# Involution algebroids: a generalisation of Lie algebroids for tangent categories

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**Abstract**

We define involution algebroids which generalise Lie algebroids to the abstract setting of tangent categories. As a part of this generalisation the Jacobi identity which appears in classical Lie theory is replaced by an identity similar to the Yang-Baxter equation. Every classical Lie algebroid has the structure of an involution algebroid and every involution algebroid in a tangent category admits a Lie bracket on the sections of its underlying bundle. As an illustrative application we take the first steps in developing the homotopy theory of involution algebroids.

## Contents



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# 1 Introduction

In this paper we generalise Lie algebroids (see [14]) using a new class of algebraic structures called *involution algebroids*. A key algebraic component of a Lie algebroid is the Lie bracket on the sections of the underlying vector bundle of the algebroid. By contrast the definition of involution algebroid does not refer to sections but instead asserts the existence of an involution on (a certain prolongation of) the total space of the bundle satisfying the axioms described in 3.2. Under this reformulation the axiom that replaces the Jacobi identity of classical Lie theory is similar to the Yang-Baxter equation (see remark 3.2.2).

The definition of involution algebroid given in 3.2 makes sense in any tangent category (see [17] and [3]) where the appropriate limits exist and are preserved by the tangent bundle functor. In particular it does not involve function spaces or indeed rely on the theory of smooth functions at all. This means that the morphisms of involution algebroids are defined in 3.2.6 as the bundle maps commuting with the involution. (By contrast the usual definition of morphisms between Lie algebroids requires the machinery of 'related sections' as described in [10].) Furthermore the appropriate reformulation of the definition of *admissible homotopy* in an involution algebroid in 5.2.1 avoids using an integral (or directly appealing to the existence of a connection) as in section 1.3 of [6].

In classical Lie theory every Lie group has associated to it a Lie algebra (see section 3.5 of [14]) that can be thought of as a linear approximation to the Lie group. Heuristically speaking the Lie bracket of this approximating Lie algebra encodes the commutator of the group multiplication. (A similar intuition applies to integrable Lie algebroids.) In 3.3 we show that every groupoid in an appropriately complete tangent category can be approximated by an involution algebroid. The correct replacement for the group commutator turns out to be the conjugation-like operation defined in 3.3. In 3.5 we show that every Lie algebroid is an example of an involution algebroid. In order to ease the calculations in this section we use the Levi-Civita connection as described in 3.1 of [1] but the final results are independent of the choice of any connection. The special case of Lie algebras is worked out in elementary terms in 3.4 where the involution map is the endo-arrow on $A \times A \times A$ defined by $\sigma : (v, w_H, w_V) \mapsto (w_H, v, w_V + [v, w_H])$. In the other direction, in section

4 we describe how to define a Lie bracket on the set of sections of an involution algebroid in a tangent category. Furthermore under the additional assumption that the tangent category has a line object $R$ we demonstrate that the Leibniz law (a part of the structure of a Lie algebroid) holds for this bracket also.

The construction of the Lie bracket on the sections of an involution algebroid in 4 and the definition of the involution algebroid structure on a Lie algebroid 3.5 use the same equation. This implies an injection on objects (see in A.2) from the category of Lie algebroids to the category of involution algebroids in the category of smooth manifolds. Therefore a natural question arises: is the category of Lie algebroids a full subcategory of the category of involution algebroids? Although we do not answer this question in this paper we take the first step in this direction by working out some of the homotopy theory of involution algebroids following the theory in [6]. The idea is that section 5.1 of [6] gives an equivalence of categories

$$\begin{tikzcd} LieAlgd \arrow[r, shift left, "w"] \arrow[r, phantom, "\perp"] & LocGpd \arrow[l, shift left, "alg"] \end{tikzcd}$$

where *LocGpd* is the category of local Lie groupoids. Here $w$ is the *Weinstein local groupoid* construction which takes the quotient of a special kind of path (*admissible* paths) by a special type of homotopy (*admissible* homotopies). Therefore one approach to understanding the morphisms in *LieAlgd* is to understand the homotopy theory of Lie algebroids. In 5 we define admissible paths and admissible homotopies in an involution algebroid, describe how to transport elements of $A$ along these paths and homotopies and work out two special cases that arise in the composite $alg \circ w$. The logical next step would be to identify the conditions on involution algebroids (and indeed the tangent categories they live in) that allow us to take the quotient involved in the Weinstein groupoid construction. We leave this as future work.

Late in the preparation of this paper the authors became aware of section 4 of [7] and proposition 1 in [16] which describe an involution similar to the one we are proposing here. It is unclear to the authors of the present paper whether the two involutions are the same and in particular whether the one defined in [7] satisfies our 'flip' axiom (the Yang-Baxter style equation in remark 3.2.2). Therefore it is possible that [7] might provide an alternative to the work presented in 3.5 which shows that all Lie algebroids are examples of involution algebroids. We keep 3.5 unchanged because we need this explicit form for comparison to our work in 4 and also because we make no assumptions involving the existence of dual bundles and differentials that are required in 4.1 of [7].

# 2 Background on tangent categories

We formulate most of the ideas in this paper using the axiomatic system given by the theory of tangent categories which was introduced in [17] and further developed in [3]. A tangent category is a category $\mathbb{X}$ equipped with an endofunctor $T$ that behaves in an analogous way to the tangent bundle endofunctor on the category of smooth manifolds.

As such this paper is a part of the body of work that reformulates various parts of differential geometry in the language of tangent categories. For instance in 2.2 we recall the definition of *differential bundle* introduced in [5] which is the appropriate generalisation of the definition of smooth vector bundle for tangent categories. In addition in 2.5 we present a modified version of a *curve object* (see section 5 of [4]) that allows one to talk about the solutions to dynamical systems in a tangent category.

In this section we very briefly recall the definitions of tangent category, differential bundle and curve object in order to fix notation and to call attention to the results that we require in the sequel. For more details and examples see [3].

## 2.1 Additive bundles and tangent categories

The tangent space at a point $m$ of a smooth manifold $M$ is regarded as a linear approximation to the region of the space $M$ that is close to $m$. In the classical case these approximations are represented by a vector spaces which (as we smoothly vary the base point $m$) assemble to form a vector bundle. In a tangent category we instead use the more general structure of an additive bundle. The following definition and remarks are contained in 2.1 and 2.2 of [3].

**Definition 2.1.1** (Additive bundle)**.** If $M$ is an object of a category $\mathbb{X}$ then an *additive bundle $q$ over $M$* is a commutative monoid in the slice category $\mathbb{X}/M$.

*Remark* 2.1.2. Included in this data is a total space $E$, a bundle projection $q : E \to M$, a zero section $\zeta : M \to E$ and an addition $+_q : E_q{\times}_q E \to E$.

*Remark* 2.1.3. A *morphism $\phi : q \Rightarrow q'$ of additive bundles* is a pair of arrows $(\phi_1, \phi_0)$ making

$$\begin{array}{ccc} E & \xrightarrow{\phi_1} & E' \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q'} \\ M & \xrightarrow{\phi_0} & M' \end{array}$$

commute preserving zero and addition in the fibres.

If $f : M \to N$ is a smooth map between smooth manifolds then it lifts to a smooth map $T(f) : T(M) \to T(N)$ between the tangent bundles. Categorically speaking we can encode various properties of the derivative (such as linearity) in terms of natural transformations between various iterates and limits of the functor $T$.

**Definition 2.1.4** (Tangent category)**.** A *tangent category* is a category $\mathbb{X}$ equipped with an endofunctor $T$ on $\mathbb{X}$ and natural transformations

$$p : T \Rightarrow id \qquad \text{(projection onto base)}$$
$$0 : id \Rightarrow T \qquad \text{(zero section)}$$
$$+ : T_p{\times}_p T \Rightarrow T \qquad \text{(addition in tangent spaces)}$$
$$l : T \Rightarrow T^2 \qquad \text{(vertical lift)}$$
$$c : T^2 \Rightarrow T^2 \qquad \text{(canonical flip)}$$

where we assume that all pullback powers of $p$ (e.g. $T_p\times_p T$ etc..) exist and:-

- all pullback powers of $p$ are preserved by $T$
- 0 is a section of $p$
- $cc = id$ and $T(c)cT(c) = cT(c)c$
- $cl = l$, $T(l)l = ll$ and $cT(c)l = T(l)c$
- if $M \in \mathbb{X}$ then $+_M$ and $0_M$ makes each $p_M$ an additive bundle
- if $M \in \mathbb{X}$ then $(l_M, 0_M) : p \Rightarrow T(p)$ is an additive bundle morphism
- if $M \in \mathbb{X}$ then $(c, id) : T(p) \Rightarrow p$ is an additive bundle morphism
- the following diagram is an equaliser:

$$T(M)_p\times_p T(M) \xrightarrowtail{\mu} T^2(M) \underset{0pp}{\overset{T(p)}{\rightrightarrows}} T(M)$$

  where $\mu(a, b) = 0b + la$.

Examples of tangent categories include the category of smooth manifolds and the infinitesimally linear objects in a well-adapted model of synthetic differential geometry (see [11]). For more details see [3].

In this paper we assume that the tangent bundle functor $T : \mathbb{X} \to \mathbb{X}$ associated to a tangent category $\mathbb{X}$ preserves certain limits in $\mathbb{X}$. Rather than assert $T$ preserves all limits (which does not hold in the category of smooth manifolds) we instead assert that $T$ preserves certain specific limits at various points in the text. In fact even the general assumption that $T$ preserves all limits is not an unreasonable one due to the main result in [9] which shows that every tangent category embeds into another tangent category for which $T$ is representable and hence preserves all limits.

## 2.2 Differential bundles

Recall that every Lie algebroid can be obtained by placing extra structure on a smooth vector bundle. Accordingly it turns out that in order to define involution algebroids in a tangent category we first need to understand *differential bundles* which are the appropriate generalisation of smooth vector bundles to this setting. The following is definition 2.3 of [5].

**Definition 2.2.1** (Differential bundle)**.** A *differential bundle* is an additive bundle $q : E \to M$ equipped with a *lift map* $\lambda : E \to TE$ such that:

- $(\lambda, 0)$ is an additive bundle morphism
- $(\lambda, \zeta)$ is an additive bundle morphism
- $T(\lambda)\lambda = l\lambda$
- the following diagram is an equaliser:

$$E_q\times_q E \xrightarrowtail{\mu} T(E) \underset{0qp}{\overset{T(q)}{\rightrightarrows}} T(M)$$

  where $\mu(a, b) = 0b +_q \lambda a$

*Example* 2.2.2. If $q : E \to M$ is a smooth vector bundle and $m \in M$ then there is an isomorphism $\psi : E_m \times E_m \to T(E_m)$ because $E_m \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$. In this case we can define $\lambda(e) = \psi(0_{qe}, e)$. In addition the map $\mu(a, b) = \psi(b, 0_m) +_q \psi(0_m, a) \cong \psi(b, a)$ is the lift defined on page 55 of [12] where $m = qa = qb$.

**Definition 2.2.3** (Morphism of differential bundles)**.** A *morphism* $\phi : q \Rightarrow q'$ *of differential bundles* is a pair $(\phi_1, \phi_0)$ of arrows making

$$\begin{tikzcd} E \arrow[r, "\phi_1"] \arrow[d, "q"] & E' \arrow[d, "q'"] \\ M \arrow[r, "\phi_0"] & M' \end{tikzcd}$$

commute. Furthermore a *linear differential bundle morphism* is a differential bundle morphism that preserves the lift: $T(\phi_1)\lambda = \lambda'\phi_1$.

Next we recall two results about differential bundles that we require in the sequel.

**Lemma 2.2.4** (Pullbacks of differential bundles)**.** Let

$$(q_0, +_0, \xi_0, \lambda_0)\ ,\ (q_1, +_1, \xi_1, \lambda_1) \text{ and } (q_2, +_2, \xi_2, \lambda_2)$$

be differential bundles with total spaces $E_0$, $E_1$ and $E_2$ and base spaces $M_0$, $M_1$ and $M_2$ respectively. If $\phi : q_1 \Rightarrow q_0$ and $\psi : q_2 \Rightarrow q_0$ are linear differential bundle morphisms then

$$q_1 {}_\phi\!\times_\psi q_2 := (q_1 \times q_2, +_1 \times +_2, \xi_1 \times \xi_2, \lambda_1 \times \lambda_2)$$

is a differential bundle with total space $E_1 {}_{\phi_1}\!\times_{\psi_1} E_2$ and base space $M_1 {}_{\phi_0}\!\times_{\psi_0} M_2$. Moreover $q_1 {}_\phi\!\times_\psi q_2$ is the pullback of $\phi$ and $\psi$ in the category of differential bundles.

*Proof.* (Sketch.) First recall that we assume all of the limits involved exist and are preserved by $T$. Second recall 2.16 of [5] which states that if $\phi$ is a linear morphism of differential bundles then $\phi$ automatically preserves the addition and zero. It is then a lengthy but straightforward calculation to check that the differential bundle axioms hold. As a representative example the addition is unital because:

$$\begin{aligned} \xi q + id &= (\xi_1 q_1 \times \xi_2 q_2) + (id \times id) \\ &= (\xi_1 q_1 +_1 id) \times (\xi_2 q_2 +_2 id) \\ &= id \times id = id \end{aligned}$$

where we have written $+$ for $+_1 \times +_2$, $\xi = \xi_1 \times \xi_2$ and $q = q_1 \times q_2$. Note also that

$$(E_1 {}_{\phi_1}\!\times_{\psi_1} E_2)_q\!\times_q (E_1 {}_{\phi_1}\!\times_{\psi_1} E_2) \overset{\mu}{\rightarrowtail} T(E_1 {}_{\phi_1}\!\times_{\psi_1} E_2) \underset{0qp}{\overset{T(q)}{\rightrightarrows}} T(M_1 {}_{\phi_0}\!\times_{\psi_0} M_2)$$

is an equaliser because limits commute with limits. $\square$

The following lemma is 2.5 in [5].

**Lemma 2.2.5.** If $(E, q, +_q, \xi, \lambda)$ is a differential bundle then

$$(T(E), T(q), T(+_q), T(\xi), cT(\lambda))$$

is a differential bundle.

Recall that a vector bundle over the singleton base space is a vector space. As in [5] we define a *differential object* to be a differential bundle over the terminal object. The following proposition (3.4 in [5]) gives a more elementary characterisation of differential objects.

**Proposition 2.2.6.** *Let $\mathbb{X}$ be a tangent category, the following are equivalent:*

1. $(E, \oplus_E, \zeta_E, \lambda_E)$ *is a differential bundle over the terminal object.*
2. *We have a commutative monoid object* $(E, \oplus_E, \zeta_E)$ *so that*
   (a) *We have the following biproduct diagram in the category of commutative monoids in* $\mathbb{X}$

$$\begin{tikzcd}
E \arrow[rr, equal] \arrow[dr, "\lambda"'] & & E \\
& T(E) \arrow[ur, "\hat{p}"'] \arrow[dr, "p"] & \\
E \arrow[ur, "0"] \arrow[rr, equal] & & E
\end{tikzcd}$$

   (b) *The biproduct structure is compatible addition on the bundle:*

$$\begin{tikzcd}
E \arrow[r, "!_E"] \arrow[d, "0_E"'] & 1 \arrow[d, "\zeta"] & T_2E \arrow[r, "\langle \hat{p}\pi_0, \hat{p}\pi_1 \rangle"] \arrow[d, "+_p"] & E \times E \arrow[d, "+_A"] \\
T(E) \arrow[r, "\hat{p}"'] & A & T(A) \arrow[r, "\hat{p}"'] & E
\end{tikzcd}$$

   (c) *The biproduct structure is coherent with the lift $\ell$:*

$$\begin{tikzcd}
TE \arrow[r, "\ell"] \arrow[d, "\hat{p}"'] & T^2E \arrow[d, "\hat{p}"] \\
E & TE \arrow[l, "\hat{p}"]
\end{tikzcd}$$

## 2.3 Affine bundles

The definition of the Lie bracket may be simplified using the *affine structure* of the second tangent bundle. If $V$ is a differential object, we say that $A$ is *affine* over $V$ if there is an action of $V$ on $A$

$$\hat{+} : A \times V \to A$$

which is linear in $V$, so that $(v+v')\hat{+}a = v\hat{+}(v'\hat{+}a)$. There is also a *strong difference*

$$\div : A \times A \to V$$

so that $a\hat{+}(a \div a') = a$. We can find a similar structure on the tangent bundle of a differential bundle; the following proposition lays out the algebraic necessities to find this structure.

**Proposition 2.3.1.** *Let $\pi : A \to M$ be a differential bundle. There are two additive bundle structures on $T(A)$, $+_\pi^T := T(+_\pi)$ and $+_A$ that satisfy the following identities:*

1. *Interchange: $(x +_\pi^T y) +_A (w +_\pi^T z) = (x +_A w) +_\pi^T (y +_A z)$.*
2. *If $v : A$, then*
$$(\lambda(v) +_\pi^T 0p(a)) +_A a = (\lambda(v) +_A \zeta\pi(a)) +_\pi^T a$$
3. *If $p(x) = p(y), T(\pi)(x) = T(\pi)(y)$, then*
$$(x -_A y) -_\pi^T 0py = (x -_\pi^T y) -_A \zeta\pi y$$
4. - $T(\pi)((x -_A y) -_\pi^T 0py = 0\pi py$
   - $p((x -_\pi^T y) -_A T(\zeta)T(\pi)y) = \zeta pT(\pi)y$
5. *$p((x -_A y) -_\pi^T 0py) = 0ppx, T(\pi)(x -_A y) -_\pi^T 0py = T(\pi)(x)$*

*Proof.* (1) Lemma 2.8 in [5].

(2) Calculate:

$$\begin{aligned}(\lambda(v) +_\pi^T 0pa) +_A a &= (\lambda(v) +_\pi^T 0pa) +_A (T(\zeta)T(\pi)a +_\pi^T a) \\ &= (\lambda(v) +_A T(\zeta)T(\pi)a) +_\pi^T (0pa +_A a) \\ &\qquad\qquad\text{(interchange)} \\ &= (\lambda(v) +_A T(\zeta)T(\pi)a) +_\pi^T a\end{aligned}$$

(3) Calculate:

$$\begin{aligned}(x -_A y) -_\pi^T 0py &= (x +_A -_A y) +_\pi^T -_\pi^T 0py \\ &= (x +_A -_A y) +_\pi^T (-_\pi^T (y) +_A -_A -_\pi^T (y)) \\ &= (x +_\pi^T -_\pi^T (y)) +_A (-_A y +_\pi^T -_A -_\pi^T (y)) \\ &= (x +_\pi^T -_\pi^T (y)) +_A (-_A y +_\pi^T -_\pi^T -_A y) \\ &= (x -_\pi^T y) -_A T(\zeta)T(\pi)y\end{aligned}$$

(4) Observe that:

$$\begin{aligned}T(\pi)((x -_A y) -_\pi^T 0py) &= T(\pi)0py \\ &= 0\pi py\end{aligned}$$

and

$$\begin{aligned}p((x -_\pi^T y) -_A T(\zeta)T(\pi)y) &= pT(\zeta)T(\pi)y \\ &= \zeta pT(\pi)y\end{aligned}$$

□

**Definition 2.3.2** (Affine bundle)**.** Let $\pi : A \to M$ be a differential bundle. We say that $q : B \to Q$ is *affine* over $\pi$ if there are maps:

- Strong difference: $\div : B_q \times_q B \to A$.
- Strong Sum: $\hat{+} : B \times A \to B$

So that:

- Associativity: if $\pi(v) = \pi(w)$ $(a \hat{+} v) \hat{+} w = a \hat{+} (v +_\pi w)$
- Inverse: if $q(a) = q(b)$ then

$$a \hat{+} (a \div b) = a$$

*Example* 2.3.3 (The tangent bundle of a differential bundle). Consider the tangent bundle of a differential bundle $\pi : A \to M$. By the earlier lemma, we may define the bundle:

$$q : T(A) \to A_\pi \times_p T(M)$$

and we have an affine structure induced by:

- $a \hat{+} v := (\lambda(v) +^T_\pi 0pa) +^T_A a$
- $a \div b := \{(a -_A b) -^T_\pi 0pb\}$

*Remark* 2.3.4. Consider the affine structure on the second tangent bundle of some object $M$. The Lie bracket $[X, Y]_M$ is precisely

$$T(X)Y \div cT(Y)X$$

## 2.4 Units and function algebras

The function algebra $C^\infty(M)$ is used in the classical definition of a Lie algebroid over a manifold $M$. However, in a tangent category there need not be any ring object $R$, let alone one satisfying the various universal properties of the real numbers. For this section we shall introduce the notion of a *unit object* to play the place of $\mathbb{R}$ in a tangent category - tangent categories with a unit object will be explored in work to appear later [8].

**Definition 2.4.1** (Unit Object)**.** A unit in a tangent category is a differential object $R$ with a point $e : 1 \to R$ with the universal property that for every morphism of differential objects

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "\langle 1, e \rangle"'] & E \\ (V \times W) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

there is a unique $\hat{f}$ such that

- $\hat{f}$ is linear in $R$.
- If $f$ is linear is $W$, then $\hat{f}$ is still linear in $W$.

The representable unit plays the role of the the real numbers in the category of smooth manifolds, or the distinguished ring in a model of synthetic differential geometry. We state the following results:

**Lemma 2.4.2.** In a tangent category (with negatives) with a scalar object $R$:

- The object $R$ is a commutative ring (if the tangent category does not negatives, this will be a commutative rig).

- The category of differential objects and linear maps is a full subcategory of $R$-modules: that is every differential object has a canonical $R$-module structure, and a morphism is linear if and only if it is an $R$-module morphism.
- The unique map $\hat{f}$ induced in the definition reducts to a map

$$\hat{f}((v,w),r) = f(v,w) \bullet_E r.$$

- We may rewrite $\lambda_E(e) = (\lambda(u) \bullet^T 0e)$.
- The action of $T(R)$ on $T(V)$ may be rewritten in terms of the isomorphism $\nu : T(V) \to V$ and the map $\omega((s,r),(u,v)) = (r \bullet_V u \oplus_V s \bullet_V v, r \bullet_V v)$.

$$\begin{tikzcd}
T(R) \times T(V) \arrow[r, "T(\bullet_V)"] \arrow[d, "\nu \times \nu"] & T(V) \arrow[d, shift left, "\nu"] \\
(R \times R) \times (V \times V) \arrow[r, "\omega"'] & V \times V \arrow[u, shift left, "\nu"]
\end{tikzcd}$$

We extend this notion to a *fibered unit* - since this is the structure that will be used throughout the paper we will generally omit the adjective "fibered". In a display tangent category - in this paper we will only consider the case where our tangent category is complete and all limits are preserved by $T$ - the category of display maps over $M$ is a tangent category. The tangent functor $T_M$ is defined by pullback:

$$\begin{tikzcd}
T_M(A) \arrow[r, "\bar{0}"] \arrow[d, "T_M(q)"] & TA \arrow[d, "T(p)"] \\
M \arrow[r, "0"] & T(M)
\end{tikzcd}$$

Recall the universal property of $\nu$ for a differential object:

$$\begin{tikzcd}
A_2 \arrow[r, "\nu"] \arrow[d, "\pi\pi_i"] & TA \arrow[d, "T(\pi)"] \\
M \arrow[r, "0"] & TM
\end{tikzcd}$$

This leads to the following proposition, which is 5.12 in [5].

**Theorem 2.4.3.** *In a complete tangent category where $T$ preserves all limits (or more generally, a display tangent category), differential bundles over $M$ are differential objects in the slice tangent category over $M$.*

**Definition 2.4.4.** We say a unit $R$ is *fibered* whenever:

- The trivial bundle $\begin{tikzcd} R \times M \arrow[d, "\pi_1"] \\ M \end{tikzcd}$ is a unit in the slice tangent category over $M$.

- Multiplication is preserved by substitution functors - for $f : M \to N$

$$\begin{tikzcd}
R \times (f^*A, f^*\pi) \arrow[r, "f^*_\pi"] \arrow[d, "f^*(\bullet_q)=" left, "\bullet_{f^*\pi}" right] & R \times (A, \pi) \arrow[d, "\bullet_\pi"] \\
(f^*A, f^*\pi) \arrow[r, "f^*_\pi"] & (A, \pi)
\end{tikzcd}$$

We may also relate $R$-module bundles and differential bundles:

**Proposition 2.4.5.** *In a tangent category with a fibered unit, the category of differential objects is a full subcategory of $R$-module bundles.*

We record the following coherence with respect to the map $T(\bullet_M)$ (which we will generally write $\bullet^T_M$.

**Theorem 2.4.6.** *Consider a differential bundle* $\begin{array}{c} A \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$ *in a category with a fibered unit $(R, e)$. Then for any generalized elements $X \xrightarrow{\langle r,v \rangle} R \times A$ , we have:*

$$(\lambda_R r) \bullet^T_\pi (v) = (\ell_\pi(r \bullet_\pi pv)) +_{T\pi} (T(0p)v)$$

*Proof.* First, set $m(r, v) := (\lambda r) \bullet^T_\pi v$. Observe that $m$ satisfies the equalizer diagram:

$$\begin{tikzcd}
R \times TA \arrow[drr, "m"] & & & \\
A_2 \arrow[rr, "\nu"] & & TA \arrow[r, shift left, "0\pi p"] \arrow[r, shift right, "p"'] & A
\end{tikzcd}$$

by calculating:

$$\begin{aligned}
pm(r, v) &= p((\lambda r) \bullet^T_M v) \\
&= (p\lambda r) \bullet_M (pv) \\
&= (\zeta_R !_R) \bullet_M (pv) = \zeta_R \pi p.
\end{aligned}$$

Consider diagram:

$$\begin{tikzcd}
R \times TA \arrow[r, "\lambda\times 1"] \arrow[drr, dotted, "\hat{m}"] & TR \times TA \arrow[r, "T(\bullet_M)"] & TA \\
R \times A_2 \arrow[u, "1\times\nu"] \arrow[r, "\lambda\times 1"'] & TR \times A_2 \arrow[u, "1\times\nu"] \arrow[r, "T_M(\bullet_M)"'] & A_2 \arrow[u, "\nu"]
\end{tikzcd}$$

Because we are now in the fiber tangent category above $M$, $\pi$ is a differential object so we may use our earlier remark and rewrite $T_M(\bullet_M)$ as $\omega((s, r), (u, v)) = (r \bullet_V u \oplus_V s \bullet_V v, r \bullet_V v)$. Note that this gives:

$$\omega(\nu_R \lambda r, \nu_\pi(u, v)) = \omega((r, 0), (u, v)) = (r \bullet v, 0).$$

Now, precompose the above diagram with the unit of $R$, and call this composite $n$.

$$\begin{tikzcd}
TA \arrow[r, "\langle e!,1\rangle"] \arrow[rrrd, dotted, "\hat{n}"] & R \times TA \arrow[r, "\lambda\times 1"] & TR \times TA \arrow[r, "T(\bullet_M)"] & TA \\
A_2 \arrow[u, "\nu"] \arrow[r, "\langle u!,1\rangle"'] & R \times T_2(M) \arrow[u, "1\times\nu"] \arrow[r, "\lambda\times 1"'] & TR \times A_2 \arrow[u, "1\times\nu"] \arrow[r, "T_M(\bullet_M)"'] & A_2 \arrow[u, "\nu"']
\end{tikzcd}$$

But not that the composite of the bottom is precisely $n^*(u, v) = (v, 0)$, so $\hat{n}\nu(u, v) = \ell(v)$, so we have $\hat{n}v = (pv, vpp0)$. Then by universality of $R$ have have that $\hat{m}(r, v) = (r \bullet_M pv, vpp0)$ - the result follows from universality of multiplication by $R$. □

We complete this section by introducing tangent categories version of the $C^\infty$ functor, associating each manifold to its algebra of smooth functions into $\mathcal{R}$.

**Definition 2.4.7.** For every object $M$ of tangent category with a scalar, there is a ring $C^\infty(M)$ defined to be $\mathbb{X}(M, R)$ with pointwise multiplication and addition. This determines a functor $C^\infty : \mathbb{X} \to R - Alg$. Note that we may consider Lie algebra of *derivations* on $C^\infty(M)$, $R$-module morphisms $d : C^\infty(M) \to C^\infty(M)$ so that

$$d(f \cdot g) = df \cdot g + f \cdot dg$$

where the Lie bracket of $d, d'$ is defined

$$[d, d'] = dd' - d'd$$

## 2.5 Curve objects

In this section we describe the axiomatic system we use to perform integration in a tangent category. The approach to integration we choose to generalise involves finding integral curves (solutions) of vector fields. As such we follow the presentation in section 5 of [4] but make some modifications concerning the solutions of linear vector fields.

In this section we first describe the classical results on integration that underpin the axiomatic system that we use in the sequel. Therefore (with the exception of the final definition 2.5.9) in this section $M$ denotes a smooth manifold and all our arrows are smooth functions between manifolds. The following definition is 5.15 in [4].

**Definition 2.5.1** (Dynamical system)**.** A *dynamical system* $(x_0, X)$ *on* $M$ consists of an initial condition $x_0 : 1 \to M$ and a section $X : M \to T(M)$ of the tangent bundle $p : T(M) \to M$.

*Example* 2.5.2 (Unit dynamical system). If $S$ is an interval in $\mathbb{R}$ containing 0 then we denote by $\partial : S \to T(S)$ the vector field $\partial : x \mapsto (x, 1)$. The *unit dynamical system* $\mathbb{U}S$ is the dynamical system $(0, \partial)$.

**Definition 2.5.3** (Morphism of dynamical system)**.** If $\mathbb{X} = (x_0, X)$ and $\mathbb{Y} = (y_0, Y)$ are dynamical systems on $M$ and $N$ respectively then a *morphism* $f : \mathbb{X} \Rightarrow \mathbb{Y}$ *of dynamical systems* is an arrow $f : M \to N$ making

$$\begin{tikzcd} 1 \arrow[r, "x_0"] \arrow[dr, "y_0"'] & M \arrow[r, "X"] \arrow[d, "f"] & T(M) \arrow[d, "T(f)"] \\ & N \arrow[r, "Y"] & T(N) \end{tikzcd}$$

commute.

**Definition 2.5.4** (Local solutions)**.** If $(x_0, X)$ is a dynamical system then a *local solution* $\gamma$ *to* $(x_0, X)$ is a morphism $\gamma : \mathbb{U}S \Rightarrow \mathbb{X}$ of dynamical systems for some interval $S$ containing 0.

The following lemma corresponds to the key classical result concerning the existence and uniqueness of solutions to differential equations.

**Lemma 2.5.5** (Maximal solution)**.** If $(x_0, X)$ is a dynamical system on $M$ then there exists an open interval $(a, b)$ of $\mathbb{R}$ containing 0 and an arrow $\gamma : (a, b) \to M$ such that:-

- $\gamma$ is a solution to $(x_0, X)$
- if $S$ is an interval in $\mathbb{R}$ strictly containing $(a, b)$ then there is no solution $S \to M$ to $(x_0, X)$
- if $\gamma'$ is another solution to $(x_0, X)$ with domain $(a, b)$ then $\gamma = \gamma'$.

**Definition 2.5.6** (Completeness)**.** A *complete solution to a dynamical system* $(x_0, X)$ is a solution with domain $\mathbb{R}$. A *complete vector field* $X$ is a vector field such that for all initial conditions $x_0$ there exists a complete solution to $(x_0, X)$.

Next we describe how to lift complete solutions to solutions of dynamical systems that are linear bundle morphisms. The following is definition 5.18 in [4] and also definition 3.4.1 in [14].

**Definition 2.5.7.** A *linear vector field* is a morphism of vector bundles

$$\begin{tikzcd} E \arrow[r, "X^E"] \arrow[d, "q"] & T(E) \arrow[d, "T(q)"] \\ M \arrow[r, "X^M"] & T(M) \end{tikzcd}$$

such that $X^E$ and $X^M$ are vector fields. A dynamical system $(x_0, X^E)$ is *linear over another dynamical system* $(m_0, X^M)$ iff $(X^E, X^M)$ is a linear vector field and $q(x_0) = m_0$.

**Proposition 2.5.8.** *If $X^E$ is a linear vector field over $X^M$ and $x_0 \in E$ such that $(q(x_0), X^M)$ has a complete solution then $(x_0, X^E)$ has a complete solution.*

*Proof.* See A.1. $\square$

Now we present our modification of definition 5.19 in [4].

**Definition 2.5.9** (Complete curve object)**.** A *complete curve object* $\mathbb{I}$ in a tangent category $\mathbb{X}$ is a dynamical system

$$\mathbb{I} = (0_I : 1 \to I, \partial : I \to T(I))$$

such that:-

- there is another point $1_I : 1 \to I$
- if $f, g : \mathbb{I} \Rightarrow \mathbb{X}$ are morphisms of dynamical systems then $f = g$
- if $(x_0, X^E)$ is linear over $(y_0, X^M)$ and $(y_0, X^M)$ has a complete solution then $(x_0, X^E)$ has a complete solution.

*Example* 2.5.10. The work in A.1 shows that the manifold $\mathbb{R}$ is a complete curve object in the tangent category of smooth manifolds.

# 3 Involution algebroids and examples

If $X$ and $Y$ are two smooth vector fields on a smooth manifold $M$ then their Lie bracket can be expressed as an algebra-theoretic commutator (see lemma 4.12 in [13]) or alternatively as a group theoretic commutator (see section I.9 of [11]). In the same way the Lie bracket on a Lie algebra (or integrable Lie algebroid) may be obtained as a commutator of infinitesimally small elements of the Lie group (or groupoid) that integrates the algebra. For the definition of an involution algebroid we instead axiomatise a structure that corresponds to operation similar to conjugation on infinitesimal elements of a groupoid. In this way we replace the Lie bracket of a Lie algebroid (which acts on sections) with an involution which acts on a certain prolongation of the algebroid. Under this replacement the Jacobi identity of classical Lie theory is replaced by an identity that is formally similar to the Yang-Baxter equation.

In 3.2 we define involution algebroids in the abstract setting of a tangent category. Then we move on to giving classes of example of involution algebroids. In 3.3 we show how to obtain a involution algebroid as an linear approximation of a groupoid in a tangent category. In 3.5 we show that classical Lie algebroids with a chosen Riemannian metric are examples of involution algebroids and in 3.4 we work through the special case of Lie algebras without a choice of Riemannian metric.

## 3.1 Anchored bundles and prolongations

Every Lie algebroid $\pi : A \to M$ is obtained by placing additional structure on a smooth vector bundle. One part of this additional structure is an *anchor* map $\varrho : A \Rightarrow T(M)$. The anchor map specifies how $A$ behaves like a generalised tangent bundle: an element $a \in A$ 'sits above' the element $\varrho a$ in an analogous way to how an arrow $g$ in a groupoid $s, t : G \rightrightarrows M$ 'sits above' the pair $(sg, tg)$. Since a differential bundle is the appropriate analogue of a smooth vector bundle in a tangent category we now define anchored bundles in terms of differential bundles.

**Definition 3.1.1** (Anchored bundle)**.** An *anchored bundle* is a differential bundle $\pi : A \to M$ equipped with a linear bundle morphism $\varrho : A \to TM$.

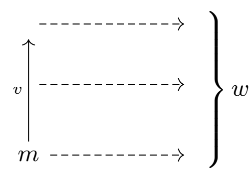
**Definition 3.1.2** (Morphism of anchored bundles)**.** If $A$ and $B$ are anchored bundles with anchors $\varrho_A$ and $\varrho_B$ respectively then *a morphism* $f : A \to B$ *of anchored bundles* is a morphism $(f_0, f_1) : A \to B$ of the underlying differential bundles preserving the anchor: $T(f_0)\varrho_A = \varrho_B f_1$.

For our definition of involution algebroid we replace the Lie bracket of a Lie algebroid (which acts on sections) with an involution which acts on a certain prolongation of the algebroid. We now describe various differential bundles that may be described as prolongations of an anchored bundle. We learnt of the following construction in section 3 of [15].

**Definition 3.1.3** (Total space of prolongation)**.** If $A$ is an anchored bundle the *total space of a prolongation of* $A$ is the pullback

$$\begin{tikzcd}
A_\varrho\times_{T(\pi)}T(A) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & T(A) \arrow[d, "T(\pi)"] \\
A \arrow[r, "\varrho"] & A
\end{tikzcd}$$

*Remark* 3.1.4. A heuristic picture of an element $(v, w) \in A_\varrho\times_{T(\pi)}T(A)$ of the prolongation is




where all the dashed arrows together represent a single element $w \in T(A)$, the solid arrow represents an element $v \in A$ and $m = \pi v = \pi p w$. Therefore the object $A_\varrho\times_{T(\pi)}T(A)$ plays an analogous role to the role played by the object of composable pairs in a groupoid.

Next we describe two differential bundle structures on $A_\varrho\times_{T(\pi)}T(A)$.

*Remark* 3.1.5 (Bundles on the prolongation). If $\pi : A \to M$ is an anchored bundle then the pullbacks

$$\begin{tikzcd}
(A_\varrho\times_{T(\pi)}T(A), p\pi_1) \arrow[r] \arrow[d] & (T(A), p) \arrow[d, "{(T(\pi),\pi)}"'] \\
(A, \pi) \arrow[r, "{(\varrho,id)}"] & (T(M), p)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A_\varrho\times_{T(\pi)}T(A), \pi_0) \arrow[r] \arrow[d] & (T(A), T(\pi)) \arrow[d, "{(T(\pi),id)}"] \\
(A, id) \arrow[r, "{(\varrho,\varrho)}"] & (T(M), id)
\end{tikzcd}$$

in the category of differential bundles (see 2.2.4) describe differential bundle structures with total space $A_\varrho\times_{T(\pi)}T(A)$.

*Remark* 3.1.6. The first differential bundle in 3.1.5 corresponds to the bundle defined in section 3 of [15]. We need both the first and second bundles in 3.1.5 to define involution algebroids. Note that by 2.2.4 and 2.2.5 the lifts of $(A_\varrho\times_{T(\pi)}T(A), \pi_0)$ and $(A_\varrho\times_{T(\pi)}T(A), p\pi_1)$ are $0 \times cT(\lambda)$ and $\lambda \times l$ respectively.

## 3.2 Definition of involution algebroids

The Lie bracket on the elements of a Lie algebra specifies the multiplication of the Lie group integrating the Lie algebra. Roughly speaking it does this by specifying the commutator of two elements of the Lie group that are infinitesimally close to an identity element. A similar intuition can be applied to the Lie bracket on the sections of a Lie algebroid in the case that there exists a Lie groupoid integrating the Lie algebroid.

An involution algebroid will encode the multiplication of a groupoid by specifying which composable pairs compose to the same element. Now we present the axioms defining an involution algebroid and in the next section demonstrate how such a structure arises on the elements of a groupoid that are infinitesimally close to an identity element.

**Definition 3.2.1** (Involution algebroid)**.** An *involution algebroid* is an anchored bundle $\pi : A \to M$ equipped with an arrow $\alpha : A_\varrho\!\times_{T(\pi)} T(A) \to T(A)$ such that $(\alpha, id) : (A_\varrho\!\times_{T(\pi)} T(A), \pi_0) \Rightarrow (T(A), p)$ and $(\alpha, \varrho) : (A_\varrho\!\times_{T(\pi)} T(A), p\pi_1) \Rightarrow (T(A), T(\pi))$ are linear bundle morphisms and:-

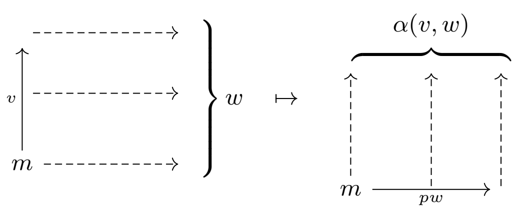
$$T(\varrho)\alpha = cT(\varrho)\pi_1 \qquad \text{(inv. algd. target)}$$

$$\alpha(p\pi_1, \alpha) = \pi_1 \qquad \text{(inv. algd. inv.)}$$

$$(T\alpha)(\alpha(\pi_0, \pi_1), \pi_2) = c(T\alpha)(\alpha(\pi_0, p\pi_2), c(T\alpha)(\pi_1, c\pi_2)) \qquad \text{(inv. algd. flip)}$$

where $\varrho\pi_0 = T(\pi)\pi_1$ and $cT^2(\pi)\pi_2 = T(\varrho)\pi_1$.

A heuristic picture of the action of $(\pi_1, \alpha)$ on an element $(v, w) \in A_\varrho\!\times_{T(\pi)} T(A)$ of the prolongation is:




where on each side the three dashed arrows each represent a single element of $T(A)$, the solid arrows represent elements of $A$ and $m = \pi v = \pi p w$.

*Remark* 3.2.2 (Yang-Baxter). We think of (inv. algd. inv.) as asserting that the endomap $\sigma = (p\pi_1, \alpha)$ on $A_\varrho\!\times_{T(\pi)} T(A)$ is an involution. When (inv. algd. flip) is rephrased in terms of $\sigma$ one obtains a Yang-Baxter style identity:

$$(\sigma \times c)(id \times T(\sigma))(\sigma \times c) = (id \times T(\sigma))(\sigma \times c)(id \times T(\sigma))$$

which we use in section 3.5 when proving classical Lie algebroids are examples of involution algebroids.

*Remark* 3.2.3 (Inv. algd. source and projection). Applying $T(p)$ to both sides of (inv. algd. target) we obtain

$$T(\pi)\alpha = T(p)cT(\varrho)\pi_1 = pT(\varrho)\pi_1 = \varrho p\pi_1 \qquad \text{(inv. algd. source)}$$

which describes the source of the element $\alpha(v, w)$. Immediately from the fact that $(\alpha, id)$ is a bundle homomorphism we have $p\alpha = \pi_0$ which we call (inv. algd. 0).

*Example* 3.2.4 (Tangent bundles). If $M$ is an object in a tangent category then $p : T(M) \to M$ is an involution algebroid with anchor $\varrho = id$ and $\alpha : T(M)_{id}\times_{T(p)} T^2(M) \to T^2(M)$ given by $c\pi_1$.

*Example* 3.2.5 (Lie algebroids). In 3.5 we show directly how Lie algebroids are examples of involution algebroids. Now we describe another way of obtaining an involution algebroid from a Lie algebroid. In 3.3 we show how to produce an involution algebroid from a groupoid in a tangent category. Now the constructions in 3.3 only rely on infinitesimal data and so can be carried out without modification to produce an involution algebroid starting from any local Lie groupoid (defined in for instance section 2.1 of [2]). However the category of Lie algebroids is equivalent to the category of local Lie groupoids (theorem 1.1 of [2]). Therefore to every classical Lie algebroid we can associate an involution algebroid.

By contrast with morphisms of Lie algebroids we may define morphisms of involution algebroids simply as the structure preserving maps. The following definition is not used in the remainder of this paper as we are mainly concerned with constructions on a single fixed involution algebroid. It will be important in future work when we compare the category of involution algebroids internal to the category of smooth manifolds to the category of Lie algebroids.

**Definition 3.2.6** (Morphism of involution algebroids)**.** If $A$ and $B$ are involution algebroids with involutions $\alpha_A$ and $\alpha_B$ respectively then *a morphism $f : A \to B$ of involution algebroids* is a morphism $(f_1, f_0) : A \to B$ of the underlying anchored bundles such that:

$$\begin{array}{ccc} A_{\varrho_A}\times_{T(\pi_A)} T(A) & \xrightarrow{f_1\times T(f_1)} & B_{\varrho_B}\times_{T(\pi_B)} T(B) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ T(A) & \xrightarrow{f_1} & T(B) \end{array}$$

commutes.

## 3.3 The involution algebroid of a groupoid

In this section we start with a groupoid $G$ in a tangent category and produce an involution algebroid that is the natural linear approximation to $G$. The process begins in an analogous fashion to the process described in section 3.5 of [14] but in the end we construct an involution on the prolongation rather than a bracket on the sections. As usual we start by constructing the underlying anchored bundle which in this case is the bundle of source constant tangent vectors to $G$ that are based at an identity element.

**Definition 3.3.1** (Anchored bundle of groupoid)**.** The *anchored bundle $\pi : A \to M$ associated to $G$* has underlying differential bundle given by the pullback

$$\begin{tikzcd}
(A,\pi) \arrow[r, "(\iota,e)"] \arrow[d, "(\pi,id)"] & (TG,p) \arrow[d, "((Ts,p),(s,id))"] \\
(M,id) \arrow[r, "((0,e),(id,e))"] & (TM\times G, p\times id)
\end{tikzcd}$$

in the category of differential bundles in $\mathbb{X}$ as described in 2.2.4. The anchor is given by $T(t)\iota : A \to TM$.

**Notation 3.3.2.** In the remainder of this section we fix $v \in A$, $w \in T(A)$ and $x \in T^2(A)$ such that $\varrho v = T(\pi)w$ and $cT(\varrho)w = T^2(\pi)x$. For notational convenience we regard:-

- $v \in T(G)$ such that $pv = em$ and $T(s)v = 0m$
- $w \in T^2(G)$ such that $T(p)w = T(e)T(t)v$ and $T^2(s)w = T(0)T(t)v$

where $m = \pi v = \pi pw = \pi ppx$. In 3.3.3, 3.3.4 and 3.3.7 use $\circ$ as infix notation for the composition $T^2(\mu) : T^2(G)_{T^2(t)}\times_{T^2(s)} T^2(G) \to T^2(G)$ and in 3.3.8 we use $\circ$ as infix notation for the composition $T^3(\mu)$. In this notation the lift of the differential bundle $(A,\pi)$ of 3.3.1 acts on elements $v \in T(G)$ as $v \mapsto lv$. (Here we have used 2.2.4 to see $\lambda = 0 \times l$.) Indeed $lv$ factors through $T(A)$ because $T(p)lv = 0pv = 0em = T(e)0m$.

Now we show how to obtain an involution algebroid from a groupoid in a tangent category. The flip map we use is given by conjugation:

$$\alpha(v,w) = cw \circ 0v \circ (c0pw)^{-1} \tag{1}$$

which we think of as the following extension:

$$\begin{tikzcd}
\cdot \arrow[r, "cw"] & \cdot \\
m \arrow[u, "0v"] \arrow[r, "c0pw"] & \cdot \arrow[u, dashed, "\alpha(v,w)"']
\end{tikzcd}$$

which corresponds to the intuition described below 3.2.1. Before we verify the involution algebroid axioms we must first show that the composite in (1) is well-typed.

**Lemma 3.3.3** (Composition well-typed)**.** The arrow $\alpha$ in (1) is well-typed with respect to the composition in $T(G)$.

*Proof.* The composition makes sense because the equalities

$$\begin{aligned}
T^2(s)0v &= 0T(s)v \\
&= 00m \\
&= 00tpv \\
&= c0pT(0)T(t)v \\
&= c0pT^2(s)w \\
&= T^2(s)c0pw \\
&= T^2(t)(c0pw)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
T^2(s)cw &= cT^2(s)w \\
&= cT(0)T(t)v \\
&= 0T(t)v = T^2(t)0v
\end{aligned}$$

hold. $\square$

The rest of this section is devoted to justifying definition 3.3.3. We show that the $\alpha$ defined in 3.3.3 is well-typed, a linear differential bundle morphism and satisfies the involution algebroid axioms of 3.2.1.

**Lemma 3.3.4.** The $\alpha$ in 3.3.3 has domain $T(A)$.

*Proof.* We check that $T(p)\alpha$ is a variation of identity arrows:

$$\begin{aligned}
T(p)\alpha(v,w) &= T(p)(cw \circ 0v \circ (c0pw)^{-1}) \\
&= T(p)cw \circ T(p)0v \circ (T(p)c0pw)^{-1} \\
&= pw \circ 0pv \circ (pw)^{-1} \\
&= pw \circ T(e)0 \circ (pw)^{-1} && \text{(definition of } v\text{)} \\
&= T(e)\varrho pw
\end{aligned}$$

as required. $\square$

**Lemma 3.3.5** (Involution is linear I)**.** The $\alpha$ defined in (1) is a linear bundle morphism from $(A_\varrho\times_{T(\pi)}T(A), \pi_0)$ to $(T(A), p)$.

*Proof.* On the one hand

$$l\alpha(v,w) = lcw \circ l0v \circ (lc0pw)^{-1}$$

and on the other hand

$$T(\alpha)(0v, cT(\lambda)w) = T(c)cT(l)w \circ T(0)0v \circ (T(c0p)cT(l)w)^{-1}$$

where we replace $\lambda$ with $l$ using 3.3.2. Now $l0 = T(0)0$ because $(l, 0)$ is a morphism of differential bundles. For the right term we check:

$$\begin{aligned}
T(c)T(0)T(p)cT(l)w &= T(c)T(0)pT(l)w \\
&= T^2(0)lpw \\
&= lc0pw
\end{aligned}$$

and for the left term:

$$\begin{aligned}
cT(c)lcw &= T(l)ccw && \text{(tangent category axiom)} \\
&= T(l)w \\
&= cT(c)T(c)cT(l)w
\end{aligned}$$

and so $lcw = T(c)cT(l)w$ because $cT(c)$ is an isomorphism. Therefore $l\alpha(v,w) = T(\alpha)(0v, cT(\lambda)w)$. $\square$

**Lemma 3.3.6** (Involution is linear II)**.** The $\alpha$ defined in (1) is a linear bundle morphism from $(A_\varrho\times_{T(\pi)}T(A), p\pi_1)$ to $(T(A), T(\pi))$.

*Proof.* On the one hand

$$cT(\lambda)\alpha(v,w) = cT(l)cw \circ cT(l)0v \circ (cT(l)c0pw)^{-1}$$

and on the other hand

$$T(\alpha)(\lambda v, lw) = T(c)lw \circ T(0)lv \circ (c0plw)^{-1}$$

where we replace $\lambda$ with $l$ using 3.3.2. The left terms are equal by a tangent category axiom and the middle terms are equal because 0 is a natural transformation. For the right term:

$$\begin{aligned}
cT(l)c0pw &= T(c)l0pw && \text{(tangent category axiom)}\\
&= T(c)lpT(0)w\\
&= T(c)0pT(0)w\\
&= T(c)00pw\\
&= 00pw\\
&= c00pw\\
&= c0plw
\end{aligned}$$

as required. □

**Lemma 3.3.7.** The $\alpha$ in 3.3.3 satisfies the first three axioms of being an involution algebroid:

$$\begin{aligned}
p\alpha &= \pi_0 && \text{(inv. algd. 0)}\\
T(\varrho)\alpha &= cT(\varrho)\pi_1 && \text{(inv. algd. target)}\\
\alpha(p\pi_1, \alpha) &= \pi_1 && \text{(inv. algd. inv.)}
\end{aligned}$$

*Proof.* First

$$\begin{aligned}
p\alpha(v,w) &= p(cw \circ 0v \circ (c0pw)^{-1})\\
&= pcw \circ v \circ (pc0pw)^{-1}\\
&= T(p)w \circ v \circ (0pT(p)w)^{-1}\\
&= T(e)\varrho v \circ v \circ (T(e)0m)^{-1}\\
&= v
\end{aligned}$$

Second

$$\begin{aligned}
T(\varrho)\alpha(v,w) &= T^2(t)(cw \circ 0v \circ (c0pw)^{-1})\\
&= T^2(t)cw\\
&= cT^2(t)w\\
&= cT(\varrho)w
\end{aligned}$$

Third

$$\begin{aligned}
\alpha(pw, \alpha(v, w)) &= c\alpha(v, w) \circ 0pw \circ (c0p\alpha(v, w))^{-1} \\
&= c\alpha(v, w) \circ 0pw \circ (c0v)^{-1} \qquad \text{(inv. algd. 0)} \\
&= c(cw \circ 0v \circ (c0pw)^{-1}) \circ 0pw \circ (c0v)^{-1} \\
&= w \circ c0v \circ (0pw)^{-1} \circ 0pw \circ c0v \\
&= w
\end{aligned}$$

□

**Lemma 3.3.8.** The $\alpha$ above satisfies the flip axiom:

$$T(\alpha)(\alpha(v, w), x) = T(\alpha)(\alpha(v, px), cT(\alpha)(w, cx))$$

*Proof.* The LHS is:

$$\begin{aligned}
&T(\alpha)(\alpha(v, w), x) \\
&\quad = T(c)x \circ T(0)\alpha(v, w) \circ (T(c0p)x)^{-1} \\
&\quad = T(c)x \circ T(0) \left(cw \circ 0v \circ (c0pw)^{-1}\right) \circ (T(c0p)x)^{-1} \\
&\quad = T(c)x \circ T(0)cw \circ T(0)0v \circ (T(0)T(0)pw)^{-1} \circ (T(c0p)x)^{-1}
\end{aligned}$$

and the RHS is:

$$\begin{aligned}
&cT(\alpha)(\alpha(v, px), cT(\alpha)(w, cx)) \\
&\quad = cT(c)cT(\alpha)(w, cx) \circ cT(0)\alpha(v, px) \circ (cT(c0p)cT(\alpha)(w, cx))^{-1}
\end{aligned}$$

We expand each of the three arrows in the composition on the RHS separately. The left term is:

$$\begin{aligned}
&cT(c)cT(\alpha)(w, cx) \\
&\quad = cT(c)c(T(c)cx \circ T(0)w \circ (T(c0p)x)^{-1}) \\
&\quad = cT(c)cT(c)cx \circ cT(c)cT(0)w \circ (cT(c)cT(c)T(0)T(p)x)^{-1} \\
&\quad = T(c)x \circ T(0)cw \circ (0cpx)^{-1}
\end{aligned}$$

The middle term is:

$$\begin{aligned}
&cT(0)\alpha(v, px) \\
&\quad = cT(0)(cpx \circ 0v \circ (c0ppx)^{-1}) \\
&\quad = cT(0)cpx \circ cT(0)0v \circ (cT(0)c0ppx)^{-1} \\
&\quad = 0cpx \circ T(0)0v \circ (0T(0)ppx)^{-1}
\end{aligned}$$

The right term is:

$$\begin{aligned}
&(cT(c0p)cT(\alpha)(w, cx))^{-1} \\
&\quad = (T^2(0)c\alpha(pw, pcx))^{-1} \\
&\quad = (T^2(0)c(cpcx \circ 0pw \circ (c0ppcx)^{-1}))^{-1} \\
&\quad = (T^2(0)ccpcx \circ T^2(0)c0pw \circ (T^2(0)cc0ppcx)^{-1})^{-1} \\
&\quad = T^2(0)cc0ppcx \circ (T^2(0)c0pw)^{-1} \circ (T^2(0)ccpcx)^{-1} \\
&\quad = 0T(0)ppx \circ (T(0)T(0)pw)^{-1} \circ (T(c0p)x)^{-1}
\end{aligned}$$

Now the result follows easily. □

## 3.4 The involution algebra of a Lie algebra

Recall that a Lie algebra is a vector space $A$ equipped with a bilinear and anti-symmetric bracket $[-,-] : A \times A \to A$ such that

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \tag{Jacobi}$$

where $x, y, z \in A$. Therefore a Lie algebra is a Lie algebroid with the trivial base space $\mathbb{R}^0$. In this section we show that Lie algebras are examples of involution algebroids with trivial base space. Recall that a *differential object* (2.2.6) is a differential bundle over the terminal object.

**Definition 3.4.1** (Involution algebra)**.** An *involution algebra* is a differential object $A$ equipped with a map $\alpha : A \times T(A) \to T(A)$ such that

$$p\alpha = \pi_0 \tag{inv. alg. 0}$$

$$\alpha(p\pi_1, \alpha) = \pi_1 \tag{inv. alg. inv.}$$

$$T(\alpha)(\alpha(v, w), x) = cT(\alpha)(\alpha(v, px), cT(\alpha)(w, cx)) \tag{inv. alg. flip}$$

where $v \in A$, $w \in T(A)$ and $x \in T^2(A)$ such that $\varrho v = T(\pi)w$ and $T(\varrho)w = cT^2(\pi)x$.

*Remark* 3.4.2. Involution algebras are precisely the involution algebroids with trivial base space. Note that the target condition of 3.2.1 is vacuous.

Now we show that every classical Lie algebra is an involution algebra in the category of smooth manifolds. The flip $\alpha : A \times T(A) \to T(A)$ that we use is:

$$\alpha(v, w) = 0v +_A (w +_p \lambda[v, pw])$$

where $v \in A$ and $w \in T(A)$ such that $\varrho v = T(\pi)w$ and we write $+_A$ for the addition induced by the vector space $A$ and $+_p$ for the addition induced by the tangent bundle structure. Actually we can rewrite $\alpha$ by using the fact that the tangent space $T(A)$ of a differential object is a product. If we write $w = (w_H, w_V)$ then $\alpha$ becomes:

$$\alpha(v, w_H, w_V) = (v, w_V + [v, w_H]) \tag{2}$$

which is the form that we now use to check the involution algebra axioms. (To avoid unnecessary subscripts in this section $+$ will be the same as $+_A$.) The axiom (inv. alg. 0) immediately follows from (2).

**Lemma 3.4.3** (Inv. alg. inv.)**.** $\alpha(p\pi_1, \alpha) = \pi_1$

*Proof.* Using the definition of $\alpha$ twice:

$$\begin{aligned} \alpha(w_H, \alpha(v, w_H, w_V)) &= \alpha(w_H, v, w_V + [v, w_H]) \\ &= (w_H, w_V + [v, w_H] + [w_H, v]) \\ &= (w_H, w_V) \end{aligned}$$

therefore (inv. alg. inv.) holds. ☐

**Lemma 3.4.4** (Inv. alg. flip)**.**

$$T(\alpha)(\alpha(v, w), x) = cT(\alpha)(\alpha(v, px), cT(\alpha)(w, cx))$$

*Proof.* First we note that by differentiating the definition of $\alpha$ we obtain

$$T(\alpha)(a,b,c,d,e,f) = (a, d + [a,c], b, f + [a,e] + [b,c])$$

where $a,b,c,d,e,f \in A$. Now we prove the axiom holds. On the one hand:

$$\begin{aligned}
&T(\alpha)(\alpha(v,w_H,w_V),x_H,x_V,x'_H,x'_V) \\
&= T(\alpha)(v,w_v + [v,w_h],x_h,x_v,x'_H,x'_V) \\
&= (v,x_v + [v,x_H],w_V + [v,w_H],x'_V + [v,x'_H] + [w_V,x_H] + [[v,w_H],x_H])
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
&cT(\alpha)(\alpha(v,x_H,x_V),cT(\alpha)(w_H,w_V,x_H,x'_H,x_V,x'_V)) \\
&= cT(\alpha)(v,x_V + [v,x_H],c(w_H,x'_H + [w_H,x_H],w_V,x'_V + [w_H,x_V] + [w_V,x_H])) \\
&= cT(\alpha)(v,x_V + [v,x_H],w_H,w_V,x'_H + [w_H,x_H],x'_V + [w_H,x_V] + [w_V,x_H]) \\
&= c(v,w_V + [v,w_H],x_V + [v,x_H],z) \\
&= (v,x_V + [v,x_H],w_V + [v,w_H],z)
\end{aligned}$$

where

$$\begin{aligned}
z &= x'_V + [w_H,x_V] + [w_V,x_H] + [v,x'_H + [w_H,x_H]] + [x_V + [v,x_H],w_H] \\
&= x'_V + [v,x'_H] + [w_V,x_H] + [[v,w_H],x_H]
\end{aligned}$$

where the last equality uses the anti-symmetry property and Jacobi identity of the Lie bracket. Now the required equality follows easily. $\square$

## 3.5 The involution algebroid of a Lie algebroid

Recall that a Lie algebroid is a smooth vector bundle $\pi : A \to M$ equipped with a vector bundle morphism $\varrho : A \Rightarrow T(M)$ and a Lie bracket $[-,-] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ on the sections $\Gamma(A)$ of $\pi$ such that the following Leibniz law holds:

$$[X, f \cdot Y] = f \cdot [X,Y] + (\varrho X)(f) \cdot Y$$

where $X,Y \in \Gamma(A)$, $f \in C^\infty(M)$ and $(\varrho X)(f)$ denotes the Lie derivative of $f$ in the direction specified by $\varrho X$. In this section we show that all Lie algebroids are examples of involution algebroids. To ease the calculations we use the Levi-Civita connection on $\pi$ as described in for instance in section 3.1 of [1]. The following definitions are 3.2, 4.5 and in [4].

**Definition 3.5.1** (Vertical connection)**.** A *vertical connection on* $\pi$ is a section $K$ of $\lambda : A \to T(A)$ such that $(K,p) : T(\pi) \Rightarrow \pi$ and $(K,\pi) : p \Rightarrow \pi$ are linear bundle morphisms.

**Definition 3.5.2** (Horizontal connection)**.** A *horizontal connection on* $\pi$ is a section $H$ of $(p, T(\pi)) : T(A) \to A_\pi{\times}_p T(M)$ such that $(H, id) : \pi^*(p) \Rightarrow p$ and $(H, id) : p^*(\pi) \Rightarrow T(\pi)$ are linear bundle morphisms.

**Definition 3.5.3** (Full connection)**.** A *(full) connection* on $\pi$ consists of a vertical connection $K$ and horizontal connection $H$ on $\pi$ such that $KH = \xi\pi\pi_0$ and $H(p, T(\pi)) +_p (\lambda K +_{T(\pi)} 0p) = id$.

If $(K, H)$ is a full connection then we have the following decomposition which is proposition 5.8 of [4].

**Proposition 3.5.4.** *If $\pi$ has a full connection then the arrow*

$$(p, T(\pi), K) : T(A) \to A_\pi{\times}_p T(M)_\pi{\times}_p A$$

*has inverse $H(\pi_0, \pi_1) +_p (\lambda\pi_2 +_{T(\pi)} 0\pi_0)$.*

We show that a Lie algebroid is an example of an involution algebroid by using the following flip map:

$$\alpha(v, w) = \alpha_L +_{T(\pi)} \alpha_R \tag{3}$$

where

$$\alpha_L = T(\overline{v})\varrho pw -_p (0v +_{T(\pi)} \lambda[\overline{v}, \overline{pw}])$$

and

$$\alpha_R = T(\xi)\varrho pw -_p T(\xi)\varrho v +_p (w -_{T(\pi)} T(\overline{pw})\varrho v)$$

where $\overline{a}$ denotes any section of $\pi$ such that $\overline{a}\pi a = a$. In terms of an arbitrary (full) connection $(H, K)$ we have that

$$\alpha(v, w) = H(v, \varrho pw) +_p (\lambda\alpha_2 +_{T(\pi)} 0v)$$

where $\alpha_2 = KT(\overline{v})\varrho pw - KT(\overline{pw})\varrho v + Kcw - [\overline{pw}, \overline{v}]$. Note that (3) shows that this definition of $\alpha$ is independent of the choice of connection.

If we additionally assume that $K$ is the vertical part of the Levi-Civita connection then $[X, Y] = KT(Y)\varrho X -_\pi KT(X)\varrho Y$ as described in section 3.1 of [1]. Then the definition of $\alpha$ simplifies to:

$$\alpha(v, w) = H(v, \varrho pw) +_p (\lambda Kw +_{T(\pi)} 0v) \tag{4}$$

which also shows that (3) is independent of the choice of sections extending $v$ and $pw$. In fact under the isomorphism of 3.5.4 $\alpha \cong (\pi_0, \varrho\pi_1, \pi_2)$ and therefore $\sigma \cong (\pi_1, \pi_0, \pi_2)$. (This final description of $\sigma$ as a permutation is used in 3.5.9 where it is easy to apply the isomorphism 3.5.4 to all the other terms.) Now we verify that every Lie algebroid is an involution algebroid with $\alpha$ defined as in (3).

**Lemma 3.5.5** (Bundle morphism I)**.** The $\alpha$ of (4) is a linear bundle morphism $(A_\varrho{\times}_{T(\pi)} T(A), \pi_0) \Rightarrow (T(A), p)$.

*Proof.* It is routine to check that $\alpha$ is a bundle morphism. We now check that $\alpha$ is linear:

$$\begin{aligned}
&T(\alpha)(0v, cT(\lambda)w) \\
&= T(H)(0v, T(\varrho p)cT(\lambda)w) +_{T(p)} (T(\lambda K)cT(\lambda)w +_{T^2(\pi)} T(0)0v) \\
&= T(H)(0v, T(\varrho)\lambda pw) +_{T(p)} (T(\lambda)\lambda Kw +_{T^2(\pi)} T(0)0v) \quad \text{(3.3 (d) in [4])} \\
&= T(H)(0v, T(\varrho)\lambda pw) +_{T(p)} (l\lambda Kw +_{T^2(\pi)} l0v) \quad \text{(diff. bundle axioms)} \\
&= T(H)(0v, l\varrho pw) +_{T(p)} (l\lambda Kw +_{T^2(\pi)} l0v) \quad (\varrho \text{ linear}) \\
&= lH(v, \varrho pw) +_{T(p)} (l\lambda Kw +_{T^2(\pi)} l0v) \quad (H \text{ linear}) \\
&= l\alpha(v, w) \quad (l \text{ linear})
\end{aligned}$$

as required. □

**Lemma 3.5.6** (Bundle morphism II)**.** The $\alpha$ of (4) is a linear bundle morphism $(A_\varrho\times_{T(\pi)}T(A), p\pi_1) \Rightarrow (T(A), T(\pi))$.

*Proof.* It is routine to check that $\alpha$ is a bundle morphism. We now check that $\alpha$ is linear:

$$\begin{aligned}
&T(\alpha)(\lambda v, lw) \\
&\quad = T(H)(\lambda v, T(\varrho p)0w) +_{T(p)} (T(\lambda K)lw +_{T^2(\pi)} T(0)\lambda v) \\
&\quad = cT(\lambda)H(v, \varrho pw) +_{T(p)} (T(\lambda)\lambda Kw +_{T^2(\pi)} cT(\lambda)0v) \qquad (H \text{ is linear}) \\
&\quad = cT(\lambda)H(v, \varrho pw) +_{T(p)} (cT(\lambda)\lambda Kw +_{T^2(\pi)} cT(\lambda)0v) \\
&\quad = cT(\lambda)\alpha(v, w)
\end{aligned}$$

as required. □

**Lemma 3.5.7** (Inv. algd. inv.)**.** The $\alpha$ of (4) satisfies $\alpha(p\pi_1, \alpha) = \pi_1$.

*Proof.* Since

$$K\alpha(v, w) = K(H(v, \varrho pw) +_p (\lambda Kw +_{T(\pi)} 0v)) = Kw$$

we can use definition of $\alpha$ twice:

$$\begin{aligned}
\alpha(pw, \alpha(v, w)) &= H(pw, \varrho p\alpha(v, w)) +_p (\lambda K\alpha(v, w) +_{T(\pi)} 0pw) \\
&= H(pw, \varrho v) +_p (\lambda Kw +_{T(\pi)} 0pw) \\
&= w \qquad \text{(full connection)}
\end{aligned}$$

as required. □

**Lemma 3.5.8** (Inv. ald. target)**.** $T(\varrho)\alpha = cT(\varrho)\pi_1$

*Proof.* We use (3) and treat the $\alpha_L$ and $\alpha_R$ terms separately. We use a torsion-free connection $\hat{p}$ to compare the two sides of the equality in the statement of the lemma. First

$$\begin{aligned}
T(\varrho)\alpha_L &= T(\varrho)(T(\overline{v})\varrho pw -_p (0v +_{T(\pi)} \lambda[\overline{v}, \overline{pw}])) \\
&= T(\varrho\overline{v})\varrho pw -_p (T(\varrho)0v +_{T(p)} T(\varrho)\lambda[\overline{v}, \overline{pw}]) \\
&= T(\varrho\overline{v})\varrho pw -_p (0\varrho v +_{T(p)} l[\varrho\overline{v}, \varrho\overline{pw}]) \qquad \text{(Leibniz law)} \\
&= T(\varrho\overline{v})\varrho pw -_p (cT(\varrho\overline{pw})\varrho v -_p T(\varrho\overline{v})\varrho pw) \\
&\qquad\qquad \text{(bracket of vector fields)} \\
&= cT(\varrho\overline{pw})\varrho v
\end{aligned}$$

and so $pT(\varrho)\alpha_L = \varrho v$, $T(p)T(\varrho)\alpha_L = \varrho pw$ and $\hat{p}T(\varrho)\alpha_L = \hat{p}T(\varrho\overline{pw})\varrho v$. Second

$$\begin{aligned}
T(\varrho)\alpha_R &= T(\varrho)\left(T(\xi)\varrho pw -_p T(\xi)\varrho v +_p (w -_{T(\pi)} T(\overline{pw})\varrho v)\right) \\
&= T(\varrho\xi)\varrho pw -_p T(\varrho\xi)\varrho v +_p (T(\varrho)w -_{T(p)} T(\varrho\overline{pw})\varrho v) \\
&= T(0)\varrho pw -_p T(0)\varrho v +_p (T(\varrho)w -_{T(p)} T(\varrho\overline{pw})\varrho v)
\end{aligned}$$

and so $pT(\varrho)\alpha_R = 0\pi v$, $T(p)T(\varrho)\alpha_R = \varrho pw$ and $\hat{p}T(\varrho)\alpha_R = \hat{p}T(\varrho)w -_p \hat{p}T(\varrho\overline{pw})\varrho v$. Now

$$T(\varrho)\alpha = T(\varrho)\alpha_L +_{T(p)} T(\varrho)\alpha_R$$

and so $pT(\varrho)\alpha = \varrho v$, $T(p)T(\varrho)\alpha = \varrho pw$ and $\hat{p}T(\varrho)\alpha = \hat{p}T(\varrho)w$. However $pcT(\varrho)w = T(\pi)w = \varrho v$, $T(p)cT(\varrho)w = \varrho pw$ and $\hat{p}cT(\varrho)w = \hat{p}T(\varrho)w$. Therefore $cT(\varrho)w = T(\varrho)\alpha$ because the image of both sides under the isomorphism $(p, T(\pi), \hat{p})$ are the same. □

**Lemma 3.5.9** (Inv. algd. flip)**.** If $\sigma = (\pi_1, \alpha)$ for $\alpha$ defined in (4) then

$$(\sigma \times c)(id \times T(\sigma))(\sigma \times c) = (id \times T(\sigma))(\sigma \times c)(id \times T(\sigma))$$

*Proof.* Note that under the isomorphism of 3.5.4 the object $A_\varrho \times_{T(\pi)} T(A)$ is isomorphic to $A_7 \cong A_\pi \times_\pi A_\pi \times_\pi A_\pi \times_\pi A_{\pi pi} \times_{A\ \pi} \times_\pi A_\pi \times_\pi A$. Next we convert the arrows $\sigma \times c$ and $id \times T(\sigma)$ into elements of the symmetric group $S_7$ corresponding to how they permute the factors of $A_7$. Since

$$(\sigma \times c)(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, a_0, a_2, a_3, a_5, a_4, a_6)$$

the arrow $\sigma \times c$ corresponds to the permutation (01)(45). Similarly since

$$(id \times T(\sigma))(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (a_0, a_3, a_4, a_1, a_2, a_5, a_6)$$

the arrow $id \times T(\sigma)$ corresponds to the permutation (13)(24). Therefore we check

$$\begin{aligned}
(\sigma \times c)(id \times T(\sigma))(\sigma \times c) &\leftrightarrow (01)(45)(13)(24)(01)(45) \\
&= (01)(13)(01)(45)(24)(45) \\
&= (03)(25) \\
&= (13)(01)(13)(24)(45)(24) \\
&= (13)(24)(01)(45)(13)(24) \\
&\leftrightarrow (id \times T(\sigma))(\sigma \times c)(id \times T(\sigma))
\end{aligned}$$

and conclude that the flip axiom holds. □

# 4 The Lie algebroid of an involution algebroid

In this section we show that every involution algebroid has a Lie bracket on the set of sections of its underlying bundle. We begin by constructing a monomorphism $\Phi$ from sections of the involution algebroid to vector fields on the total space of the bundle, and show that this preserves addition. Given a pair of sections of an involution algebroid with flip map $\alpha$, we define the bracket

$$[X, Y] := T(X)\varrho Y \div \alpha(X, T(Y)\varrho X)$$

and show that the morphism $\Phi$ preserves the bracket. Furthermore, under the further assumption of a ring object $R$ as described in [8] and a certain unitary condition on the flip map, this bracket satisfies the Liebniz law that is one of the axioms for a classical Lie algebroid.

## 4.1 The Lie bracket

In this section we will work in a tangent category without any specified ring object $R$. We will set an involution algebroid $(\pi : A \to M, \varrho, \alpha)$ throughout this section. The Lie bracket structure on sections of $\pi$, denoted $\Gamma(\pi)$ will be induced using the lie bracket on section of the total space $\chi(A)$. We begin by using the involution to define a monic additive map $\Gamma(\pi) \to \chi(A)$.

**Definition 4.1.1.** Define the morphism:

$$\alpha_{(-)} : \Gamma(\pi) \to \mathbb{X}(A, T(A))$$

to be:

$$\alpha_X = \alpha\langle \mathsf{id}, T(X)\rho \rangle$$

**Lemma 4.1.2.** This morphism sends sections to sections, that is:

$$\alpha_{(-)} : \Gamma(\pi) \to \chi(A)$$

*Proof.*

$$\begin{aligned} p\alpha_X &= p\alpha\langle \mathsf{id}, T(X)\rho \rangle \\ &= p\pi_0\langle \mathsf{id}, T(X)\rho \rangle \\ &= \mathsf{id} \end{aligned}$$

□

**Lemma 4.1.3.** The morphism $\alpha_{(-)}$ is additive and monic, and if $\mathbb{X}$ has a universal ring object $R$, preserves the action of $[M, R]$ by sending $f : M \to R$ to $f\pi : A \to R$.

*Proof.* The map preserves addition:

$$\begin{aligned} \alpha_{X+_A Y} v &= \alpha(v, T(X +_\pi Y)\varrho v) \\ &= \alpha(v, (T(X)\varrho v) +^T_\pi (T(Y)\varrho v)) \\ &= \alpha(v, T(X)\varrho v) +_A \alpha(v, T(Y)\varrho v) && \text{(bilinear)} \\ &= \alpha_X +_A \alpha_Y \end{aligned}$$

and zero:

$$\begin{aligned} \alpha_\zeta v &= \alpha(v, T(\zeta)\varrho v) \\ &= \alpha(1, T(\zeta))(v, \varrho v) \\ &= 0v && \text{(bilinear)} \end{aligned}$$

We see the map preserves the $\mathbb{X}(M, R)$-action by:

$$\begin{aligned} \alpha_{f \bullet_\pi X} &= \alpha\langle 1, T(X \bullet_\pi f)\varrho \rangle \\ &= \alpha\langle 1, (T(X)\varrho) \bullet^T_\pi (T(f)\varrho) \rangle \\ &= \alpha\langle 1, (T(X)\varrho) \bullet_{T(\pi)} (pT(f)\varrho) \rangle && \text{(by ???)} \\ &= (pT(f)\varrho) \bullet_A \alpha\langle 1, T(X)\varrho \rangle && \text{(bilinearity-mult)} \\ &= (fp\varrho) \bullet_A \alpha\langle 1, T(X)\varrho \rangle \\ &= (f\pi) \bullet_A \alpha\langle 1, T(X)\varrho \rangle \end{aligned}$$

The map is monic: consider $X, Y$ so that $\alpha_X = \alpha_Y$. Then:

$$\alpha\langle p\pi_1, \alpha\rangle\langle 1, T(X)\rho\rangle = \alpha\langle p\pi_1, \alpha\rangle\langle 1, T(Y)\rho\rangle$$

Thus:

$$pT(X)\rho v = pT(Y)\rho v$$

and we may compute:

$$pT(X)\rho = Xp\rho = X\pi$$

and similarly, $pT(Y)\rho = Y\pi$. Because $\pi$ is epi, we have that $X = Y$. $\square$

We may define a bracket operation on $\Gamma(\pi)$ using the involution algebroid structure:

**Definition 4.1.4.** Let $(\pi, \varrho, \alpha)$ be an involution algebroid and $X, Y \in \Gamma(\pi)$. We may define the Lie bracket on sections as:

$$[X, Y]_\alpha := T(X)\varrho Y \dot{-} \alpha_Y X$$

**Lemma 4.1.5.** The Lie bracket on $\alpha$ is well defined.

*Proof.* Observe:

$$pT(X)\varrho Y = Xp\varrho Y = X\pi Y = X$$

and we have already shown $p\alpha_Y X = X$.

Next, see that

$$T(\pi)T(X)\varrho Y = T(\pi X)\varrho Y = \varrho Y$$

and

$$\begin{aligned} T(\pi)\alpha_Y X &= T(p\rho)\alpha(X, T(Y)\rho X) \\ &= T(p)cT(\varrho)T(Y)\varrho X \\ &= pT(\varrho)T(Y)\varrho X \\ &= \varrho Y p\varrho X = \varrho Y \pi X = \varrho Y \end{aligned}$$

$\square$

**Proposition 4.1.6.** *The map $\alpha_{(-)}$ preserves the bracket:*

$$\alpha_{[X,Y]_\alpha} = [\alpha_X, \alpha_Y]_A$$

*Proof.* We begin with the left hand side:

$$\ell[\alpha_X, \alpha_Y]_A v = \ell(T(\alpha_X)\alpha_Y \dot{-} cT(\alpha_Y)\alpha_X)v$$

We restrict our attention to the second term:

$$\begin{aligned} &= cT(\alpha_Y)\alpha_X v \\ &= cT(\alpha)(\alpha_X v, T^2(Y)T(\varrho)\alpha_X v) \\ &= cT(\alpha)(\alpha(v, T(X)\varrho v), T^2(Y)T(\varrho)\alpha_X v) \\ &= T(\alpha)(\alpha(v, T^2(Y)T(\varrho)\alpha_X v), cT(\alpha)(T(X)\varrho v, cT^2(Y)T(\varrho)\alpha_X v)) \qquad \text{(flip)} \end{aligned}$$

Now restrict our attention to the first argument of $T(\alpha)$:

$$\begin{aligned}\alpha(v, pT^2(Y)T(\varrho)\alpha_X v) &= \alpha(v, T(Y)\varrho p\alpha_X v)\\ &= \alpha(v, T(Y)\varrho v) = \alpha_Y(v)\end{aligned}$$

now the second argument:

$$\begin{aligned}cT(\alpha)(T(X)\varrho v, cT^2(Y)T(\varrho)\alpha_X v) &= cT(\alpha)(T(X)\varrho v, T^2(Y)cT(\varrho)\alpha(v, T(X)\varrho v))\\ &= cT(\alpha)(T(X)\varrho v, T^2(Y)ccT(\varrho)T(X)\varrho v)\\ &= cT(\alpha)\langle 1, T^2(Y)T(\varrho)\rangle T(X)\varrho v\\ &= cT(\alpha_Y)T(X)\varrho v\end{aligned}$$

The term is now:

$$T(\alpha)(\alpha_Y(v), cT(\alpha_Y)T(X)\varrho v)$$

If we return to our original expression, we now have:

$$\ell(T(\alpha_X)\alpha_Y \div cT(\alpha_Y)\alpha_X)v$$

simplifies to

$$\begin{aligned}&\big(T(\alpha)(\alpha_Y v, T^2(X)T(\varrho)\alpha_Y v) -_{TA} T(\alpha)(\alpha_Y v, T(\alpha)(\alpha_Y(v), cT(\alpha_Y)T(X)\varrho v))\big)\\ &\qquad -^T_A 0pT(\alpha)(\alpha_Y v, T^2(X)T(\varrho)\alpha_Y v)\end{aligned}$$

Simplify the inner term:

$$\begin{aligned}&T(\alpha)(\alpha_Y v, T^2(X)T(\varrho)\alpha_Y v) -_{TA} T(\alpha)(\alpha_Y v, T(\alpha)(\alpha_Y(v), cT(\alpha_Y)T(X)\varrho v))\\ =&T(\alpha)(\alpha_Y v -_A \alpha_Y v, T^2(X)T(\varrho)\alpha_Y v -_{TA} cT(\alpha_Y)T(X)\varrho v))\\ =&T(\alpha)(0v, T^2(X)cT(\varrho)T(Y)\varrho v -_{TA} cT(\alpha_Y)T(X)\varrho v))\\ =&T(\alpha)(0v, cT^2(X)T(\varrho)T(Y)\varrho v -_{TA} cT(\alpha_Y)T(X)\varrho v))\\ =&T(\alpha)(0v, c(T^2(X)T(\varrho)T(Y)\varrho v -^T_A T(\alpha_Y)T(X)\varrho v)))\end{aligned}$$

The outside term may also be simplified:

$$\begin{aligned}0pT(\alpha)(\alpha_Y v, T^2(X)T(\varrho)\alpha_Y v) &= 0\alpha(p\alpha_Y v, pT^2(X)T(\varrho)\alpha_Y v)\\ &= T(\alpha)(0v, 0T(X)\varrho v)\end{aligned}$$

Now we may apply bilinearity once again:

$$
\begin{aligned}
&T(\alpha)(0v, c(T^2(X)T(\varrho)T(Y)\varrho v -_A^T T(\alpha_Y)T(X)\varrho v))) -_A^T T(\alpha)(0v, 0T(X)\varrho v) \\
=&T(\alpha)(0v, c(T^2(X)T(\varrho)T(Y)\varrho v -_A^T T(\alpha_Y)T(X)\varrho v) -_\pi^{T^2} 0T(X)\varrho v) \\
=&T(\alpha)(0v, c(T^2(X)T(\varrho)T(Y)\varrho v -_A^T T(\alpha_Y)T(X)\varrho v) -_\pi^{T^2} cT(0X)\varrho v) \\
=&T(\alpha)(0v, c\left((T^2(X)T(\varrho)T(Y)\varrho v -_A^T T(\alpha_Y)T(X)\varrho v) -_\pi^{T^2} T(0X)\varrho v\right)) \\
=&T(\alpha)(0v, c\left((T^2(X)T(\varrho)T(Y) -_A^T T(\alpha_Y)T(X)) -_\pi^{T^2} T(0X)\right)\varrho v) \\
=&T(\alpha)(0v, cT\left((T(X)\varrho Y -_A \alpha_Y X) -_\pi^T 0X\right)\varrho v) \\
=&T(\alpha)(0v, cT(\lambda)T(T(X)\varrho Y \div \alpha_Y X)\varrho v) \\
=&\ell\alpha(v, T([X,Y]_\alpha)\varrho v) \\
=&\ell\alpha_{[X,Y]_\alpha}
\end{aligned}
$$

As $\ell$ is monic, we've shown the desired equality holds. ☐

**Theorem 4.1.7.** *Given an involution algebroid $(\pi, \rho, \alpha)$, the bracket $[-,-]_\alpha$ gives a Lie algebroid.*

*Proof.*

To show that the bracket is bilinear, observe that:

$$
\begin{aligned}
\alpha_{[X+Y,Z]} &= [\alpha_{X+Y}, \alpha_Z]_A \\
&= [\alpha_X + \alpha_Y, \alpha_Z]_A \\
&= [\alpha_X, \alpha_Z]_A + [\alpha_Y, \alpha_Z]_A \\
&= \alpha_{[X,Z]} + \alpha_{[Y,Z]} \\
&= \alpha_{[X,Z]+[Y,Z]}
\end{aligned}
$$

Alternating:

$$
\begin{aligned}
\alpha_{[X,Y]} &= [\alpha_X, \alpha_Y] \\
&= -[\alpha_Y, \alpha_X] \\
&= -\alpha_{[Y,X]} = \alpha_{-[Y,X]}
\end{aligned}
$$

Jacobi identity:

$$
\begin{aligned}
&\alpha_{[X,[Y,Z]+[Z,[X,Y]]+[Y,[Z,X]]} \\
=&\alpha_{[X,[Y,Z]]} + \alpha_{[Z,[X,Y]} + \alpha_{[Y,[Z,X]} \\
=&[\alpha_X, \alpha_{[Y,Z]}]_A + [\alpha_Z, \alpha_{[X,Y]}]_A + [\alpha_Y, \alpha_{[Z,X]}]_A \\
=&[\alpha_X, [\alpha_Y, \alpha_Z]_A]_A + [\alpha_Z, [\alpha_X, \alpha_Y]_A]_A + [\alpha_Y, [\alpha_Z, \alpha_X]_A]_A \\
=&0_A = \alpha_\zeta
\end{aligned}
$$

☐

## 4.2 The Liebniz law

In this section, we prove that the Lie bracket on an involution algebroid $(\pi : A \to M, \rho, \alpha)$ satisfies the Liebniz law. We will make an assumption that our involution algebroids satisfy a certain *unitary* condition:

$$\alpha(\zeta\pi v, \lambda v) = \lambda v$$

we can see this holds in the case for groupoids, as we have:

$$\alpha(\zeta\pi v, \lambda v) = (c\ell v) \circ (0\zeta\pi v) \circ (p\ell v)^{-1} = (\ell v) \circ \mathsf{id} \circ \mathsf{id}$$

furthermore, if we have a Levi-Civita connection $(H, K)$, the associated involution algebroid has

$$\alpha(\zeta\pi v, \lambda v) = H(\zeta\pi v, \varrho p\lambda v) +_p ((\lambda K\lambda v) +_p^T (0\zeta\pi v)) = H(\zeta\pi v, 0\pi v) +_p (\lambda v +_p^T 0p\lambda v) = \lambda v$$

At this point it is unclear as to whether the unitary condition is independent of the axioms of an involution algebroid. In this section, all involution algebroids will be unitary.

**Proposition 4.2.1.** *Let $(\pi, \varrho, \alpha)$ be a unitary involution algebroid. Then for any $X \in \Gamma(\pi), \psi \in C^\infty(M)$*

$$\alpha_{\psi \bullet X} = (\psi\pi) \bullet \alpha_X +_p \lambda_A(\hat{p}T(\psi)\varrho \bullet_\pi X\pi)$$

*Proof.* Calculate:

$$\begin{aligned}
\alpha_{\psi \bullet X} &= \alpha\langle 1, T(\psi \bullet_\pi X)\varrho\rangle \\
&= \alpha\langle 1, (\lambda\hat{p}T(\psi)\varrho \bullet_\pi^T T(X)\varrho) +_\pi^T (0pT(\psi)\varrho \bullet_\pi^T T(X)\varrho)\rangle \\
&= \alpha\langle 1, \lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T T(X)\varrho)\rangle +_p \alpha\langle 1(\psi\pi \bullet_{T\pi} T(X)\varrho)\rangle \\
&= \alpha\langle 1, \lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T pT(X)\varrho) +_p T(\zeta\pi)T(X)\varrho\rangle +_p \alpha\langle 1(\psi\pi \bullet_{T\pi} T(X)\varrho)\rangle \\
&= \alpha\langle 1, \lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T X\pi) +_p T(\zeta)\varrho\rangle +_p \psi\pi \bullet_p \alpha\langle 1, T(X)\varrho\rangle \\
&= (\alpha\langle 0, \lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T X\pi)\rangle +_\pi \alpha_\zeta) +_p \psi\pi \bullet_p \alpha\langle 1, T(X)\varrho\rangle \\
&\quad \alpha \text{ is unitary} \\
&= (\lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T X\pi) +_\pi^T 0_A) +_p (\psi\pi \bullet_p \alpha\langle 1, T(X)\varrho\rangle +_\pi^T T(\zeta)) \\
&\quad \text{interchange law for addition} \\
&= (\lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T X\pi) +_p \psi\pi \bullet_p \alpha\langle 1, T(X)\varrho\rangle) +_\pi^T (0 +_p T(\zeta)) \\
&= \lambda(\hat{p}T(\psi)\varrho \bullet_\pi^T X\pi) +_p \psi\pi \bullet_p \alpha\langle 1, T(X)\varrho\rangle
\end{aligned}$$

□

Naively, one might think that the morphism $\alpha_{(-)}$ should be a morphism between $C^\infty(M)$ modules, where we give a $C^\infty(M)$ structure to $\chi(A)$ via precomposition with $\pi$:

$$\psi \bullet' v := \psi\pi \bullet_\pi v$$

but as the above calculation shows, this is not the case. However, we may now prove the Liebniz rule directly:

**Proposition 4.2.2.** *Let $(\pi : A \to M, \varrho, \alpha)$ be a unitary involution algebroid. Then the associated Lie bracket on $\Gamma(\pi)$ satisfies the Liebniz law.*

*Proof.* Consider $X, Y \in \Gamma(\pi)$, and $\psi : M \to R$. We may compute:

$$\begin{aligned}
&(\lambda[X, \psi \bullet Y]v +_{\pi}^{T} 0X) -_p \lambda(L_{X\varrho}\psi \bullet_\pi Y) \\
&= (\alpha_{\psi \bullet Y} X -_p T(X)\varrho(\psi \bullet Y)v) -_p \lambda(L_{X\varrho}\psi \bullet_\pi Y) \\
&= (((\psi\pi) \bullet \alpha_Y +_p \lambda_A(\hat{p}T(\psi)\varrho \bullet_\pi Y\pi))X -_p \psi \bullet_p T(X)\varrho Y) \\
&= ((\psi \bullet_p \alpha_Y X +_p \lambda_A(L_{\varrho X} \bullet_\pi Y)) -_p \psi \bullet_p T(X)\varrho Y) -_p \lambda(L_{X\varrho}\psi \bullet_\pi Y) \\
&= \psi \bullet_p (\alpha_Y X -_p T(X)\varrho Y)
\end{aligned}$$

It follows that

$$\lambda[X, \psi \bullet Y] = \lambda(\psi \bullet_p [X, Y] + L_{X\varrho}\psi \bullet_\pi Y)$$

thus, the Liebniz rule holds for unitary Involution algebroids. □

# 5 The homotopy theory of involution algebroids

In section 5.1 of [6] the *Weinstein local groupoid* of a Lie algebroid is constructed in an analogous way to how the fundamental groupoid is constructed from a manifold. The appropriate definition of paths and homotopies in a Lie algebroid required for this construction are the *admissible paths* and *admissible homotopies* of section 1 in [6]. In fact this construction gives an equivalence of categories

$$\begin{tikzcd} LieAlgd \arrow[r, shift left=1.5ex, "w"] \arrow[r, phantom, "\perp"] & LocGpd \arrow[l, shift left=1.5ex, "alg"] \end{tikzcd}$$

where $alg$ is the extension of section 3.5 of [14] to local Lie groupoids and $w$ is the Weinstein local groupoid construction. Therefore one approach to understanding the morphisms in *LieAlgd* is to understand the homotopy theory of Lie algebroids.

In this section we describe the homotopy theory associated to involution algebroids. Moreover we analyse some special cases of paths and homotopies in involution algebroids that arise in the composite $alg \circ w$ in the diagram above. In a future paper we complete this picture by examining the extra assumptions required to integrate involution algebroids to groupoids. In this section we will assume the existence of various function spaces as we need them. Again this assumption is not an unreasonable one due to the embedding theorem of [9].

In 5.1 we describe how to transport elements of an algebroid along an infinitesimal variation of A-paths. This construction will require the use of a complete curve object as described in 2.5. Our definition of admissible path in 5.1.1 is the direct translation of the classical idea found in 1.1 of [6]. By contrast to define an A-homotopy in an involution algebroid we use the flip map $\alpha : A_{\varrho}\times_{T(\pi)}T(A) \to T(A)$ directly. In this way we avoid using an integral (or directly appealing to the existence of a connection) as in 1.3 of [6]. In 5.2 we describe how to transport elements of an algebroid along an infinitesimal variation of A-homotopies. Here we also demonstrate that this transport is 'path independent' in an analogous way to how parallel transport with respect to a flat connection is independent of the path taken. In 5.3 we specialise the construction of 5.1 to 'infinitesimal A-paths' which arise in the composite $alg \circ w$ above. We show that an infinitesimal A-path in $A$ is the same as a path in a fibre of $A$ starting at zero. In 5.4 we specialise the construction of 5.2 to the 'infinitesimal A-homotopies' which arise in the composite $alg \circ w$. Again we show that an infinitesimal A-homotopy in $A$ is the same as a homotopy in a fibre of $A$ which starts at zero. We leave for future work the natural next step of taking the quotient of the A-paths by the A-homotopies and identifying the extra assumptions necessary to carry out the Weinstein groupoid construction in this context.

## 5.1 Transport along A-paths

An A-path (as described in for instance 1.1 of [6]) is a path in the total space of an algebroid for which the anchor coincides with the projection of the derivative of the source. Intuitively we think of this condition as saying that it is possible to post-compose an element $a(x_0)$ with the element $a(x_0 + h)$ where $a$ is an A-path and $h$ infinitesimally small.

Now we describe how to transport an element $a \in A$ along a variation of A-paths (a tangent vector to the space of A-paths). Roughly speaking the $\alpha : A_{\varrho}\times_{T(\pi)}T(A) \to T(A)$ of an involution algebroid gives a way of 'infinitesimally transporting' an element $a \in A$ along an element $w \in T(A)$ to produce an element of $T(A)$. Then we use a complete curve object as described in 2.5 to extend this infinitesimal transport to a full transport along a variation of A-paths. The following definition is in 1.1 of [6].

**Definition 5.1.1** (Admissible path)**.** An *admissible path in $A$ (or A-path)* is an arrow $a : I \to A$ such that $\varrho a = T(\pi)T(a)\partial$. We write $APath$ for the object of admissible paths of $A$.

**Definition 5.1.2** (Tangent to admissible paths)**.** The object $APath^D$ is the subobject of $T(A)^I$ of $\phi \in T(A)^I$ such that $T(\varrho)\phi = cT^2(\pi)T(\phi)\partial$.

At this stage we cannot immediately define the transport of A-paths as a solution to a vector field of type $X : A \to T(A)$. This would require being able to extend a variation of A-paths $\phi$ to a vector field on $A$. Classically we can extend $\phi$ to a time-dependent vector field on $A$ but in a tangent category we cannot guarantee that this extension exists. However if we assume the existence of a line object $R$ with unit $u$ and zero $0_R$ we can produce the vector field we want on the following pullback.

**Definition 5.1.3** (Bundle of composables)**.** If $\phi \in APath^D$ then the *the* *bundle $A_\phi$ of arrows post-composable with $\phi$* is the pullback

$$\begin{tikzcd}
(A_\phi, p\pi_0) \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & (I_{T(\pi)\phi}\times_\pi A, T(\pi)) \arrow[d, "(id\times\varrho, id)"] \\
(I\times R, \pi_0) \arrow[r, "((\pi_0, \pi_1\cdot T(\pi)\phi\pi_0), id)"] & (I_{T(\pi)\phi}\times_p T(M), \pi_0)
\end{tikzcd}$$

in the category of differential bundles (the lift is $0\times l\times\lambda$).

**Lemma 5.1.4.** The arrow $X : A_\phi \to T(A_\phi)$ defined by

$$X = (\partial\pi_0, 0\pi_1, \alpha(\pi_2, \pi_1\cdot\phi\pi_0))$$

is a well-defined vector field.

*Proof.* It is routine to check that $X$ is a vector field and that the expression $\alpha(\pi_2, \pi_1\cdot\phi\pi_0)$ is well-typed. Now we check $X$ has codomain $T(A_\phi)$:

$$\begin{aligned}
T(\varrho)\alpha(\pi_2, \pi_1\cdot\phi\pi_0) &= cT(\varrho)(\pi_1\cdot\phi\pi_0) \\
&= \pi_1\cdot cT(\varrho)\phi\pi_0 && (c \text{ and } T(\varrho) \text{ linear}) \\
&= \pi_1\cdot T^2(\pi)T(\phi)\partial\pi_0
\end{aligned}$$

and so $X$ is well-defined. □

**Lemma 5.1.5** (Linearity of infinitesimal composition)**.** The vector field $X$ of 5.1.4 is linear over $\partial : I \to T(I)$ in the sense of 2.5.7.

*Proof.* It is routine to check that $X$ and $\partial$ commute with the projections. For linearity we check

$$\begin{aligned}
&T(X)(0\times l\times\lambda) \\
&= (T(\partial)0\pi_0, T(0)l\pi_1, T(\alpha)(\lambda\pi_2, l\pi_1\cdot T(\phi)0\pi_0)) \\
&= (0\partial, T(0)l\pi_1, T(\alpha)(\lambda\pi_2, l(\pi_1\cdot\phi\pi_0))) && (\cdot \text{ linear}) \\
&= (0\partial, cT(l)0\pi_1, cT(\lambda)\alpha(\pi_2, \pi_1\cdot\phi\pi_0)) && (\alpha \text{ linear}) \\
&= (0\times cT(l)\times cT(\lambda))X
\end{aligned}$$

and so $X$ is linear over $\partial$. □

Since $X$ is linear over $\partial$ and $\partial$ is a complete vector field with solution $id : I \to I$ we can apply 2.5.9 to obtain a complete solution for $X$. So let $a \in A$ and $\phi \in APath^D$ such that $\varrho a = T(\pi)\phi 0_I$. Further let $\Psi_a$ be the solution of $X$ starting at $(0_I, u, a)$. Now we work out the properties of $\pi_0\Psi_a$ and $\pi_1\Psi_a$. First

$$T(\pi_0\Psi_a)\partial = \pi_0 T(\Psi_a)\partial = \pi_0 X\Psi_a = \partial\pi_0\Psi_a$$

and $\pi_0\Psi_a 0_I = 0_I$ therefore $\pi_0\Psi_a = id$. Second

$$T(\pi_1\Psi_a)\partial = \pi_1 T(\Psi_a)\partial = \pi_1 X\Psi_a = 0\pi_1\Psi_a$$

and $\pi_1\Psi_a 0_I = u$ so $\pi_1\Psi_a = u$. The projection $\pi_2\Psi_a$ is the transport that we want.

**Definition 5.1.6** (Infinite composition)**.** If $a \in A$ and $\phi \in APath^D$ then *the infinite composite* $\psi_a : I \to A$ *of* $\phi$ *starting at* $a$ is $\pi_2 \Psi_a$. In other words $\psi_a$ is the solution to $X$ starting at $(0, u, a)$ followed by $\pi_2$.

The following lemma presents the properties of $\psi_a$ that we use in the sequel.

**Lemma 5.1.7.** If $a \in A$ and $\phi \in APath^D$ then

$$\psi_a(0_I) = a \qquad \text{(initial condition)}$$
$$T(\psi_a)\partial = \alpha(\psi_a, \phi) \qquad \text{(solution to vector field)}$$

because $\psi_a$ is defined using a solution to a vector field.

*Proof.* Since $\Psi_a$ is the solution to $X$ starting at $(0, u, a)$ then

$$\begin{aligned} T(\psi_a)\partial &= \pi_2 T(\Psi_a)\partial \\ &= \pi_2 X \Psi_a \\ &= \alpha(\pi_2 \Psi_a, \pi_1 \Psi_a \cdot \phi \pi_0 \Psi_a) \\ &= \alpha(\psi_a, u \cdot \phi) \end{aligned}$$

where the last line uses the characterisation of $\pi_0 \Psi_a$ and $\pi_1 \Psi_a$ established previously. For the initial condition: $\psi_a 0_I = \pi_2 \Psi_a 0_I = a$. □

The next lemma confirms that $\psi_a$ has the anchor we would expect of a transport of $a$ along $\phi$.

**Lemma 5.1.8.** If $a \in A$ and $\phi \in APath^D$ then $\varrho \psi_a = T(\pi)\phi$.

*Proof.* We check

$$\begin{aligned} \varrho \psi_a &= pT(\varrho)T(\psi_a)\partial && (\partial \text{ is section}) \\ &= pT(\varrho)\alpha(\psi_a, \phi) && \text{(soln to vector field)} \\ &= pcT(\varrho)\phi && \text{(inv. algd. target)} \\ &= T(\pi)\phi \end{aligned}$$

as required. □

## 5.2 Transport along A-homotopies

Next we develop the theory analogous to that developed in 5.1 with A-homotopies in place of A-paths. In 5.2.1 we define an A-homotopy as a map from $I \times I$ into the total space of an algebroid that is both horizontally and vertically an $A$-path and moreover satisfies a commutativity condition that is analogous to the classical condition determining a flat connection on a manifold. In classical Lie theory (for instance in [6]) an A-homotopy in an integrable Lie algebroid $A$ corresponds to a map $\nabla(I \times I) \to \mathbb{G}$ into the groupoid $\mathbb{G}$ integrating $A$. We have used $\nabla(I \times I)$ to denote the pair (or chaotic, or indiscrete) groupoid on $I \times I$. In a future paper we show that the commutativity condition in 5.2.1 precisely corresponds to the fact that $\nabla(I \times I)$ is commutative. As before transport along A-homotopies requires a complete curve object $I$ as described in 2.5.

Therefore we show that if we start with an infinitesimal variation $h$ of A-homotopies and an element $a \in A$ such that $\varrho a$ is the same as the 'source' of the A-homotopy then we can transport $a$ along $h$ to obtain a map $\chi : I \times I \to A$. In the case of A-homotopies it seems like we have a choice of how to obtain this $\chi$. We could either integrate the horizontal A-path first and the vertical A-path second or the other way around. In 5.2.6 we show that our construction is independent of the choice of path we use.

**Definition 5.2.1** (Admissible homotopy)**.** An *admissible homotopy in A (or A-homotopy)* is an arrow $h : I \times I \to A_2 \cong A_\pi \times_\pi A$ such that

$$\varrho h_0 = T(\pi)T(h_0)(\partial \times 0) \qquad \text{(hor. A-path)}$$
$$\varrho h_1 = T(\pi)T(h_1)(0 \times \partial) \qquad \text{(vert. A-path)}$$
$$\alpha(h_0, T(h_1)(\partial \times 0)) = T(h_0)(0 \times \partial) \qquad \text{(A-homotopy ctd.)}$$

where we have used $h_i$ to denote $\pi_i h$ for $i \in \{0, 1\}$.

**Definition 5.2.2** (Tangent to admissible homotopies)**.** The object $AHtpy^D$ is the subobject of $T(A_2)^{I\times I}$ of $h \in T(A_2)^{I\times I}$ such that

$$T(\varrho)h_0 = cT^2(\pi)T(h_0)(\partial \times 0) \qquad \text{(hor. A-path)}$$
$$T(\varrho)h_1 = cT^2(\pi)T(h_1)(0 \times \partial) \qquad \text{(vert. A-path)}$$
$$T(\alpha)(h_0, cT(h_1)(\partial \times 0)) = cT(h_0)(\partial \times 0) \qquad \text{(A-homotopy ctd.)}$$

where we regard $h_0 \in T(A)^{I\times I}$ and $h_1 \in T(A)^{I\times I}$ where $h_i = \pi_i h$.

**Notation 5.2.3** (Horizontal and vertical infinite composition)**.** If $h$ is an A-homotopy and $a \in A$ such that $\varrho a = T(\pi)h_0(0_I, 0_I)$ then the arrows $\psi_0$, $\psi_1$, $\Phi_0$ and $\Phi_1$ are the unique infinite composites satisfying the following conditions:-

- $\psi_0 : I \to A$, $\psi_0 0_I = a$ and $T(\psi_0)\partial = \alpha(\psi_0, h_0(id, 0_I))$
- $\psi_1 : I \to A$, $\psi_1 0_I = a$ and $T(\psi_1)\partial = \alpha(\psi_1, h_1(0_I, id))$
- $\Phi_0 : I \times I \to A$, $\Phi_0(0_I, id) = \psi_1$ and $T(\Phi_0)(\partial \times 0) = \alpha(\Phi_0, h_0)$
- $\Phi_1 : I \times I \to A$, $\Phi_1(id, 0_I) = \psi_0$ and $T(\Phi_1)(0 \times \partial) = \alpha(\Phi_1, h_1)$

where in the latter two cases we create solutions $I \to A^I$ and apply the hom-tensor adjunction.

The following lemma serves to introduce the proof technique that we use repeatedly in this section and also is useful in its own right.

**Lemma 5.2.4.**

$$\Phi_0(id, 0_I) = \psi_0$$

*Proof.* We use the uniqueness condition in the definition of curve object. First we check that both sides have the same initial condition:

$$\Phi_0(id, 0_I)0_I = \Phi_0(0_I, 0_I) = \psi_1 0_I = a = \psi_0 0_I$$

Second we check that both sides solve the same vector field. On the one hand:

$$
\begin{aligned}
T(\Phi_0(id, 0_I))\partial &= T(\Phi_0)(id, T(0_I))\partial \\
&= T(\Phi_0)(\partial \times 0)(id, 0_I) \\
&= \alpha(\Phi_0(id, 0_I), h_0(id, 0_I)) && (\Phi_0 \text{ soln.})
\end{aligned}
$$

and on the other hand:

$$
T(\psi_0)\partial = \alpha(\psi_0, h_0(id, 0_I)) \qquad (\psi_0 \text{ soln.})
$$

as required. □

Now we show that the transport along A-homotopies is independent of the path chosen. In other words we show $\Phi_0 = \Phi_1$. Recall that $\Phi_0$ transports $a$ vertically then horizontally. First we measure the vertical infinitesimal variation of $\Phi_0$.

**Lemma 5.2.5.**

$$
T(\Phi_0)(0 \times \partial) = \alpha(\Phi_0, h_1)
$$

*Proof.* We use the uniqueness condition in the definition of curve object. First we check that both sides have the same initial condition:

$$
\begin{aligned}
\alpha(\Phi_0, h_1)(0_I, id) &= \alpha(\Phi_0(0_I, id), h_1(0_I, id)) \\
&= \alpha(\psi_1, h_1(0_I, id)) && (\text{init. ctd. } \Phi_0) \\
&= T(\psi_1)\partial && (\psi_1 \text{ soln.}) \\
&= T(\Phi_0(0_I, id))\partial && (\text{init. ctd. } \Phi_0) \\
&= T(\Phi_0)(T(0_I)\partial, \partial) \\
&= T(\Phi_0)(0 \times \partial)(0_I, id)
\end{aligned}
$$

Second we check that both sides solve the same vector field. On the one hand:

$$
\begin{aligned}
T(T(\Phi_0)(0 \times \partial))(\partial \times 0) &= cT(T(\Phi_0)(\partial \times 0))(0 \times \partial) \\
&= cT(\alpha(\Phi_0, h_0))(0 \times \partial) && (\Phi_0 \text{ is soln.}) \\
&= cT(\alpha)(T(\Phi_0)(0 \times \partial), T(h_0)(0 \times \partial))
\end{aligned}
$$

and on the other hand:

$$
\begin{aligned}
T(\alpha(\Phi_0, h_1))(\partial \times 0) &= T(\alpha)(T(\Phi_0)(\partial \times 0), T(h_1)(\partial \times 0)) \\
&= T(\alpha)(\alpha(\Phi_0, h_0), T(h_1)(\partial \times 0)) && (\Phi_0 \text{ soln.}) \\
&= cT(\alpha)(\alpha(\Phi_0, h_1), cT(\alpha)(h_0, cT(h_1)(\partial \times 0))) \\
& && (\text{inv. algd. flip}) \\
&= cT(\alpha)(\alpha(\Phi_0, h_1), T(h_0)(0 \times \partial)) \\
& && (\text{A-homotopy ctd.})
\end{aligned}
$$

as required. □

**Proposition 5.2.6.**

$$\Phi_0 = \Phi_1$$

*Proof.* We use the uniqueness condition in the definition of curve object. First we check that both sides have the same initial condition:

$$\begin{aligned} \Phi_1(id, 0_I) &= \psi_0 && \text{(init. ctd. } \Phi_1) \\ &= \Phi_0(id, 0_I) && (5.2.4) \end{aligned}$$

Second we check that both sides solve the same vector field. On the one hand $T(\Phi_0)(0 \times \partial) = \alpha(\Phi_0, h_1)$ by 5.2.5. On the other hand $T(\Phi_1)(0 \times \partial) = \alpha(\Phi_1, h_1)$ because of the characterisation of $\Phi_1$ as a solution to a vector field. □

## 5.3 Infinitesimal A-paths

In this section study the homotopy theory of involution algebroids when applied to paths and homotopies that are appropriately 'infinitesimally close to an identity element'. These infinitesimal paths and homotopies arise in a natural way in the method of integrating Lie algebroids described in [6] and in a future paper we show how the theory presented in that paper can be translated to apply to involution algebroids in tangent categories. In this section we treat infinitesimal A-paths and in the next infinitesimal A-homotopies. The key results are 5.3.5 and 5.4.5 which show that any infinitesimal A-path or A-homotopy in an involution algebroid $A$ can be viewed as a path or homotopy respectively in a single fibre of $A$. Both of these results require a complete curve object as described in 2.5.

First we define infinitesimal A-paths and make precise what we mean by a path in a fibre of an involution algebroid. Then we show how to use the flip $\alpha : A_\varrho \times_{T(\pi)} T(A) \to T(A)$ to create an infinitesimal A-path $\vee\psi$ from a path in a fibre $\psi$. In the other direction we combine $\alpha$ with solutions obtained from complete curve object $I$ (see 2.5) to obtain a path $\wedge\phi$ in a fibre from an infinitesimal A-path $\phi$. Then in 5.3.5 we show that $\vee$ and $\wedge$ are inverses.

**Definition 5.3.1** (Infinitesimal A-paths)**.** The object $alg(wA)_1$ is the subobject of $T(A)^I$ consisting of the $\phi \in T(A)^I$ such that:-

$$\begin{aligned} p\phi &= \xi m && \text{(starts at zero)} \\ T(\pi)\phi 0_I &= 0m && \text{(source constant)} \\ T(\varrho)\phi &= cT^2(\pi)T(\phi)\partial && \text{(variation of A-paths)} \end{aligned}$$

where $m = \pi p \phi 0_I$.

**Definition 5.3.2** (Paths in fibres)**.** The object $(A^I)^\pi_M$ is the subobject of $A^I$ consisting of the $\chi \in A^I$ such that:-

$$\begin{aligned} \chi 0_I &= \xi m && \text{(starts at zero)} \\ \pi\chi &= m && (\pi \text{ constant}) \end{aligned}$$

where $m = \pi \chi 0_I$.

**Lemma 5.3.3** (Differentiation to A-path)**.** The arrow $\vee : (A^I)^\pi_M \to alg(wA)_1$ defined by $\chi \mapsto \alpha(\xi m, T(\chi)\partial)$ is well-typed.

*Proof.* First the base $m$ is preserved:

$$\begin{aligned} \pi p(\vee\chi)0_I &= \pi p\alpha(\xi m, T(\chi)\partial)0_I \\ &= \pi\xi\pi\chi 0_I && \text{(inv. algd. 0)} \\ &= \pi\chi 0_I = m \end{aligned}$$

Second $\vee\chi = \alpha(\xi\pi\chi, T(\chi)\partial)$ starts at zero:

$$\begin{aligned} p\alpha(\xi m, T(\chi)\partial) &= \xi\pi\chi && \text{(inv. algd. 0)} \\ &= \xi m && (\pi \text{ constant}) \end{aligned}$$

Third $\vee\chi$ is source constant:

$$\begin{aligned} T(\pi)\alpha(\xi m, T(\chi)\partial)0_I &= \varrho p T(\chi)\partial 0_I && \text{(inv. algd. source)} \\ &= \varrho\chi 0_I && (\partial \text{ section}) \\ &= \varrho\xi m && \text{(starts at zero)} \\ &= 0m \end{aligned}$$

Fourth $\vee\chi$ is a variation of A-paths:

$$\begin{aligned} cT^2(\pi)T(\alpha(\xi m, T(\chi)\partial))\partial &= cT(T(\pi)\alpha(\xi m, T(\chi)\partial))\partial \\ &= cT(\varrho p T(\chi)\partial)\partial && \text{(inv. algd. source)} \\ &= cT(\varrho)T(\chi)\partial && (\partial \text{ section}) \\ &= T(\varrho)\alpha(\xi m, T(\chi)\partial) && \text{(inv. algd. target)} \end{aligned}$$

Therefore $\vee\chi \in alg(wA)_1$. □

**Lemma 5.3.4** (Integration to path in fibre)**.** The arrow $\wedge : alg(wA)_1 \to (A^I)^\pi_M$ defined as the unique infinite composite satisfying $(\wedge\phi)0_I = \xi m$ and $T(\wedge\phi) = \alpha(\wedge\phi, \phi)$ is well-typed.

*Proof.* First the base point $m$ is preserved:

$$\pi(\wedge\phi)0_I = \pi\xi m = m$$

Second $\wedge\phi$ is $\pi$ constant:

$$\begin{aligned} \pi(\wedge\phi) &= pT(\pi)T(\wedge\phi)\partial && (\partial \text{ section}) \\ &= pT(\pi)\alpha(\wedge\phi, \phi) && (\wedge\phi \text{ soln.}) \\ &= p\varrho p\phi && \text{(inv. algd. source)} \\ &= \pi p\phi \\ &= \pi\xi m && \text{(starts at zero)} \\ &= m \end{aligned}$$

Third $\wedge\phi$ starts at zero by the initial condition defining $\wedge\phi$. Therefore $\wedge\phi \in (A^I)^\pi_M$. □

**Proposition 5.3.5.** *The arrows $\vee$ and $\wedge$ are inverses.*

*Proof.* First

$$
\begin{aligned}
\vee(\wedge\phi) &= \alpha(\xi m, T(\wedge\phi)\partial) \\
&= \alpha(\xi m, \alpha(\wedge\phi, \phi)) && (\wedge\phi \text{ is soln.}) \\
&= \phi && (\text{inv. algd. inv.})
\end{aligned}
$$

because $p\phi = \xi m$. Second $\wedge(\vee\chi)$ is the unique infinite composite satisfying $T(\wedge\vee\chi)\partial = \alpha(\wedge\vee\chi, \vee\chi)$ and $(\wedge\vee\chi)0_I = \xi m$. But $\chi$ satisfies these equations. Indeed $\chi 0_I = \xi m$ because $\chi$ starts at zero and

$$
\begin{aligned}
\alpha(\chi, \vee\chi) &= \alpha(\chi, \alpha(\xi m, T(\chi)\partial)) \\
&= T(\chi)\partial && (\text{inv. algd. inv.})
\end{aligned}
$$

Therefore $\vee$ and $\wedge$ are inverses. □

## 5.4 Infinitesimal A-homotopies

Now we prove the analogous result to 5.3.5 with A-homotopies in place of A-paths. So roughly speaking we show that an infinitesimal A-homotopy in an involution algebroid $A$ is the same as a homotopy in a fibre of $A$. First we define infinitesimal A-homotopies in the appropriate way and make precise what we mean by an homotopy in a fibre of $A$. Then we show how to use the flip $\alpha : A_\varrho\times_{T(\pi)} T(A) \to T(A)$ to create an infinitesimal A-homotopy $\vee\chi$ from an homotopy in a fibre $\chi$. In the other direction we combine $\alpha$ with solutions obtained from complete curve object $I$ (see 2.5) to obtain an homotopy $\wedge h$ in a fibre from an infinitesimal A-homotopy $h$. Then in 5.4.5 we show that $\vee$ and $\wedge$ are inverses.

**Definition 5.4.1.** The object $alg(wA)_2$ is the subobject of $T(A_2)^{I\times I}$ on the elements $h \in T(A_2)^{I\times I}$ such that:-

$$
\begin{aligned}
ph_i &= \xi m && (\text{starts at zero}) \\
T(\pi)h_i(0_I, 0_I) &= 0m && (\text{source constant}) \\
T(\varrho)h_0 &= cT^2(\pi)T(h_0)(\partial\times 0) && (\text{hor. A-path}) \\
T(\varrho)h_1 &= cT^2(\pi)T(h_1)(0\times\partial) && (\text{vert. A-path}) \\
cT(h_0)(0\times\partial) &= T(\alpha)(h_0, cT(h_1)(\partial\times 0)) && (\text{A-homotopy ctd.})
\end{aligned}
$$

where $m = \pi\pi_0 ph(0_I, 0_I)$ and $h_i = T(\pi_i)h : I\times I \to T(A)$ for $i \in \{0,1\}$.

**Definition 5.4.2.** The object $(A^{I\times I})^\pi_M$ is the subobject of $A^{I\times I}$ on the elements $\eta \in A^{I\times I}$ such that:-

$$
\begin{aligned}
\eta(0_I, 0_I) &= \xi m && (\text{starts at zero}) \\
\pi\eta &= m && (\pi \text{ constant})
\end{aligned}
$$

where $m = \pi\eta(0_I, 0_I)$.

**Definition 5.4.3** (Differentiation to admissible homotopies)**.** The arrow $\vee : (A^{I\times I})^\pi_M \to alg(wA)_2$ defined by

$$
\eta \mapsto (\alpha(\xi m, T(\eta)(\partial\times 0)), \alpha(\xi m, T(\eta)(0\times\partial)))
$$

is well-typed.

*Proof.* First the base point $m$ is preserved:

$$\begin{aligned}\pi\pi_0 p(\vee\eta)(0_I,0_I) &= \pi p\alpha(\xi m, T(\eta)(\partial\times 0))(0_I,0_I)\\ &= m && \text{(inv. algd. 0)}\end{aligned}$$

Second $\vee\eta$ starts at zero:

$$\begin{aligned}p(\vee\eta)_0 &= p\alpha(\xi m, T(\eta)(\partial\times 0))\\ &= \xi m && \text{(inv. algd. 0)}\end{aligned}$$

Third $\vee\eta$ is source constant:

$$\begin{aligned}T(\pi)(\vee\eta)_0(0_I,0_I) &= T(\pi)\alpha(\xi m, T(\eta)(\partial\times 0))(0_I,0_I)\\ &= \varrho p T(\eta)(\partial\times 0)(0_I,0_I) && \text{(inv. algd. source)}\\ &= \varrho\eta(0_I,0_I) && (\partial\text{ section})\\ &= \varrho\xi m && \text{(starts at zero)}\\ &= 0m\end{aligned}$$

Fourth $(\vee\eta)_0$ is a horizontal A-path:

$$\begin{aligned}&cT^2(\pi)T(\vee\eta)(\partial\times 0)\\ &\quad= cT^2(\pi)T(\alpha(\xi m, T(\eta)(\partial\times 0)))(\partial\times 0)\\ &\quad= cT(T(\pi)\alpha(\xi m, T(\eta)(\partial\times 0)))(\partial\times 0)\\ &\quad= cT(\varrho p T(\eta)(\partial\times 0))(\partial\times 0) && \text{(inv. algd. source)}\\ &\quad= cT(\varrho\eta)(\partial\times 0) && (\partial\text{ section})\\ &\quad= T(\varrho)\alpha(\xi m, T(\eta)(\partial\times 0)) && \text{(inv. algd. target)}\end{aligned}$$

and similarly for the identities involving $(\vee\eta)_1$. Fifth $\vee\eta$ satisfies the A-homotopy condition:

$$\begin{aligned}&T(\alpha)((\vee\eta)_0, T((\vee\eta)_1)(\partial\times 0))\\ &\quad= T(\alpha)(\alpha(\xi m, T(\eta)(\partial\times 0)), cT(\alpha(\xi m, T(\eta)(0\times\partial)))(\partial\times 0))\\ &\quad= T(\alpha)(\alpha(\xi m T(\eta)(\partial\times 0)), cT(\alpha)(T(\xi)T(m), T^2(\eta)T(0\times\partial))(\partial\times 0))\\ &\quad= T(\alpha)(\alpha(\xi m, T(\eta)(\partial\times 0), cT(\alpha)(T(\xi)T(m), T^2(\eta)T(0\times\partial)(\partial\times 0))))\\ &\quad= cT(\alpha)(\alpha(\xi m, T(\xi)T(m)), cT^2(\eta)T(0\times\partial)(\partial\times 0)) && \text{(inv. algd. flip)}\\ &\quad= cT(\alpha)(\alpha(\xi m, T(\xi)T(m)), T^2(\eta)T(\partial\times 0)(0\times\partial))\\ &\quad= cT(\alpha)(\alpha(\xi m, T(\xi)T(m)), T^2(\eta)T(\partial\times 0))(0\times\partial)\\ &\quad= cT(\alpha(\xi m, T(\eta)(\partial\times 0)))(0\times\partial)\\ &\quad= cT((\vee\eta)_0)(0\times\partial)\end{aligned}$$

Therefore $\vee\eta \in alg(wA)_2$. □

**Definition 5.4.4** (Integration to homotopy in fibre)**.** The arrow $\wedge$ : $alg(wA)_2 \to (A^{I\times I})^{\pi}_{M}$ defined by $h \mapsto \Phi_0 = \Phi_1$ of 5.2.3 is well-typed.

*Proof.* As usual $\wedge$ preserves the base point $m$:

$$\begin{aligned} \pi\Phi_0(0_I, 0_I) &= \pi\psi_1 0_I && \text{(init. ctd. } \Phi_0) \\ &= \pi\xi m && \text{(init. ctd. } \psi_1) \\ &= m \end{aligned}$$

First $\Phi_0$ starts at zero:

$$\begin{aligned} \Phi_0(0_I, 0_I) &= \psi_1 0_I && \text{(init. ctd. } \Phi_0) \\ &= \xi m && \text{(init. ctd. } \psi_1) \end{aligned}$$

Second $\Phi_0$ is $\pi$ constant:

$$\begin{aligned} \pi\Phi_0 &= pT(\pi)T(\Phi_0)(\partial \times 0) && (\partial \text{ section}) \\ &= pT(\pi)\alpha(\Phi_0, h_0) && (\Phi_0 \text{ soln.}) \\ &= p\varrho p h_0 && \text{(inv. algd. source)} \\ &= p\varrho\xi m = m \end{aligned}$$

Therefore $\wedge h \in (A^{I\times I})^{\pi}_{M}$. $\square$

**Proposition 5.4.5.** *The arrows $\vee$ and $\wedge$ are inverses.*

*Proof.* On the one hand note that

$$\vee \wedge h = (\alpha(\xi m, T(\wedge h)(\partial \times 0)), \alpha(\xi m, T(\wedge h)(0 \times \partial)))$$

Now treating the zeroth projection:

$$\begin{aligned} \alpha(\xi m, T(\wedge h)(\partial \times 0)) &= \alpha(\xi m, T(\Phi_0)(\partial \times 0)) \\ &= \alpha(\xi m, \alpha(\Phi_0, h_0)) && (\Phi_0 \text{ soln.}) \\ &= h_0 && \text{(inv. algd. inv.)} \end{aligned}$$

because $ph_0 = \xi m$. Similarly $(\vee \wedge h)_1 = h_1$ and therefore $\vee \wedge h = h$. On the other hand $\wedge \vee \eta$ is the unique infinite composite satisfying

$$\begin{aligned} T(\wedge \vee \eta)(\partial \times 0) &= \alpha(\wedge \vee \eta, (\vee\eta)_0) \\ (\wedge \vee \eta)(0_I, id) &= \psi_1 \end{aligned}$$

which using the definition of $\psi_1$ is equivalent to satisfying the following three conditions:-

$$\begin{aligned} T(\wedge \vee \eta)(\partial \times 0) &= \alpha(\wedge \vee \eta, (\vee\eta)_0) \\ T(\wedge \vee \eta)(0 \times \partial) &= \alpha(\wedge \vee \eta, (\vee\eta)_1) \\ (\wedge \vee \eta)(0_I, 0_I) &= \xi m \end{aligned}$$

But $\eta$ satisfies these conditions. Indeed $\eta(0_I, 0_I) = \xi m$ because $\eta$ starts at zero and

$$\begin{aligned} \alpha(\eta, (\vee\eta)_0) &= \alpha(\eta, \alpha(\xi m, T(\eta)(\partial \times 0))) \\ &= T(\eta)(\partial \times 0) && \text{(inv. algd. inv.)} \end{aligned}$$

and similarly for the remaining condition involving $0\times\partial$. Therefore $\wedge\vee\eta = \eta$ and $\wedge$ and $\vee$ are inverses. $\square$

# A Appendix

## A.1 Linear vector fields

**Lemma A.1.1** (Linear differential equations)**.** If $(x_0, X)$ is a dynamical system on $\mathbb{R}^n$ for some $n \in \mathbb{N}$ such that $\pi_1 X : \mathbb{R}^n \to \mathbb{R}^n$ is a linear then $X$ is complete.

*Proof.* This is a classical result: use $\gamma(t) = (\exp(t\pi_1 X))x_0$ where exp denotes the matrix exponential. □

We split the proof of the lifting result into two steps. The first concerns a vector field on a Euclidean space.

**Lemma A.1.2.** If $X : \mathbb{R} \times \mathbb{R}^n \to T(\mathbb{R} \times \mathbb{R}^n)$ is the vector field defined by

$$X : (x, \vec{v}) \mapsto (x, \vec{v}, 1, A_x(\vec{v}))$$

where $A_x(\vec{v})$ is smooth in $x$ and linear in $\vec{v}$ then $X$ is complete.

*Proof.* Let $\gamma : (a, b) \to \mathbb{R} \times \mathbb{R}^n$ be the maximal solution to $(x_0, X)$ for some initial condition $x_0 \in \mathbb{R} \times \mathbb{R}^n$. Suppose (for the purpose of obtaining a contradiction) that $b < \infty$. Let $M = sup_{x \in [0,b]} \|A_x\|$ where $\| - \|$ denotes the operator norm. Then the vector field

$$\hat{X} : (x, \vec{v}) \mapsto (x, \vec{v}, 1, M\vec{v})$$

is linear and so has a complete solution $\hat{\gamma} : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n$. Now by construction $|\gamma(y)| \leq |\hat{\gamma}(y)|$ for all $y \in [0, b)$. Therefore $\gamma[0, b)$ is bounded because $\hat{\gamma}[0, b]$ is bounded. But now we can extend $\gamma$ smoothly to $[0, b]$:

$$\gamma(b) := \lim_{n \to \infty} \gamma\left(b - \frac{1}{n}\right)$$

which exists because $\gamma[0, b)$ is bounded. This extension of $\gamma$ is still a solution to $(x_0, X)$ because the derivative of $\gamma$ is smooth. However this means that $(a, b)$ wasn't the domain for the maximal solution to $(x_0, X)$ and so $b = \infty$. Similarly $a = -\infty$. □

The second step is to use the complete solution of $X^M$ to reduce the general case to the above vector field on a Euclidean space.

**Proposition A.1.3.** *If $X^E$ is a linear vector field over $X^M$ and $x_0 \in E$ such that $(q(x_0), X^M)$ has a complete solution then $(x_0, X^E)$ has a complete solution.*

*Proof.* If $f : \mathbb{R} \to M$ is the solution to $X^M$ then we can restrict $X^E$ to a section $X : \mathbb{R} \times \mathbb{R}^n \to T(\mathbb{R} \times \mathbb{R}^n)$ by pulling back along $f$:

$$\begin{tikzcd}
T(\mathbb{R} \times \mathbb{R}^n) \arrow[rrr, "T(p_1)"] \arrow[ddd, "T(\pi_0)"] & & & T(E) \arrow[ddd, "T(q)"] \\
& \mathbb{R} \times \mathbb{R}^n \arrow[ul, dashed, "X"] \arrow[r, "p_1"] \arrow[d, "\pi_0"] & E \arrow[ur, "X^E"] \arrow[d, "q"] & \\
& \mathbb{R} \arrow[r, "f"] \arrow[dl, "\partial"] & M \arrow[dr, "X^M"] & \\
T(\mathbb{R}) \arrow[rrr, "T(f)"] & & & T(M)
\end{tikzcd}$$

where the middle (and outer) squares are pullbacks because $\mathbb{R}$ is contractible. Now $X$ is given in coordinates by:

$$X : (x, \vec{v}) \mapsto \left(x, \vec{v}, 1, X^E_{f(x)}(\vec{v})\right)$$

and so the result follows from A.1.2. $\square$

## A.2 Injection on objects

In this section we confirm that the work in sections 4 and 3.5 imply an injection on objects from the category *LieAlgd* of Lie algebroids and the category *SInvAlgd* of involution algebroids in the category of smooth manifolds.

Let $\Phi : SInvAlgd_0 \to LieAlgd_0$ be the function defined in 4. Therefore if $A$ is an involution algebroid the bracket on $\Phi(A)$ is given by:

$$\lambda[X, Y] = (T(X)\varrho Y -_p \alpha(X, T(Y)\varrho X)) +_{T(\pi)} 0X \tag{5}$$

and let $\Psi : LieAlgd_0 \to SInvAlgd$ be the function defined in 3.5. Therefore if $A$ is a Lie algebroid $\Psi(A)$ has involution given by:

$$\alpha(v, w) = \alpha_L +_{T(\pi)} \alpha_R$$

where

$$\alpha_L = T(\overline{v})\varrho pw -_p (0v +_{T(\pi)} \lambda[\overline{v}, \overline{pw}])$$

and

$$\alpha_R = T(\xi)\varrho pw -_p T(\xi)\varrho v +_p (w -_{T(\pi)} T(\overline{pw})\varrho v)$$

where $\overline{a}$ denotes any section of $\pi$ such that $\overline{a}\pi a = a$.

We now show that $\Psi$ is a section of $\Phi$ (i.e. $\Phi\Psi = id$). Since in $LieAlgd_0$ we are only required to define the bracket on sections (not elements of the prolongation) we may assume there exists a section $Y$ such that $T(Y)\varrho v = w$. In this case

$$\alpha_R = T(\xi)\varrho pw -_p T(\xi)\varrho v +_p (w -_{T(\pi)} T(Y)\varrho v) = T(\xi)(\varrho pw -_p \varrho v)$$

and so $\alpha = \alpha_L$. But now if we substitute $\alpha_L$ into (5) with $v = Xm$ and $w = T(Y)\varrho Xm$ we see that the bracket on $\Phi\Psi(A)$ is the same as the bracket on $A$.

## A.3 Pullback of differential bundles

Sketch proof in ...

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