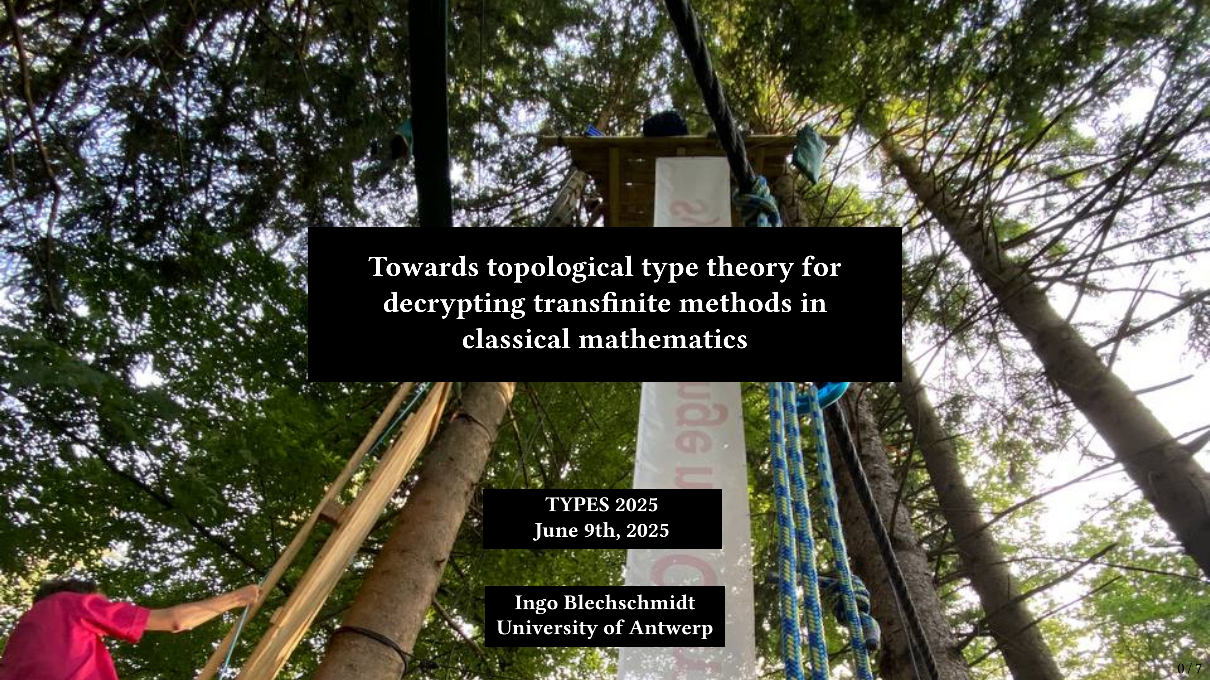

Towards topological type theory for decrypting transfinite methods in classical mathematics

TYPES 2025
June 9th, 2025

Ingo Blechschmidt
University of Antwerp

A case study in Hilbert's program

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there merely exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
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Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. □

offensive?

$$7\,, \quad 4\,, \quad 3\,, \quad 1\,, \quad 8\,, \quad 2\,, \quad \ldots$$

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Key stability results

Assuming LEM and DC, …

Dickson: If X and Y are well, so is $X \times Y$.

Higman: If X is well, so is List X.

Kruskal: If X is well, so is Tree X.

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$$\frac{p : \mathrm{Good}\,\sigma}{\mathrm{now}\,p : G\,\sigma} \qquad \frac{f : (x : X) \to G\,(\sigma ::^r x)}{\mathrm{later}\,f : G\,\sigma}$$

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Is there a procedure for reinterpreting **classical proofs** regarding well$_\infty$ as **blueprints for constructive proofs** regarding well$_{\text{ind}}$?

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Central insight: A quasi-order X is well$_{\text{ind}}$ iff $\Box \forall \alpha : \mathbb{N} \to X.\, \exists i < j.\, \alpha\, i \leq \alpha\, j$.

Missing functions in the type of all functions?

Behold: A transfinite tool …

Lemma. LEM Let X be well$_\infty$. Let $\alpha : \mathbb{N} \to X$. Then there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.

Proof.

1. The type $I :\equiv \sum_{i : \mathbb{N}} \neg \sum_{j : \mathbb{N}} i < j \times \alpha\, i \leq \alpha\, j$ is not in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would not be good.
2. By LEM, the type I is finite.
3. Any index i_0 larger than all the indices in I is a suitable starting point for an increasing subsequence. $\square$

… implying a concrete consequence

Cor. LEM Let X and Y be well$_\infty$. Then $X \times Y$ is well$_\infty$.

Proof.

1. Let a sequence $\gamma : \mathbb{N} \to X \times Y$ be given. Write $\gamma\, k = (\alpha\, k, \beta\, k)$.
2. By the lemma, there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \cdots$.
3. Because Y is well, there are indices $n < m$ such that $\beta\, i_n \leq \beta\, i_m$.
4. As also $\alpha\, i_n \leq \alpha\, i_m$, the sequence γ is good. $\square$

We cannot trust LEM*-provided sequences to be available in the type $\mathbb{N} \to X$. Similarly with* DC.

Where do many cherished inductive definitions come from?

In mathematics, we routinely enlarge structures:

- Pass from $\mathbb{Q}$ to $\mathbb{R}$, to embrace irrationals.
- Pass from $\mathbb{R}$ to $\mathbb{C}$, to obtain $\sqrt{-1}$.
- Pass from $\mathbb{C}$ to $\mathbb{C}[X]$, to obtain a “generic number”.

In set and type theory, we can also enlarge **the mathematical universe**:

- Force a **generic sequence** $\mathbb{N} \to X$.
- Force a **generic enumeration** $\mathbb{N} \twoheadrightarrow X$ (even if X is uncountable).
- Force a **generic prime ideal** of a given ring.

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Central observations of the multiversal yoga:

- A quasi-order X is well$_{\text{ind}}$ iff the generic sequence $\mathbb{N} \to X$ is good.
- A relation is well-founded iff for the generic descending chain, $\bot$.
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The mystery of nongeometric sequents:

- The generic ring is a field.
- For the generic surjection $f : \mathbb{N} \twoheadrightarrow \mathbb{R}$, $\neg\neg\exists n : \mathbb{N}.\, f(n) = \pi \wedge f(n+1) = e$.

A modal language for harnessing the multiverse

Def. A statement φ holds …

- **everywhere** $(\Box\varphi)$ iff it holds **in every topos** (over the current base).
- **somewhere** $(\Diamond\varphi)$ iff it holds **in some positive topos**.
- **proximally** $(\Diamond\!\!\!\diamond\,\varphi)$ iff it holds **in some positive overt topos**.

We then have:

1. $\mathrm{Well}_{\mathrm{ind}}(X, \leq) \iff \Box\,\mathrm{Well}_{\infty}(X, \leq)$.
2. $(\Diamond\varphi) \iff \varphi$, if φ is a geometric implication ("$\forall \ldots \forall(\% \Rightarrow \%)$", with no $\forall$ nor $\Rightarrow$ in $\%$).
3. $(\Diamond\!\!\!\diamond\,\varphi) \iff \varphi$, if φ is a bounded first-order statement.
4. For every inhabited type X, $\Diamond\!\!\!\diamond\,\Box\,\mathrm{Countable}(X)$,
   where $\mathrm{Countable}(X) :\equiv (\exists f : \mathbb{N} \to X.\, \mathrm{Surjective}(f))$.
   So being countable is a **button**.
5. $\Diamond$ LEM. (Baby Barr / Friedman's trick / nontrivial exit continuation)
   In fact, LEM is a **switch**: $\Box((\Diamond\,\text{LEM}) \wedge (\Diamond\,\neg\,\text{LEM}))$
6. ZORN $\Rightarrow \Box\Diamond$ AC. (Great Barr)

The plain combinatorics of toposes

1. Realize a generic gadget as some kind of limit of **approximations** from the base universe. For the generic function $f_0 : \mathbb{N} \to X$:

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L : Set
L = List X
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3. Be prepared to evolve approximations (⚙).

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data ∇ (P : L → Prop) : L → Prop where
  now : {σ : L} → P σ → ∇ P σ
  later : {σ : L} → ((x : X) → ∇ P (σ ::ʳ x)) → ∇ P σ
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   - For a stage-dependent proposition $P : L \to \mathrm{Prop}$, $\nabla\ P\ \sigma$ expresses that no matter how σ will evolve over a time to a better approximation τ, eventually $P\ \tau$ will hold.
   - That f_0 is defined on input n can be expressed as $\nabla\ P\ [\,]$ where $P\ \sigma :\equiv (\mathrm{length}\ \sigma > n)$.

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4. Crucially, this interpretation is **sound** with respect to constructive reasoning.

Formal metatheory

✓ *There are type-theoretic multiverses,* such as

- the collection of all $\mathrm{PSh}(\mathcal{C} \times \mathcal{B})$, where $\mathcal{B}$ ranges over cube categories and $\mathcal{C}$ over arbitrary small categories, and their corresponding sheaf models

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✗ *Accessing the multiverse from within intensional type theory is tricky:*

- Given a model of $\mathfrak{s}$CIC and a category $\mathcal{C}$ in it, we have a syntactic presheaf model of CIC.

  Coquand, Jaber. "A note on forcing and type theory". *Fundamenta Informaticae 100* (2010).
  Jaber, Lewertowski, Pédrot, Sozeau, Tabareau. "The definitional side of the forcing". *Proceedings of LICS '16* (2016).
  Pédrot. "Russian constructivism in a prefascist theory". *Proceedings of LICS '20* (2020).

- Given a suitable lex modality, we have a syntactic sheaf model (model of modal types).

  Coquand, Ruch, Sattler. "Constructive sheaf models of type theory." *Math. Struct. Comput. Sci.* 31.9 (2021).
  Escardó, Xu. "Sheaf models of type theory in type theory". Unpublished (2016).
  Quirin. "Lawvere–Tierney sheafification in Homotopy Type Theory". PhD thesis (2016).

- (I believe) we have syntactic sheaf models in certain special cases, when no coherence issues arise in defining the notion of presheaves.

Note: We can use ∇ even without a proper metatheoretic backing.

Increasing subsequences as convenient fictions

Let X be a quasi-order. Let $B : \mathsf{List}\ X \to \mathsf{Prop}$ be a monotone predicate.

Classical blueprint

Thm. LEM If X is well_∞ and if every increasing sequence $\alpha : \mathbb{N} \to X$ has a prefix validating B, then every sequence has a prefix validating B.

Proof. Let $\alpha : \mathbb{N} \to X$ be a sequence. By the lemma, there is an increasing subsequence $\alpha\ i_0 \leq \alpha\ i_1 \leq \cdots$. By assumption, this subsequence has a prefix validating B. This prefix is part of a prefix of the original sequence α. Hence we conclude by monotonicity. □

Constructive reimagination

Thm. If X is well_{ind} and if $\nabla^{\nearrow} B\,[\,]$, then $\nabla\ B\,[\,]$.

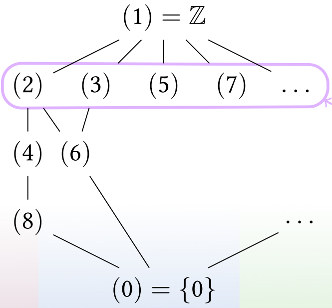
Proof. Let $\alpha : \mathbb{N} \to X$ be a sequence (in an arbitrary topos). *Somewhere,* LEM holds. *There* X is still well_∞, so that we have an increasing subsequence $\alpha\ i_0 \leq \alpha\ i_1 \leq \cdots$. By assumption, this subsequence has a finite prefix validating B. This prefix is part of a prefix of the original sequence α. Hence α has a prefix validating B by monotonicity *there*. So *somewhere* there is a finite prefix validating B. Thus there actually is a finite prefix validating B. □

Maximal ideals as convenient fictions

Let A be a commutative ring with unit.

Thm. Let M be a surjective matrix with more rows than columns over A. Then $1 = 0$ in A.

Classical proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix M is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$



maximal among the proper ideals

- $\neg(1 \in \mathfrak{m})$
- $\neg\big(1 \in \mathfrak{m} + (x)\big) \Rightarrow x \in \mathfrak{m}$

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Multiversal constructive proof. We may work *somewhere* where LEM holds. So assume not. *Proximally*, there is a maximal ideal $\mathfrak{m}$ (⚙). The matrix M is still surjective *there*, and also over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra. $\square$

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Unrolled constructive proof (special case). Write $M = \begin{pmatrix} x \\ y \end{pmatrix}$. By surjectivity, have u, v with

$$u \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. $\square$

Stabilization as convenient fiction

A commutative ring A with unit is …

- Noetherian$_\infty$ iff for every sequence $x_0, x_1, \ldots$ of ring elements, there is a number $n : \mathbb{N}$ such that $x_n, x_{n+1}, x_{n+2}, \ldots \in (x_0, \ldots, x_{n-1})$.
- Noetherian$_{\mathrm{RS}}$ iff for every sequence $x_0, x_1, \ldots$ of ring elements, there is a number $n : \mathbb{N}$ such that $x_n \in (x_0, \ldots, x_{n-1})$.
- Noetherian$_{\mathrm{ind}}$ iff a certain inductively defined condition holds [Coquand–Persson].

We then have:

$$\begin{aligned} \mathsf{Noetherian}_{\mathrm{ind}}(A) &\Longleftrightarrow \Box\,\mathsf{Noetherian}_{\mathrm{RS}}(A) \\ &\Longleftrightarrow \Box\big(\mathrm{LEM} \Rightarrow \mathsf{Noetherian}_{\infty}(A)\big) \\ &\Longleftrightarrow \Box\Diamond\,\mathsf{Noetherian}_{\infty}(A). \end{aligned}$$

Hence, from the corresponding classical proofs, we can extract constructive proofs of:

1. Hilbert's basis theorem: If A is Noetherian$_{\mathrm{ind}}$, then so is $A[X]$.
2. Locality: If $1 = f_1 + \ldots + f_n$ and if each ring $A[f_i^{-1}]$ is Noetherian$_{\mathrm{ind}}$, then so is A. (As a consequence, A is Noetherian$_{\mathrm{ind}}$ iff $\mathcal{O}_{\mathrm{Spec}(A)}$ is.)

Welcome — Let's play Ag

lets-play-agda.quasicoherent.io

Download raw

Let's play Agda Beta

6c81e93

Welcome

- ✅ About this project
- ✅ Getting Agda
- ✅ References
- ✅ A mathematical Rosetta Stone

Agda as a programming language

- ☑️ Booleans
- ✅ Operators
- ☑️ Higher-order functions
- 🗃️ Natural numbers
- ⬜ Dependent functions
- ✅ Syntactic sugar
- ☑️ Lists
- ☑️ Vectors

Agda as a proof language

- ✅ Propositions as types
- ☑️ Predicates on natural numbers
- ☑️ Negation
- 🗃️ Equality
- 🗃️ Logical connectives
- 🗃️ Well-founded recursion

Verified algorithms

- ⬜ Post-hoc verification
- ⬜ Correct by construction
- ⬜ Case study: Binary representation of natural numbers
- 🗃️ Case study: Insertion sort
- ⬜ Case study: Compiler and interpreter

Cubical Agda 🚧

Explorations

- ☑️ Sets as trees
- 🗃️ Uncountability
- ⬜ Numbers larger than infinity
- 🗃️ Forcing

Computational content of classical logic

- ⬜ Intro
- ⬜ Sarcastic interpretation of classical logic
- 🗃️ Classical blueprints
- 🗃️ Convenient fictions
- ⬜ Case study: Dickson's lemma
- ⬜ Case study: Pigeonhole principle

Recreational mathematics

Current streak: 1 day

Longest streak: 10 days

Let's play Agda: running abstract mathematical proofs as programs

The purpose of this website is to help you learn Agda, the functional proof language. It was created for a 2025 course at the University of Padova.

Start here: `Agda as a programming language`

👋 Padova students: **Transcripts and links** accompanying our sessions

1. In a nutshell: Agda is a programming language…

- 🧊 **purely functional**, free of mutable state
- **similar to Haskell** in syntax and vibe
- 🧱 **statically typed**, so that many errors are caught at compile-time
- 🌳 **familiar types** like integers, lists, …
- 🧭 **type inference**, so that we can focus on those types which matter

For instance, an implementation of insertion sort might look like this:

```
sort : List ℕ → List ℕ
sort []       = []
sort (x ∷ xs) = insert x (sort xs)
```

And we can imagine functions with type signatures such as…

```
fibonacci : ℕ → ℕ                              -- Fibonacci numbers
pi        : ℝ                                  -- arbitrary-precision constant
size      : {X : Set} → Tree X → ℕ             -- size of binary tree
replicate : {X : Set} (len : ℕ) → X → Vector len X  -- vector of identical elements
```

2. …and simultaneously: a proof language…

- 🧾 **types of witnesses** for unifying the activities of proving and programming
- ✨ **context assistance** for interactively constructing proofs
- 🎨 sufficiently **expressive** for vast parts of mathematics
- **hole-driven development**

In Agda, we prove a proposition by constructing a program which computes a suitable witness. This approach is the celebrated *propositions as types* philosophy:

```
grande-teorema   : 2 + 2 ≡ 4
binomial-theorem : (x : ℕ) (y : ℕ) → (x + y) ² ≡ x ² + 2 · x · y + y ²
sort-correct     : (xs : List ℕ) → IsSorted (sort xs)
```

For instance, there is a type of witnesses…

- that 2 + 2 equals 4 (this type is inhabited);
- that 2 + 2 equals 5 (this type is empty);
- that there are prime numbers larger than 42 (this type contains infinitely many values, for instance the pair (43 , p), where p is a witness that 43 is a prime larger than 42)
- that there are infinitely many prime numbers;
- that a given sorting function `sort` works correctly (this type contains functions which read an arbitrary list `xs` as input and output a witness that `sort xs` is ascendingly ordered)
- that the continuum hypothesis holds;
- and so on.

Workshop: AI Transforms

uni-augsburg.de/en/fakultaet/mntf/math/prof/geom/workshop-aimath/

University: Faculties: Faculty of Mathematics, Natural Sciences, and Materials Engineering: Institute of Mathematics: Chairs: Analysis and Geometry: Workshop: AI Transforms Math Research

Workshop: AI Transforms Math Research

TOPIC

The workshop will center around the topic Artificial Intelligence in Mathematics Research and aims to

- Highlight examples of current or past usage of AI in Math Research, as well as
- Provide a forum for mathematicians to discuss future such usage.

Here the broad term AI is purposefully used vaguely to include all forms of mechanical reasoning, be it by proof assistants, classical machine learning, or generative AI.

OVERVIEW

Date

25.th - 29.th of August 2025

3377 - conferences.cirm-

conferences.cirm-math.fr/3377.html

CONFERENCE

INTRANET FOR ORGANIZERS

Synthetic mathematics, logic-affine computation and efficient proof systems

Mathématiques synthétiques, calcul logique affine et systèmes de preuve efficients

8 – 12 September, 2025

Scientific Committee Comité scientifique

Andrej Bauer (University of Ljubljana)

Olivia Caramello (IHES)

Maria Emilia Maietti (Universita degli Studi di Padova)

Michael Rathjen (University of Leeds)

Invited Speakers conférenciers invités

Ulrik Buchholtz (University of Nottingham)

Laura Fontanella (Université Paris Est Créteil)

Joost Joosten (Universitat de Barcelona)

Dominik Kirst (INRIA)

Etienne Miquey (I2M)

Pierre-Marie Pédrot (INRIA)

Emily Riehl (Johns Hopkins University)

Talia Ringer (University of Illinois Urbana-Champaign) on videoconference

Alex Simpson (University of Ljubljana)

Time schedule

Abstracts

Participants

Audiovisual Mathematics Library (after the event)

Backup slides.

Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

In the forcing extension V^{∇}, there will then be a **generic filter** (ideal object).

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For the generic surjection $\mathbb{N} \twoheadrightarrow X$

Use **finite lists** $\sigma \in X^*$ as forcing conditions, where $\tau \preccurlyeq \sigma$ iff σ is an initial segment of τ, and be prepared to grow σ to …

(a) one of $\{\sigma x \mid x \in X\}$, to make σ more defined

(b) one of $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$, for any $a \in X$, to make σ more surjective

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For the generic prime ideal of a ring A

Use **f.g. ideals** as forcing conditions, where $\mathfrak{b} \preccurlyeq \mathfrak{a}$ iff $\mathfrak{b} \supseteq \mathfrak{a}$, and be prepared to grow $\mathfrak{a}$ to …

(a) one of $\emptyset$, if $1 \in \mathfrak{a}$, to make $\mathfrak{a}$ more proper

(b) one of $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$, if $xy \in \mathfrak{a}$, to make $\mathfrak{a}$ more prime

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
- Good $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i \leq x_j$ (for some preorder $\leq$ on X)

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We then define "$\nabla P \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $\nabla P \sigma$.
2. If $\nabla P \tau$ for all $\tau \in R$, where R is some covering of σ, then $\nabla P \sigma$.

So $\nabla P \sigma$ expresses in a **direct inductive fashion**:

> "No matter how σ evolves to a better approximation τ, eventually $P \tau$ will hold."

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We use quantifier-like notation: "$\nabla(\tau \preccurlyeq \sigma).\, P\tau$" means "$\nabla\, P\, \sigma$".

Proof translations

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{ccclrcl} \exists & \rightsquigarrow & \exists^{\mathrm{cl}}, & \text{where} & \exists^{\mathrm{cl}} & :\equiv & \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\mathrm{cl}}, & \text{where} & \alpha \vee^{\mathrm{cl}} \beta & :\equiv & \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\mathrm{cl}}, & \text{where} & s =^{\mathrm{cl}} t & :\equiv & \neg\neg(s = t). \end{array}$$

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When we say: some statement "holds in $V^{\neg\neg}$",
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Ex. As $\neg\neg(\varphi \vee \neg\varphi)$ is a theorem of IQC, the law of excluded middle holds in $V^{\neg\neg}$.

The ∇-translation

For bounded first-order formulas over the (large) first-order signature which has

1. one sort $\underline{X}$ for each set X in the base universe,
2. one n-ary function symbol $\underline{f} : \underline{X_1} \times \cdots \times \underline{X_n} \to \underline{Y}$ for each map $f : X_1 \times \cdots \times X_n \to Y$,
3. one n-ary relation symbol $\underline{R} \hookrightarrow \underline{X_1} \times \cdots \times \underline{X_n}$ for each relation $R \subseteq X_1 \times \cdots \times X_n$, and
4. an additional unary relation symbol $G \hookrightarrow \underline{L}$ (for the *generic filter* of L),

we recursively define:

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

Finally, we say that φ "holds in V^∇" iff for all $\sigma \in L$, $\sigma \models \varphi$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma \in X^*).\, \forall(a \in X).\, \nabla(\tau \preccurlyeq \sigma).\, \exists(n \in \mathbb{N}).\, \tau[n] = a$.

The ∇-translation

$$
\begin{array}{llll}
\sigma \models s = t & \text{iff} \quad \nabla\sigma.\, [\![s]\!] = [\![t]\!]. & \sigma \models \underline{R}(s_1, \ldots, s_n) & \text{iff} \quad \nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!]). \\
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map $\mathbb{N} \to X$	“the gen. sequence is good”	Good $\mid [\,]$.
frame of opens	“every complex number has a square root”	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	“$x \neq 0 \Rightarrow x$ inv.”	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.

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big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.
little Zariski	"every f.g. vector space does *not not* have a basis"	**Grothendieck's generic freeness lemma**

Outlook

Passing to and from extensions

Thm. Let φ be a **bounded first-order formula** not mentioning G. In each of the following situations, we have that φ holds in V^∇ iff φ holds in V:

1. L and all coverings are inhabited (proximality).
2. L contains a top element, every covering of the top element is inhabited, and φ is a coherent implication (positivity).

The mystery of nongeometric sequents

The **generic ideal** of a ring is maximal:

$$(x \in \mathfrak{a} \Rightarrow 1 \in \mathfrak{a}) \Longrightarrow 1 \in \mathfrak{a} + (x).$$

The **generic ring** is a field:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow (\exists y.\, xy = 1).$$

Traveling the multiverse ...

LEM is a **switch** and **holds positively**; being countable is a **button**.

Every instance of DC **holds proximally**.

A geometric implication is provable iff it holds **everywhere**.

... upwards, but always keeping ties to the base.

More on forcing notions

Def. A **forcing notion** consists of a preorder L of **forcing conditions**, and for every $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of **coverings** of σ such that: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

	preorder L	coverings of an element $\sigma \in L$	filters of L
1	X^*	$\{\sigma x \mid x \in X\}$	maps $\mathbb{N} \to X$
2	X^*	$\{\sigma x \mid x \in X\}$, $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$ for each $a \in X$	surjections $\mathbb{N} \twoheadrightarrow X$
3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
5	opens	$\mathcal{U}$ such that $\sigma = \bigcup \mathcal{U}$	points
6	$\{\star\}$	$\{\star \mid \varphi\} \cup \{\star \mid \neg\varphi\}$	witnesses of LEM

Def. A *filter* of a forcing notion (L, Cov) is a subset $F \subseteq L$ such that

1. F is upward-closed: if $\tau \preccurlyeq \sigma$ and if $\tau \in F$, then $\sigma \in F$;
2. F is downward-directed: F is inhabited, and if $\alpha, \beta \in F$, then there is a common refinement $\sigma \preccurlyeq \alpha, \beta$ such that $\sigma \in F$; and
3. F splits the covering system: if $\sigma \in F$ and $R \in \mathrm{Cov}(\sigma)$, then $\tau \in F$ for some $\tau \in R$.