I wanted to try out making a proof assistant corresponding to the double category **Rel**.

You have a calculus of "values" corresponding to one edge of **Rel** and a sort of relational calculus of "predicates" corresponding to the other edge. Squares ought to correspond to judgements stating a value satisfies a predicate.

The language of "predicates" is based off the Simply Typed Lambda Calculus (in category theory terms is closed/has exponential objects) and the language of "values" has product types (in category theory terms is Cartesian.)

This is the core framework. I've been thinking about further extensions but I want to see how far I can get characterizing **Rel** with as few language features as possible. Later on I give a few possible extensions.

$$\begin{tikzcd} \top \arrow[r, "\sigma_v"] \arrow[d, "\sigma_p"'] \arrow[dr, phantom, "v \models p\,[\sigma_v, \sigma_p]"] & \prod_{x\colon t! \in \Gamma_v} t \arrow[d, "v"] \\ \sum_{x\colon t? \in \Gamma_p} t \arrow[r, "p"'] & t \end{tikzcd}$$

Core Calculus

Types	$t ::= \top \mid t \times t$
Sorts	$s ::= t? \mid t!$
Values	$v ::= x \mid \mathbf{tt} \mid \mathbf{fst}(v) \mid \mathbf{snd}(v) \mid (v, v)$
Predicates	$p ::= x \mid \mathbf{fail}_t(p) \mid p\,p \mid \mu(x\colon t?).p$
Environment	$\Gamma ::= \cdot \mid \Gamma,\, x\colon s$
Substitutions	$\sigma ::= \cdot \mid \sigma,\, v \models x$

I need a better name for the abstraction for predicates. It's a little like the μ abstraction from the $\bar{\lambda}\mu\tilde{\mu}$ calculus but different.

Typing judgments.

$$\frac{}{\Gamma \vdash \mathbf{tt}\colon \top!}$$
$$\frac{\Gamma \vdash v\colon t_0 \times t_1!}{\Gamma \vdash \mathbf{fst}(v)\colon t_0!}$$
$$\frac{\Gamma \vdash v\colon t_0 \times t_1!}{\Gamma \vdash \mathbf{snd}(v)\colon t_1!}$$
$$\frac{\Gamma \vdash v_0\colon t_0! \quad \Gamma \vdash v_1\colon t_1!}{\Gamma \vdash (v_0, v_1)\colon t_0 \times t_1!}$$
$$\frac{\Gamma \vdash p\colon \top?}{\Gamma \vdash \mathbf{fail}_t(p)\colon t?}$$
$$\frac{\Gamma \vdash p_0\colon t_0 \times t_1? \quad \Gamma \vdash p_1\colon t_0?}{\Gamma \vdash p_0\,p_1\colon t_1?}$$
$$\frac{\Gamma,\, x\colon t_0? \vdash p\colon t_1?}{\Gamma \vdash \mu(x\colon t_0?).p\colon t_0 \times t_1?}$$

When a value satisfies a predicate (I need a better symbol here.)

$$\frac{v_0 \models p \quad [\sigma]}{\mathbf{fst}(v_0, v_1) \models p \quad [\sigma]}$$
$$\frac{v_1 \models p \quad [\sigma]}{\mathbf{snd}(v_0, v_1) \models p \quad [\sigma]}$$
$$\frac{v \models \mathbf{fail}_t(p) \quad [\sigma]}{\mathbf{tt} \models p\ [\sigma]}$$
$$\frac{v \models [x \leftarrow p_1]p_0 \quad [\sigma]}{v \models (\mu(x\colon t?).p_0)\,p_1 \quad [\sigma]}$$
$$\frac{v_1 \models p \quad [\sigma,\, v_0 \models x]}{(v_0, v_1) \models \mu(x\colon t?).p \quad [\sigma]}$$

Examples

Pattern matching on equality

$$(v, v) \models \mu(x\colon t?).x$$

Transposition

$$\mu(p\colon t \times t \times \top)(x\colon t)(y\colon t).p\,y\,x$$

Disjoint Unions

I am fairly confident in a simple extension to disjoint unions of sets which are sum types in the value calculus and product types in the predicate calculus.

Types $t ::= \ldots \mid \bot \mid t + t$

Values $v ::= \ldots \mid \mathbf{absurd}_t(v) \mid \mathbf{inl}_t(v) \mid \mathbf{inr}_t(v) \mid \mathbf{match}\ v \begin{cases} v & \leftarrow \mathbf{inl}(x) \\ v & \leftarrow \mathbf{inr}(x) \end{cases}$

Predicates $p ::= \ldots \mid \mathbf{false} \mid \mathbf{left}(p) \mid \mathbf{right}(p) \mid [p; p]$

Typing judgments.

$$\frac{\Gamma \vdash v \colon \bot!}{\Gamma \vdash \mathbf{absurd}_t(v) \colon t!}$$

$$\frac{\Gamma \vdash v \colon t_0!}{\Gamma \vdash \mathbf{inl}_{t_1}(v) \colon t_0 + t_1!}$$

$$\frac{\Gamma \vdash v \colon t_1!}{\Gamma \vdash \mathbf{inr}_{t_0}(v) \colon t_0 + t_1!}$$

$$\frac{\Gamma \vdash v_0 \colon t_0 + t_1! \; \Gamma,\, x_0 \colon t_0! \vdash v_1 \colon t_2! \; \Gamma,\, x_1 \colon t_1! \vdash v_2 \colon t_2!}{\Gamma \vdash \mathbf{match}\ v_0 \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \colon t_2!}$$

$$\frac{}{\Gamma \vdash \mathbf{false} \colon \bot?}$$

$$\frac{\Gamma \vdash p \colon t_0 + t_1?}{\Gamma \vdash \mathbf{left}(p) \colon t_0?}$$

$$\frac{\Gamma \vdash p \colon t_0 + t_1?}{\Gamma \vdash \mathbf{right}(p) \colon t_1?}$$

$$\frac{\Gamma \vdash p_0 \colon t_0? \quad \Gamma \vdash p_1 \colon t_1?}{\Gamma \vdash [p_0; p_1] \colon t_0 + t_1?}$$

Satisfies

$$\frac{[x_0 \leftarrow v_0]v_1 \models p \quad [\sigma]}{\mathbf{match}\ \mathbf{inl}_t(v_0) \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \models p\ [\sigma]}$$

$$\frac{[x_1 \leftarrow v_0]v_2 \models p \quad [\sigma]}{\mathbf{match}\ \mathbf{inr}_t(v_0) \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \models p\ [\sigma]}$$

$$\frac{v \models p_0\ [\sigma]}{v \models \mathbf{left}([p_0; p_1]) \quad [\sigma]}$$

$$\frac{v \models p_1\ [\sigma]}{v \models \mathbf{right}([p_0; p_1]) \quad [\sigma]}$$

$$\frac{\mathbf{absurd}_t(v) \models p \quad [\sigma]}{v \models \mathbf{false}\ [\sigma]}$$

$$\frac{v \models p_0\ [\sigma]}{\mathbf{inl}_t(v) \models [p_0; p_1] \quad [\sigma]}$$

$$\frac{v \models p_1\ [\sigma]}{\mathbf{inr}_t(v) \models [p_0; p_1] \quad [\sigma]}$$

Dependent Sums

If product of sets becomes an exponential in the predicate calculus then dependent sums ought to become like Π types. So the predicate calculus effectively becomes like System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Types $t ::= \ldots \mid x \mid \mathbf{head}(v) \mid \Sigma(x\colon\ *).t$

Sorts $s ::= \ldots \mid *$

Values $v ::= \ldots \mid \mathbf{tail}(v) \mid \langle x \leftarrow t, v \rangle$

Predicates $p ::= \ldots \mid p\, t \mid \mathbf{M}(x\colon\ *).p$

Substitutions $\sigma ::= \ldots \mid \sigma,\, t \models x$

Not really good at the typing judgements for dependent sum types.

$$\frac{\Gamma \vdash v \colon \Sigma(x \colon *).t!}{\Gamma \vdash \mathbf{head}(v) \colon *}$$

$$\frac{\Gamma \vdash v \colon \Sigma(x \colon *).t!}{\Gamma \vdash \mathbf{tail}(v) \colon [x \leftarrow \mathbf{head}(v)]t!}$$

$$\frac{\Gamma \vdash t_0 \colon * \quad \Gamma,\, x \colon * \vdash v \colon t_1!}{\Gamma \vdash \langle x \leftarrow t_0, v\rangle \colon \Sigma(x \colon *).t_0!}$$

$$\frac{\Gamma \vdash p \colon \Sigma(x \colon *).t_1? \quad \Gamma \vdash t_0 \colon *}{\Gamma \vdash p_0\, t_0 \colon [x \leftarrow t_0]t_1?}$$

$$\frac{\Gamma,\, x \colon * \vdash p \colon t?}{\Gamma \vdash \mathbf{M}(x \colon *).p \colon \Sigma(x \colon *).t?}$$

I can't figure out satisfaction at all.

$$\frac{t \models p \quad [\sigma]}{\mathbf{head}(\langle x \leftarrow t, v\rangle) \models p \quad [\sigma]}$$

$$\frac{[x \leftarrow t]v \models p \quad [\sigma]}{\mathbf{tail}(\langle x \leftarrow t, v\rangle) \models p \quad [\sigma]}$$

$$\frac{v \models [x \leftarrow t]p \quad [\sigma]}{v \models (\mathbf{M}(x \colon *).p)\, t \quad [\sigma]}$$

$$\frac{v \models p\ [\sigma,\, t \models x]}{\langle x \leftarrow t, v\rangle \models \mathbf{M}(x \colon *).p \quad [\sigma]}$$

The Future?

Satisifies judgments correspond to thin squares. Moving to more generally categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.