bmatrix} d\tau$$

$$\therefore W(0,1) = \left[\begin{matrix} \cancel{-\tfrac{1}{4}e^{-4\tau} \quad -\tfrac{1}{5}e^{-5\tau}} \\ -\tfrac{1}{5}e^{-5\tau} \quad -\tfrac{1}{6}e^{-6\tau} \end{matrix}\right]_0^1 \simeq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/4 & 1/5 \\ 1/5 & 1/6 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/5 \\ 1/5 & 1/6 \end{bmatrix}$$

$$W^{-1}(0,1) = \frac{1}{\frac{1}{24} - \frac{1}{25}} \begin{bmatrix} 1/6 & -1/5 \\ -1/5 & 1/4 \end{bmatrix} = \begin{bmatrix} \frac{25\times 24}{6} & -\frac{25\times 24}{5} \\ -\frac{25\times 24}{5} & \frac{25\times 24}{4} \end{bmatrix} = \begin{bmatrix} 100 & -120 \\ -120 & 150 \end{bmatrix}$$

correct result 13

$$\Rightarrow u = -[1 \ \ 1]\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 100 & -120 \\ -120 & 150 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -[1 \ \ 1]\begin{bmatrix} -20e^{-2t} \\ 30e^{-3t} \end{bmatrix} = \underline{20e^{-2t} - 30e^{-3t}}$$

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2) For the system below, verify if there are initial states that can, or cannot, be controlled to zero in finite time. If that is the case, express the system into controllable and uncontrollable parts and justify the results.

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 5 \\ 8 & 4 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}_{B} u$$

$$P = \begin{array}{cc} B & AB \\ \downarrow & \downarrow \end{array} \begin{bmatrix} -2 & 8 \\ 2 & -8 \end{bmatrix} \longrightarrow \det(P) = 0 \longrightarrow \text{Uncontrollable!}$$

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$$P x_n = 0 \longrightarrow \begin{bmatrix} -2 & 8 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} x_{n1} \\ x_{n2} \end{bmatrix} = 0 \longrightarrow -2x_{n1} + 8x_{n2} = 0 \rightarrow x_{n1} = +4x_{n2}$$

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$\Longrightarrow$ Any initial state of the form $\boxed{x_0 = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$ cannot be controlled to zero in finite time.

15 = 8 + 8

$$\text{Let } T = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$$

any vector that is independent of 1'st column

$$x = Tz \longrightarrow \dot{x} = T\dot{z} = Ax + Bu \longrightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$T^{-1} = \frac{1}{-2}\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$\therefore \dot{z} = \begin{bmatrix} -4 & 4 \\ 0 & 9 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

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Thus 2nd equation shows that z_2 cannot be controlled (disconnected from u)

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$$\cancel{\Longrightarrow x_0 = \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightsquigarrow z_0 = T^{-1}x_0 = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \alpha \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} \frac{1}{2} \\ 5 \end{bmatrix}}$$

3) Consider an n-dimensional state space system given by $\dot{x} = Ax$ with an unknown initial condition $x(0) = x_0$. The m-dimensional measured output vector is given by $y = Cx$. Assume that we can measure $y(0)$ and its time derivatives at $t = 0$, i.e., $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, ..., $y^{n-1}(0)$. Obtain an equation and condition(s) indicating how the initial condition $x(0)$, and hence $x(t)$, can be determined uniquely.

$$\dot{x} = Ax \rightsquigarrow y = Cx \longrightarrow \dot{y} = C\dot{x} = CAx$$

$$\ddot{y} = CA\dot{x} = CA^2x$$

$$\vdots$$

$$y^{(n-1)} = CA^{n-1}x$$

$$\Longrightarrow \quad y(0) = CAx_0$$

$$\dot{y}(0) = CA^2x_0$$

$$\vdots$$

$$y^{(n-1)}(0) = CA^{n-1}x_0$$

$$\Longrightarrow \underbrace{\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix}}_{(m\times n)\times 1 \text{ vector}} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q,\ (n\times m)\times n \text{ matrix}} \underbrace{x_0}_{n\times 1 \text{ vector}}$$

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If the rank of matrix Q is n, we can uniquely obtain x_0 for any $y(0), \dot{y}(0), \ldots y^{(n-1)}(0)$ and vice-versa

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Once x_0 is known we can use the following eq. to obtain $x(t)$:

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$$x(t) = e^{At}x_0 + \cancel{\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau}^{\,0}$$

$$x(t) = e^{At}x_0$$