Optimization methods of image inpainting

Name: Shengnan Miao RIN: 661969553

Abstract

1. Problems

Figure: Left: A contaminated image. Right: Ground truth.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be an image contaminated in a domain Ω^c. Reconstruct the image based on a

the correct intensity of f on Ω by solving the following convex optimization

$$\min_u R(u), \text{ s.t. } u(x) = f(x), \ \forall x \in \Omega$$

where R can be chosen as wavelet tight frame transformation or total variation

2. Methodology

2.1 Model

Since we can choose different objective functions as regularization $R(u)$ and they can have different performance, here I list 3 methods.

1. Wavelet tight frame transformation: build a sparse vector Wu as $R(u)$
   Idea: To smooth the contaminated area of the image, we want to find a sparse vector as Wu under certain overcomplete basis, in this assumption, we need do $\min_u \|Wu\|_0$, s.t. $Au = b$.
   Since computation of $\min_u \|Wu\|_0$ is complicated, we can relax the problem to
   $\min_u \|Wu\|_1$, s.t. $Au = b$,
   here W is wavelet tight frame transformation, u is the image vector, whose magnitude represents an image. $Au = b$ exhibits the constraint.

2. Total variation: 1-norm of Gradient defined as $R(u)$
   Idea: To smooth the contaminated area of the image, we can minimize the total variation of the image. Here we define the total variation as the integration of $\|\nabla u\|_1$ on the whole image.

$\min_u \int_{\Omega+\Omega_c} \|\nabla u(x)\|_1 dx$, s.t. $u(x) = f(x),\ \forall x \in \Omega$
here $u(x)$ is the image function $\mathbb{R}^2 \to \mathbb{R}$, whose value represents an image.

3. Total variation: 2-norm of Gradient defined as $R(u)$
Idea: To smooth the contaminated area of the image, we can minimize the total variation of the image. Here we define the total variation as the integration of $\|\nabla u\|_2{}^2$ on the whole image.
$\min_u \int_{\Omega+\Omega_c} \|\nabla u(x)\|_2{}^2 dx$, s.t. $u(x) = f(x),\ \forall x \in \Omega$
here $u(x)$ is the image function $\mathbb{R}^2 \to \mathbb{R}$, whose value represents an image.

2.2 Algorithm

Method 1: Wavelet tight frame transformation

In this method, I used Augmented Lagrangian method and Alternative directional minimization method (ADMM)[1][2] for the outer loop, steepest gradient and proximal gradient descent[1][3] in the inner loop. The details of derivation of algorithm for method 1 is as following:

First, variable splitting: introduce an auxiliary variable $Q = Wx$, define a problem: $\min_{x,Q} \|Q\|_1$ s.t. $Ax = b$ & $Wx = Q$.

$L_\mu(x, Q; v_1, v_2) = \|Q\|_1 + < v_1, b - Ax > + \frac{\mu}{2}\|Ax - b\|_2{}^2 + + < v_2, Q - Wx > + \frac{\mu}{2}\|Wx - Q\|_2{}^2;$

Outer loop: Use ADMM method: $\begin{cases} x^{k+1} \leftarrow arg \min_x L_\mu\left(x; Q^k; {v_1}^k, {v_2}^k\right) \\ Q^{k+1} \leftarrow arg \min_Q L_\mu\left(x^{k+1}; Q; {v_1}^k, {v_2}^k\right) \\ {v_1}^{k+1} = {v_1}^k + \rho(b - Ax^{k+1}) \\ {v_2}^{k+1} = {v_2}^k + \rho(Q^{k+1} - Wx^{k+1}) \end{cases}$

Inner loop-- x-sub problem: Apply iterative method to solve $Hx = C$, where $H = \mu A^T A + \mu I$, $C = A^T({v_1}^k + \mu b) + W^T({v_2}^k + \mu Q^k)$. So we need to do $\min_x \frac{1}{2} x^T Hx - x^T C$, then I apply steepest gradient descent method.

Steepest gradient descent: $\begin{cases} r^k = Hx^k - C \\ \alpha^k = \left\|r^k\right\|_2^2 / (r^k)^T Hr^k \\ x^{k+1} = x^k - \alpha^k r^k \end{cases}$

Q-sub problem: $Q^{k+1} = arg\min_\alpha \|Q\|_1 + \frac{\mu}{2}\left\|Q - (Wx^{k+1} - \frac{1}{\mu}v_2^k)\right\|_2^2 = shrink(Wx^{k+1} - \frac{1}{\mu}v_2^k, \frac{1}{\mu})$.

***Pseudo code:**

Step 0. Set $x^0, Q^0, {v_1}^0, {v_2}^0$ as 0 vector. Choose μ, ρ.

(Outer loop) for L=0 to 100

Step 1. While $\left\|r^k\right\|_\infty > tol = 1e^{-6}$ (Inner loop)

$r^{k,L} = Hx^{k,L} - C,$

$\alpha^{k,L} = \|r^{k,L}\|_2^2 / (r^{k,L})^T H r^{k,L}$,

$x^{k+1} = x^{k,L} - \alpha^{k,L} r^{k,L}$.

End

$x^{1,L+1} = x^{k,L}$

Step 2. $Q^{L+1} = shrink(Wx^{1,L+1} - \frac{1}{\mu} v_2^L, \frac{1}{\mu})$.

Step 3. $v_1{}^{L+1} = v_1{}^L + \rho(b - Ax^{L+1})$;

$v_2{}^{L+1} = v_2{}^L + \rho(Q^{L+1} - Wx^{L+1})$;

end while

Method 2: Total variation—1 norm of gradient

In this method, I used Augmented Lagrangian method and Alternative directional minimization method (ADMM) for the outer loop, steepest gradient and proximal gradient in the inner loop. The details of derivation of algorithm for method 2 is as following:

First, variable splitting: introduce an auxiliary variable $Q = (D_1 - D_2)x$, define a problem: $\min_{x,Q} \|Q\|_1$ s.t. $Ax = b$ & $(D_1 - D_2)x = Q$. Where x is a 256×256 row, 1 column-image vector, Q is a reshaped vector consists x-gradient and y-gradient generated by the Kernel[4] $[1 \quad -1]$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, convoluted with the image matrix.

$L_\mu(x, Q; v_1, v_2) = \|Q\|_1 + < v_1, b - Ax > + \frac{\mu}{2}\|Ax - b\|_2{}^2 + + < v_2, Q - (D_1 - D_2)x > + \frac{\mu}{2}\|(D_1 - D_2)x - Q\|_2{}^2$;

Outer loop: Use ADMM method: $\begin{cases} x^{k+1} \leftarrow arg\min_x L_\mu(x; Q^k; v_1{}^k, v_2{}^k) \\ Q^{k+1} \leftarrow arg\min_Q L_\mu(x^{k+1}; Q; v_1{}^k, v_2{}^k) \\ v_1{}^{k+1} = v_1{}^k + \rho(b - Ax^{k+1}) \\ v_2{}^{k+1} = v_2{}^k + \rho(Q^{k+1} - (D_1 - D_2)x^{k+1}) \end{cases}$

Inner loop-- x-sub problem: Apply iterative method to solve $Hx = C$, where $H = \mu A^T A + \mu(D_1 - D_2)^T(D_1 - D_2)$, $C = A^T(v_1{}^k + \mu b) + (D_1 - D_2)^T(v_2{}^k + \mu Q^k)$. So we need to do $\min_x \frac{1}{2}x^T H x - x^T C$, then I apply steepest gradient descent method.

Steepest gradient descent: $\begin{cases} r^k = Hx^k - C \\ \alpha^k = \|r^k\|_2^2 / (r^k)^T H r^k \\ x^{k+1} = x^k - \alpha^k r^k \end{cases}$

Q-sub problem: $Q^{k+1} = arg\min_\alpha \|Q\|_1 + \frac{\mu}{2}\left\|Q - ((D_1 - D_2)x^{k+1} - \frac{1}{\mu}v_2^k)\right\|_2^2 = shrink((D_1 - D_2)x^{k+1} - \frac{1}{\mu}v_2^k, \frac{1}{\mu})$.

***Pseudo code:**

Step 0. Set $x^0, Q^0, {v_1}^0, {v_2}^0$ as 0 vector. Choose μ, ρ.

(Outer loop) for L=0 to 20

Step 1. While $\|r^k\|_\infty > tol = 1e^{-6}$ (Inner loop)

$r^{k,L} = Hx^{k,L} - C,$

$\alpha^{k,L} = \|r^{k,L}\|_2^2 / \left(r^{k,L}\right)^T Hr^{k,L},$

$x^{k+1} = x^{k,L} - \alpha^{k,L} r^{k,L}.$

End

$x^{1,L+1} = x^{k,L}$

Step 2. $Q^{L+1} = shrink((D_1 - D_2)x^{1,L+1} - \frac{1}{\mu} v_2^L, \frac{1}{\mu}).$

Step 3. ${v_1}^{L+1} = {v_1}^L + \rho(b - Ax^{L+1});$

${v_2}^{L+1} = {v_2}^L + \rho(Q^{L+1} - (D_1 - D_2)x^{L+1});$

end while

Method 3: Total variation—2 norm of gradient

In this method, I used Augmented Lagrangian method and Alternative directional minimization method (ADMM) for the outer loop and steepest gradient in the inner loop. The details of derivation of algorithm for method 3 is as following:

First, variable splitting: introduce an auxiliary variable $Q = (D_1 - D_2)x$, define a problem: $\min_{x,Q} \|Q\|_2{}^2$ s.t. $Ax = b$ & $(D_1 - D_2)x = Q$. Where x is a 256×256 row, 1 column-image vector, Q is a reshaped vector consists x-gradient and y-gradient generated by the Kernel $[1 \quad -1]$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, convoluted with the image matrix.

$L_\mu(x, Q; v_1, v_2) = \|Q\|_2{}^2 + < v_1, b - Ax > + \frac{\mu}{2}\|Ax - b\|_2{}^2 + + < v_2, Q - (D_1 - D_2)x > + \frac{\mu}{2}\|(D_1 - D_2)x - Q\|_2{}^2;$

Outer loop: Use ADMM method: $\begin{cases} x^{k+1} \leftarrow arg \min\limits_x L_\mu\left(x; Q^k; {v_1}^k, {v_2}^k\right) \\ Q^{k+1} \leftarrow arg \min\limits_Q L_\mu\left(x^{k+1}; Q; {v_1}^k, {v_2}^k\right) \\ {v_1}^{k+1} = {v_1}^k + \rho(b - Ax^{k+1}) \\ {v_2}^{k+1} = {v_2}^k + \rho(Q^{k+1} - (D_1 - D_2)x^{k+1}) \end{cases}$

Inner loop-- x-sub problem: Apply iterative method to solve $Hx = C$, where $H = \mu A^T A + \mu(D_1 - D_2)^T(D_1 - D_2)$, $C = A^T({v_1}^k + \mu b) + (D_1 - D_2)^T({v_2}^k + \mu Q^k)$. So we need to do $\min_x \frac{1}{2}x^T Hx - x^T C$, then I apply steepest gradient descent method.

Steepest gradient descent: $\begin{cases} r^k = Hx^k - C \\ \alpha^k = \|r^k\|_2^2/(r^k)^T Hr^k \\ x^{k+1} = x^k - \alpha^k r^k \end{cases}$

Q-sub problem: $Q^{k+1} = arg\min_{\alpha}\|Q\|_2{}^2 + \frac{\mu}{2}\left\|Q - ((D_1 - D_2)x^{k+1} - \frac{1}{\mu}v_2^k)\right\|_2^2 = \frac{2}{2+\mu}(\mu(D_1 - D_2)x^{k+1} - v_2^k)$

***Pseudo code:**

Step 0. Set $x^0, Q^0, {v_1}^0, {v_2}^0$ as 0 vector. Choose μ, ρ.

(Outer loop) for L=0 to 5

Step 1. While $\|r^k\|_\infty > tol = 1e^{-6}$ (Inner loop)

$r^{k,L} = Hx^{k,L} - C,$

$\alpha^{k,L} = \|r^{k,L}\|_2^2/(r^{k,L})^T Hr^{k,L},$

$x^{k+1} = x^{k,L} - \alpha^{k,L} r^{k,L}.$

End

$x^{1,L+1} = x^{k,L}$

Step 2. $Q^{L+1} = \frac{2}{2+\mu}(\mu(D_1 - D_2)x^{L+1} - v_2^L).$

Step 3. ${v_1}^{L+1} = {v_1}^L + \rho(b - Ax^{L+1});$

${v_2}^{L+1} = {v_2}^L + \rho(Q^{L+1} - (D_1 - D_2)x^{L+1});$

end while

3 Results

Now use our model and algorithm to do the inpainting, these are the original contaminated images. The performance of each method is as followings.

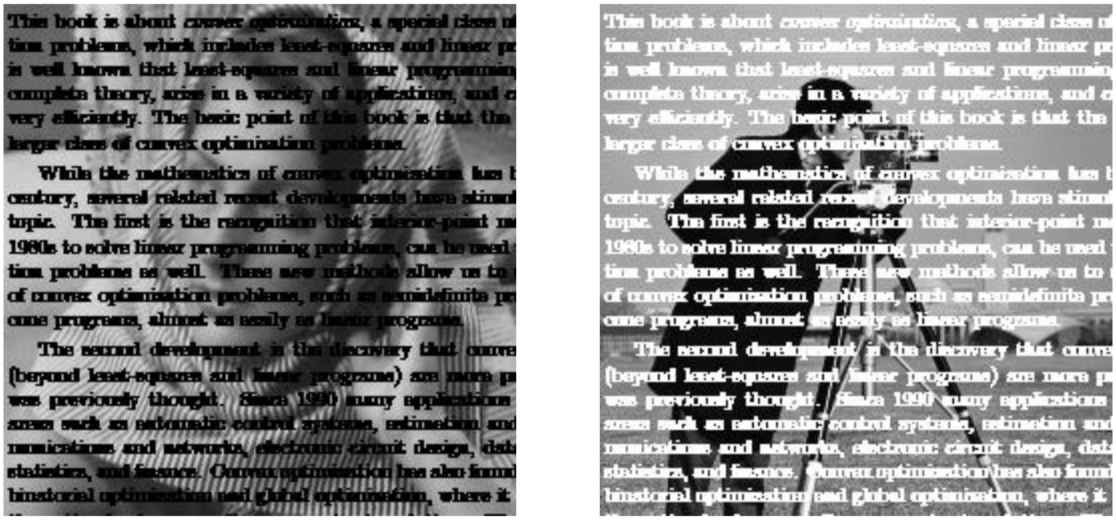

Figure 2: Left:Barbara, Right: Cameraman.

Method 1: Wavelet tight frame transformation

The computation time for these figures are 128.504828s and 108.504828s.

Method 2: Total variation—1 norm of gradient

The computation time for these figures are 78.504828s and 70.365029s.

Method 3: Total variation—2 norm of gradient

The computation time for these figures are 30.866046s and 24.085837s.

4 Observation and Conclusions

From the results of three methods, we can see that:

1. For the method 1, which uses wavelet tight frame transformation to build a sparse vector Wu as $R(u)$:
   Advantages: it will not lose the sharpness of the image.
   Disadvantages: the contamination was not totally removed; the iteration number is large and thus computation time is long comparing to other methods.

To some degree, the performance of method 1 depends on the choice of initialization and iteration number.

2. For the method 2, which defines total variation as 1-norm of Gradient and use it as $R(u)$:
   Advantages: the contamination was removed cleanly; the image is smoother and the computation is fast.
   Disadvantages: it loses some sharpness of the image.
   The performance of method 2 is not sensitively dependent on the choice of initialization and iteration number.
3. For the method 3, which defines total variation as 1-norm of Gradient and use its square as $R(u)$:
   Advantages: the contamination was removed cleanly; the loss of sharpness is not obvious, and the computation is very efficient.
   The performance of method 3 is stable and not sensitively dependent on the choice of initialization and iteration number.

Conclusion: we can do good inpainting image if we choose suitable regularization $R(u)$ as objective functions and use the knowledge of convex optimization to design the algorithm.

References

[1] Mokhtar S. Bazaraa, Hanif D. Sherali, C. M. Shetty--*Nonlinear Programming: Theory and Algorithm*

[2] Lecture note 6 from "*Introduction of optimization*" by Prof. Rongjie Lai.

[3] Lecture note 3 from "*Introduction of optimization*" by Prof. Rongjie Lai.

[4] Wikipedia: Kernel (image processing)

https://en.wikipedia.org/wiki/Kernel_(image_processing)

Appendix

(see attached codes)