Second-order expansion of the law of motion of a perturbed distribution

Maximilian Weiß

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In this note, I develop analytically the second-order expansion of the law of motion of a perturbed distribution. With this, I clarify the justified critique by Bhandari et al. (2023) that the solution for perturbed distributions so far in the heterogeneous agent literature misses a second order (or higher order) term. The focus on lotteries is, however, a red herring. Instead, it is simply the case that the literature so far has only considered models where the law of motion of the perturbed distribution is approximated up to the first order. When the model is solved up to higher orders, this introduces an error. The second-order expansion that I derive yields a second-order correction term that can be implemented within standard state-space solution methods, where the distribution is discretized over a grid.

To rigorously analyze perturbed distributions, I borrow a concept from the robust estimation literature (see Hampel 1974): the influence function (or influence curve). Intuitively, the influence function yields the marginal effect of perturbing a distribution infinitissimally at one point on a functional over that distribution. Here, the functional will be a mapping of a distribution in period t to the cumulative probability of a given point in period $t+1$. I analyze the general case where the functional is an integral over a possibly continuous distribution. However, I find that the result does not change if everything is discrete during the derivation. This clarifies that the missing second-order term does not stem from premature discretization, but from a simple neglect of higher order terms in the law of motion of the perturbed distribution.

1 Definitions

Let a distribution be characterized by its cdf $F(k)$ and its pdf $f(k)$, where k is an idiosyncratic state on the support $[0, k_{max}] \subset \mathbb{R}$. Let $\phi_{k'} : F \mapsto \phi(F)(k')$ be the functional that maps distribution F_t to $F_{t+1}(k')$. It is defined by

$$\phi(F)(k') := \int_0^{g^{-1}(k'|F)} \Pi(k, k' \mid g(\cdot \mid F)) dF(k). \tag{1}$$

Here, $g(k \mid F)$ is a function that maps k to the *optimal policy.* The optimal policy may depend on the distribution F (next to other aggregate states that I abstract from here). I assume that the optimal policy function is continuously differentiable in all arguments, monotonly increasing in k, and that $g(0 \mid F) = 0$ (the last assumption is for convenience only).

$\Pi(k, k' \mid g(\cdot \mid F))$ is the *transition probability* of an agent of going from idiosyncratic state k to k', given her optimal policy function g. In general, these functions will depend on other dimensions of the agent's state, like her income. I do not account for this explicitly for ease of exposition. $\Pi(\cdot, k' \mid g(\cdot \mid F))$ has to be continuously differentiable almost everywhere. This assumption holds in particular for the case of lotteries over a grid, which I consider as a special case later on.

The *influence function* is defined as

$$IF(k \mid \phi_{k'}, F) := \lim_{\epsilon \to 0} \frac{\phi_{k'}((1-\epsilon)F + \epsilon\delta_k) - \phi_{k'}(F)}{\epsilon}, \tag{2}$$

where δ_k denotes the Dirac measure $\delta_k(x) = \mathbb{I}_{\{x=k\}}$.

2 Law of motion of the perturbed distribution

Let $\bar{F}$ denote the distribution in a non-stochastic (with respect to aggregate variables) steady state, and F_t the distribution in period t. I define $\hat{F}_t := F_t - \bar{F}$ as the *perturbed distribution* in period t. The main observation is that the distribution F_{t+1} can be approximated around $\bar{F}$ using a Taylor-expansion and the influence function:

$$F_{t+1}(k') \approx \bar{F}(k') + \int IF(k \mid \phi_{k'}, \bar{F}) d(F_t - \bar{F})(k) + \frac{1}{2} \int IF_k(k \mid \phi_{k'}, \bar{F}) d(F_t - \bar{F})^2(k) \tag{3}$$

Two comments are in order. First, while Hampel (1974) only considers a first-order Taylor expansion of the perturbed distribution, I extend it to the second order. Second, and relatedly, I have to define the second-order derivative of the functional $\phi_{k'}$ by the perturbation of the distribution at point k. I define the second-order derivative as the marginal change of IF when marginally changing perturbation point k, and call this function IF_k.

Note that equation (3) can be reformulated as a *second-order approximation of the law of motion of the perturbed distribution*:

$$\hat{F}_{t+1}(k') \approx \int IF(k \mid \phi_{k'}, \bar{F}) d\hat{F}_t(k) + \frac{1}{2} \int IF_k(k \mid \phi_{k'}, \bar{F}) d\hat{F}_t^2(k) \tag{4}$$

3 Computation of the influence function

In this section, I analytically compute $IF(k \mid \phi_{k'}, F)$. In order to save space, I define $F_\epsilon := (1-\epsilon)F + \epsilon\delta_k$ as the distribution that is perturbed at point k.

First, observe that

$$\phi_{k'}(F_\epsilon) - \phi_{k'}(F) = \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))((1-\epsilon)f(x) + \epsilon\delta_k(x))dx \tag{5}$$

$$- \int_0^{g^{-1}(k'|F)} \Pi(x, k' \mid g(\cdot \mid F))f(x)dx \tag{6}$$

$$= \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))\epsilon(\delta_k(x) - f(x))dx \tag{7}$$

$$+ \int_0^{g^{-1}(k'|F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon))f(x)dx - \int_0^{g^{-1}(k'|F)} \Pi(x, k' \mid g(\cdot \mid F))f(x)dx \tag{8}$$

I take the limit $\lim_{\epsilon \to 0} \frac{(\cdot)}{\epsilon}$ separately for the integral (7) and the integrals in (8). For the integral (7), I get[1]

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^{g^{-1}(k' \mid F_\epsilon)} \Pi(x, k' \mid g(\cdot \mid F_\epsilon)) \epsilon(\delta_k(x) - f(x)) dx \tag{9}$$

$$= \int_0^{g^{-1}(k' \mid F)} \Pi(x, k' \mid g(\cdot \mid F))(\delta_k(x) - f(x)) dx = \Pi(k, k' \mid g(\cdot \mid F)) - \phi_{k'}(F). \tag{10}$$

For the integrals in (8), I observe that taking their limit is the same as differentiating the functional $\phi_{k'}(F)$. Using Leibniz' rule, I get

$$\frac{\partial}{\partial F} \phi_{k'}(F) = \Pi(g^{-1}(k' \mid F), k' \mid g(\cdot \mid F)) f(g^{-1}(k' \mid F))(g_F^{-1})(k' \mid F) \tag{11}$$

$$+ \int_0^{g^{-1}(k' \mid F)} \Pi_g(x, k' \mid g(\cdot \mid F)) g_F(x \mid F) f(x) dx. \tag{12}$$

In sum, I obtain

$$IF(k \mid \phi_{k'}, F) = \underbrace{\Pi(k, k' \mid g(\cdot \mid F))}_{=:A(k)} - \phi_{k'}(F) + \Pi(g^{-1}(k' \mid F), k' \mid g(\cdot \mid F)) f(g^{-1}(k' \mid F))(g_F^{-1})(k' \mid F)$$
$$+ \int_0^{g^{-1}(k' \mid F)} \Pi_g(x, k' \mid g(\cdot \mid F)) g_F(x \mid F) f(x) dx \tag{13}$$

I define B such that $IF(k \mid \phi_{k'}, F) = A(k) + B$. Importantly, only term $A(k)$ depends on the perturbation point k. Hence, the second derivative of the functional by the perturbation of the distribution at point k is given by

$$IF_k(k \mid \phi_{k'}, F) = \Pi_k(k, k' \mid g(\cdot \mid F)). \tag{14}$$

3.1 Lotteries over grid points

Consider the important special case where the transition probability function is given by[2]

$$\Pi(k, k_i \mid g(\cdot \mid F)) = \mathbb{I}_{\{g(k|F) \in (k_{i-1}, k_i]\}} \frac{g(k \mid F) - k_{i-1}}{k_i - k_{i-1}} + \mathbb{I}_{\{g(k|F) \in (k_i, k_{i+1}]\}} \frac{k_{i+1} - g(k \mid F)}{k_{i+1} - k_i}, \tag{15}$$

where $\{k_i\}_{i=1..N}$ denotes the list of N grid points. This transition probability function is differentiable in k except at the points $\{g^{-1}(k_i \mid F)\}_{i=1..N}$, which is a zero-probability set. The first derivative is given by

$$\Pi_k(k, k_i \mid g(\cdot \mid F)) = g_k(k \mid F) \left(\mathbb{I}_{\{g(k|F) \in (k_{i-1}, k_i]\}} \frac{1}{k_i - k_{i-1}} - \mathbb{I}_{\{g(k|F) \in (k_i, k_{i+1}]\}} \frac{1}{k_{i+1} - k_i} \right). \tag{16}$$

It is easy to see that in order to capture differences in the approximation of the law of motion of the distribution for third or higher orders, one needs a numerical representation of the optimal policy function with non-zero curvature.

[1]Note that $\Pi(k, k' \mid g(\cdot \mid F)) = 0$ if $k > g^{-1}(k' \mid F)$.

[2]Again, note that I abstract from other states or idiosyncratic stochastic shocks for expositional clarity, knowing that in general one would have them to obtain non-degenerate distributions.

4 Computing the law of motion of the distribution

I substitute the above results into equation (4) to get

$$\hat{F}_{t+1}(k') \approx \int \Pi(k, k' \mid g(\cdot \mid \bar{F})) + B d\hat{F}_t(k) + \frac{1}{2} \int \Pi_k(k, k' \mid g(\cdot \mid \bar{F})) d\hat{F}_t^2(k) \tag{17}$$

$$= \int \Pi(k, k' \mid g(\cdot \mid \bar{F})) d\hat{F}_t(k) + B \int d\hat{F}_t(k) + \frac{1}{2} \int \Pi_k(k, k' \mid g(\cdot \mid \bar{F})) d\hat{F}_t^2(k) \tag{18}$$

$$= \int \Pi(k, k' \mid g(\cdot \mid \bar{F})) d\hat{F}_t(k) + \frac{1}{2} \int \Pi_k(k, k' \mid g(\cdot \mid \bar{F})) d\hat{F}_t^2(k), \tag{19}$$

where I use in the last line that $\int d\hat{F}_t(k) = 0$, since $\hat{F}_t(k_{max}) = F_t(k_{max}) - \bar{F}(k_{max}) = 1 - 1 = 0$.

Note that up to first order, the law of motion of the perturbed distribution has the well-known shape of the Fokker-Planck equation. The second order-term is an integral over the first derivative of the optimal policy function. Intuitively, when computing the change in the probability weight of point k' from changes in the probability weights of points k, one should account for the speed with which agents go from k to k'.

4.1 Implementation over a finite grid

Let μ denote a row vector of size N where $\mu_i = F(k_i) - F(k_{i-1})$, and $F(k_0) := 0$. Let $\bar{\Pi}$ denote the transition probability matrix in steady state, where $\bar{\Pi}_{ij} = \Pi(k_i, k_j \mid g(\cdot \mid \bar{F}))$, and let $\bar{\Psi}$ denote the *marginal* transition probability matrix in steady state, with $\bar{\Psi}_{ij} = \Pi_k(k_i, k_j \mid g(\cdot \mid \bar{F}))$. The second order approximation of the law of motion of the perturbation in the histogram $\hat{\mu}$ is given by

$$\hat{\mu}_{t+1} = \hat{\mu}_t \bar{\Pi} + \frac{1}{2} \sum_{l=1}^{N} \hat{\mu}_{l,t}^2 \bar{\Psi}_{l \cdot}. \tag{20}$$

5 References

Bhandari, Anmol, Bourany, Thomas, Evans, David, and Mikhail Golosov (2023): “A Perturbational Approach for Approximating Heterogeneous Agent Models”, mimeo

Hampel, Frank R. (1974): “The Influence Curve and Its Role in Robust Estimation”, Journal of the American Statistical Association, 346(69), 383-393