Second-order expansion of the law of motion of a perturbed distribution

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In this note, I develop analytically the second-order expansion of the law of motion of a perturbed distribution. With this, I clarify the critique by Bhandari et al. (2023) that the solution for perturbed distributions so far in the heterogeneous agent literature misses second order (and higher order) terms. I find their focus on lotteries to be misleading. Instead, it appears to be the case that the literature so far has only considered models where the law of motion of the perturbed distribution is approximated up to the first order, and where the grid is fine enough. When the model is solved up to higher orders, or if it is solved up to first order but on a coarse grid, biases emerge. The second-order expansion that I derive yields first- and second-order correction terms that can be implemented within standard state-space solution methods, where the distribution is discretized over a grid.

To rigorously analyze perturbed distributions, I borrow a concept from the robust estimation literature (see Hampel 1974): the influence function (or influence curve). Intuitively, the influence function yields the marginal effect of perturbing a distribution infinitissimally at one point on a functional over that distribution. Here, the functional will be a mapping of a distribution in period t to the cumulative probability of a given point in period $t+1$. I analyze the general case where the functional is an integral over a possibly continuous distribution. However, the missing second-order terms do not stem from premature discretization, but from an apparent neglect of higher order terms in the law of motion of the perturbed distribution, while the missing first-order terms vanish for a fine enough grid.

1 Definitions

Let a distribution be characterized by its cdf $F(k)$ and its pdf $f(k)$, where k is an idiosyncratic state on the support $[0, k_{max}] \subset \mathbb{R}$. Let $\phi_{k'} : F \mapsto \phi(F)(k')$ be the functional that maps distribution F_t to $F_{t+1}(k')$. It is defined by

$$\phi(F)(k') := \int_0^{g^{-1}(k'|F)} \Pi(k, k' \mid g(\cdot \mid F))dF(k). \tag{1}$$

Here, $g(k \mid F)$ is a function that maps k to the *optimal policy.* The optimal policy generally depends on the distribution F (next to other aggregate states that I abstract from here). I make the following assumption:

Assumption 1.1. *The optimal policy function depends on the distribution only through some average over it:*

$$g(k \mid F) = g(k \mid K_\alpha(F)) \text{ with } K_\alpha(F) := \int \alpha(k)dF(k), \tag{2}$$

for some $\alpha : \mathbb{R} \to \mathbb{R}$.

In typical economic models, assumption (1.1) holds with $\alpha(x) = x$, through a market-clearing condition. I assume that the optimal policy function is continuously differentiable in all arguments, monotonly increasing in k, and that $g(0 \mid F) = 0$ (the last assumption is for convenience only).

$\Pi(k, k' \mid g(\cdot \mid F))$ is the *transition probability* of an agent of going from idiosyncratic state k to k', given her optimal policy function g. In general, these functions will depend on other dimensions of the agent's state, like her income. I do not account for this explicitly for ease of exposition. $\Pi(\cdot, k' \mid g(\cdot \mid F))$ has to be continuously differentiable almost everywhere. This assumption holds in particular for the case of lotteries over a grid, which I consider as a special case later on.

The *influence function* is defined as

$$IF(k \mid \phi_{k'}, F) := \lim_{\epsilon \to 0} \frac{\phi_{k'}((1-\epsilon)F + \epsilon\delta_k) - \phi_{k'}(F)}{\epsilon}, \tag{3}$$

where δ_k denotes the Dirac measure $\delta_k(x) = \mathbb{I}_{\{x=k\}}$.

2 Law of motion of the perturbed distribution

Let $\bar{F}$ denote the distribution in a non-stochastic (with respect to aggregate variables) steady state, and F_t the distribution in period t. I define $\hat{F}_t := F_t - \bar{F}$ as the *perturbed distribution* in period t. The main observation is that the distribution F_{t+1} can be approximated around $\bar{F}$ using a Taylor-expansion and the influence function:

$$F_{t+1}(k') \approx \bar{F}(k') + \int IF(k \mid \phi_{k'}, \bar{F}) d(F_t - \bar{F})(k) + \frac{1}{2} \int IF_k(k \mid \phi_{k'}, \bar{F}) d(F_t - \bar{F})^2(k) \tag{4}$$

Two comments are in order. First, while Hampel (1974) only considers a first-order Taylor expansion of the perturbed distribution, I extend it to the second order. Second, and relatedly, I have to define the second-order derivative of the functional $\phi_{k'}$ by the perturbation of the distribution at point k. I define the second-order derivative as the marginal change of IF when marginally changing perturbation point k, and call this function IF_k.

Note that equation (4) can be reformulated as a *second-order approximation of the law of motion of the perturbed distribution*:

$$\hat{F}_{t+1}(k') \approx \int IF(k \mid \phi_{k'}, \bar{F}) d\hat{F}_t(k) + \frac{1}{2} \int IF_k(k \mid \phi_{k'}, \bar{F}) d\hat{F}_t^2(k) \tag{5}$$

3 The influence function of the law of motion

In this section, I analytically compute $IF(k \mid \phi_{k'}, F)$. In order to save space, I define $F_\epsilon := (1-\epsilon)F + \epsilon\delta_k$ as the distribution that is perturbed at point k.

First, observe that

$$
\begin{aligned}
\phi_{k'}(F_\epsilon) - \phi_{k'}(F) =& \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))((1-\epsilon)f(x) + \epsilon\delta_k(x))dx && (6)\\
&- \int_0^{g^{-1}(k'|F)} \Pi(x, k' \mid g(\cdot \mid F))f(x)dx && (7)\\
=& \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))\epsilon(\delta_k(x) - f(x))dx && (8)\\
&+ \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))f(x)dx - \int_0^{g^{-1}(k'|F)} \Pi(x, k' \mid g(\cdot \mid F))f(x)dx && (9)
\end{aligned}
$$

I take the limit $\lim_{\epsilon\to 0} \frac{(\cdot)}{\epsilon}$ separately for the integral (8) and the integrals in (9). For the integral (8), I get[1]

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon} \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))\epsilon(\delta_k(x) - f(x))dx && (10)\\
&= \int_0^{g^{-1}(k'|F)} \Pi(x, k' \mid g(\cdot \mid F))(\delta_k(x) - f(x))dx = \Pi(k, k' \mid g(\cdot \mid F)) - \phi_{k'}(F). && (11)
\end{aligned}
$$

For the integrals in (9), I observe that taking their limit is the same as differentiating the functional $\phi_{k'}s(F)$ in the direction of F_ϵ. Using Leibniz' rule and assumption 1.1, I get

$$
\begin{aligned}
\frac{\partial}{\partial F}\phi_{k'}(F)\hat{F}_\epsilon =& \Pi(g^{-1}(k' \mid F), k' \mid g(\cdot \mid F))f(g^{-1}(k' \mid F))(g^{-1})_{K_\alpha}(k' \mid F)IF(k \mid K_\alpha, F) && (12)\\
&+ \int_0^{g^{-1}(k'|F)} \Pi_g(x, k' \mid g(\cdot \mid F))g_{K_\alpha}(x \mid F)IF(k \mid K_\alpha, F)f(x)dx, && (13)
\end{aligned}
$$

where $IF(k \mid K_\alpha, F)$ is the influence function of perturbing F at the point k on the functional K_α. A computation similar to the above yields that

$$
IF(k \mid K_\alpha, F) = \alpha(k) - K_\alpha(F). \tag{14}
$$

In sum, I obtain $IF(k \mid \phi_{k'}, F) = A_{k'}(k \mid F) + B_{k'}(F)$, where only $A_{k'}(k \mid F)$ depends on k. The two terms are given by

$$
A_{k'}(k \mid F) = \Pi(k, k' \mid g(\cdot \mid F)) + \alpha(k)C_{k'}(F) \text{ and } B_{k'}(F) = -\phi_{k'}(F) - K_\alpha(F)C_{k'}(F), \tag{15}
$$

where I define

$$
\begin{aligned}
C_{k'}(F) :=& \Pi(g^{-1}(k' \mid F), k' \mid g(\cdot \mid F))f(g^{-1}(k' \mid F))(g^{-1})_{K_\alpha}(k' \mid F) && (16)\\
&+ \int_0^{g^{-1}(k'|F)} \Pi_g(x, k' \mid g(\cdot \mid F))g_{K_\alpha}(x \mid F)f(x)dx
\end{aligned}
$$

The second derivative of the functional by the perturbation of the distribution at point k is given by

$$
IF_k(k \mid \phi_{k'}, F) = \Pi_k(k, k' \mid g(\cdot \mid F)) + \alpha_k(k)C_{k'}(F). \tag{17}
$$

[1] Note that $\Pi(k, k' \mid g(\cdot \mid F)) = 0$ if $k > g^{-1}(k' \mid F)$.

3.1 Lotteries over grid points

Consider the important special case where the transition probability function is given by[2]

$$\Pi(k, k_i \mid g(\cdot \mid F)) = \mathbb{I}_{\{g(k|F)\in(k_{i-1},k_i]\}}\frac{g(k \mid F) - k_{i-1}}{k_i - k_{i-1}} + \mathbb{I}_{\{g(k|F)\in(k_i,k_{i+1}]\}}\frac{k_{i+1} - g(k \mid F)}{k_{i+1} - k_i}, \tag{18}$$

where $\{k_i\}_{i=1..N}$ denotes the list of N grid points. This transition probability function is differentiable in k except at the points $\{g^{-1}(k_i \mid F)\}_{i=1..N}$, which is a zero-probability set. The first derivative is given by

$$\Pi_k(k, k_i \mid g(\cdot \mid F)) = g_k(k \mid F)\left(\mathbb{I}_{\{g(k|F)\in(k_{i-1},k_i]\}}\frac{1}{k_i - k_{i-1}} - \mathbb{I}_{\{g(k|F)\in(k_i,k_{i+1}]\}}\frac{1}{k_{i+1} - k_i}\right). \tag{19}$$

It is easy to see that in order to capture differences in the approximation of the law of motion of the distribution for third or higher orders, one needs a numerical representation of the optimal policy function with non-zero curvature.

4 Computing the law of motion of the distribution

I define $\bar{B}_{k'} := B_{k'}(\bar{F})$, $\bar{C}_{k'} := C_{k'}(\bar{F})$ and $\hat{K}_{\alpha,t} := K_\alpha(F_t) - K_\alpha(\bar{F})$ for notational ease. Substituting the above results into equation (5) yields

$$\hat{F}_{t+1}(k') \approx \int \Pi(k, k' \mid g(\cdot \mid \bar{F})) + \alpha(k)\bar{C}_{k'} + \bar{B}_{k'}d\hat{F}_t(k) + \frac{1}{2}\int \Pi_k(k, k' \mid g(\cdot \mid \bar{F})) + \alpha_k(k)\bar{C}_{k'}d\hat{F}_t^2(k) \tag{20}$$

$$= \int \Pi(k, k' \mid g(\cdot \mid \bar{F}))d\hat{F}_t(k) + \bar{C}_{k'}\int \alpha(k)d\hat{F}_t(k) + \frac{1}{2}\int \Pi_k(k, k' \mid g(\cdot \mid \bar{F})) + \alpha_k(k)\bar{C}_{k'}d\hat{F}_t^2(k) \tag{21}$$

$$= \int \Pi(k, k' \mid g(\cdot \mid \bar{F}))d\hat{F}_t(k) + \underbrace{\frac{1}{2}\int \Pi_k(k, k' \mid g(\cdot \mid \bar{F}))d\hat{F}_t^2(k)}_{=:SO} \tag{22}$$

$$+ \underbrace{\bar{C}_{k'}\left(\hat{K}_{\alpha,t} + \frac{1}{2}\int \alpha_k(k)d\hat{F}_t^2(k)\right)}_{=:GE},$$

where $\bar{B}_{k'}$ vanishes in the second line as $\int d\hat{F}_t(k) = 0$, since $\hat{F}_t(k_{max}) = F_t(k_{max}) - \bar{F}(k_{max}) = 1 - 1 = 0$.

Comparing the resulting second-order approximation of the law of motion of the perturbed distribution, (22), to the Fokker-Planck equation that governs the dynamic of the distribution in typical heterogeneous agent models, there are two types of corrections: SO is a "pure" second order-correction, which is an integral over the first derivative of the optimal policy function. Intuitively, when computing the change in the probability weight of point k' from changes in the probability weights of points k, one should account for the speed with which agents go from k to k'.

GE is a general equilibrium effect: it captures the impact of a change in the distribution on optimal policies, which *in the full model* – which I did not spell out here – works through market-clearing and prices. Interestingly, my analysis shows that a change in K_α – which for example is the aggregate amount of capital in the economy – has a *first-order effect* on the change in the distribution next period. The strength of the GE-effect – and thus the magnitude of the bias when leaving it out – is governed by $\bar{C}_{k'}$.

[2] Again, note that I abstract from other states or idiosyncratic stochastic shocks for expositional clarity, knowing that in general one would have them to obtain non-degenerate distributions.

4.1 Implementation over a finite grid

Let μ denote a row vector of length $N-1$ where $\mu_i = F(k_{i+1}) - F(k_i), i = 1..N-1$. Let $\bar{\Pi}$ denote the transition probability matrix in steady state, where $\bar{\Pi}_{ij} = \Pi(k_i, k_j \mid g(\cdot \mid \bar{F}))$, and let $\bar{\Psi}$ denote the *marginal* transition probability matrix in steady state, with $\bar{\Psi}_{ij} = \Pi_k(k_i, k_j \mid g(\cdot \mid \bar{F}))$.

Let $\Delta_N \bar{C}$ denote a row vector of size $N-1$ with $\Delta_N \bar{C}_i := \bar{C}_{k_{i+1}} - \bar{C}_{k_i}, i = 1..N-1$. Assuming a smooth optimal policy function, $k' \mapsto \bar{C}_{k'}$ is a continuous function. It follows immediately that $\Delta_N \bar{C} \to [00..0]$ for $N \to \infty$, given that k_{max} stays constant. Assuming $\alpha(x) = x$, I define $K_t := \sum_{i=1}^{N-1} k_i \mu_{i,t}$ and $\hat{K}_t = K_t - \bar{K}$ analogously.

The second order approximation of the law of motion of the perturbation in the histogram $\hat{\mu}$ around the steady state is then given by

$$\hat{\mu}_{t+1} = \hat{\mu}_t \bar{\Pi} + \frac{1}{2} \sum_{l=1}^{N} \hat{\mu}_{l,t}^2 \bar{\Psi}_{l\cdot} + \Delta_N \bar{C} \left(\hat{K}_t + \frac{1}{2} \sum_{l=1}^{N} \hat{\mu}_{l,t}^2 \right) \tag{23}$$

4.2 Remark: Interaction terms with other variables

There is a difference between the non-linear effects that are caused by changes in the distribution *only*, and non-linear effects caused by changes in the distribution that *interact* with changes in other variables. In this note, I only analyzed the former (remember that also the *GE*-effect is only caused by a change in the distribution itself). Since perturbations in other aggregate variables are abstracted from, the second-order approximation around the steady-state distribution is sufficient.

In order to implement the second-order approximation as in equation (23) in a full-sized model, the matrices $\bar{\Pi}$ and $\bar{\Psi}$ and the vector $\Delta_N \bar{C}$ have to be substituted for their counterparts at period t, i.e. Π_t, Ψ_t, and $\Delta_N C_t$. Interactions of perturbations in these objects (due to changes in the optimal policy function) with perturbations in the distribution cause additional non-linearities that have to be accounted for once the model is solved to the second or a higher order.

5 References

Bhandari, Anmol, Bourany, Thomas, Evans, David, and Mikhail Golosov (2023): “A Perturbational Approach for Approximating Heterogeneous Agent Models”, mimeo

Hampel, Frank R. (1974): “The Influence Curve and Its Role in Robust Estimation”, Journal of the American Statistical Association, 346(69), 383-393