I wanted to try figuring out a sort of internal language corresponding to the double category **Rel**.

You have a calculus of "values" corresponding to the **Set** edge of **Rel** and a sort of relational calculus of "predicates" corresponding to the other edge. Squares ought to correspond to judgements stating a value satisfies a predicate.

The language of "values" handling Cartesian product of sets has product types (in category theory terms is Cartesian.)

The language of "predicates" which we call the "context calculus" ought to be more complicated. **Rel** is a closed monoidal category over Cartesian product of sets. One has an isomorphism $\mathbf{Rel}(A, B \otimes C) \sim \mathbf{Rel}(A \otimes B, C)$. So some sort of linear type theory is required.

The language of "squares" is confusing. It ought to correspond to commands in $\bar{\lambda}\mu\tilde{\mu}$.

This is the core framework. I've been thinking about further extensions but I want to see how far I can get characterizing **Rel** with as few language features as possible. Later on I give a few possible extensions.

Core Calculi

Types	$t ::= \top \mid t \times t$
Terms	$v ::= x \mid \mathbf{!} \mid \pi_1(v) \mid \pi_2(v) \mid v, v$
Contexts	$E ::= x_t \mid \mathbf{true} \mid E \gg E \mid E\, E \mid \forall(x\colon t).E$
Environment	$\Gamma ::= \cdot \mid \Gamma,\, x\colon t$
Command Environment	$\Delta ::= \cdot \mid \Delta,\, x \mid v$

The core context calculus is the linear lambda calculus with a few symbol changes. The core value calculus has only product types.

I'm not really sure small steps semantics make sense with respect to the context calculus but the value calculus corresponding to **Set** ought to have simple deterministic semantics.

Context Calculus

The context calculus has a linear variable rule. As a hack to make mechanization easier linear variables are explicitly indexed by their types and variable binding is more imperative.

$\boxed{\text{V}}\ \cdot,\, x\colon t \vdash x_t\colon t$

$\boxed{\top\text{I}}\ \cdot \vdash \mathbf{true}\colon \top$

$\boxed{\top\text{E}}\ \dfrac{\Gamma_0 \vdash p_0\colon \top \qquad \Gamma_1 \vdash p_1\colon t}{\Gamma_0,\, \Gamma_1 \vdash E_0 \gg E_1\colon t}$

$\boxed{\times\text{I}}\ \dfrac{\Gamma \vdash E\colon t_1 \qquad x\colon t_0 \in \Gamma}{\Gamma \setminus x \vdash \forall(x\colon t_0).E\colon t_0 \times t_1}$

$\boxed{\times\text{E}}\ \dfrac{\Gamma_0 \vdash E_0\colon t_0 \times t_1 \qquad \Gamma_1 \vdash E_1\colon t_1}{\Gamma_0,\, \Gamma_1 \vdash E_0\, E_1\colon t_1}$

$\boxed{\top\beta}\ \mathbf{true} \gg E \rightsquigarrow E$

$\boxed{\times\beta}\ (\forall(x\colon t).E_0)\, E_1 \rightsquigarrow [x_t := E_1]E_0$

Term Calculus

$\boxed{\text{V}}\ \dfrac{x\colon t \in \Gamma}{\Gamma \vdash x\colon t}$

$\boxed{\top\text{I}}\ \Gamma \vdash \mathbf{!}\colon \top$

$\boxed{\times\text{E}_1}\ \dfrac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_1(v)\colon t_0}\qquad \boxed{\times\text{E}_2}\ \dfrac{\Gamma \vdash v\colon t_0 \times t_1}{\Gamma \vdash \pi_2(v)\colon t_1}$

$\boxed{\times\text{I}}\ \dfrac{\Gamma \vdash v_0\colon t_0 \qquad \Gamma \vdash v_1\colon t_1}{\Gamma \vdash v_0, v_1\colon t_0 \times t_1}$

$\boxed{\times\beta_1}\ \pi_1(v_0, v_1) \rightsquigarrow v_0 \qquad \boxed{\times\beta_2}\ \pi_2(v_0, v_1) \rightsquigarrow v_1$

Command Calculus

$\boxed{\text{V}}\ \cdot,\, x_t \otimes v \vdash x_t \otimes v$

$\boxed{\text{vR}}\ \dfrac{\Delta \vdash E \otimes v_0 \qquad v_0 \rightsquigarrow v_1}{\Delta \vdash E \otimes v_1}$

$\boxed{\text{pR}}\ \dfrac{\Delta \vdash E_0 \otimes v \qquad E_0 \rightsquigarrow E_1}{\Delta \vdash E_1 \otimes v}$

$\boxed{\top\text{I}}\ \Delta \vdash \mathbf{!} \otimes \mathbf{true}$

$\boxed{\times\text{I}}\ \dfrac{\Delta,\, x_t \otimes v_0 \vdash E \otimes v_1}{\Delta \vdash \forall(x\colon t).E \otimes v_0, v_1}$

Examples

Pattern matching on equality

$$\dfrac{\dfrac{\top}{\Delta,\, x_t \otimes v \vdash x_t \otimes v}\ (\text{V})}{\Delta \vdash \forall(x\colon t).x_t \otimes v, v}\ (\times\text{I})$$

Transposition

$$\forall(p\colon t \times t \times \top)(x\colon t)(y\colon t).p_{(t \times t \times \top)}\, y_t\, x_t$$

Disjoint Unions

Disjoint unions in **Set** become Cartesian product/coproduct in **Rel**. It might make sense to use different notations for coproduct in **Rel**.

I decided against rules for mapping from **Set** to **Rel** because I wanted to see how far **Rel** could go on its own and also wanted to see if these could be defined later.

I have a hunch it is proper for the combination of product/coproduct to introduce nondeterminism in the operational semantics but I need to think more about the issue.

As a technical hack **false** is explicitly indexed by the environment it ignores. This hack also requires inferring the environment in certain reduction rules.

Types	$t ::= \ldots \mid \emptyset \mid t + t$
Terms	$v ::= \ldots \mid \mathbf{abort}_t(v) \mid \mathbf{i_1}_t(v) \mid \mathbf{i_2}_t(v) \mid$
	$\quad \mathbf{m}(v, x.v, x.v)$

Contexts $E ::= \ldots \mid \mathbf{abort}_t(E) \mid \mathbf{i}_{1t}(E) \mid \mathbf{i}_{2t}(E) \mid$
$\mathbf{m}(v, E.v, x.E) \mid \mathbf{false}_\Gamma \mid \mathbf{l}(E) \mid \mathbf{r}(E) \mid E; E$

Term Calculus

$$\boxed{\emptyset \mathrm{E}} \ \frac{\Gamma \vdash v \colon \emptyset}{\Gamma \vdash \mathbf{abort}_t(v) \colon t}$$

$$\boxed{+\mathrm{I}_1} \ \frac{\Gamma \vdash v \colon t_0}{\Gamma \vdash \mathbf{i}_{1 t_1}(v) \colon t_0 + t_1}$$

$$\boxed{+\mathrm{I}_2} \ \frac{\Gamma \vdash v \colon t_1}{\Gamma \vdash \mathbf{i}_{2 t_0}(v) \colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}} \ \frac{\Gamma \vdash v_0 \colon t_0 + t_1 \qquad \Gamma, x_0 \colon t_0 \vdash v_1 \colon t_2 \qquad \Gamma, x_1 \colon t_1 \vdash v_2 \colon t_2}{\Gamma \vdash \mathbf{m}(v_0, x_0.v_1, x_1.v_2) \colon t_2}$$

$$\boxed{+\beta_1} \ \mathbf{m}(\mathbf{i}_{1t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_0 := v_0]v_1$$

$$\boxed{+\beta_2} \ \mathbf{m}(\mathbf{i}_{2t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_1 := v_0]v_2$$

Context Calculus

$$\boxed{\emptyset \, \mathrm{E}^{\mathrm{T}}} \ \frac{\Gamma \vdash E \colon \emptyset}{\Gamma \vdash \mathbf{abort}_t(E) \colon t}$$

$$\boxed{+\mathrm{I}_1^{\mathrm{T}}} \ \frac{\Gamma \vdash E \colon t_0}{\Gamma \vdash \mathbf{i}_{1 t_1}(E) \colon t_0 + t_1}$$

$$\boxed{+\mathrm{I}_2^{\mathrm{T}}} \ \frac{\Gamma \vdash E \colon t_1}{\Gamma \vdash \mathbf{i}_{2 t_0}(E) \colon t_0 + t_1}$$

$$\boxed{+\mathrm{E}^{\mathrm{T}}} \ \frac{\Gamma \vdash E_0 \colon t_0 + t_1 \qquad \Gamma, x_0 \colon t_0 \vdash E_1 \colon t_2 \qquad \Gamma, x_1 \colon t_1 \vdash E_2 \colon t_2}{\Gamma \vdash \mathbf{m}(E_0, x_0.E_1, x_1.E_2) \colon t_2}$$

$$\boxed{\emptyset \mathrm{I}} \ \Gamma \vdash \mathbf{false}_\Gamma \colon \emptyset$$

$$\boxed{+\mathrm{E}_1} \ \frac{\Gamma \vdash E \colon t_0 + t_1}{\Gamma \vdash \mathbf{l}(E) \colon t_0} \qquad \boxed{+\mathrm{E}_2} \ \frac{\Gamma \vdash E \colon t_0 + t_1}{\Gamma \vdash \mathbf{r}(E) \colon t_1)}$$

$$\boxed{+\mathrm{I}} \ \frac{\Gamma \vdash E_0 \colon t_0 \qquad \Gamma \vdash E_1 \colon t_1}{\Gamma \vdash E_0; E_1 \colon t_0 + t_1}$$

$$\boxed{+\beta_1^{\mathrm{T}}} \ \mathbf{m}(\mathbf{i}_{1t}(E_0), x_0.E_1, x_1.E_2) \rightsquigarrow [x_0 := E_0]E_1$$

$$\boxed{+\beta_2^{\mathrm{T}}} \ \mathbf{m}(\mathbf{i}_{2t}(E_0), x_0.E_1, x_1.E_2) \rightsquigarrow [x_1 := E_0]E_2$$

$$\boxed{+\beta_1} \ \mathbf{l}(E_0; E_1) \rightsquigarrow E_0 \qquad \boxed{+\beta_2} \ \mathbf{r}(E_0; E_1) \rightsquigarrow E_1$$

$$\boxed{+\beta\beta_1^{\mathrm{T}}} \ \mathbf{l}(\mathbf{i}_{1t}(E)) \rightsquigarrow E \qquad \boxed{+\beta\beta_2^{\mathrm{T}}} \ \mathbf{r}(\mathbf{i}_{2t}(E)) \rightsquigarrow E$$

$$\boxed{+\mathrm{RR}_3^{\mathrm{T}}} \ \frac{\Gamma \vdash E \colon t}{\mathbf{l}(\mathbf{i}_{2t}(E)) \rightsquigarrow \mathbf{abort}_t(\mathbf{false}_\Gamma)}$$

$$\boxed{+\mathrm{RR}_4^{\mathrm{T}}} \ \frac{\Gamma \vdash E \colon t}{\mathbf{r}(\mathbf{i}_{1t}(E)) \rightsquigarrow \mathbf{abort}_t(\mathbf{false}_\Gamma)}$$

$$\boxed{+\mathrm{R}^{\mathrm{T}}\mathrm{R}_1} \ \mathbf{m}(E_0; E_1, x_0.E_2, x_1.E_3) \rightsquigarrow [x_0 := E_0]E_2$$

$$\boxed{+\mathrm{R}^{\mathrm{T}}\mathrm{R}_2} \ \mathbf{m}(E_0; E_1, x_0.E_2, x_1.E_3) \rightsquigarrow [x_1 := E_1]E_3$$

Command Calculus

$$\boxed{+\mathrm{I}_1} \ \frac{\Delta \vdash E_0 \otimes v}{\Delta \vdash E_0; E_1 \otimes \mathbf{i}_{1t}(v)}$$

$$\boxed{+\mathrm{I}_2} \ \frac{\Delta \vdash E_1 \otimes v}{\Delta \vdash E_0; E_1 \otimes \mathbf{i}_{2t}(v)}$$

$$\boxed{\emptyset \mathrm{I}} \ \frac{\Delta \vdash E \otimes \mathbf{abort}_t(v)}{\Delta \vdash \mathbf{false}_\Gamma \otimes v}$$

Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to become a little like Π types. So the predicate calculus effectively becomes like a linear System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

Types $\qquad t ::= \ldots \mid x \mid * \mid \mathbf{h}(v) \mid \Sigma(x \colon *).t$
Terms $\qquad v ::= \ldots \mid \mathbf{t}(v) \mid \langle x := t, v \rangle$
Contexts $\qquad E ::= \ldots \mid E\,t \mid \Pi(x \colon *).E$
Command Environment $\qquad \Delta ::= \ldots \mid \Delta, x \otimes t$

Not really good at the typing judgements for dependent sum types.

Terms

$$\boxed{\Sigma \mathrm{E}_1} \ \frac{\Gamma \vdash E \colon \Sigma(x \colon *).t}{\Gamma \vdash \mathbf{h}(v) \colon *}$$

$$\boxed{\Sigma \mathrm{E}_2} \ \frac{\Gamma \vdash v \colon \Sigma(x \colon *).t}{\Gamma \vdash \mathbf{t}(v) \colon [x := \mathbf{h}(v)]t}$$

$$\boxed{\Sigma \mathrm{I}} \ \frac{\Gamma \vdash t_0 \colon * \qquad \Gamma, x \colon * \vdash v \colon t_1}{\Gamma \vdash \langle x := t_0, v \rangle \colon \Sigma(x \colon *).t_0}$$

$$\boxed{\Sigma \beta_1} \ \mathbf{h}(\langle x := t, v \rangle) \rightsquigarrow t$$

$$\boxed{\Sigma \beta_2} \ \mathbf{t}(\langle x := t, v \rangle) \rightsquigarrow [x := t]v$$

Context Calculus

$$\boxed{\Sigma \mathrm{E}} \ \frac{\Gamma_0 \vdash E \colon \Sigma(x \colon *).t_1 \qquad \Gamma_1 \vdash t_0 \colon *}{\Gamma_0, \Gamma_1 \vdash E_0\,t_0 \colon [x := t_0]t_1}$$

$$\boxed{\Sigma \mathrm{I}} \ \frac{\Gamma, x \colon * \vdash E \colon t}{\Gamma \vdash \Pi(x \colon *).E \colon \Sigma(x \colon *).t}$$

$$\boxed{\Sigma \beta} \ (\Pi(x \colon *).E)\,t \rightsquigarrow [x := t]E$$

Command Calculus

I can't figure out satisfaction at all.

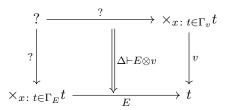$$\boxed{\Sigma \mathrm{I}} \ \frac{\Delta, x \otimes t \vdash E \otimes v}{\Delta \vdash \Pi(x \colon *).E \otimes \langle x := t, v \rangle}$$

Categorical Semantics

The intent is to create calculi encoding the core features of the double category **Rel**.

If this is successful then terms and types ought to map to **Rel** as follows.

I think you want pullbacks for the environment arrows?

I have no idea about universe issues and such. Dependent sum is probably wrong.

Really need to think about denotation again.

$$[\![\top]\!] = \{\emptyset\}$$
$$[\![t_0 \times t_1]\!] = [\![t_0]\!] \times [\![t_1]\!]$$
$$[\![\emptyset]\!] = \emptyset$$
$$[\![t_0 + t_1]\!] = [\![t_0]\!] \sqcup [\![t_1]\!]$$
$$[\![\Sigma(x\colon *).t(x)]\!] = \bigsqcup_{x \in *} [\![t(x)]\!]$$

$$[\![!]\!] = \emptyset$$
$$[\![v_0, v_1]\!] = ([\![v_0]\!], [\![v_1]\!])$$

$$[\![\mathbf{true}]\!](y) = \top$$
$$[\![E_0\, E_1]\!](y) = \exists z, [\![E_1]\!](z) \wedge [\![E_0]\!](z, y)$$
$$[\![\forall(x\colon t).E]\!](\langle x, y\rangle) = [\![E(x)]\!](y)$$

Synthetic Category Theory

A category is a monad in Span. Once this system has been generalized to Span we can define monads internal to Span.

This is not fully internal but a simple approach to defining an equivalence relation might be something like:

$$\boxed{\text{object}}\ \cdot \vdash O\colon *$$
$$\boxed{\text{arrow}}\ \cdot \vdash R\colon O \times O$$
$$\boxed{\text{refl}}\ \frac{\cdot \vdash o\colon O}{\cdot \vdash R \otimes o, o}$$
$$\boxed{\text{trans}}\ \frac{\cdot \vdash \forall(x\colon O).R\,(R\,x) \otimes o_0, o_1 \qquad \cdot \vdash o_0, o_1\colon O \times O}{\cdot \vdash R \otimes o_0, o_1}$$
$$\boxed{\text{sym}}\ \frac{\cdot \vdash R \otimes o_0, o_1 \qquad \cdot \vdash o_1, o_0\colon O \times O}{\cdot \vdash R \otimes o_1, o_0}$$

Generalizing to a constructive interpretation in terms of spans and groupoids is future work.

The Future?

Satisifies judgments correspond to thin squares. Moving to more general categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.