I wanted to try figuring out a sort of internal language corresponding to the double category **Rel**.

You have a calculus of "values" corresponding to the **Set**edge of **Rel**and a sort of relational calculus of "predicates" corresponding to the other edge. Squares ought to correspond to judgements stating a value satisfies a predicate.

The language of "values" handling Cartesian product of sets has product types (in category theory terms is Cartesian.)

The language of "predicates" ought to be more complicated. **Rel**is a closed monoidal category over Cartesian product of sets. One has an isomorphism $\mathbf{Rel}(A, B \otimes C) \sim \mathbf{Rel}(A \otimes B, C)$. So some sort of linear type theory is required.

The language of "squares" is confusing. It ought to correspond to control expressions in $\bar{\lambda}\mu\tilde{\mu}$.

This is the core framework. I've been thinking about further extensions but I want to see how far I can get characterizing **Rel**with as few language features as possible. Later on I give a few possible extensions.

Core Calculi

Types $t ::= \top \mid t \times t$

Values $v ::= x \mid \mathbf{!} \mid \pi_1(v) \mid \pi_2(v) \mid v, v$

Predicates $p ::= x_t \mid \mathbf{success} \mid p \gg p \mid p\, p \mid \forall(x\colon t).p$

Environment $\Gamma ::= \cdot \mid \Gamma,\, x\colon t$

Substitutions $\sigma ::= \cdot \mid \sigma,\, \langle v \mid x\rangle$

The core relational calculus is the linear lambda calculus with a few symbol changes. The core value calculus has only product types.

I'm not really sure small steps semantics make sense with respect to the relational calculus but the value calculus corresponding to **Set**ought to have simple deterministic semantics.

Relational Calculus

The relational calculus has a linear variable rule. As a hack to make mechanization easier linear variables are explicitly indexed by their types and variable binding is more imperative.

$\boxed{\mathrm{V}}$ $\cdot,\, x\colon t \vdash x_t\colon t$

$\boxed{\top\mathrm{I}}$ $\cdot \vdash \mathbf{success}\colon \top$

$\boxed{\top\mathrm{E}}$ $\dfrac{\Gamma \vdash p_0\colon \top \qquad \Delta \vdash p_1\colon t}{\Gamma,\, \Delta \vdash p_0 \gg p_1\colon t}$

$\boxed{\times\mathrm{I}}$ $\dfrac{x\colon t_0 \in \Gamma \qquad \Gamma \vdash p\colon t_1}{\Gamma \setminus x \vdash \forall(x\colon t_0).p\colon t_0 \times t_1}$

$\boxed{\times\mathrm{E}}$ $\dfrac{\Gamma \vdash p_0\colon t_0 \times t_1 \qquad \Delta \vdash p_1\colon t_1}{\Gamma,\, \Delta \vdash p_0\, p_1\colon t_1}$

$\boxed{\top\mathrm{R}}$ $\mathbf{success} \gg p \rightsquigarrow p$

$\boxed{\times\mathrm{R}}$ $(\forall(x\colon t).p_0)\, p_1 \rightsquigarrow [x_t := p_1]p_0$

Value Calculus

$\boxed{\mathrm{V}}$ $\dfrac{x\colon t \in \Gamma}{\Gamma \vdash x\colon t}$

$\boxed{\top\mathrm{I}}$ $\Gamma \vdash \mathbf{!}\colon \top$

$\boxed{\times\mathrm{E}_1}$ $\dfrac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_1(v)\colon t_0}$ $\boxed{\times\mathrm{E}_2}$ $\dfrac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_2(v)\colon t_1}$

$\boxed{\times\mathrm{I}}$ $\dfrac{\Gamma \vdash v_0\colon t_0 \qquad \Gamma \vdash v_1\colon t_1}{\Gamma \vdash v_0, v_1\colon t_0 \times t_1}$

$\boxed{\times\mathrm{R}_1}$ $\pi_1(v_0, v_1) \rightsquigarrow v_0$

$\boxed{\times\mathrm{R}_2}$ $\pi_2(v_0, v_1) \rightsquigarrow v_1$

Satisfaction Judgements

$\boxed{\text{VAR-S}}$ $\dfrac{\langle v \mid x_t\rangle \in \sigma}{\langle v \mid x_t\rangle\ [\sigma]}$

$\boxed{\text{vR-S}}$ $\dfrac{v_0 \rightsquigarrow v_1 \qquad \langle v_0 \mid p\rangle\ [\sigma]}{\langle v_1 \mid p\rangle\ [\sigma]}$

$\boxed{\text{pR-S}}$ $\dfrac{p_0 \rightsquigarrow p_1 \qquad \langle v \mid p_0\rangle\ [\sigma]}{\langle v \mid p_1\rangle\ [\sigma]}$

$\boxed{\top\mathrm{S}}$ $\langle \mathbf{!} \mid \mathbf{success}\rangle\ [\sigma]$

$\boxed{\times\mathrm{S}}$ $\dfrac{\langle v_1 \mid p\rangle\ [\sigma,\, \langle v_0 \mid x_t\rangle]}{\langle v_0, v_1 \mid \forall(x\colon t).p\rangle\ [\sigma]}$

Examples

Pattern matching on equality

$$\dfrac{\dfrac{\top}{\langle v \mid x_t\rangle\ [\sigma,\, \langle v \mid x_t\rangle]}\ (\text{VAR-S})}{\langle (v, v) \mid \forall(x\colon t).x_t\rangle\ [\sigma]}\ (\times\mathrm{S})$$

Transposition

$\forall(p\colon t \times t \times \top)(x\colon t)(y\colon t).p_{(t\times t\times\top)}\, y_t\, x_t$

Disjoint Unions

Disjoint unions in **Set**become Cartesian product/coproduct in **Rel**. It might make sense to use different notations for coproduct in **Rel**.

I decided against rules for mapping from **Set**to **Rel**because I wanted to see how far **Rel**could go on its own and also wanted to see if these could be defined later.

I have a hunch it is proper for the combination of product/coproduct to introduce nondeterminism in the operational semantics but I need to think more about the issue.

As a technical hack **fail** is explicitly indexed by the environment it ignores. This hack also requires inferring the environment in certain reduction rules.

Types $t ::= \ldots \mid \emptyset \mid t + t$

Values $v ::= \ldots \mid \mathbf{abort}_t(v) \mid \mathbf{i}_{1t}(v) \mid \mathbf{i}_{2t}(v) \mid \mathbf{m}(v, x.v, x.v)$

Predicates $p ::= \ldots \mid \mathbf{abort}_t(p) \mid \mathbf{i}_{1t}(p) \mid \mathbf{i}_{2t}(p) \mid \mathbf{m}(v, x.v, x.v) \mid \mathbf{fail}_\Gamma \mid \mathbf{l}(p) \mid \mathbf{r}(p) \mid [p; p]$

Values

$$\boxed{\emptyset\mathrm{E}}\ \frac{\Gamma \vdash v\colon \emptyset}{\Gamma \vdash \mathbf{abort}_t(v)\colon t}$$

$$\boxed{+\mathrm{I}_1}\ \frac{\Gamma \vdash v\colon t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(v)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{I}_2}\ \frac{\Gamma \vdash v\colon t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(v)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}}\ \frac{\Gamma \vdash v_0\colon t_0 + t_1 \qquad \Gamma,\, x_0\colon t_0 \vdash v_1\colon t_2 \qquad \Gamma,\, x_1\colon t_1 \vdash v_2\colon t_2}{\Gamma \vdash \mathbf{m}(v_0, x_0.v_1, x_1.v_2)\colon t_2}$$

$$\boxed{+\mathrm{R}_1}\ \mathbf{m}(\mathbf{i}_{1t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_0 := v_0]v_1$$

$$\boxed{+\mathrm{R}_2}\ \mathbf{m}(\mathbf{i}_{2t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_1 := v_0]v_2$$

Predicates

$$\boxed{\emptyset\,\mathrm{E}^\mathrm{T}}\ \frac{\Gamma \vdash p\colon \emptyset}{\Gamma \vdash \mathbf{abort}_t(v)\colon t}$$

$$\boxed{+\mathrm{I}_1^\mathrm{T}}\ \frac{\Gamma \vdash p\colon t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(v)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{I}_2^\mathrm{T}}\ \frac{\Gamma \vdash p\colon t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(v)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}^\mathrm{T}}\ \frac{\Gamma \vdash p_0\colon t_0 + t_1 \qquad \Gamma,\, x_0\colon t_0 \vdash p_1\colon t_2 \qquad \Gamma,\, x_1\colon t_1 \vdash p_1\colon t_2}{\Gamma \vdash \mathbf{m}(p_0, x_0.p_1, x_1.p_2)\colon t_2}$$

$$\boxed{\emptyset\mathrm{I}}\ \Gamma \vdash \mathbf{fail}_\Gamma\colon \emptyset$$

$$\boxed{+\mathrm{E}_1}\ \frac{\Gamma \vdash p\colon t_0 + t_1}{\Gamma \vdash \mathbf{l}(p)\colon t_0} \qquad \boxed{+\mathrm{E}_2}\ \frac{\Gamma \vdash p\colon t_0 + t_1}{\Gamma \vdash \mathbf{r}(p)\colon t_1)}$$

$$\boxed{+\mathrm{I}}\ \frac{\Gamma \vdash p_0\colon t_0 \qquad \Gamma \vdash p_1\colon t_1}{\Gamma \vdash [p_0; p_1]\colon t_0 + t_1}$$

$$\boxed{+\mathrm{R}_1^\mathrm{T}}\ \mathbf{m}(\mathbf{i}_{1t}(p_0), x_0.p_1, x_1.p_2) \rightsquigarrow [x_0 := p_0]p_1$$

$$\boxed{+\mathrm{R}_2^\mathrm{T}}\ \mathbf{m}(\mathbf{i}_{2t}(p_0), x_0.p_1, x_1.p_2) \rightsquigarrow [x_1 := p_0]p_2$$

$$\boxed{+\mathrm{R}_1}\ \mathbf{l}([p_0; p_1]) \rightsquigarrow p_0 \quad \boxed{+\mathrm{R}_2}\ \mathbf{r}([p_0; p_1]) \rightsquigarrow p_1$$

$$\boxed{+\mathrm{RR}_1^\mathrm{T}}\ \mathbf{l}(\mathbf{i}_{1t}(p)) \rightsquigarrow p \qquad \boxed{+\mathrm{RR}_2^\mathrm{T}}\ \mathbf{r}(\mathbf{i}_{2t}(p)) \rightsquigarrow p$$

$$\boxed{+\mathrm{RR}_3^\mathrm{T}}\ \frac{\Gamma \vdash p\colon t}{\mathbf{l}(\mathbf{i}_{2t}(p)) \rightsquigarrow \mathbf{abort}_t(\mathbf{fail}_\Gamma)}$$

$$\boxed{+\mathrm{RR}_4^\mathrm{T}}\ \frac{\Gamma \vdash p\colon t}{\mathbf{r}(\mathbf{i}_{1t}(p)) \rightsquigarrow \mathbf{abort}_t(\mathbf{fail}_\Gamma)}$$

$$\boxed{+\mathrm{R}^\mathrm{T}\mathrm{R}_1}\ \mathbf{m}([p_0; p_1], x_0.p_2, x_1.p_3) \rightsquigarrow [x_0 := p_0]p_2$$

$$\boxed{+\mathrm{R}^\mathrm{T}\mathrm{R}_2}\ \mathbf{m}([p_0; p_1], x_0.p_2, x_1.p_3) \rightsquigarrow [x_1 := p_1]p_3$$

Satisfies

$$\boxed{+\mathrm{S}_1}\ \frac{\langle v \mid p_0\rangle\ [\sigma]}{\langle \mathbf{i}_{1t}(v) \mid [p_0; p_1]\rangle\ [\sigma]}$$

$$\boxed{+\mathrm{S}_2}\ \frac{\langle v \mid p_1\rangle\ [\sigma]}{\langle \mathbf{i}_{2t}(v) \mid [p_0; p_1]\rangle\ [\sigma]}$$

$$\boxed{\emptyset\mathrm{S}}\ \frac{\langle \mathbf{abort}_t(v) \mid p\rangle\ [\sigma]}{\langle v \mid \mathbf{fail}_\Gamma\rangle\ [\sigma]}$$

Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to become a little like Π types. So the predicate calculus effectively becomes like a linear System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Types $t ::= \ldots \mid x \mid * \mid \mathbf{h}(v) \mid \Sigma(x\colon *).t$

Values $v ::= \ldots \mid \mathbf{t}(v) \mid \langle x := t, v\rangle$

Predicates $p ::= \ldots \mid p\, t \mid \Pi(x\colon *).p$

Substitutions $\sigma ::= \ldots \mid \sigma,\, \langle t \mid x\rangle$

Not really good at the typing judgements for dependent sum types.

Values

$$\boxed{\Sigma\mathrm{E}_1}\ \frac{\Gamma \vdash v\colon \Sigma(x\colon *).t}{\Gamma \vdash \mathbf{h}(v)\colon *}$$

$$\boxed{\Sigma\mathrm{E}_2}\ \frac{\Gamma \vdash v\colon \Sigma(x\colon *).t}{\Gamma \vdash \mathbf{t}(v)\colon [x := \mathbf{h}(v)]t}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Gamma \vdash t_0\colon * \qquad \Gamma,\, x\colon * \vdash v\colon t_1}{\Gamma \vdash \langle x := t_0, v\rangle\colon \Sigma(x\colon *).t_0}$$

$$\boxed{\Sigma\mathrm{S}_1}\ \mathbf{h}(\langle x := t, v\rangle) \rightsquigarrow t$$

$$\boxed{\Sigma\mathrm{S}_2}\ \mathbf{t}(\langle x := t, v\rangle) \rightsquigarrow [x := t]v$$

Predicates

$$\boxed{\Sigma\mathrm{E}}\ \frac{\Gamma \vdash p\colon \Sigma(x\colon *).t_1 \qquad \Delta \vdash t_0\colon *}{\Gamma,\, \Delta \vdash p_0\, t_0\colon [x := t_0]t_1}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Gamma,\, x\colon * \vdash p\colon t}{\Gamma \vdash \Pi(x\colon *).p\colon \Sigma(x\colon *).t}$$

$$\boxed{\Sigma\mathrm{R}}\ (\Pi(x\colon *).p)\, t \rightsquigarrow [x := t]p$$

Satisfaction

I can't figure out satisfaction at all.

$$\boxed{\Sigma\mathrm{S}}\ \frac{\langle v \mid p\rangle\ [\sigma,\, \langle t \mid x\rangle]}{\langle \langle x := t, v\rangle \mid \Pi(x\colon *).p\rangle\ [\sigma]}$$

Categorical Semantics

The intent is to create calculi encoding the core features of the double category **Rel**.

If this is successful then terms and types ought to map to **Rel**as follows.

$$\begin{tikzcd}
\top \arrow[r, "\sigma_v"] \arrow[d, "\sigma_p"'] \arrow[dr, phantom, "\Downarrow \langle v|p\rangle\,[\sigma_v,\sigma_p]"] & \times_{x\colon t\in\Gamma_v} t \arrow[d, "v"] \\
\times_{x\colon t\in\Gamma_p} t \arrow[r, "p"'] & t
\end{tikzcd}$$

I have no idea about universe issues and such. Dependent sum is probably wrong.

Really need to think about denotation again.

$$
\begin{gathered}
[\![\top]\!] = \{\emptyset\} \\
[\![t_0 \times t_1]\!] = [\![t_0]\!] \times [\![t_1]\!] \\
[\![\emptyset]\!] = \emptyset \\
[\![t_0 + t_1]\!] = [\![t_0]\!] \sqcup [\![t_1]\!] \\
[\![\Sigma(x\colon\ *).t(x)]\!] = \bigsqcup_{x \in *} [\![t(x)]\!]
\end{gathered}
$$

$$
\begin{gathered}
[\![\mathbf{!}]\!] = \emptyset \\
[\![(v_0, v_1)]\!] = ([\![v_0]\!], [\![v_1]\!])
\end{gathered}
$$

$$
\begin{gathered}
[\![\mathbf{success}]\!](y) = \top \\
[\![p_0 p_1]\!](y) = \exists z, [\![p_1]\!](z) \wedge [\![p_0]\!](z, y) \\
[\![\forall(x\colon t).p]\!](\langle x, y\rangle) = [\![p(x)]\!](y)
\end{gathered}
$$

Synthetic Category Theory

A category is a monad in Span. Once this system has been generalized to Span we can define monads internal to Span.

This is not fully internal but a simple approach to defining an equivalence relation might be something like:

$\boxed{\text{object}}\ \cdot \vdash O\colon\ *$

$\boxed{\text{arrow}}\ \cdot \vdash R\colon O \times O$

$\boxed{\text{refl}}\ \dfrac{\cdot \vdash o\colon O}{\langle o, o \mid R\rangle\ [\cdot]}$

$\boxed{\text{trans}}\ \dfrac{\cdot \vdash o_0, o_1\colon O \times O \qquad \langle o_0, o_1 \mid \forall(x\colon O).R\,(R\,x)\rangle\ [\cdot]}{\langle o_0, o_1 \mid R\rangle\ [\cdot]}$

$\boxed{\text{sym}}\ \dfrac{\cdot \vdash o_1, o_0\colon O \times O \qquad \langle o_0, o_1 \mid R\rangle\ [\cdot]}{\langle o_1, o_0 \mid R\rangle\ [\cdot]}$

Generalizing to a constructive interpretation in terms of spans and groupoids is future work.

The Future?

Satisifies judgments correspond to thin squares. Moving to more general categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.