Internal Language of Rel

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I wanted an internal language corresponding to the double category **Rel**. Recall **Rel** is the double category with sets as objects, relations as horizontal arrows and functions as vertical arrows.

I present three different calculi: a substructural "context calculus" corresponding to the horizontal edge of **Rel**; a "term calculus" corresponding to **Set**, the vertical edge category of **Rel**; and a simple "command" calculus encoding commuting squares of **Rel**.

First, I give the basic framework of the calculi. Then I discuss semantics and applications. And later I list a few possible extensions such as coproduct and depedent sum types.

Core Calculi

The core context calculus is the linear lambda calculus with a few symbol changes and models the closed monoidal structure of the horizontal edge of **Rel**.

The core term calculus is a little language with only product types and models the Cartesian structure of **Set**.

The core command calculus models commuting squares in **Rel**.

Types	$t ::= * \mid t \times t$
Contexts	$E ::= x_t \mid E\,E \mid \forall(x\colon t).E$
Terms	$v ::= x \mid \pi_1(v) \mid \pi_2(v) \mid v, v$
Environment	$\Gamma ::= \cdot \mid \Gamma,\, x\colon t$
Command Env	$\Delta ::= \cdot \mid \Delta,\, x \mid v$

Context Calculus

$$\boxed{\mathrm{V}}\ \cdot,\, x\colon t \vdash x_t\colon t$$

$$\boxed{\times\mathrm{E}}\ \frac{\Gamma_0 \vdash E_0\colon t_0 \times t_1 \qquad \Gamma_1 \vdash E_1\colon t_0}{\Gamma_0,\, \Gamma_1 \vdash E_0\, E_1\colon t_1}$$

$$\boxed{\times\beta}\ (\forall(x\colon t).E_0)\, E_1 \rightsquigarrow [x_t := E_1]E_0$$

Term Calculus

$$\boxed{\mathrm{V}}\ \frac{x\colon t \in \Gamma}{\Gamma \vdash x\colon t}$$

$$\boxed{\times\mathrm{E}_1}\ \frac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_1(v)\colon t_0} \qquad \boxed{\times\mathrm{E}_2}\ \frac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_2(v)\colon t_1}$$

$$\boxed{\times\mathrm{I}}\ \frac{\Gamma \vdash v_0\colon t_0 \qquad \Gamma \vdash v_1\colon t_1}{\Gamma \vdash v_0, v_1\colon t_0 \times t_1}$$

$$\boxed{\times\beta_1}\ \pi_1(v_0, v_1) \rightsquigarrow v_0 \qquad \boxed{\times\beta_2}\ \pi_2(v_0, v_1) \rightsquigarrow v_1$$

Command Calculus

$$\boxed{\mathrm{V}}\ \cdot,\, x_t \otimes v \vdash x_t \otimes v$$

$$\boxed{\mathrm{v}}\ \frac{\Delta \vdash E \otimes v_0 \qquad v_0 \rightsquigarrow v_1}{\Delta \vdash E \otimes v_1} \qquad \boxed{\mathrm{E}}\ \frac{\Delta \vdash E_0 \otimes v \qquad E_0 \rightsquigarrow E_1}{\Delta \vdash E_1 \otimes v}$$

$$\boxed{\times\mathrm{I}}\ \frac{\Delta,\, x_t \otimes v_0 \vdash E \otimes v_1}{\Delta \vdash \forall(x\colon t).E \otimes v_0, v_1}$$

$$\boxed{\times\mathrm{E}}\ \frac{\Delta_0 \vdash \forall(x\colon t).E_0 \otimes v_0, v_1 \qquad \Delta_1 \vdash E_1 \otimes v_0}{\Delta_0,\, \Delta_1 \vdash [x := E_1]E_0 \otimes v_1}$$

The context calculus has a linear variable rule. To make mechanization easier linear variables are explicitly indexed by their types and variable binding is more imperative and so ensuring the typing environment can be directly inferred from context expressions.

Discussion

Examples

Identity:

$$\dfrac{\dfrac{\top}{\Delta,\, x_t \otimes v \vdash x_t \otimes v}\ (\mathrm{V})}{\Delta \vdash \forall(x\colon t).x_t \otimes v, v}\ (\times\mathrm{I})$$

Transposition:

$$\mathbf{trans} = \forall(p\colon t_0 \times t_1 \times t_2)(x\colon t_1)(y\colon t_0).p_{(t_0 \times t_1 \times t_2)}\, y_t\, x_t$$

Categorical Semantics

The intent is to create calculi encoding the core features of the double category **Rel**.

If this is successful then terms and types ought to map to **Rel** as follows.

I think you want pullbacks for the environment arrows?

$$\begin{tikzcd}
? \arrow[r, "?"] \arrow[d, "?"'] \arrow[dr, phantom, "\Downarrow \Delta \vdash E \otimes v"] & \times_{x\colon t \in \Gamma_v} t \arrow[d, "v"] \\
\times_{x\colon t \in \Gamma_E} t \arrow[r, "E"'] & t
\end{tikzcd}$$

I have no idea about universe issues and such. Dependent sum is probably wrong.

Really need to think about denotation again.

$$[\![t_0 \times t_1]\!] = [\![t_0]\!] \times [\![t_1]\!]$$

$$\begin{aligned}
[\![x]\!](\sigma) &= \sigma(x) \\
[\![v_0, v_1]\!](\sigma) &= ([\![v_0]\!](\sigma), [\![v_1]\!](\sigma))
\end{aligned}$$

$$\begin{aligned}
[\![x_t]\!](\sigma, s) &= \sigma(x, s) \\
[\![E_0\, E_1]\!](\sigma, s_0) &= \exists s_1, [\![E_1]\!](\sigma, s_1) \wedge [\![E_0]\!](\sigma, (s_1, s_0)) \\
[\![\forall(x\colon t).E_0]\!](\sigma, (s_0, s_1)) &= \forall E_1, [\![E_1]\!](\sigma, s_0) \to [\![E_0]\!]([x := s \mapsto [\![E_1]\!](\sigma, s)]\sigma, s_1)
\end{aligned}$$

Synthetic Category Theory

A category is a monad in **Span**. Once this system has been generalized to **Span** we can define monads internal to **Span**.

This is not fully internal but a simple approach to defining an equivalence relation might be something like:

$\boxed{\text{object}}\ \cdot \vdash O\colon *$ $\qquad\qquad$ $\boxed{\text{arrow}}\ \cdot \vdash R\colon O \times O$

$$\boxed{\text{refl}}\ \frac{\cdot \vdash o\colon O}{\cdot \vdash R \otimes o, o}$$

$$\boxed{\text{trans}}\ \frac{\cdot \vdash \forall(x\colon O).R\,(R\,x) \otimes o_0, o_1}{\cdot \vdash R \otimes o_0, o_1}\ \{\cdot \vdash o_1, o_0\colon O \times O\}$$

$$\boxed{\text{sym}}\ \frac{\cdot \vdash R \otimes o_0, o_1}{\cdot \vdash R \otimes o_1, o_0}\ \{\cdot \vdash o_1, o_0\colon O \times O\}$$

Generalizing to a constructive interpretation in terms of spans and groupoids is future work.

Unit Type

$$
\begin{array}{ll}
\textbf{Types} & t ::= \ldots \mid 1 \\
\textbf{Contexts} & E ::= \ldots \mid \textbf{true} \mid E \gg E \\
\textbf{Terms} & v ::= \ldots \mid \textbf{!}
\end{array}
$$

Context Calculus

$$\boxed{\top\mathrm{I}}\ \cdot \vdash \textbf{true}\colon\ \top$$

$$\boxed{\top\mathrm{E}}\ \frac{\Gamma_0 \vdash p_0\colon\ \top \qquad \Gamma_1 \vdash p_1 : t}{\Gamma_0,\ \Gamma_1 \vdash E_0 \gg E_1 : t}$$

$$\boxed{\times\mathrm{I}}\ \frac{\Gamma \vdash E\colon t_1}{\Gamma \setminus x \vdash \forall(x\colon t_0).E\colon t_0 \times t_1}\ \{x\colon t_0 \in \Gamma\}$$

Term Calculus

$$\boxed{\top\mathrm{I}}\ \Gamma \vdash \textbf{!}\colon\ \top$$

Command Calculus

$$\boxed{\top\mathrm{I}}\ \cdot \vdash \textbf{true} \otimes \textbf{!}$$

$$\boxed{\top\mathrm{E}}\ \frac{\Delta_0 \vdash \textbf{true} \otimes \textbf{!} \qquad \Delta_1 \vdash E \otimes v}{\Delta_0,\ \Delta_1 \vdash E \otimes v}$$

Disjoint Unions

Disjoint unions in **Set** become Cartesian products/coproducts in **Rel**.

As a technical hack **false** is explicitly indexed by the environment it ignores. This hack also requires inferring the environment in certain reduction rules.

$$
\begin{array}{ll}
\textbf{Types} & t ::= \ldots \mid \emptyset \mid t + t \\
\textbf{Contexts} & E ::= \ldots \mid \textbf{abort}_t(E) \mid \mathbf{i_1}_t(E) \mid \mathbf{i_2}_t(E) \mid \\
 & \quad \mathbf{m}(v, E.v, x.E) \mid \textbf{false}_\Gamma \mid \mathbf{l}(E) \mid \mathbf{r}(E) \mid E; E \\
\textbf{Terms} & v ::= \ldots \mid \textbf{abort}_t(v) \mid \mathbf{i_1}_t(v) \mid \mathbf{i_2}_t(v) \mid \mathbf{m}(v, x.v, x.v)
\end{array}
$$

Context Calculus

$$\boxed{\emptyset\,\mathrm{E}^{\mathrm{T}}}\ \frac{\Gamma \vdash E\colon \emptyset}{\Gamma \vdash \mathbf{abort}_t(E)\colon t}$$

$$\boxed{+\mathrm{I}_1^{\mathrm{T}}}\ \frac{\Gamma \vdash E\colon t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(E)\colon t_0 + t_1} \qquad \boxed{+\mathrm{I}_2^{\mathrm{T}}}\ \frac{\Gamma \vdash E\colon t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(E)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}^{\mathrm{T}}}\ \frac{\Gamma \vdash E_0\colon t_0 + t_1 \qquad \Gamma,\, x_0\colon t_0 \vdash E_1\colon t_2 \qquad \Gamma,\, x_1\colon t_1 \vdash E_1\colon t_2}{\Gamma \vdash \mathbf{m}(E_0, x_0.E_1, x_1.E_2)\colon t_2}$$

$$\boxed{\emptyset\mathrm{I}}\ \Gamma \vdash \mathbf{false}_\Gamma\colon\ \emptyset$$

$$\boxed{+\mathrm{E}_1}\ \frac{\Gamma \vdash E\colon t_0 + t_1}{\Gamma \vdash \mathbf{l}(E)\colon t_0} \qquad \boxed{+\mathrm{E}_2}\ \frac{\Gamma \vdash E\colon t_0 + t_1}{\Gamma \vdash \mathbf{r}(E)\colon t_1)}$$

$$\boxed{+\mathrm{I}}\ \frac{\Gamma \vdash E_0\colon t_0 \qquad \Gamma \vdash E_1\colon t_1}{\Gamma \vdash E_0; E_1\colon t_0 + t_1}$$

$$\boxed{+\beta_1^{\mathrm{T}}}\ \mathbf{m}(\mathbf{i}_{1t}(E_0), x_0.E_1, x_1.E_2) \rightsquigarrow [x_0 := E_0]E_1$$

$$\boxed{+\beta_2^{\mathrm{T}}}\ \mathbf{m}(\mathbf{i}_{2t}(E_0), x_0.E_1, x_1.E_2) \rightsquigarrow [x_1 := E_0]E_2$$

$$\boxed{+\beta_1}\ \mathbf{l}(E_0; E_1) \rightsquigarrow E_0 \qquad \boxed{+\beta_2}\ \mathbf{r}(E_0; E_1) \rightsquigarrow E_1$$

$$\boxed{+\beta\beta_1^{\mathrm{T}}}\ \mathbf{l}(\mathbf{i}_{1t}(E)) \rightsquigarrow E \qquad \boxed{+\beta\beta_2^{\mathrm{T}}}\ \mathbf{r}(\mathbf{i}_{2t}(E)) \rightsquigarrow E$$

$$\boxed{+\mathrm{RR}_3^{\mathrm{T}}}\ \frac{\Gamma \vdash E\colon t}{\mathbf{l}(\mathbf{i}_{2t}(E)) \rightsquigarrow \mathbf{abort}_t(\mathbf{false}_\Gamma)} \qquad \boxed{+\mathrm{RR}_4^{\mathrm{T}}}\ \frac{\Gamma \vdash E\colon t}{\mathbf{r}(\mathbf{i}_{1t}(E)) \rightsquigarrow \mathbf{abort}_t(\mathbf{false}_\Gamma)}$$

$$\boxed{+\mathrm{R}^{\mathrm{T}}\mathrm{R}_1}\ \mathbf{m}(E_0; E_1, x_0.E_2, x_1.E_3) \rightsquigarrow [x_0 := E_0]E_2$$

$$\boxed{+\mathrm{R}^{\mathrm{T}}\mathrm{R}_2}\ \mathbf{m}(E_0; E_1, x_0.E_2, x_1.E_3) \rightsquigarrow [x_1 := E_1]E_3$$

Term Calculus

$$\boxed{\emptyset\mathrm{E}}\ \frac{\Gamma \vdash v\colon\ \emptyset}{\Gamma \vdash \mathbf{abort}_t(v)\colon t}$$

$$\boxed{+\mathrm{I}_1}\ \frac{\Gamma \vdash v\colon t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(v)\colon t_0 + t_1} \qquad \boxed{+\mathrm{I}_2}\ \frac{\Gamma \vdash v\colon t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(v)\colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}}\ \frac{\Gamma \vdash v_0\colon t_0 + t_1 \qquad \Gamma,\, x_0\colon t_0 \vdash v_1\colon t_2 \qquad \Gamma,\, x_1\colon t_1 \vdash v_1\colon t_2}{\Gamma \vdash \mathbf{m}(v_0, x_0.v_1, x_1.v_2)\colon t_2}$$

$$\boxed{+\beta_1}\ \mathbf{m}(\mathbf{i}_{1t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_0 := v_0]v_1$$

$$\boxed{+\beta_2}\ \mathbf{m}(\mathbf{i}_{2t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_1 := v_0]v_2$$

Command Calculus

$$\boxed{+\mathrm{I}_1}\ \frac{\Delta \vdash E_0 \otimes v}{\Delta \vdash E_0; E_1 \otimes \mathbf{i}_{1t}(v)} \qquad \boxed{+\mathrm{I}_2}\ \frac{\Delta \vdash E_1 \otimes v}{\Delta \vdash E_0; E_1 \otimes \mathbf{i}_{2t}(v)}$$

$$\boxed{\emptyset\mathrm{I}}\ \frac{\Delta \vdash E \otimes \mathbf{abort}_t(v)}{\Delta \vdash \mathbf{false}_\Gamma \otimes v}$$

Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to become a little like Π types. So the predicate calculus effectively becomes like a linear System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Not really good at the typing judgements for dependent sum types.

Types	$t ::= \ldots \mid x \mid * \mid \mathbf{h}(v) \mid \Sigma(x\colon *).t$
Contexts	$E ::= \ldots \mid E\,t \mid \Pi(x\colon *).E$
Terms	$v ::= \ldots \mid \mathbf{t}(v) \mid \langle x := t, v\rangle$
Command Env	$\Delta ::= \ldots \mid \Delta,\, x \otimes t$

Context Calculus

$$\boxed{\Sigma\mathrm{E}}\ \frac{\Gamma_0 \vdash E\colon \Sigma(x\colon *).t_1 \qquad \Gamma_1 \vdash t_0\colon *}{\Gamma_0,\, \Gamma_1 \vdash E_0\, t_0\colon [x := t_0]t_1}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Gamma,\, x\colon * \vdash E\colon t}{\Gamma \vdash \Pi(x\colon *).E\colon \Sigma(x\colon *).t}$$

$$\boxed{\Sigma\beta}\ (\Pi(x\colon *).E)\, t \rightsquigarrow [x := t]E$$

Term Calculus

$$\boxed{\Sigma\mathrm{E}_1}\ \frac{\Gamma \vdash E\colon \Sigma(x\colon *).t}{\Gamma \vdash \mathbf{h}(v)\colon *}$$

$$\boxed{\Sigma\mathrm{E}_2}\ \frac{\Gamma \vdash v\colon \Sigma(x\colon *).t}{\Gamma \vdash \mathbf{t}(v)\colon [x := \mathbf{h}(v)]t}$$

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Gamma \vdash t_0\colon * \qquad \Gamma,\, x\colon * \vdash v\colon t_1}{\Gamma \vdash \langle x := t_0, v\rangle\colon \Sigma(x\colon *).t_0}$$

$$\boxed{\Sigma\beta_1}\ \mathbf{h}(\langle x := t, v\rangle) \rightsquigarrow t \qquad\qquad \boxed{\Sigma\beta_2}\ \mathbf{t}(\langle x := t, v\rangle) \rightsquigarrow [x := t]v$$

Command Calculus

$$\boxed{\Sigma\mathrm{I}}\ \frac{\Delta,\, x \otimes t \vdash E \otimes v}{\Delta \vdash \Pi(x\colon *).E \otimes \langle x := t, v\rangle}$$

I can't figure out commands here at all.

The Future?

Satisifies judgments correspond to thin squares. Moving to more general categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.