Phase transition in a simple food web

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Abstract

This is a simple mean field (in the sense that lack spatial structure) model for a food web of one prey species and one predator species. I designed it having in mind the principal properties of the one in the draft but in a simplified version, in particular the evolution part is absent. The model shows a phase transition in the predator (probability) density π at a critical value λ_c of the resources generation rate λ.

1 The model

To build the model I refer to the properties of the one of the draft, in particular I think the resources distribution among preys and predators and their role in the reproductive and death rate is a relevant feature to observe the emergence of the transition with respect to the parameter λ which controls the growth rate of the resources. Therefore I keep track of the distribution of resources present 'inside' the preys r_p and 'inside' the predators r_π as independent variable with respect to the density of preys p and predators π.

The dynamical variables of the problem are

- r_f, the density of free resources
- r_p, the density of resources collected by preys
- r_π, the density of the resources collected by predators
- p, the density of preys
- π, the density predators

We have a free parameter λ, which is the constant *growth* rate at which new free resources are generated, and fixed parameters which are the *voracity* v_p and v_π, which regulates how good prey and predators are able to capture respectively free resources and preys and the *metabolisms* m_p and m_π which controls both the reproduction and death rates of prey and predators. The master equation reads

$$\dot{r}_f = \lambda - v_p r_f p \tag{1}$$

$$\dot{r}_p = r_f v_p p - m_p r_p - v_\pi r_p \pi \tag{2}$$

$$\dot{r}_\pi = v_\pi r_p \pi - m_\pi r_\pi \tag{3}$$

$$\dot{p} = r_p m_p - m_p p - v_\pi \pi p \tag{4}$$

$$\dot{\pi} = r_\pi m_\pi - m_\pi \pi. \tag{5}$$

The free resources r_f increase with constant rate λ and get consumed with rate $v_p r_f p$ which accounts for how good are the preys to collect the free resources and how abundant are both, we obtain Eq. (1). The resources of the preys r_p increase consistently with the decrease of the free resources and they are depleted by the metabolism of the preys $m_p(r_p/p)p = m_p r_p$ and by the amount that the predators take from the preys $v_\pi \pi (r_p/p)p = v_\pi \pi r_p$ and one gets Eq. (2). Similar reasoning brings to Eq. (3). The density of preys is increased or decreased by their metabolism dependently on the abundance of resources they collected, they reproduce if there is more then one resource per individual and they die if there is less, the term $(r_p/p - 1)m_p p = m_p r_p - m_p p$ accounts for this. Moreover the density of preys is reduced by predation which is accounted for with the term $v_\pi \pi$, Eq. (4) follows. Equation (5) is derived with the same arguments.

2 Stationary solution

If we look for a stationary solution we obtain from Eq. (1) $r_f^s = \lambda / v_p p$, from Eq. (5) $r_\pi^s = \pi$ and then from Eq. (3) $r_p^s = m_\pi / v_\pi$. We are left with

$$0 = \frac{m_\pi m_p}{v_\pi} - m_p p - v_\pi \pi p \tag{6}$$

$$0 = \lambda - \frac{m_p m_\pi}{v_\pi} - m_\pi \pi. \tag{7}$$

From the first one we obtain

$$\pi = \frac{m_\pi m_p}{v_\pi^2 p} - \frac{m_p}{v_\pi}, \tag{8}$$

and from the second one

$$p^s = \frac{m_\pi^2 m_p}{\lambda v_\pi^2}. \tag{9}$$

Therefore, for given value of the fixed parameters, we have a stable finite solution for the density of the preys for every λ. For the predators this is not true and we can see that, substituting now Eq. (9) in the solution for π we obtain

$$\pi^s = \frac{1}{m_\pi}\Big(\lambda - \frac{m_p m_\pi}{v_\pi}\Big). \tag{10}$$

A meaningful finite stationary solution for the predator density exist only if $\lambda > \lambda_c$, with the critical value given by

$$\lambda_c = \frac{m_p m_\pi}{v_\pi}. \tag{11}$$

References