

# Calcul numérique des solides et structures non linéaires

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First course

Introduction to non-linearities  
in mechanics

# **Two types of non-linearities**

- Material non-linearities
  - Stress is no longer proportional to strain
- Geometric non-linearities
  - The hypothesis of small disturbances around a natural state is no longer valid

Purpose of the Course: To calculate with finite element methods in these cases

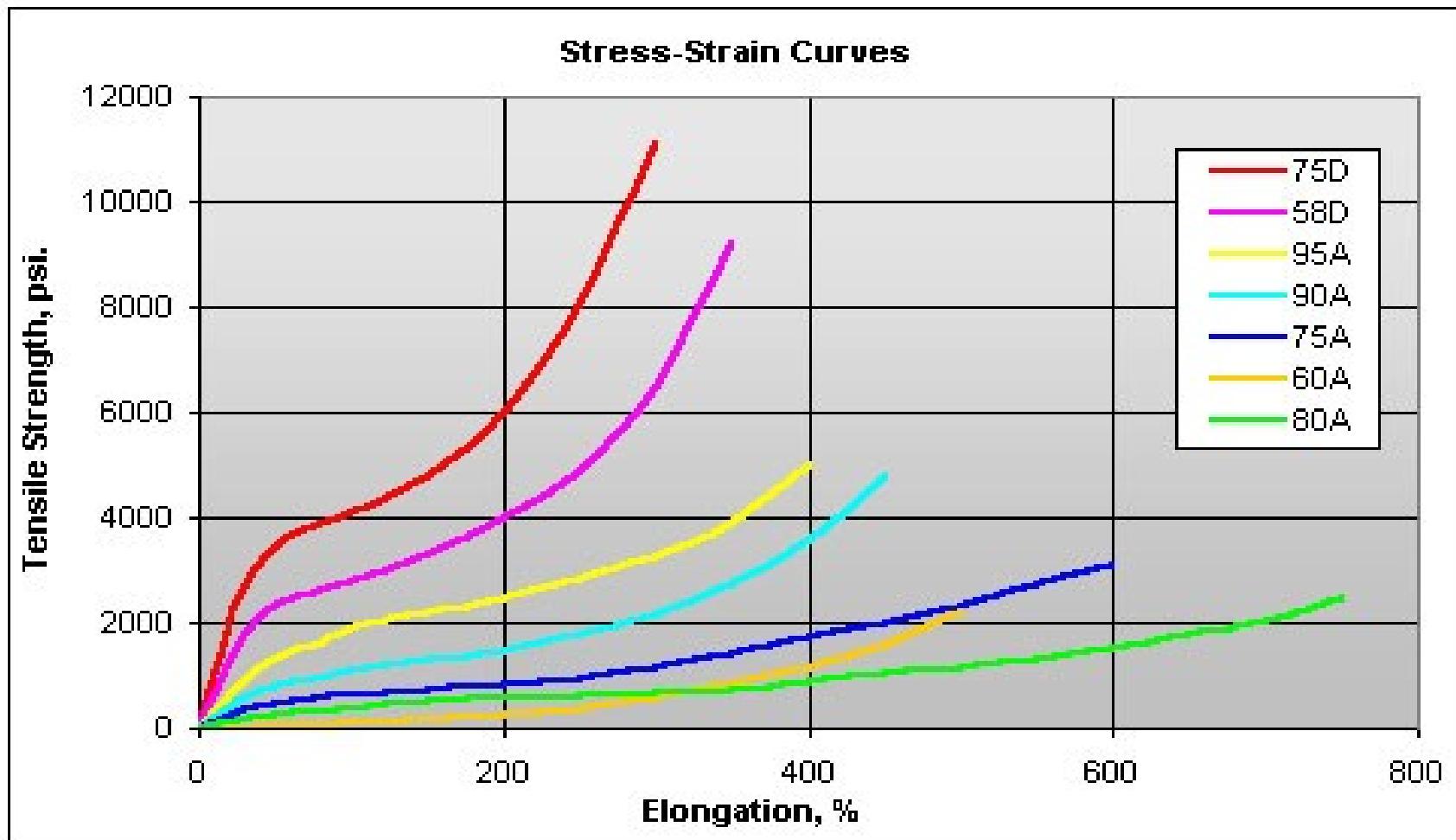
# Material Nonlinearities

- Non-linear elasticity
- Plasticity
- (Viscoelasticity and) viscoplasticity
- Damage
- Fatigue
- Dependence of materials / environment

# Non-linear elasticity

- Elastomer, rubber, wood
- Same path for loading and unloading
- No dissipation
- Existence of potential
- Deformation energy

## Nonlinear stress-strain relationships



Strain energy for the one-dimensional linear case

$$\psi = \frac{1}{2} E e^2$$

$$\sigma = \frac{\partial \psi}{\partial e} = E e$$

For the three-dimensional non-linear case, something like

$$\sigma_{ij} = \frac{\partial \psi}{\partial e_{ij}}$$

Hyperelastic material with different models: Neo-Hookean, Mooney Rivlin, ...

Can be incompressible (rubber)

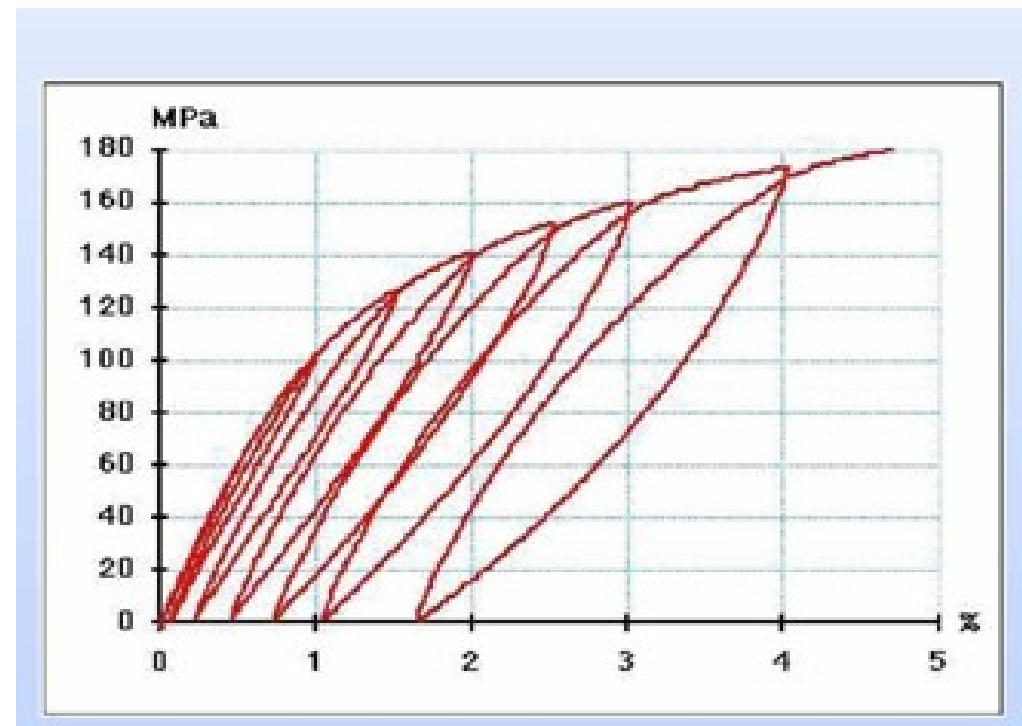
# Plasticity

- Ferrous metals
- Elastic limit
- Plasticity criterion
  - Von Mises 2nd invariant of stress deviator
  - Tresca max ( $\sigma_i - \sigma_k$ )
- Plastic flow law

## Plastic behaviour



Essai



Traction à  $45^0$  : Comportement plastique

## Example of plastic behaviour

\* When  $\sigma < \sigma_0$

$\sigma = E\epsilon$  Reversible linear elastic behaviour

\* When  $\sigma > \sigma_0$

$$\epsilon = \epsilon^e + \epsilon^p$$

$\epsilon^e$  Reversible elastic deformation

$\epsilon^p$  Plastic deformation

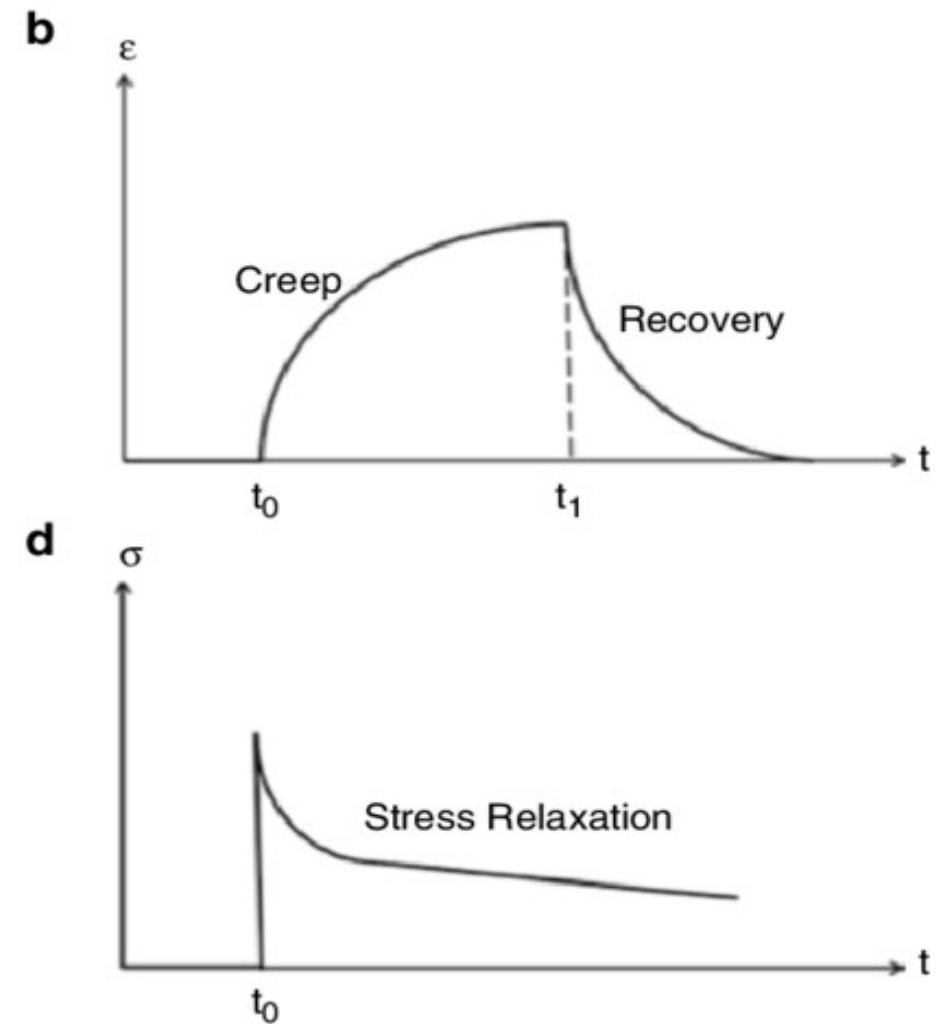
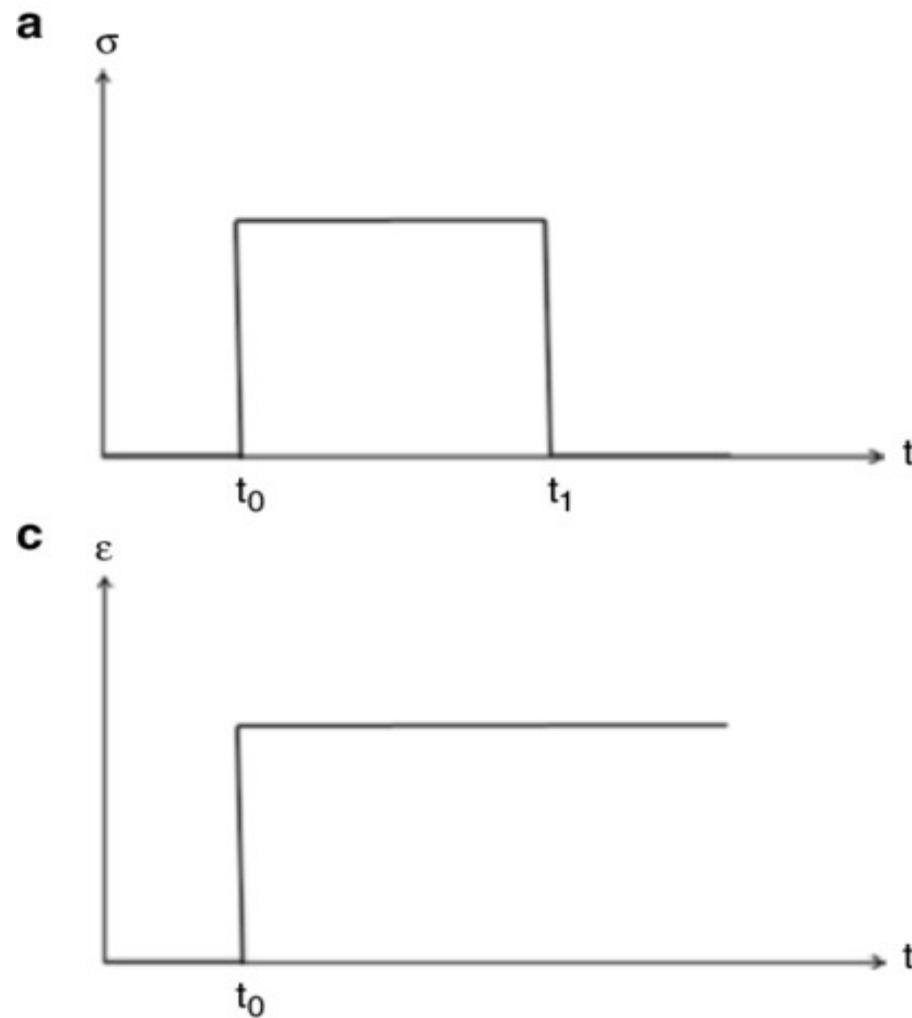
Flow law

$$\dot{\epsilon}^p = f(\sigma)$$

# Viscoelasticity - Viscoplasticity

- State of the material that evolves with time. Stress and strain.
- Creep. Constant stress, deformation increases.
- Relaxation. Constant strain, stress decreases.
- Maxwell model
- Kelvin model
- Often coupled with plasticity

# Stress strain relationship



## Viscoelasticity models

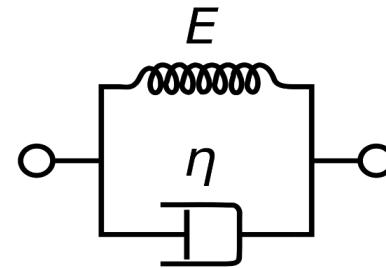
Maxwell model

$$\sigma + \frac{\eta}{E} \dot{\sigma} = \eta \dot{\epsilon}$$



Kelvin model

$$\sigma = E \epsilon + \eta \dot{\epsilon}$$



# Damage

- Metals - composites - ceramics
- Plasticity + change of elastic modulus.
- Growth of cavities in the material associated (or not) with plasticity
- Ultimate state ( $\varepsilon > 10\%$ )

# Damage



# Cracking

Crack opening and crack growth when stresses exceed a threshold

- Slow growth or
- Sudden rupture



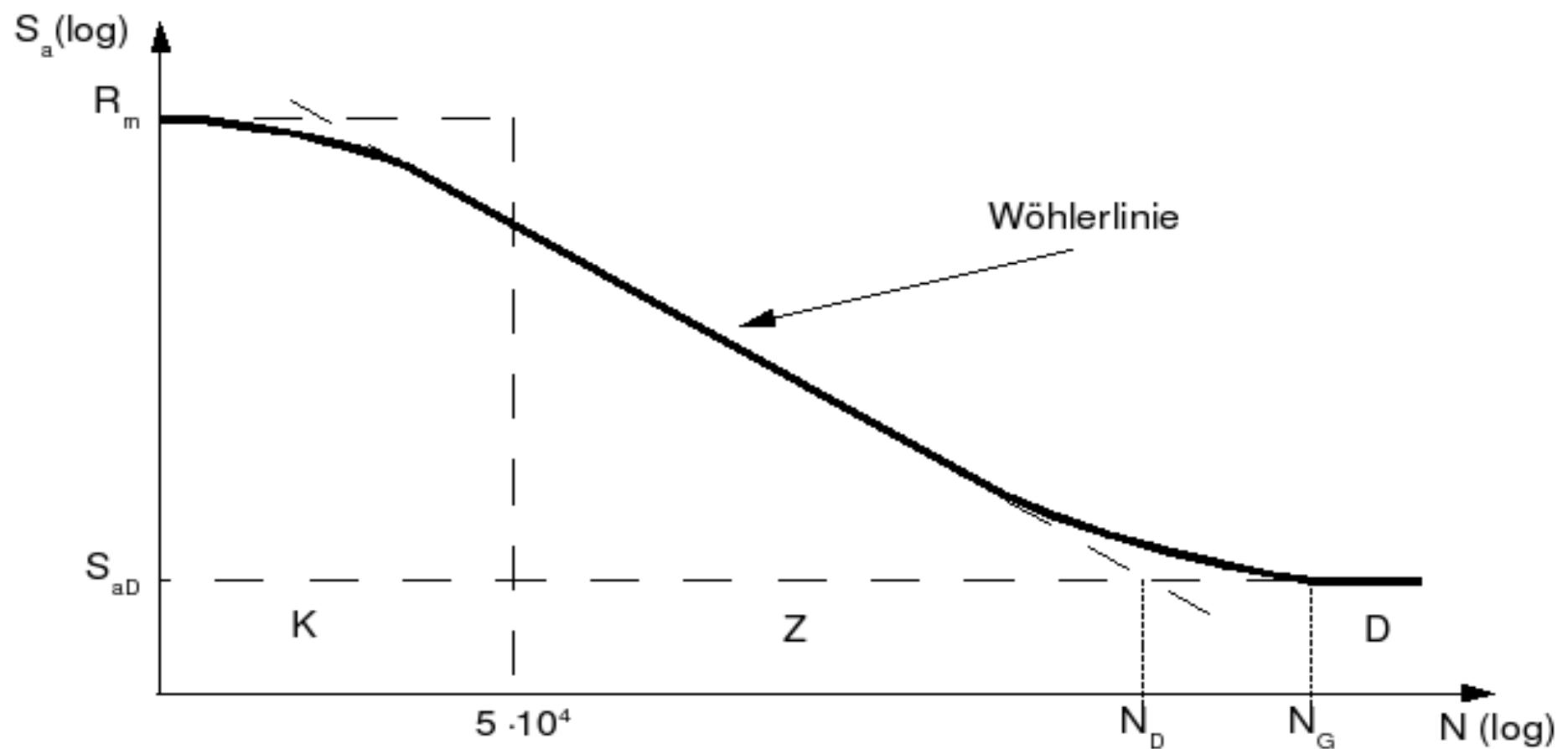
# Fatigue

- Light alloys, steel
- Aging of the material according to the number and intensity of cycles
- Effect: decreases the capacity of deformation then rupture
- Method: modification of the properties of the material established from tests

## Fatigue fracture



## S-N Curves



# Dependence of materials / environment

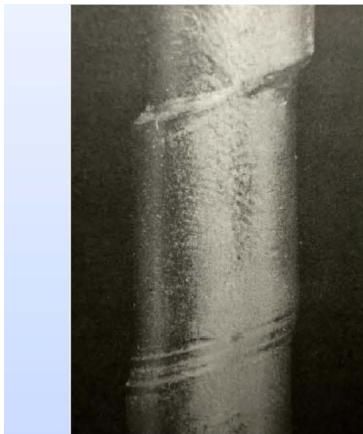
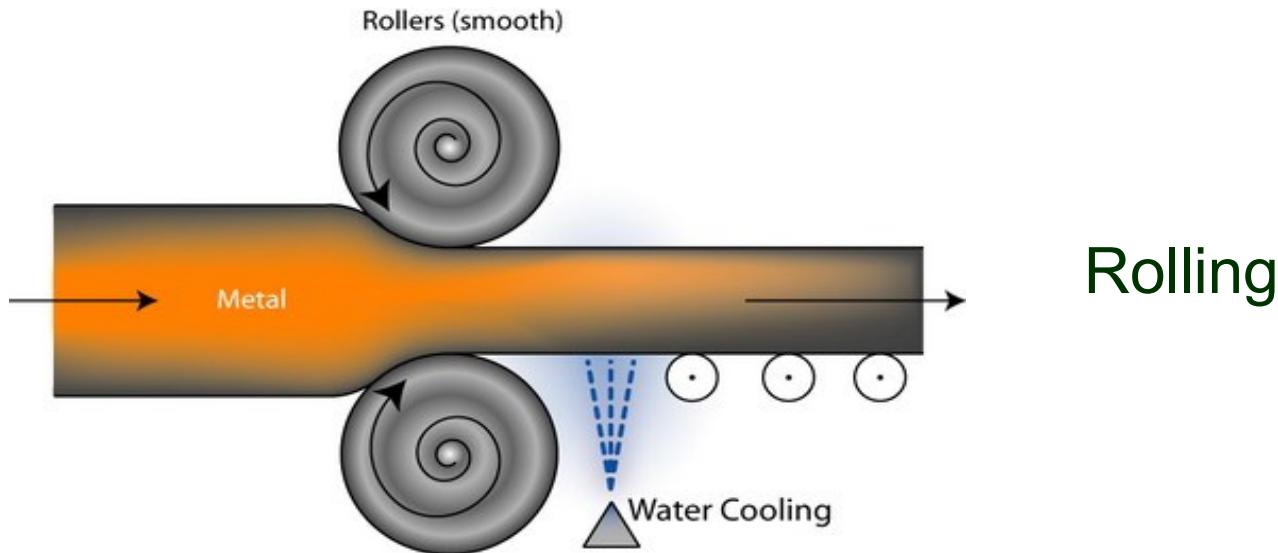
Properties function of

- Temperature
- Strain state
- Chemistry, phase change
- Aging
- Hygrometry
- ...

# Geometric non-linearities

- Large strain
- Prestressed load
- Buckling
- Contact, boundary conditions
- Loading function of the geometry
- Mechanical properties function of the temperature

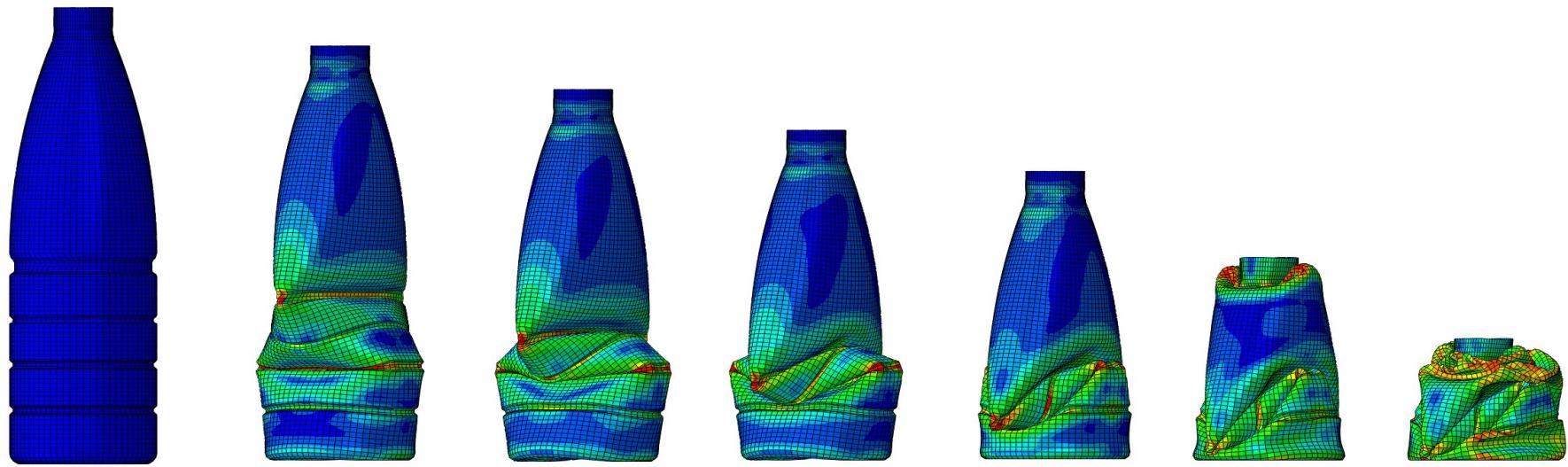
# Large strain



Traction sur  
monocristal de  
Zinc.



compression sur monocristal d'Aluminium.





crash



## Large strain

Use the strain tensor

$$e = \frac{1}{2}({}^t\nabla \phi \nabla \phi - I)$$

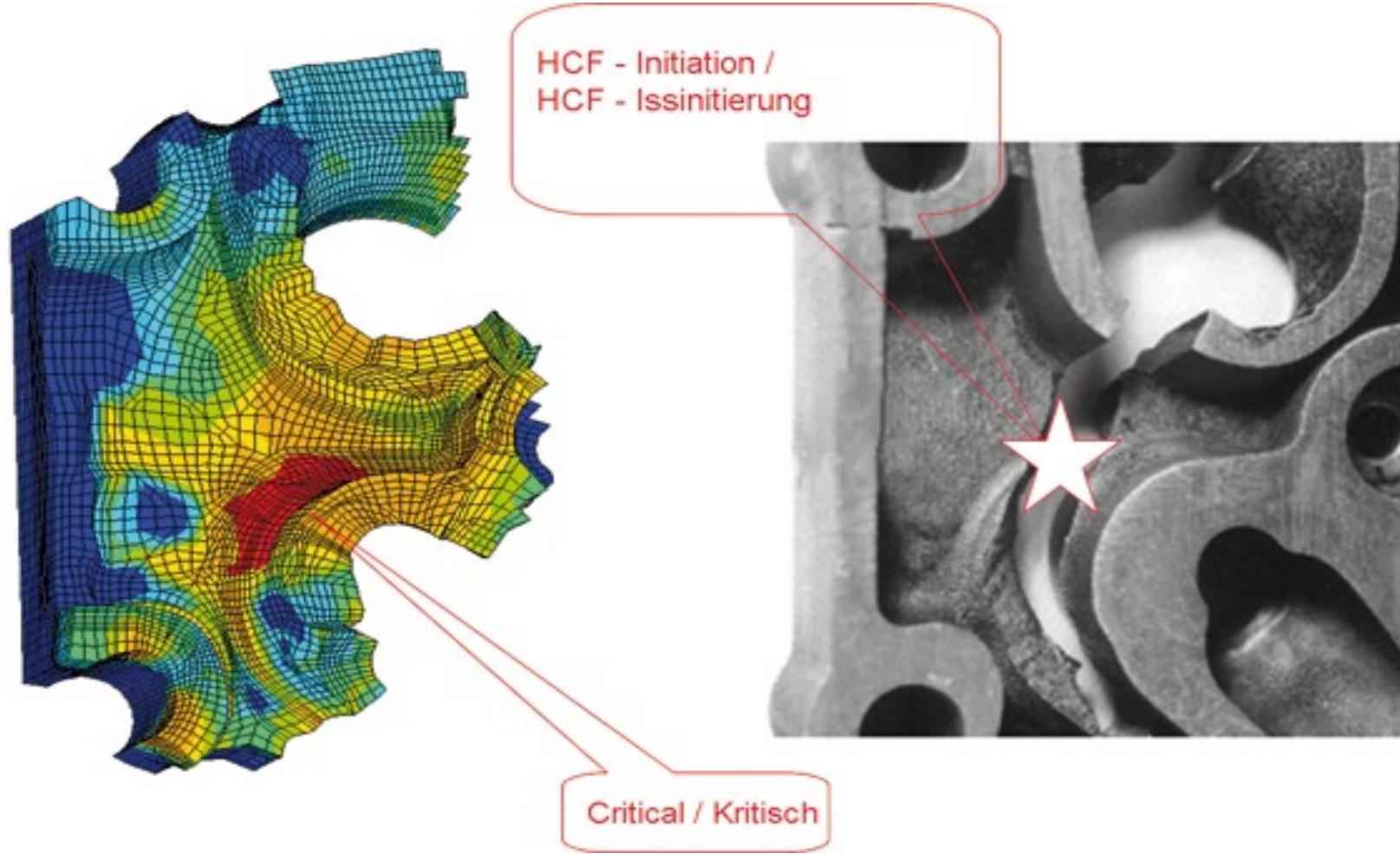
Use the correct constitutive law of the material giving  
the correct stress tensor (Piola, Cauchy) from the strain tensor  $e$

Set the equilibrium equation on the deformed  
or reference configurations

## Prestressed load



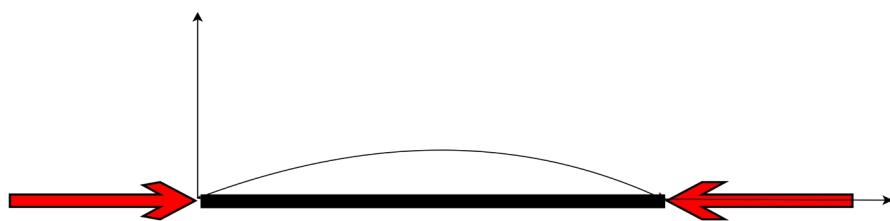
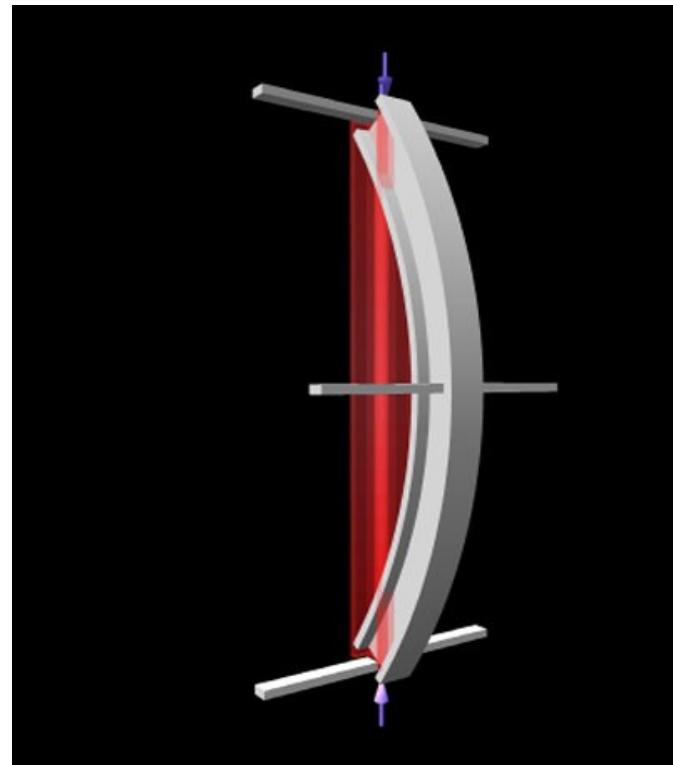
## Residual stress



## Prestressed load

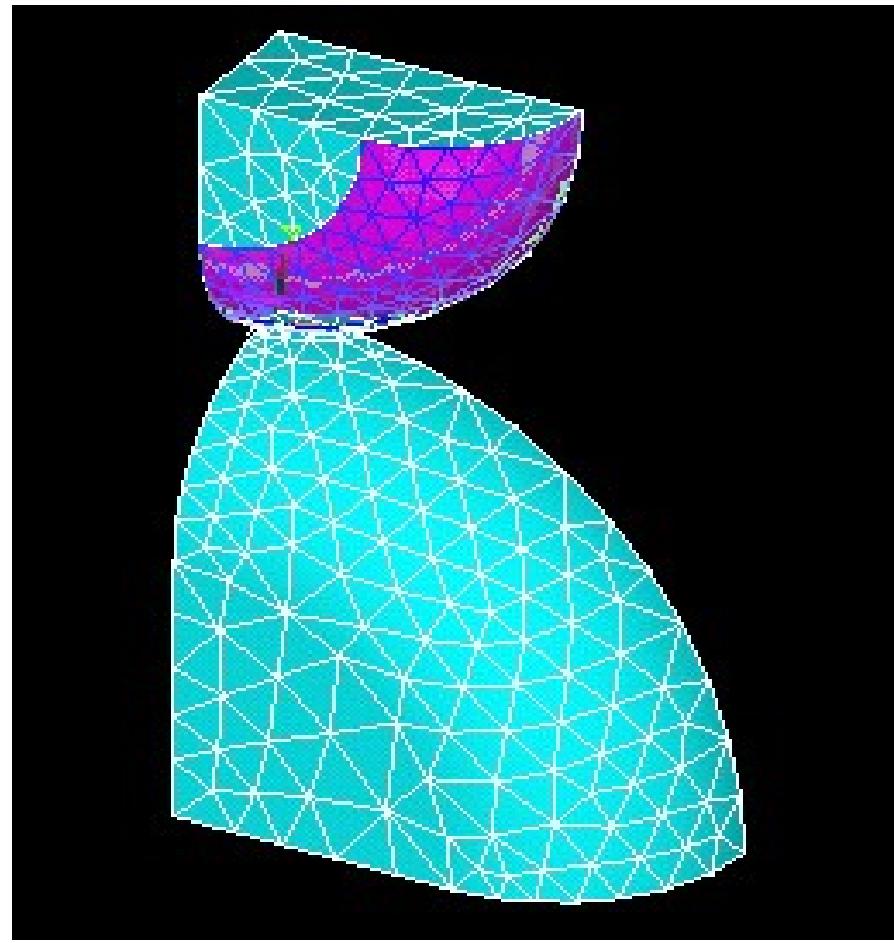
- The reference state is not a natural state
- This state is at equilibrium without external load
- A new load is applied in addition to the prestressed load
- Other example: a guitar string has a static preload and vibrations are applied in addition to the prestressed load

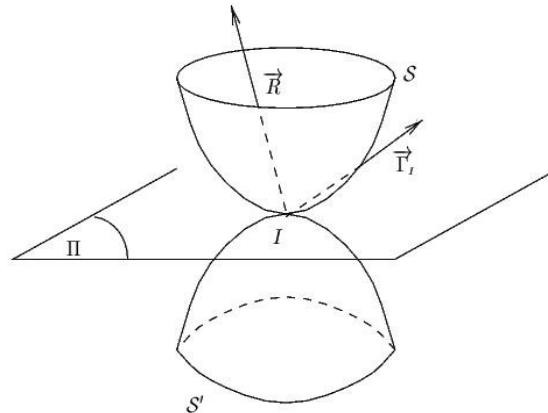
# Buckling



$$\text{When } F > \pi^2 \frac{EI}{L^2}$$

## Contact, friction





No interpenetration, only detachment is possible

$$(\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{n} \geq 0$$

Unilateral contact, only compression is possible

$$R_{n_1} = -R_{n_2} \leq 0$$

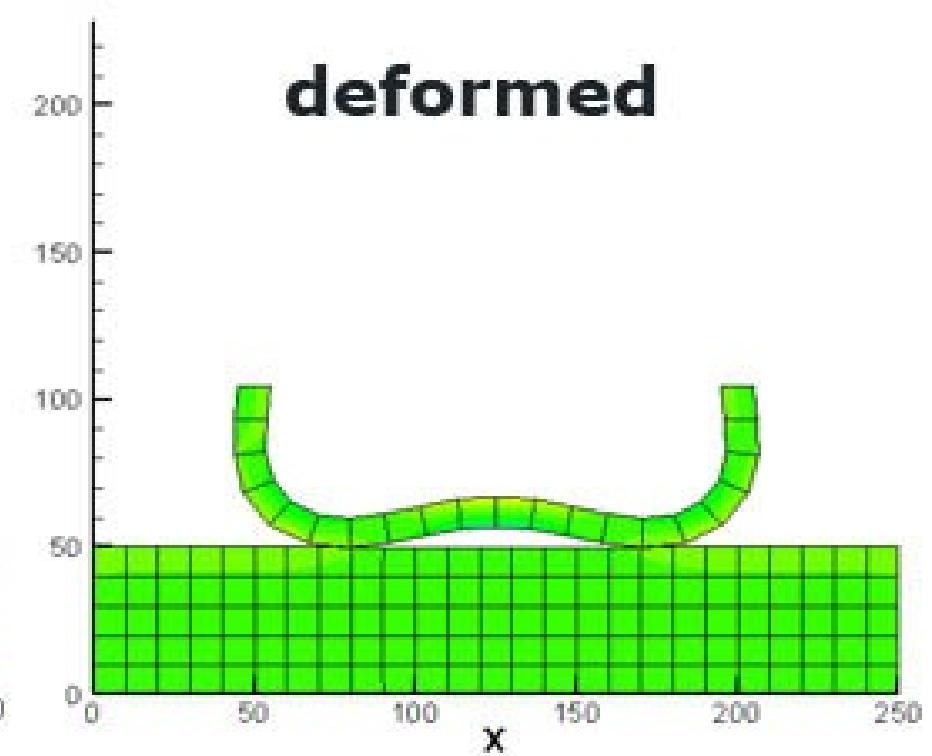
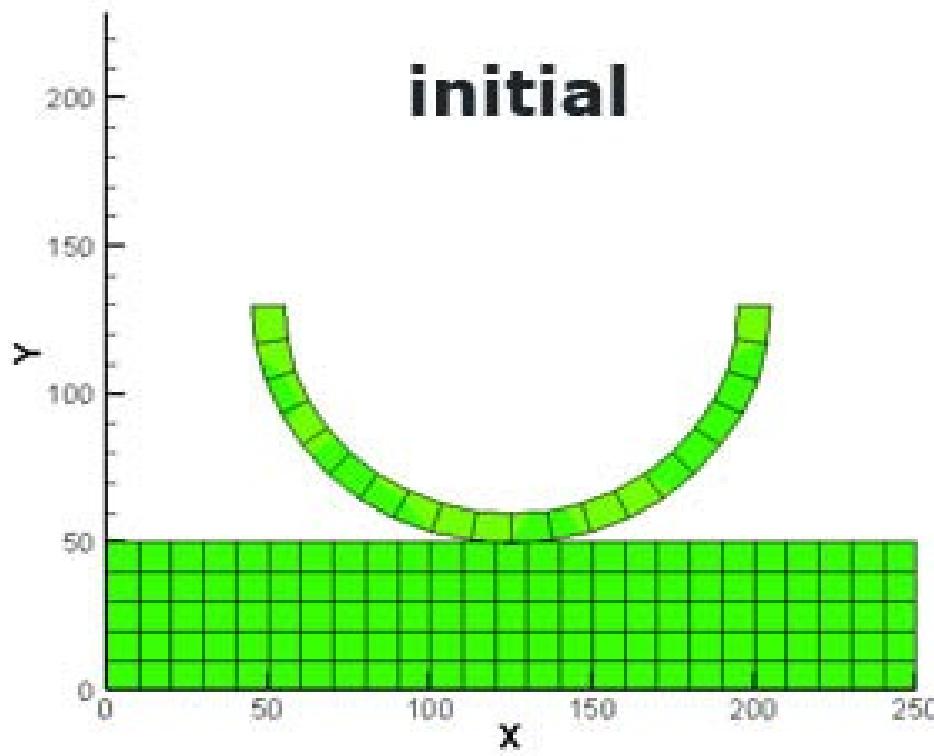
No friction, change this relation if there is friction

$$R_{t_1} = -R_{t_2} = 0$$

Either contact or detachment

$$((\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{n}) R_n = 0$$

Finding the contact zone is a part of the problem



## Non-linear boundary condition

Follower pressure in piping, tank, dawn

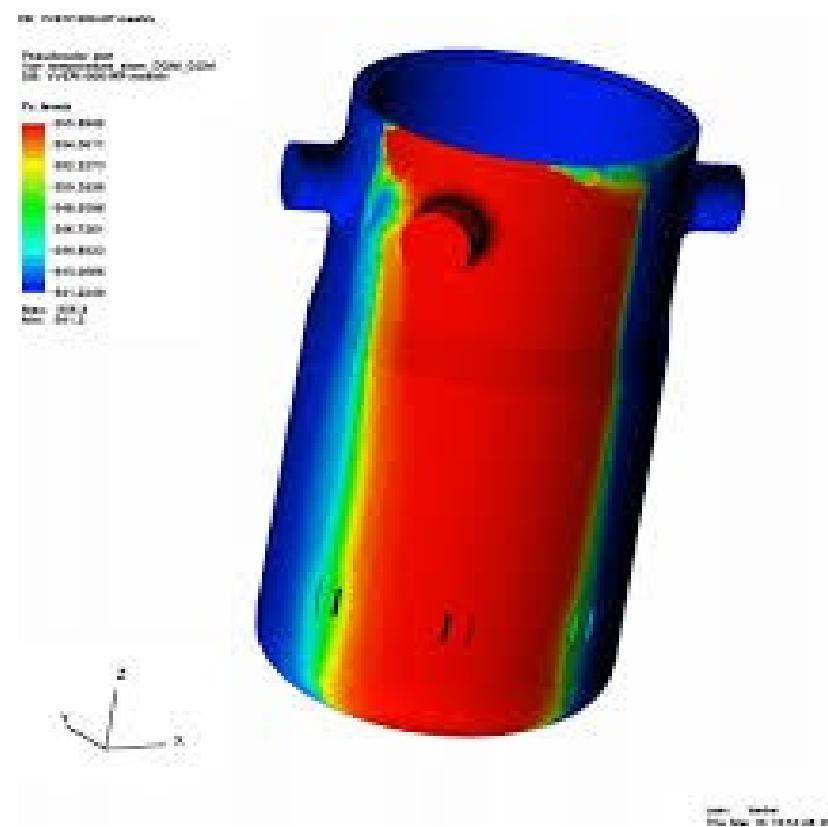
$$F = P N$$

Required for stability analysis

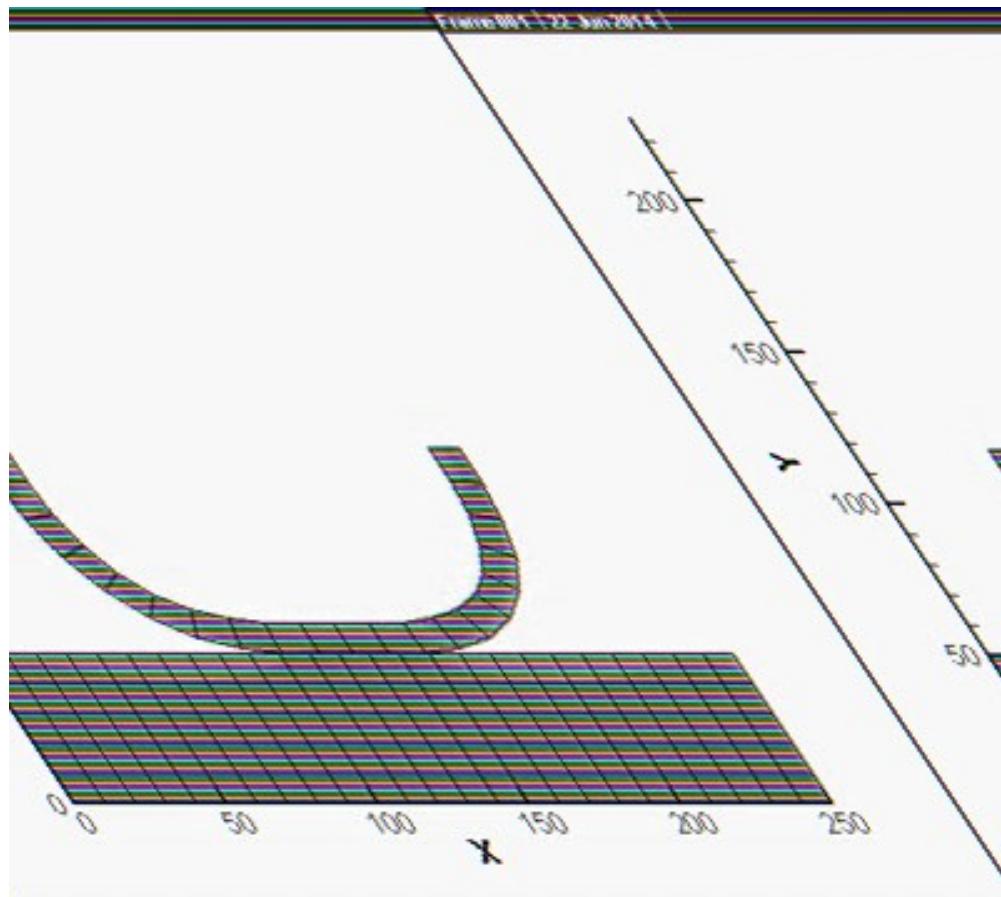


# Thermics

Position-dependent temperature, change of elastic properties

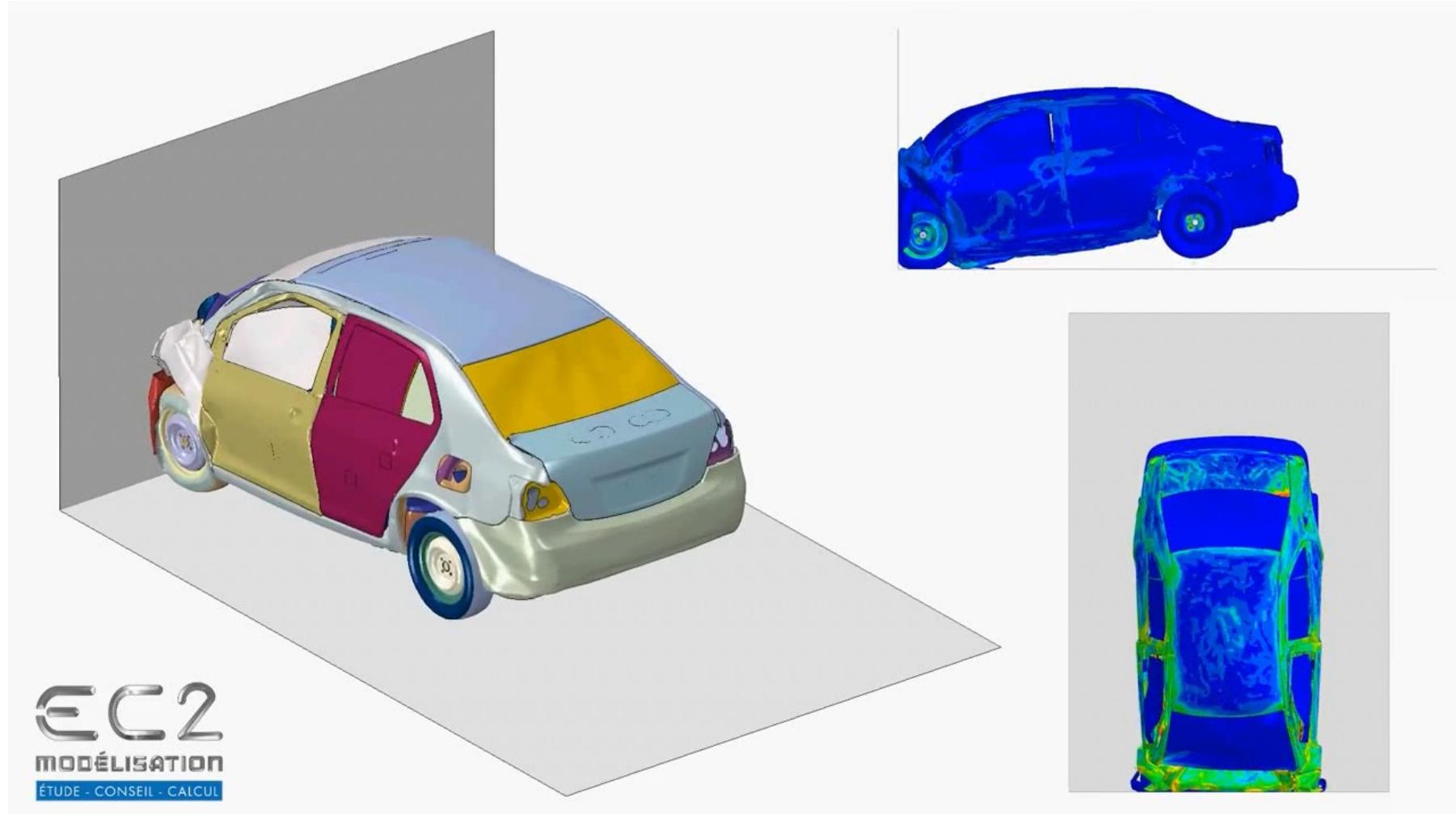


## Steps of a non linear computation



A non linear computation is an iterative process

# Non linear dynamics



# Conclusion

- Understand the mechanics and the equations
- Be able to solve with FeniCS
  - Other possibilities:
    - ✓ Abaqus, Ansys, Nastran
    - ✓ FreeFem++
    - ✓ Write the program with python, C++, matlab
- Have a critical look at the results
- Postprocessing

THE END