I wanted to try out making a proof assistant corresponding to the double category **Rel**.

You have a calculus of "values" corresponding to one edge of **Rel** and a sort of relational calculus of "predicates" corresponding to the other edge. Squares ought to correspond to judgements stating a value satisfies a predicate.

The language of "values" handling Cartesian product of sets has product types (in category theory terms is Cartesian.)

The language of "predicates" ought to be more complicated. **Rel** is a closed monoidal category over Cartesian product of sets. One has an isomorphism $\mathbf{Rel}(A, B \otimes C) \sim \mathbf{Rel}(A \otimes B, C)$. So some sort of substructural type theory ought to be the basic framework. I have a hunch explicit unification of logical variables corresponds to explicit duplication of resources but this is a guess.

This is the core framework. I've been thinking about further extensions but I want to see how far I can get characterizing **Rel** with as few language features as possible. Later on I give a few possible extensions.

$$\begin{tikzcd} \top \arrow[r, "\sigma_v"] \arrow[d, "\sigma_p"'] \arrow[dr, phantom, "v \models p\,[\sigma_v, \sigma_p]"] & \times_{x:\, t\,\mathbf{v} \in \Gamma_v} t \arrow[d, "v"] \\ \times_{x:\, t\,\mathbf{p} \in \Gamma_p} t \arrow[r, "p"'] & t \end{tikzcd}$$

Core Calculus

Types	$t ::= \top \mid t \times t$
Sorts	$s ::= t\,\mathbf{p} \mid t\,\mathbf{v}$
Values	$v ::= x \mid \mathbf{tt} \mid \mathbf{fst}(v) \mid \mathbf{snd}(v) \mid (v, v)$
Predicates	$p ::= x \mid \mathbf{pass}(p, p) \mid p\,p \mid \forall(x \colon t\,\mathbf{p}).p$
Environment	$\Gamma ::= \cdot \mid \Gamma,\, x \colon s$
Substitutions	$\sigma ::= \cdot \mid \sigma,\, v \models x$

I need a better name for the abstraction for predicates. It's a little like the μ abstraction from the $\bar{\lambda}\mu\tilde{\mu}$ calculus but different. I called it $\forall$ because it's opposite to application/composition in **Rel**which is existential quantification.

The core calculus is based off multiplicative linear type theory.

Typing judgments.

$$\frac{}{\Gamma \vdash \mathbf{tt} \colon \top\,\mathbf{v}}$$
$$\frac{\Gamma \vdash v \colon t_0 \times t_1\,\mathbf{v}}{\Gamma \vdash \mathbf{fst}(v) \colon t_0\,\mathbf{v}}$$
$$\frac{\Gamma \vdash v \colon t_0 \times t_1\,\mathbf{v}}{\Gamma \vdash \mathbf{snd}(v) \colon t_1\,\mathbf{v}}$$
$$\frac{\Gamma \vdash v_0 \colon t_0\,\mathbf{v} \quad \Gamma \vdash v_1 \colon t_1\,\mathbf{v}}{\Gamma \vdash (v_0, v_1) \colon t_0 \times t_1\,\mathbf{v}}$$
$$\frac{\Gamma \vdash p_0 \colon \top\,\mathbf{p} \quad \Delta \vdash p_1 \colon t\,\mathbf{p}}{\Gamma,\, \Delta \vdash \mathbf{pass}(p, p_1) \colon t\,\mathbf{p}}$$
$$\frac{\Gamma \vdash p_0 \colon t_0 \times t_1\,\mathbf{p} \quad \Delta \vdash p_1 \colon t_0\,\mathbf{p}}{\Gamma,\, \Delta \vdash p_0\,p_1 \colon t_1\,\mathbf{p}}$$
$$\frac{\Gamma,\, x \colon t_0\,\mathbf{p} \vdash p \colon t_1\,\mathbf{p}}{\Gamma \vdash \forall(x \colon t_0\,\mathbf{p}).p \colon t_0 \times t_1\,\mathbf{p}}$$

When a value satisfies a predicate (I need a better symbol here.)

$$\frac{v_0 \models p \quad [\sigma]}{\mathbf{fst}(v_0, v_1) \models p \quad [\sigma]}$$
$$\frac{v_1 \models p \quad [\sigma]}{\mathbf{snd}(v_0, v_1) \models p \quad [\sigma]}$$
$$\frac{v \models \mathbf{pass}(p_0, p_1) \quad [\sigma]}{\mathbf{tt} \models p_0\ [\sigma]}$$
$$\frac{v \models [x \leftarrow p_1]p_0 \quad [\sigma]}{v \models (\forall(x \colon t\,\mathbf{p}).p_0)\,p_1 \quad [\sigma]}$$
$$\frac{v_1 \models p \quad [\sigma,\, v_0 \models x]}{(v_0, v_1) \models \forall(x \colon t\,\mathbf{p}).p \quad [\sigma]}$$

Examples

Pattern matching on equality

$$(v, v) \models \forall(x \colon t\,\mathbf{p}).x$$

Transposition

$$\forall(p \colon t \times t \times \top)(x \colon t)(y \colon t).p\,y\,x$$

Disjoint Unions

Disjoint unions in set become Cartesian product/coproduct in **Rel**.

I am fairly confident in a simple extension to disjoint unions of sets which are sum types in the value calculus and product types in the predicate calculus.

Types $t ::= \ldots \mid \bot \mid t + t$
Values $v ::= \ldots \mid \mathbf{absurd}_t(v) \mid \mathbf{inl}_t(v) \mid \mathbf{inr}_t(v) \mid \mathbf{match}\ v \begin{cases} v & \leftarrow \mathbf{inl}(x) \\ v & \leftarrow \mathbf{inr}(x) \end{cases}$
Predicates $p ::= \ldots \mid \mathbf{false} \mid \mathbf{left}(p) \mid \mathbf{right}(p) \mid [p; p]$

Typing judgments.

$$\frac{\Gamma \vdash v\colon \bot\ \mathbf{v}}{\Gamma \vdash \mathbf{absurd}_t(v)\colon t\ \mathbf{v}}$$

$$\frac{\Gamma \vdash v\colon t_0\ \mathbf{v}}{\Gamma \vdash \mathbf{inl}_{t_1}(v)\colon t_0 + t_1\ \mathbf{v}}$$

$$\frac{\Gamma \vdash v\colon t_1\ \mathbf{v}}{\Gamma \vdash \mathbf{inr}_{t_0}(v)\colon t_0 + t_1\ \mathbf{v}}$$

$$\frac{\Gamma \vdash v_0\colon t_0 + t_1\ \mathbf{v} \quad \Gamma,\, x_0\colon t_0\ \mathbf{v} \vdash v_1\colon t_2\ \mathbf{v} \quad \Gamma,\, x_1\colon t_1\ \mathbf{v} \vdash v_2\colon t_2\ \mathbf{v}}{\Gamma \vdash \mathbf{match}\ v_0 \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \colon t_2\ \mathbf{v}}$$

$$\frac{}{\Gamma \vdash \mathbf{false}\colon \bot\ \mathbf{p}}$$

$$\frac{\Gamma \vdash p\colon t_0 + t_1\ \mathbf{p}}{\Gamma \vdash \mathbf{left}(p)\colon t_0\ \mathbf{p}}$$

$$\frac{\Gamma \vdash p\colon t_0 + t_1\ \mathbf{p}}{\Gamma \vdash \mathbf{right}(p)\colon t_1\ \mathbf{p}}$$

$$\frac{\Gamma \vdash p_0\colon t_0\ \mathbf{p} \quad \Gamma \vdash p_1\colon t_1\ \mathbf{p}}{\Gamma \vdash [p_0; p_1]\colon t_0 + t_1\ \mathbf{p}}$$

Satisfies

$$\frac{[x_0 \leftarrow v_0] v_1 \models p \quad [\sigma]}{\mathbf{match}\ \mathbf{inl}_t(v_0) \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \models p\ [\sigma]}$$

$$\frac{[x_1 \leftarrow v_0] v_2 \models p \quad [\sigma]}{\mathbf{match}\ \mathbf{inr}_t(v_0) \begin{cases} v_1 & \leftarrow \mathbf{inl}(x_0) \\ v_2 & \leftarrow \mathbf{inr}(x_1) \end{cases} \models p\ [\sigma]}$$

$$\frac{v \models p_0\ [\sigma]}{v \models \mathbf{left}([p_0; p_1]) \quad [\sigma]}$$

$$\frac{v \models p_1\ [\sigma]}{v \models \mathbf{right}([p_0; p_1]) \quad [\sigma]}$$

$$\frac{\mathbf{absurd}_t(v) \models p \quad [\sigma]}{v \models \mathbf{false}\ [\sigma]}$$

$$\frac{v \models p_0\ [\sigma]}{\mathbf{inl}_t(v) \models [p_0; p_1] \quad [\sigma]}$$

$$\frac{v \models p_1\ [\sigma]}{\mathbf{inr}_t(v) \models [p_0; p_1] \quad [\sigma]}$$

Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to become a little like Π types. So the predicate calculus effectively becomes like System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Types $t ::= \ldots \mid x \mid \mathbf{head}(v) \mid \Sigma(x\colon\ *).t$
Sorts $s ::= \ldots \mid *$
Values $v ::= \ldots \mid \mathbf{tail}(v) \mid \langle x \leftarrow t, v \rangle$
Predicates $p ::= \ldots \mid p\, t \mid \mathbf{M}(x\colon\ *).p$
Substitutions $\sigma ::= \ldots \mid \sigma,\, t \models x$

Not really good at the typing judgements for dependent sum types.

$$\frac{\Gamma \vdash v\colon \Sigma(x\colon *).t \ \mathbf{v}}{\Gamma \vdash \mathbf{head}(v)\colon *}$$

$$\frac{\Gamma \vdash v\colon \Sigma(x\colon *).t \ \mathbf{v}}{\Gamma \vdash \mathbf{tail}(v)\colon [x \leftarrow \mathbf{head}(v)]t \ \mathbf{v}}$$

$$\frac{\Gamma \vdash t_0\colon * \quad \Gamma, x\colon * \vdash v\colon t_1 \ \mathbf{v}}{\Gamma \vdash \langle x \leftarrow t_0, v\rangle\colon \Sigma(x\colon *).t_0 \ \mathbf{v}}$$

$$\frac{\Gamma \vdash p\colon \Sigma(x\colon *).t_1 \ \mathbf{p} \quad \Gamma \vdash t_0\colon *}{\Gamma \vdash p_0 \, t_0\colon [x \leftarrow t_0]t_1 \ \mathbf{p}}$$

$$\frac{\Gamma, x\colon * \vdash p\colon t \ \mathbf{p}}{\Gamma \vdash \mathbf{M}(x\colon *).p\colon \Sigma(x\colon *).t \ \mathbf{p}}$$

I can't figure out satisfaction at all.

$$\frac{t \models p \quad [\sigma]}{\mathbf{head}(\langle x \leftarrow t, v\rangle) \models p \quad [\sigma]}$$

$$\frac{[x \leftarrow t]v \models p \quad [\sigma]}{\mathbf{tail}(\langle x \leftarrow t, v\rangle) \models p \quad [\sigma]}$$

$$\frac{v \models [x \leftarrow t]p \quad [\sigma]}{v \models (\mathbf{M}(x\colon *).p)\, t \quad [\sigma]}$$

$$\frac{v \models p \ [\sigma, t \models x]}{\langle x \leftarrow t, v\rangle \models \mathbf{M}(x\colon *).p \quad [\sigma]}$$

The Future?

Satisifies judgments correspond to thin squares. Moving to more generally categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.