

I wanted to try figuring out a sort of internal language corresponding to the double category **Rel**.

You have a calculus of “values” corresponding to the **Set** edge of **Rel** and a sort of relational calculus of “predicates” corresponding to the other edge. Squares ought to correspond to judgements stating a value satisfies a predicate.

The language of “values” handling Cartesian product of sets has product types (in category theory terms is Cartesian.)

The language of “predicates” which we call the “context calculus” ought to be more complicated. **Rel** is a closed monoidal category over Cartesian product of sets. One has an isomorphism  $\mathbf{Rel}(A, B \otimes C) \sim \mathbf{Rel}(A \otimes B, C)$ . So some sort of linear type theory is required.

The language of “squares” is confusing. It ought to correspond to control expressions in  $\lambda\mu\tilde{\mu}$ .

This is the core framework. I’ve been thinking about further extensions but I want to see how far I can get characterizing **Rel** with as few language features as possible. Later on I give a few possible extensions.

## Core Calculi

<b>Types</b>	$t ::= \top \mid t \times t$
<b>Values</b>	$v ::= x \mid ! \mid \pi_1(v) \mid \pi_2(v) \mid v, v$
<b>Predicates</b>	$p ::= x_t \mid \mathbf{success} \mid p \gg p \mid p p \mid \forall(x:t).p$
<b>Environment</b>	$\Gamma ::= \cdot \mid \Gamma, x:t$
<b>Control Environment</b>	$\Theta ::= \cdot \mid \Theta, x \mid v$

The core context calculus is the linear lambda calculus with a few symbol changes. The core value calculus has only product types.

I’m not really sure small steps semantics make sense with respect to the context calculus but the value calculus corresponding to **Set** ought to have simple deterministic semantics.

### Context Calculus

The context calculus has a linear variable rule. As a hack to make mechanization easier linear variables are explicitly indexed by their types and variable binding is more imperative.

$\boxed{\mathbf{V}}$	$\cdot, x:t \vdash x_t:t$
$\boxed{\mathbf{\top I}}$	$\cdot \vdash \mathbf{success}: \top$
$\boxed{\mathbf{\top E}}$	$\frac{\Gamma \vdash p_0: \top \quad \Delta \vdash p_1: t}{\Gamma, \Delta \vdash p_0 \gg p_1: t}$
$\boxed{\times \mathbf{I}}$	$\frac{x: t_0 \in \Gamma \quad \Gamma \vdash p: t_1}{\Gamma \setminus x \vdash \forall(x:t_0).p: t_0 \times t_1}$
$\boxed{\times \mathbf{E}}$	$\frac{\Gamma \vdash p_0: t_0 \times t_1 \quad \Delta \vdash p_1: t_1}{\Gamma, \Delta \vdash p_0 p_1: t_1}$

$\boxed{\mathbf{\top R}}$	$\mathbf{success} \gg p \rightsquigarrow p$
$\boxed{\times \mathbf{R}}$	$(\forall(x:t).p_0) p_1 \rightsquigarrow [x_t := p_1]p_0$

### Value Calculus

$\boxed{\mathbf{V}}$	$\frac{x:t \in \Gamma}{\Gamma \vdash x: t}$
$\boxed{\mathbf{\top I}}$	$\Gamma \vdash !: \top$
$\boxed{\times \mathbf{E}_1}$	$\frac{\Gamma \vdash v: t_0 \times t_1}{\Gamma \vdash \pi_1(v): t_0}$
$\boxed{\times \mathbf{E}_2}$	$\frac{\Gamma \vdash v: t_0 \times t_1}{\Gamma \vdash \pi_2(v): t_1}$
$\boxed{\times \mathbf{I}}$	$\frac{\Gamma \vdash v_0: t_0 \quad \Gamma \vdash v_1: t_1}{\Gamma \vdash v_0, v_1: t_0 \times t_1}$
$\boxed{\times \mathbf{R}_1}$	$\pi_1(v_0, v_1) \rightsquigarrow v_0$
$\boxed{\times \mathbf{R}_2}$	$\pi_2(v_0, v_1) \rightsquigarrow v_1$

### Control Calculus

$\boxed{\mathbf{V}}$	$\cdot, x_t \otimes v \vdash x_t \otimes v$
$\boxed{\mathbf{vR}}$	$\frac{v_0 \rightsquigarrow v_1 \quad \Theta \vdash p \otimes v_0}{\Theta \vdash p \otimes v_1}$
$\boxed{\mathbf{pR}}$	$\frac{p_0 \rightsquigarrow p_1 \quad \Theta \vdash p_0 \otimes v}{\Theta \vdash p_1 \otimes v}$
$\boxed{\mathbf{\top I}}$	$\Theta \vdash ! \otimes \mathbf{success}$
$\boxed{\times \mathbf{I}}$	$\frac{\Theta, x_t \otimes v_0 \vdash p \otimes v_1}{\Theta \vdash \forall(x:t).p \otimes v_0, v_1}$

## Examples

Pattern matching on equality

$$\frac{\top}{\Theta, x_t \otimes v \vdash x_t \otimes v} (\mathbf{V})$$

$$\frac{}{\Theta \vdash \forall(x:t).x_t \otimes v, v} (\times \mathbf{I})$$

Transposition

$$\forall(p: t \times t \times \top)(x:t)(y:t).p_{(t \times t \times \top)} y_t x_t$$

## Disjoint Unions

Disjoint unions in **Set** become Cartesian product/coproduct in **Rel**. It might make sense to use different notations for coproduct in **Rel**.

I decided against rules for mapping from **Set** to **Rel** because I wanted to see how far **Rel** could go on its own and also wanted to see if these could be defined later.

I have a hunch it is proper for the combination of product/coproduct to introduce nondeterminism in the operational semantics but I need to think more about the issue.

As a technical hack **fail** is explicitly indexed by the environment it ignores. This hack also requires inferring the environment in certain reduction rules.

<b>Types</b>	$t ::= \dots \mid \emptyset \mid t + t$
<b>Values</b>	$v ::= \dots \mid \mathbf{abort}_t(v) \mid \mathbf{i}_{1t}(v) \mid \mathbf{i}_{2t}(v) \mid \mathbf{m}(v, x.v, x.v)$

**Predicates**  $p ::= \dots \mid \mathbf{abort}_t(p) \mid \mathbf{i}_{1t}(p) \mid \mathbf{i}_{2t}(p) \mid \mathbf{m}(v, x.v, x.v) \mid \mathbf{fail}_\Gamma \mid \mathbf{l}(p) \mid \mathbf{r}(p) \mid p; p$

### Value Calculus

$$\begin{array}{c}
\boxed{\emptyset E} \frac{\Gamma \vdash v: \emptyset}{\Gamma \vdash \mathbf{abort}_t(v): t} \\
\boxed{+I_1} \frac{\Gamma \vdash v: t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(v): t_0 + t_1} \\
\boxed{+I_2} \frac{\Gamma \vdash v: t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(v): t_0 + t_1} \\
\boxed{+E} \frac{\Gamma \vdash v_0: t_0 + t_1 \quad \Gamma, x_0: t_0 \vdash v_1: t_2 \quad \Gamma, x_1: t_1 \vdash v_2: t_2}{\Gamma \vdash \mathbf{m}(v_0, x_0.v_1, x_1.v_2): t_2} \\
\boxed{+R_1} \mathbf{m}(\mathbf{i}_{1t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_0 := v_0]v_1 \\
\boxed{+R_2} \mathbf{m}(\mathbf{i}_{2t}(v_0), x_0.v_1, x_1.v_2) \rightsquigarrow [x_1 := v_0]v_2
\end{array}$$

### Context Calculus

$$\begin{array}{c}
\boxed{\emptyset E^T} \frac{\Gamma \vdash p: \emptyset}{\Gamma \vdash \mathbf{abort}_t(p): t} \\
\boxed{+I_1^T} \frac{\Gamma \vdash p: t_0}{\Gamma \vdash \mathbf{i}_{1t_1}(p): t_0 + t_1} \\
\boxed{+I_2^T} \frac{\Gamma \vdash p: t_1}{\Gamma \vdash \mathbf{i}_{2t_0}(p): t_0 + t_1} \\
\boxed{+E^T} \frac{\Gamma \vdash p_0: t_0 + t_1 \quad \Gamma, x_0: t_0 \vdash p_1: t_2 \quad \Gamma, x_1: t_1 \vdash p_2: t_2}{\Gamma \vdash \mathbf{m}(p_0, x_0.p_1, x_1.p_2): t_2} \\
\boxed{\emptyset I} \Gamma \vdash \mathbf{fail}_\Gamma: \emptyset \\
\boxed{+E_1} \frac{\Gamma \vdash p: t_0 + t_1}{\Gamma \vdash \mathbf{l}(p): t_0} \quad \boxed{+E_2} \frac{\Gamma \vdash p: t_0 + t_1}{\Gamma \vdash \mathbf{r}(p): t_1} \\
\boxed{+I} \frac{\Gamma \vdash p_0: t_0 \quad \Gamma \vdash p_1: t_1}{\Gamma \vdash p_0; p_1: t_0 + t_1} \\
\boxed{+R_1^T} \mathbf{m}(\mathbf{i}_{1t}(p_0), x_0.p_1, x_1.p_2) \rightsquigarrow [x_0 := p_0]p_1 \\
\boxed{+R_2^T} \mathbf{m}(\mathbf{i}_{2t}(p_0), x_0.p_1, x_1.p_2) \rightsquigarrow [x_1 := p_0]p_2 \\
\boxed{+R_1} \mathbf{l}(p_0; p_1) \rightsquigarrow p_0 \quad \boxed{+R_2} \mathbf{r}(p_0; p_1) \rightsquigarrow p_1 \\
\boxed{+RR_1^T} \mathbf{l}(\mathbf{i}_{1t}(p)) \rightsquigarrow p \quad \boxed{+RR_2^T} \mathbf{r}(\mathbf{i}_{2t}(p)) \rightsquigarrow p \\
\boxed{+RR_3^T} \frac{\Gamma \vdash p: t}{\mathbf{l}(\mathbf{i}_{2t}(p)) \rightsquigarrow \mathbf{abort}_t(\mathbf{fail}_\Gamma)} \\
\boxed{+RR_4^T} \frac{\Gamma \vdash p: t}{\mathbf{r}(\mathbf{i}_{1t}(p)) \rightsquigarrow \mathbf{abort}_t(\mathbf{fail}_\Gamma)} \\
\boxed{+R^T R_1} \mathbf{m}(p_0; p_1, x_0.p_2, x_1.p_3) \rightsquigarrow [x_0 := p_0]p_2 \\
\boxed{+R^T R_2} \mathbf{m}(p_0; p_1, x_0.p_2, x_1.p_3) \rightsquigarrow [x_1 := p_1]p_3
\end{array}$$

### Control Calculus

$$\begin{array}{c}
\boxed{+I_1} \frac{\Theta \vdash p_0 \otimes v}{\Theta \vdash p_0; p_1 \otimes \mathbf{i}_{1t}(v)} \\
\boxed{+I_2} \frac{\Theta \vdash p_1 \otimes v}{\Theta \vdash p_0; p_1 \otimes \mathbf{i}_{2t}(v)} \\
\boxed{\emptyset I} \frac{\Theta \vdash p \otimes \mathbf{abort}_t(v)}{\Theta \vdash \mathbf{fail}_\Gamma \otimes v}
\end{array}$$

### Dependent Sums

If product of sets becomes an internal hom in the predicate calculus then dependent sums ought to

become a little like  $\Pi$  types. So the predicate calculus effectively becomes like a linear System-F.

Some things become awkward to interpret here though.

I also really can't figure out unpacking. It's messy if you don't want full dependent types.

**Types**  $t ::= \dots \mid x \mid * \mid \mathbf{h}(v) \mid \Sigma(x: *) . t$   
**Values**  $v ::= \dots \mid \mathbf{t}(v) \mid \langle x := t, v \rangle$   
**Predicates**  $p ::= \dots \mid p t \mid \Pi(x: *) . p$   
**Control Environment**  $\Theta ::= \dots \mid \Theta, x \otimes t$

Not really good at the typing judgements for dependent sum types.

### Values

$$\begin{array}{c}
\boxed{\Sigma E_1} \frac{\Gamma \vdash v: \Sigma(x: *) . t}{\Gamma \vdash \mathbf{h}(v): *} \\
\boxed{\Sigma E_2} \frac{\Gamma \vdash v: \Sigma(x: *) . t}{\Gamma \vdash \mathbf{t}(v): [x := \mathbf{h}(v)]t} \\
\boxed{\Sigma I} \frac{\Gamma \vdash t_0: * \quad \Gamma, x: * \vdash v: t_1}{\Gamma \vdash \langle x := t_0, v \rangle: \Sigma(x: *) . t_0} \\
\boxed{\Sigma S_1} \mathbf{h}(\langle x := t, v \rangle) \rightsquigarrow t \\
\boxed{\Sigma S_2} \mathbf{t}(\langle x := t, v \rangle) \rightsquigarrow [x := t]v
\end{array}$$

### Context Calculus

$$\begin{array}{c}
\boxed{\Sigma E} \frac{\Gamma \vdash p: \Sigma(x: *) . t_1 \quad \Delta \vdash t_0: *}{\Gamma, \Delta \vdash p_0 t_0: [x := t_0]t_1} \\
\boxed{\Sigma I} \frac{\Gamma, x: * \vdash p: t}{\Gamma \vdash \Pi(x: *) . p: \Sigma(x: *) . t} \\
\boxed{\Sigma R} (\Pi(x: *) . p) t \rightsquigarrow [x := t]p
\end{array}$$

### Control Calculus

I can't figure out satisfaction at all.

$$\boxed{\Sigma I} \frac{\Theta, x \otimes t \vdash p \otimes v}{\Theta \vdash \Pi(x: *) . p \otimes \langle x := t, v \rangle}$$

### Categorical Semantics

The intent is to create calculi encoding the core features of the double category **Rel**.

If this is successful then terms and types ought to map to **Relas** follows.

$$\begin{array}{ccc}
\top & \xrightarrow{\Theta_v} & \times_{x: t \in \Gamma_v} t \\
\Theta_p \downarrow & & \downarrow v \\
& \Downarrow \Theta_p, \Theta_v \vdash p \otimes v & \\
\times_{x: t \in \Gamma_p} t & \xrightarrow{p} & t
\end{array}$$

I have no idea about universe issues and such. Dependent sum is probably wrong.

Really need to think about denotation again.

$$\begin{aligned}
\llbracket \top \rrbracket &= \{\emptyset\} \\
\llbracket t_0 \times t_1 \rrbracket &= \llbracket t_0 \rrbracket \times \llbracket t_1 \rrbracket \\
\llbracket \emptyset \rrbracket &= \emptyset \\
\llbracket t_0 + t_1 \rrbracket &= \llbracket t_0 \rrbracket \sqcup \llbracket t_1 \rrbracket \\
\llbracket \Sigma(x: *) . t(x) \rrbracket &= \bigsqcup_{x \in *} \llbracket t(x) \rrbracket
\end{aligned}$$

$$\begin{aligned}
\llbracket ! \rrbracket &= \emptyset \\
\llbracket (v_0, v_1) \rrbracket &= (\llbracket v_0 \rrbracket, \llbracket v_1 \rrbracket)
\end{aligned}$$

$$\begin{aligned}
\llbracket \text{success} \rrbracket(y) &= \top \\
\llbracket p_0 p_1 \rrbracket(y) &= \exists z, \llbracket p_1 \rrbracket(z) \wedge \llbracket p_0 \rrbracket(z, y) \\
\llbracket \forall(x: t) . p \rrbracket(\langle x, y \rangle) &= \llbracket p(x) \rrbracket(y)
\end{aligned}$$

## Synthetic Category Theory

A category is a monad in  $\text{Span}$ . Once this system has been generalized to  $\text{Span}$  we can define monads internal to  $\text{Span}$ .

This is not fully internal but a simple approach to defining an equivalence relation might be something like:

$$\begin{array}{c}
\boxed{\text{object}} \quad \cdot \vdash O : * \\
\boxed{\text{arrow}} \quad \cdot \vdash R : O \times O \\
\boxed{\text{refl}} \quad \frac{\cdot \vdash o : O}{\cdot \vdash R \otimes o, o} \\
\boxed{\text{trans}} \quad \frac{\cdot \vdash o_0, o_1 : O \times O \quad \cdot \vdash \forall(x: O). R(Rx) \otimes o_0, o_1}{\cdot \vdash R \otimes o_0, o_1} \\
\boxed{\text{sym}} \quad \frac{\cdot \vdash o_1, o_0 : O \times O \quad \cdot \vdash R \otimes o_0, o_1}{\cdot \vdash R \otimes o_1, o_0}
\end{array}$$

Generalizing to a constructive interpretation in terms of spans and groupoids is future work.

## The Future?

Satisfies judgments correspond to thin squares. Moving to more general categories such as **Span** or **Prof** or **Vect** for matrix math requires an interpretation of squares carrying constructive content.