Homework 3 quantum mechanics applications, Fine and hyperfine structure of Hydrogen

April 17, 2023

- Problem 1:

  Consider the eight states n=2 labeled $|2, l, j, mj\rangle$. Find the energy of each state under the Zeeman effect for a weak magnetic field.

- Problem 2:

  Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two vectors, show:

  a.

  $$\int (\overrightarrow{a} \cdot \hat{r})(\overrightarrow{b} \cdot \hat{r}) \sin\theta d\theta d\phi = \frac{4\pi}{3}(\overrightarrow{a} \cdot \overrightarrow{b}) \tag{1}$$

  where the limits of integration of θ and ϕ are the usual in the solid angle.

  b. Use the previous result to show that, when $l = 0$:

  $$\left\langle \frac{3(\overrightarrow{I} \cdot \hat{r})(\overrightarrow{S} \cdot \hat{r}) - \overrightarrow{I} \cdot \overrightarrow{S}}{r^3} \right\rangle = 0 \tag{2}$$

  where $\overrightarrow{I}$ is the proton spin operator and $\overrightarrow{S}$ is the electron spin operator. Hint, use $\hat{r}$ in spherical coordinates.

- Problem 3. For the Spin orbit coupling term, the calculation of $\langle \frac{1}{r^3} \rangle$ is needed, this is equal to $\frac{1}{l(l+1/2)(l+1)n^3a_0^3}$. With this result, find the first order contribution to the energy generated by this term for any value of j, l and n.

- Problem 4: The Feynman-Hellmann theorem states that:

$$\frac{\partial E_n}{\partial \lambda} = \left\langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \right\rangle \tag{3}$$

  where E_n is non-degenerate or if degenerate ψ_n are linear combination of the degenerate eigenfunctions. Use this theorem to find the expection values of $\frac{1}{r}$ and $\frac{1}{r^2}$

- Problem 5 Consider a Hydrogen atom which is subject to two weak static fields. An electric field in the xy plane $\overrightarrow{\varepsilon} = \varepsilon_0(\hat{i} + \hat{j})$ and a magnetic field along the z-axis with strength B_0. Neglecting spin-orbit coupling, calculate the energy levels of the n=2 states to first order in perturbation theory.