Internal Language of Rel

Molly Stewart-Gallus

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I wanted an internal language corresponding to the double category **Rel** the double category with sets as objects, relations as horizontal arrows and functions as vertical arrows.

I present three different calculi: a substructural "context calculus" corresponding to the horizontal edge of **Rel**; a "term calculus" corresponding to **Set**, the vertical edge category of **Rel**; and a simple "command calculus" corresponding to commuting squares of **Rel**.

First, I give the basic framework of the calculi. Then I discuss semantics and applications. And finally I list a few possible extensions such as coproduct and dependent sum types.

Core Calculi

The core context calculus is the linear lambda calculus with a few symbol changes and corresponds to the closed monoidal structure of the horizontal edge of **Rel**. The core term calculus is a little language with only product types and corresponds to the Cartesian structure of **Set**. I am still figuring out the core command calculus which ought to correspond to commuting squares in **Rel**.

The context calculus has a linear variable rule. To make mechanization easier and ensure the typing environment can always be directly inferred from context expressions linear variables are explicitly indexed by their types and variable binding is more imperative.

Grammar, Syntax and Reductions

$$
\begin{array}{lll}
\textit{type},\ t & ::= & \\
 & | & \mathbf{I} \\
 & | & t \otimes t'
\end{array}
$$

$$
\begin{array}{lll}
\textit{term},\ v & ::= &
\end{array}
$$

	$\vert$	x	
	$\vert$	$\mathbf{!}$	
	$\vert$	$\pi_1\ v$	
	$\vert$	$\pi_2\ v$	
	$\vert$	v, v'	
	$\vert$	(v)	S
	$\vert$	$[x := v]v'$	M
	$\vert$	$[\![N]\!]$	M
	$\vert$	$[:=\rho]v'$	M
normal, N	$::=$		
	$\vert$	$\mathbf{!}$	
	$\vert$	N, N'	
	$\vert$	(N)	S
context, E	$::=$		
	$\vert$	$\lambda\, x.E$	bind x in E
	$\vert$	$\mathbf{!}$	
	$\vert$	E, E'	
	$\vert$	$\mathbf{neu}\ e$	
	$\vert$	(E)	S
environment, Γ	$::=$		
	$\vert$	$\bullet$	
	$\vert$	$\Gamma, x\colon\ t$	
	$\vert$	(Γ)	S
subst, ρ	$::=$		
	$\vert$	$\bullet$	M
	$\vert$	$\rho, x := v$	M
	$\vert$	(ρ)	S
usage, Δ	$::=$		
	$\vert$	$\bullet$	
	$\vert$	Δ, u	
	$\vert$	$\emptyset_n$	M
store, σ	$::=$		
	$\vert$	$\bullet$	M
	$\vert$	$x\colon\ t := N$	M
	$\vert$	σ, σ'	M
	$\vert$	$\sigma + +\sigma'$	M

$span,\ P$	$::=$		
	$\vert$	$\sigma \vdash N$	
$spans,\ P^*$	$::=$		
	$\vert$	$\bullet$	
	$\vert$	P^*, P	
$nat,\ n$	$::=$		
	$\vert$	0	
	$\vert$	$n + 1$	
	$\vert$	$\mathbf{len}\, x^*$	M
	$\vert$	$\mathbf{len}\, \Gamma$	M
	$\vert$	$\mathbf{len}\, \Delta$	M
	$\vert$	(n)	S

$\boxed{[\![N]\!]}$

$$[\![\mathbf{!}]\!] \equiv \mathbf{!}$$
$$[\![N, N']\!] \equiv [\![N]\!], [\![N']\!]$$

$\boxed{[:=\rho]v'}$

$$[:= \bullet\,]v \equiv v$$
$$[:=\rho, x := v']v \equiv [:=\rho]([x := v']v)$$

$\boxed{\textbf{jdummy}}$

$\boxed{x\colon\ t \in \Gamma}$

$$\frac{}{x\colon\ t \in \Gamma, x\colon\ t} \quad \text{mem\_eq}$$

$$\frac{\begin{array}{c} x \neq y \\ x\colon\ t \in \Gamma \end{array}}{x\colon\ t \in \Gamma, y\colon\ t'} \quad \text{mem\_ne}$$

$\boxed{\Gamma \vdash v\colon\ t}$

$$\frac{x\colon\ t \in \Gamma}{\Gamma \vdash x\colon\ t} \quad \text{Jv\_var}$$

$$\frac{}{\Gamma \vdash\ \mathbf{!}\colon \mathbf{I}} \quad \text{Jv\_tt}$$

$$\frac{\begin{array}{c} \Gamma \vdash v_1\colon\ t_1 \\ \Gamma \vdash v_2\colon\ t_2 \end{array}}{\Gamma \vdash v_1, v_2\colon\ t_1 \otimes t_2} \quad \text{Jv\_fanout}$$

$$\frac{\Gamma \vdash v\colon\ t_1 \otimes t_2}{\Gamma \vdash \pi_1\, v\colon\ t_1} \quad \text{Jv\_fst}$$

$$\frac{\Gamma \vdash v\colon\ t_1 \otimes t_2}{\Gamma \vdash \pi_2\, v\colon\ t_2} \quad \text{Jv\_snd}$$

$\boxed{\rho\colon \Gamma}$

$$\frac{}{\bullet\colon\ \bullet} \quad \text{Jp\_nil}$$

$$\frac{\begin{array}{l}\Gamma \vdash v\colon\ t \\ \rho\colon \Gamma\end{array}}{\rho, x := v\colon \Gamma, x\colon\ t} \quad \text{Jp\_cons}$$

$\boxed{v \Downarrow N}$

$$\frac{}{\mathbf{!} \Downarrow \mathbf{!}} \quad \text{big\_tt}$$

$$\frac{\begin{array}{l}v_1 \Downarrow N_1 \\ v_2 \Downarrow N_2\end{array}}{v_1, v_2 \Downarrow N_1, N_2} \quad \text{big\_fanout}$$

$$\frac{v \Downarrow N_1, N_2}{\pi_1\ v \Downarrow N_1} \quad \text{big\_fst}$$

$$\frac{v \Downarrow N_1, N_2}{\pi_2\ v \Downarrow N_2} \quad \text{big\_snd}$$

$\boxed{\emptyset_n}$

$$\begin{array}{l}\emptyset_0 \equiv \bullet \\ \emptyset_{n+1} \equiv \emptyset_n, \mathbf{f}\end{array}$$

$\boxed{\mathbf{xsof}\,\Gamma}$

$$\begin{array}{l}\mathbf{xsof}\ \bullet\ \equiv \bullet \\ \mathbf{xsof}\,(\Gamma, x\colon\ t) \equiv \mathbf{xsof}\,\Gamma, x\end{array}$$

$\boxed{x \in x^*; \Delta \to \Delta'}$

$$\frac{\mathbf{len}\, x^* = \mathbf{len}\, \Delta}{x \in x^*, x; \Delta, \mathbf{f} \to \Delta, \mathbf{s}} \quad \text{lmem\_eq}$$

$$\frac{\begin{array}{l}x \neq y \\ x \in x^*; \Delta \to \Delta'\end{array}}{x \in x^*, y; \Delta, u \to \Delta', u} \quad \text{lmem\_ne}$$

$\boxed{\Gamma; \Delta \to \Delta' \vdash e\colon\ t}$

$$\frac{\begin{array}{l}x\colon\ t \in \Gamma \\ x \in \mathbf{xsof}\,\Gamma; \Delta \to \Delta'\end{array}}{\Gamma; \Delta \to \Delta' \vdash x\colon\ t} \quad \text{infer\_var}$$

$$\frac{\Gamma; \Delta_1 \to \Delta_2 \vdash e_1 \colon t_1 \otimes t_2 \quad \Gamma; \Delta_2 \to \Delta_3 \vdash E_2 \colon t_1}{\Gamma; \Delta_1 \to \Delta_3 \vdash e_1\, E_2 \colon t_2} \text{ infer\_app}$$

$$\frac{\Gamma; \Delta_1 \to \Delta_2 \vdash e_1 \colon \mathbf{I} \quad \Gamma; \Delta_2 \to \Delta_3 \vdash E_2 \colon t}{\Gamma; \Delta_1 \to \Delta_3 \vdash e_1; E_2 \colon t \colon t} \text{ infer\_step}$$

$$\frac{\Gamma; \Delta_1 \to \Delta_2 \vdash e_1 \colon t_1 \otimes t_2 \quad \Gamma, x\colon t_1, y\colon t_2; \Delta_2, \mathbf{f}, \mathbf{f} \to \Delta_3, \mathbf{s}, \mathbf{s} \vdash E_2 \colon t_3}{\Gamma; \Delta_1 \to \Delta_3 \vdash \ \mathbf{let}\ x, y = e_1\ \mathbf{in}\ E_2 \colon t_3 \colon t_3} \text{ infer\_let}$$

$$\frac{\Gamma; \Delta \to \Delta' \vdash E\colon t}{\Gamma; \Delta \to \Delta' \vdash \mathbf{cut}\, E\, t\colon t} \text{ infer\_cut}$$

$\boxed{\Gamma; \Delta \to \Delta' \vdash E\colon t}$

$$\frac{\Gamma, x\colon t_1; \Delta, \mathbf{f} \to \Delta', \mathbf{s} \vdash E\colon t_2}{\Gamma; \Delta \to \Delta' \vdash \lambda\, x.E\colon t_1 \otimes t_2} \text{ check\_lam}$$

$$\frac{\mathbf{len}\,\Gamma = \mathbf{len}\,\Delta}{\Gamma; \Delta \to \Delta \vdash \ \mathbf{!} \colon \mathbf{I}} \text{ check\_tt}$$

$$\frac{\Gamma; \Delta_1 \to \Delta_2 \vdash E_1 \colon t_1 \quad \Gamma; \Delta_2 \to \Delta_3 \vdash E_2 \colon t_2}{\Gamma; \Delta_1 \to \Delta_3 \vdash E_1, E_2 \colon t_1 \otimes t_2} \text{ check\_fanout}$$

$$\frac{\Gamma; \Delta \to \Delta' \vdash e\colon t}{\Gamma; \Delta \to \Delta' \vdash \mathbf{neu}\, e\colon t} \text{ check\_neu}$$

$\boxed{\sigma \vdash e[N] \colon t}$

$$\frac{}{x\colon t := N \vdash x[N] \colon t} \text{ sate\_var}$$

$$\frac{\sigma \vdash e[\mathbf{!}] \colon \mathbf{I} \quad \sigma' \vdash E'[N] \colon t}{\sigma + +\sigma' \vdash (e; E' \colon t)[N] \colon t} \text{ sate\_step}$$

$$\frac{\sigma \vdash e[N_0, N_1] \colon t_1 \otimes t_2 \quad \sigma', x\colon t_1 := N_0, y\colon t_2 := N_1 \vdash E'[N_2] \colon t}{\sigma + +\sigma' \vdash (\mathbf{let}\, x, y = e\ \mathbf{in}\ E' \colon t)[N_2] \colon t} \text{ sate\_let}$$

$$\frac{\sigma \vdash e[N, N'] \colon t_1 \otimes t_2 \quad \sigma' \vdash E'[N] \colon t_1}{\sigma + +\sigma' \vdash (e\, E')[N'] \colon t_2} \text{ sate\_app}$$

$$\frac{\sigma \vdash E[N] \colon t}{\sigma \vdash (\mathbf{cut}\, E\, t)[N] \colon t} \text{ sate\_cut}$$

$\boxed{\sigma \vdash E[N] \colon t}$

$$\frac{}{\bullet \vdash \mathbf{!}[\mathbf{!}] \colon \mathbf{I}} \quad \text{satE\_tt}$$

$$\frac{\begin{array}{c}\sigma \vdash E[N] \colon t_1 \\ \sigma' \vdash E'[N'] \colon t_2\end{array}}{\sigma + +\sigma' \vdash (E, E')[N, N'] \colon t_1 \otimes t_2} \quad \text{satE\_fanout}$$

$$\frac{\sigma, x \colon t_1 := N \vdash E[N'] \colon t_2}{\sigma \vdash (\lambda\, x.E)[N, N'] \colon t_1 \otimes t_2} \quad \text{satE\_lam}$$

$$\frac{\sigma \vdash e[N] \colon t}{\sigma \vdash (\mathbf{neu}\, e)[N] \colon t} \quad \text{satE\_neu}$$

$\boxed{E[P^*] \colon t}$

$$\frac{}{E[\bullet] \colon t} \quad \text{sound\_nil}$$

$$\frac{\begin{array}{c}E[P^*] \colon t \\ \sigma \vdash E[N] \colon t\end{array}}{E[P^*, \sigma \vdash N] \colon t} \quad \text{sound\_cons}$$

Examples

Identity

$$\mathbf{id}_t = \lambda X \colon t.X \quad \dfrac{\dfrac{\top}{\bullet, X \colon t \vdash X \colon t}\,(\mathrm{V})}{\bullet \vdash \lambda X \colon t.X \colon t \otimes t}\,(\times\mathrm{I}) \qquad \dfrac{\dfrac{\top}{\bullet, X[N] \vdash X[N]}\,(\mathrm{V})}{\bullet \vdash (\lambda X \colon t.X)[N, N]}\,(\times\mathrm{I})$$

Composition

$$f \circ_t g = \lambda X \colon t.\, f\,(g\,X)$$

$$\dfrac{\dfrac{\bullet \vdash f \colon t_1 \otimes t_2 \qquad \dfrac{\bullet \vdash g \colon t_0 \otimes t_1 \qquad \dfrac{\top}{\bullet,\ X : t_0 \vdash X : t_0}\,(\times\mathrm{V})}{\bullet,\ X : t_0 \vdash g\,X : t_1}\,(\times\mathrm{E})}{\bullet,\ X : t_0 \vdash f\,(g\,X) : t_2}\,(\times\mathrm{E})}{\bullet \vdash \lambda X : t_0.\, f\,(g\,X) : t_0 \otimes t_2}\,(\times\mathrm{I})$$

$$\dfrac{\dfrac{\bullet \vdash f[N_1, N_2] \qquad \dfrac{\bullet \vdash g[N_0, N_1] \qquad \dfrac{\top}{\bullet,\ X[N_0] \vdash X[N_0]}\,(\mathrm{V})}{\bullet,\ X[N_0] \vdash g\,X[N_1]}\,(\times\mathrm{E})}{\bullet,\ X[N_0] \vdash f\,(g\,X)[N_2]}\,(\times\mathrm{E})}{\bullet \vdash \lambda X : t_0.\, f\,(g\,X)[N_0, N_2]}\,(\times\mathrm{I})$$

Categorical Semantics

The intent is to create calculi encoding the core features of the double category **Rel**. If this is successful then terms and types ought to map to **Rel** as follows. Note that defining normalization in terms of closed terms means a little workaround of multisubstition is required.

$$
\begin{aligned}
&\Gamma \vdash v \colon t \\
&\Delta \vdash E \colon t \\
&\sigma \colon \Delta \\
&\rho \colon \Gamma \\
&[:= \rho]v \Downarrow N
\end{aligned}
$$

$$
\begin{tikzcd}
\mathbf{I} \arrow[r, leftrightarrow, "\mathbf{id}"] \arrow[dd, "\sigma"'] \arrow[ddr, phantom, "\sigma \vdash E[N] \; \Rrightarrow"] & \mathbf{I} \arrow[d, "\rho"] \\
 & \Gamma \arrow[d, "v"] \\
\Delta \arrow[r, "E"'] & t
\end{tikzcd}
$$

I have no idea about universe issues and such. Dependent sum is probably wrong. Really need to think about denotation again.

$$
[\![t_0 \times t_1]\!] = [\![t_0]\!] \times [\![t_1]\!]
$$

$$
\begin{aligned}
[\![x]\!](\sigma) &= \sigma(x) \\
[\![v_0, v_1]\!](\sigma) &= ([\![v_0]\!](\sigma), [\![v_1]\!](\sigma))
\end{aligned}
$$

$$
\begin{aligned}
[\![X]\!](\sigma, s) &= \sigma(X, s) \\
[\![\mathbf{!}]\!](\bullet, \mathbf{!}) &= \top \\
[\![E_0; E_1]\!](\sigma_0 \cup \sigma_1, s) &= [\![E_0]\!](\sigma_0, \mathbf{!}) \wedge [\![E_1]\!](\sigma_1, s) \\
[\![E_0, E_1]\!](\sigma_0 \cup \sigma_1, (s_0, s_1)) &= [\![E_0]\!](\sigma_0, s_0) \wedge [\![E_1]\!](\sigma_1, s_1) \\
[\![E_0\ E_1]\!](\sigma, s_0) &= \exists s_1, [\![E_1]\!](\sigma, s_1) \wedge [\![E_0]\!](\sigma, (s_1, s_0)) \\
[\![\lambda X \colon t.E_0]\!](\sigma, (s_0, s_1)) &= \forall E_1, [\![E_1]\!](\sigma, s_0) \to [\![E_0]\!]([X := s \mapsto [\![E_1]\!](\sigma, s)]\sigma, s_1)
\end{aligned}
$$

Applications to Synthetic Category Theory

A category is a monad in **Span**. Once this system has been generalized to **Span** we can define monads internal to **Span**.

This is not fully internal but a simple approach to defining an equivalence relation might be something like:

$$\boxed{\text{object}}\ \bullet \vdash O\colon *$$

$$\boxed{\text{arrow}}\ \bullet \vdash R\colon O \times O$$

$$\boxed{\text{refl}}\ \frac{\bullet \vdash o\colon O}{\bullet \vdash R[o,o]}$$

$$\boxed{\text{trans}}\ \frac{\bullet \vdash \lambda X\colon O.R\,(R\,X)[o_0,o_1]}{\bullet \vdash R[o_0,o_1]}\ \{\bullet \vdash o_1, o_0 \colon O \times O\}$$

$$\boxed{\text{sym}}\ \frac{\bullet \vdash R[o_0,o_1]}{\bullet \vdash R[o_1,o_0]}\ \{\bullet \vdash o_1, o_0 \colon O \times O\}$$

Generalizing to a constructive interpretation in terms of spans and groupoids is future work.

Extensions

Disjoint union and dependent sum types.

Disjoint Unions

Disjoint unions in **Set** become Cartesian products/coproducts in **Rel**.

I have a hunch it is proper for the combination of product/coproduct to introduce nondeterminism in the operational semantics but I need to think more about the issue.

$$\begin{array}{ll}
\textbf{Types} & t ::= \ldots \mid \emptyset \mid t \oplus t \\
\textbf{Contexts} & E ::= \ldots \mid \mathbf{abort}_t\, E \mid \mathbf{i}_{1t}\, E \mid \mathbf{i}_{2t}\, E \mid \\
& \quad \mathbf{m}(E_0, X.E_1, Y.E_2) \mid \mathbf{false} \mid \mathbf{l}\, E \mid \mathbf{r}\, E \mid E; E' \\
\textbf{Terms} & v ::= \ldots \mid \mathbf{abort}_t\, v \mid \mathbf{i}_{1t}\, v \mid \mathbf{i}_{2t}\, v \mid \mathbf{m}(v_0, x.v_1, y.v_2)
\end{array}$$

Context Calculus

$$\boxed{\emptyset\,\mathrm{E}^{\mathrm{T}}}\ \frac{\Delta \vdash E\colon \emptyset}{\Delta \vdash \mathbf{abort}_t\, E\colon t}$$

$$\boxed{+\mathrm{I}_1^{\mathrm{T}}}\ \frac{\Delta \vdash E\colon t_0}{\Delta \vdash \mathbf{i}_{1t_1}\, E\colon t_0 \oplus t_1}$$

$$\boxed{+\mathrm{I}_2^{\mathrm{T}}}\ \frac{\Delta \vdash E\colon t_1}{\Delta \vdash \mathbf{i}_{2t_0}\, E\colon t_0 \oplus t_1}$$

$$\boxed{+\mathrm{E}^{\mathrm{T}}}\ \frac{\Delta \vdash E_0\colon t_0 \oplus t_1 \qquad \Delta,\ X_0\colon t_0 \vdash E_1\colon t_2 \qquad \Delta,\ X_1\colon t_1 \vdash E_1\colon t_2}{\Delta \vdash \mathbf{m}(E_0, x_0.E_1, x_1.E_2)\colon t_2}$$

$$\boxed{\emptyset\mathrm{I}}\ \Delta \vdash \mathbf{false}\colon \emptyset$$

$$\boxed{+\mathrm{E}_1}\ \frac{\Delta \vdash E\colon t_0 \oplus t_1}{\Delta \vdash \mathbf{l}\, E\colon t_0}$$

$$\boxed{+\mathrm{E}_2}\ \frac{\Delta \vdash E\colon t_0 \oplus t_1}{\Delta \vdash \mathbf{r}\, E\colon t_1}$$

$$\boxed{+\mathrm{I}}\ \frac{\Delta \vdash E_0\colon t_0 \qquad \Delta \vdash E_1\colon t_1}{\Delta \vdash E_0; E_1\colon t_0 \oplus t_1}$$

Term Calculus

$$\boxed{\emptyset\mathrm{E}}\ \frac{\Gamma \vdash v\colon\ \emptyset}{\Gamma \vdash \mathbf{abort}_t\, v\colon t}$$

$$\boxed{+\mathrm{I}_1}\ \frac{\Gamma \vdash v\colon t_0}{\Gamma \vdash \mathbf{i}_{1t_1}\, v\colon t_0 \oplus t_1} \qquad \boxed{+\mathrm{I}_2}\ \frac{\Gamma \vdash v\colon t_1}{\Gamma \vdash \mathbf{i}_{2t_0}\, v\colon t_0 \oplus t_1}$$

$$\boxed{+\mathrm{E}}\ \frac{\Gamma \vdash v_0\colon t_0 \oplus t_1 \qquad \Gamma,\, x_0\colon t_0 \vdash v_1\colon t_2 \qquad \Gamma,\, x_1\colon t_1 \vdash v_1\colon t_2}{\Gamma \vdash \mathbf{m}(v_0, x_0.v_1, x_1.v_2)\colon t_2}$$

$$\boxed{+\beta_1}\ \mathbf{m}(\mathbf{i}_{1t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_0 := v_0]v_1$$

$$\boxed{+\beta_2}\ \mathbf{m}(\mathbf{i}_{2t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_1 := v_0]v_2$$

Command Calculus

$$\boxed{+\mathrm{I}_1}\ \frac{\sigma \vdash E_0[N]}{\sigma \vdash (E_0; E_1)[\mathbf{i}_{1t}\, N]} \qquad \boxed{+\mathrm{I}_2}\ \frac{\sigma \vdash E_1[N]}{\sigma \vdash (E_0; E_1)[\mathbf{i}_{2t}\, N]}$$

Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to become a little like Π types. So the predicate calculus effectively becomes like a linear System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Not really good at the typing judgements for dependent sum types.

Types	$t ::= \ldots \mid x \mid * \mid \mathbf{h}(v) \mid \Sigma X\colon *.t$
Contexts	$E ::= \ldots \mid E\, t \mid \lambda X\colon *.E$
Terms	$v ::= \ldots \mid \mathbf{t}(v) \mid \langle x := t, v\rangle$
Command Env	$\sigma ::= \ldots \mid \Delta,\, X[t]$

Context Calculus

$$\boxed{\Sigma\mathrm{E}}\ \frac{\Delta_0 \vdash E\colon \Sigma X\colon *.t_1 \qquad \Delta_1 \vdash t_0\colon *}{\Delta_0,\, \Delta_1 \vdash E_0\, t_0\colon [X := t_0]t_1}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Delta,\, x\colon * \vdash E\colon t}{\Delta \vdash \lambda X\colon *.E\colon \Sigma X\colon *.t}$$

Term Calculus

$$\boxed{\Sigma\mathrm{E}_1}\ \frac{\Gamma \vdash E\colon \Sigma x\colon *.t}{\Gamma \vdash \mathbf{h}(v)\colon *}$$

$$\boxed{\Sigma\mathrm{E}_2}\ \frac{\Gamma \vdash v\colon \Sigma x\colon *.t}{\Gamma \vdash \mathbf{t}(v)\colon [x := \mathbf{h}(v)]t}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Gamma \vdash t_0 \colon\ * \qquad \Gamma,\, x \colon\ * \vdash v \colon t_1}{\Gamma \vdash \langle x := t_0, v\rangle \colon \Sigma x \colon\ * . t_0}$$

$$\boxed{\Sigma\beta_1}\ \mathbf{h}(\langle x := t, v\rangle) \rightsquigarrow t \qquad \boxed{\Sigma\beta_2}\ \mathbf{t}(\langle x := t, v\rangle) \rightsquigarrow [x := t]v$$

Command Calculus

$$\boxed{\Sigma\mathrm{I}}\ \frac{\sigma,\, X[t] \vdash E[N]}{\sigma \vdash (\lambda X \colon\ * . E)[\langle x := t, N\rangle]}$$

I can't figure out commands here at all.

The Future?

Satisifies judgments correspond to thin squares. Moving to more general categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.