

Lie's Third Theorem using an Intuitionistic Double Negation

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1 Introduction

In this paper we formulate and prove a version of Lie's third theorem in the language of categories and functors. Our main tool is the theory of synthetic differential geometry that provides a rigorous definition of infinitesimal objects, infinitesimal actions and infinitesimal transformations. Our local approximations are constructed with an intuitionistic double negation operation and our integration procedure uses a pushout in a similar manner to the first step in an enriched version of the small object argument. The main work in proving

Lie's third theorem is to make a concrete description of this particular pushout of categories and show that no new infinitesimal arrows are introduced in the integration procedure.

Recall that in classical Lie theory we can approximate a Lie group in several different ways. The smallest approximation is a linear one and produces a Lie algebra. Another larger approximation is an analytic one which produces a formal group law (see the Introduction to [7]). A yet larger approximation is that of a local Lie group germ as defined in Definition 1.2. In this way we have the following approximation functors:

$$LieAlg \xleftarrow{\simeq} FGLaw \xleftarrow{\simeq} LLGGerm \xleftarrow{\alpha} LieGrp \quad (1)$$

where $LieGrp$ is the category of Lie groups, $LieAlg$ is the category of Lie algebras, $FGLaw$ is the category of formal group laws and $LLGGerm$ is the category of local Lie group germs. In this classical situation there is no theoretical difference between using the linear, analytic or local approximations: the categories $LieAlg$, $FGLaw$ and $LLGGerm$ are all equivalent. Furthermore if one restricts to the category of simply connected Lie groups then α becomes an equivalence also.

In the reformulation of Lie theory presented in this paper the differences between the linear, analytic and local approximations become significant. This reformulation is closely related to the abstract treatment of Lie's second theorem presented in [3] and in fact in Section 5.3 we apply the results of that paper to show that the appropriate version of Lie's second theorem holds for the constructions presented in this paper also. Also the present paper is related to the approach presented in [2] and [4] but differs in two important respects. Firstly we prove Lie's third theorem rather than Lie's second theorem. Secondly we use an intuitionistic double negation operation to create local approximations whereas in [2] and [4] an analytic approximation involving nilpotent elements is used. The use of the larger approximation arising from the double negation operation is a necessary one in order to prove Lie's third theorem at this level of abstraction and arises naturally from certain counterexamples that occur when using the nilpotent approximation.

In comparison to the classical theory we make a twofold generalisation: we replace groups with categories and the category Man of smooth manifolds with a certain well adapted model of synthetic differential geometry \mathcal{E} called the Dubuc topos (see Definition 2.10). If \mathbb{C} is a category internal to \mathcal{E} then for our local approximation we use the subcategory of \mathbb{C} consisting of all the arrows that are not equal to an identity arrow (see Definition 2.18). The global structures that correspond to the classical Lie groups are not simply internal categories in \mathcal{E} . This is because we also need to impose a condition (given in Definition 5.2) that corresponds to the existence of solutions to certain left-invariant time-dependent vector fields on the arrow space of \mathbb{C} . This means that our version of the Lie adjunction then breaks down into the following composite of a coreflection and a reflection:

$$\begin{array}{ccccc}
& & \xrightarrow{i} & & \xrightarrow{(-)_{int}} \\
Cat_\infty(\mathcal{E}) & \perp & Cat(\mathcal{E}) & \perp & Cat_{int}(\mathcal{E}) \\
& \xleftarrow{(-)_\infty} & & \xleftarrow{j} &
\end{array} \tag{2}$$

Then we can state our main theorem as follows: the composite functor $(-)_\infty \circ j \circ (-)_{int} \circ i$ is a natural isomorphism. Note that by version 2 of our previous work in [3] we have shown that when we impose the appropriate connectivity conditions on the categories in $Cat_{int}(\mathcal{E})$ then the functor $(-)_\infty \circ j$ is full and faithful.

1.1 Local Lie Group Germs

Definition 1.1. A *local Lie group* consists of open sets $U_0, U \subset \mathbb{R}^n$ containing $\vec{0} \in \mathbb{R}^n$, a smooth map $\mu : U \times U \rightarrow \mathbb{R}^n$ and a smooth map $i : U_0 \rightarrow \mathbb{R}^n$ such that if $X, Y, Z \in U$ then:

- $\mu(X, 0) = X = \mu(0, X)$
- $X \in U_0 \implies \mu(X, i(X)) = \mu(i(X), X) = 0$
- $\mu(X, Y), \mu(Y, Z) \in U \implies \mu(X, \mu(Y, Z)) = \mu(\mu(X, Y), Z)$

Definition 1.2. A *germ of a local Lie group* is an equivalence class of local Lie groups where $G \sim H$ iff there exists a open neighbourhoods $V_0 \subset V$ of $\vec{0} \in \mathbb{R}^n$ such that $\mu_G|V = \mu_H|V$ and $i_G|V_0 = i_H|V_0$.

2 Infinitesimals in a Topos

2.1 The Dubuc Topos

In this paper we will work in a particular well-adapted model of synthetic differential geometry called the Dubuc topos. In this section we sketch its basic properties and refer to [5] and Section III of [9] for more details. In future work it would be interesting to see how much of the theory presented in this paper can be adapted to the theory of synthetic differential topology whose primary model is the Dubuc topos.

Notation 2.1. We use $C^\infty(\mathbb{R}^n, \mathbb{R})$ to denote the ring of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$. If I is an ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$ then we use $Z(I)$ to denote the zero set $\{\vec{x} \in \mathbb{R}^n : \forall i \in I. i(\vec{x}) = 0\}$. We use x_i to denote the i th coordinate projection $\mathbb{R}^n \rightarrow \mathbb{R}$. We use Man to denote the category of smooth, paracompact and Hausdorff manifolds.

Definition 2.2. The *category \mathcal{C} of smooth affine schemes* has as objects pairs $[n, I]$ where $n \in \mathbb{N}_{\geq 0}$ and I is an ideal in $C^\infty(\mathbb{R}^n, \mathbb{R})$. The arrows $[n, I] \rightarrow [m, J]$ in \mathcal{C} are equivalence classes of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that for all $j \in J$ the arrow $jf = 0 \pmod{I}$ and $f \sim g$ iff $f \equiv g \pmod{I}$.

Lemma 2.3. *There is a full and faithful embedding of $\mathcal{M}an$ into \mathcal{C} .*

Proof. Theorem 6.15 of [10] states that every smooth n -dimensional manifold M with or without boundary admits a proper smooth embedding $\iota_M : M \rightarrow \mathbb{R}^{2n+1}$. Section 1.2 of [2] shows that it is possible to choose the embeddings ι_M coherently so that the map $M \mapsto [2n+1, I_M]$ is a full and faithful embedding $\mathcal{M}an \rightarrow \mathcal{C}$ where I_M is the ideal of functionals vanishing on $\iota_M(M)$. \square

Definition 2.4. If $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth functions and $\vec{x} \in \mathbb{R}^n$ then f and g have the same germ at \vec{x} iff there exists an open subset U containing \vec{x} such that $f|_U = g|_U$.

Remark 2.5. Let $[n, I]$ be an object of \mathcal{C} and $i_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function such that for all $\vec{x} \in Z(I)$ there exists $i_{\vec{x}} \in I$ with the same germ as i_0 at \vec{x} . Then it is immediate from Definition 2.2 that the presheaf $\mathcal{C}(-, [n, I])$ is isomorphic to the presheaf $\mathcal{C}(-, [n, I + (i_0)])$. By contrast the presheaf $\mathcal{C}([n, I], -)$ can be different to $\mathcal{C}([n, I + (i_0)], -)$. (To see this consider the functions $i_0 : [n, I] \rightarrow [1, (0)]$ and $i_0 : [n, I + (i_0)] \rightarrow [1, (0)]$ when $i_0 \notin I$.)

Definition 2.6. If I is an ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$ then I is *germ-determined* iff for all $i_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for all $\vec{x} \in Z(I)$ there exists $i_{\vec{x}} \in I$ with the same germ as i_0 then in fact $i_0 \in I$ also.

Remark 2.7. The ideals I_M defined in Lemma 2.3 are germ-determined.

Notation 2.8. We write \mathcal{C}_{germ} for the full subcategory of \mathcal{C} on the objects $[n, I]$ for which I is germ-determined.

Definition 2.9. If $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ then the *Dubuc open set* U_χ associated to χ is the arrow $(x_1, x_2, \dots, x_n) : [n+1, (\chi(x_1, x_2, \dots, x_n)x_{n+1} - 1)] \rightarrow [n, (0)]$ in \mathcal{C} . A *Dubuc open set* is any arrow of the form U_χ or any pullback of an arrow of the form U_χ .

Definition 2.10. The *Dubuc coverage* is the Grothendieck coverage on \mathcal{C} generated by the Dubuc open sets. The *Dubuc topos* \mathcal{E} is the Grothendieck topos generated by the site $(\mathcal{C}_{germ}, J_{germ})$ where J_{germ} is the restriction of the Dubuc coverage on \mathcal{C} to \mathcal{C}_{germ} .

Lemma 2.11. *There is a full and faithful embedding ι of the category $\mathcal{M}an$ of smooth, paracompact, Hausdorff manifolds into the Dubuc topos \mathcal{E} .*

Proof. Recall from [5] and Lemma 1.3 in III.1 of [12] that the site generating the Dubuc topos is subcanonical. Then the result follows from Remark 2.7. \square

2.2 The Infinitesimal Part of a Category

In this section we define the local approximation to a category that we will use throughout this paper. The central feature of this definition is the use of an intuitionistic double negation operator.

Notation 2.12. We use \mathcal{E} to denote an arbitrary topos. We use \mathbb{C} to denote a category internal to \mathcal{E} with object space M and arrow space C . We denote by s , t and e respectively the source, target and identity maps of \mathbb{C} .

Following Penon in [13] we make the following definition:

Definition 2.13. An arrow $c \in \mathbb{C}^2$ is *infinitesimal* iff $\neg\neg(esc = c)$.

Remark 2.14. If c is an infinitesimal arrow then $\neg\neg(sc = tc)$.

Remark 2.15. An arrow c is infinitesimal iff $\neg\neg(etc = c)$. Also an arrow c is infinitesimal iff $\exists m \in M. \neg\neg(em = c)$.

Notation 2.16. We use $\iota_{\mathbb{C}}^{\infty} : \mathbb{C}_{\infty}^2 \rightarrow \mathbb{C}^2$ to denote the subobject

$$\{c \in \mathbb{C}^2 : \neg\neg(esc = c)\} \rightarrow \mathbb{C}^2$$

of infinitesimal arrows in \mathbb{C} . It is immediate that $e : M \rightarrow \mathbb{C}^2$ factors through \mathbb{C}_{∞}^2 and we use e_{∞} to denote the unique arrow such that $\iota_{\mathbb{C}}^{\infty} e_{\infty} = e$. Similarly we write $s_{\infty} = s\iota_{\mathbb{C}}^{\infty}$ and $t_{\infty} = t\iota_{\mathbb{C}}^{\infty}$.

Lemma 2.17. *The reflexive graph*

$$\mathbb{C}_{\infty}^2 \begin{array}{c} \xrightarrow{s_{\infty}} \\ \xleftarrow{e_{\infty}} \\ \xrightarrow{t_{\infty}} \end{array} M$$

is a category when equipped with the multiplication μ_{∞} that is the restriction of the multiplication μ of \mathbb{C} .

Proof. Let $c, d \in \mathbb{C}_{\infty}^2$ such that $tc = sd$. By Definition 2.13 $\neg\neg(esc = c)$ and so $\neg\neg(es(dc) = c)$. If $d_0 \in \mathbb{C}$ then in general $(d = d_0) \implies \mu(c, d) = \mu(c, d_0)$ and so in particular $\neg\neg(esd = d) \implies \neg\neg(\mu(esd, c) = \mu(d, c))$. Therefore $\neg\neg(es(\mu(d, c)) = \mu(d, c))$ as required. \square

Definition 2.18. If \mathbb{C} is a category internal to \mathcal{E} then the *infinitesimal part* \mathbb{C}_{∞} of \mathbb{C} is the subcategory of $\iota_{\mathbb{C}} : \mathbb{C}_{\infty} \rightarrow \mathbb{C}$ defined in Lemma 2.17 and Notation 2.16.

2.3 The Infinitesimal Part is a Coreflection

In this section we extend the infinitesimal part construction of Definition 2.18 to an endofunctor on the category $Cat(\mathcal{E})$ of internal categories in a topos \mathcal{E} . Furthermore we show that this endofunctor defines a special type of coreflective subcategory of $Cat(\mathcal{E})$.

Notation 2.19. We use \mathcal{E} to denote an arbitrary topos. We use \mathbb{C} to denote a category internal to \mathcal{E} with object space M and arrow space C . We denote by s , t and e respectively the source, target and identity maps of \mathbb{C} . We use \mathbb{C}_{∞} to denote the infinitesimal part of \mathbb{C} constructed in Definition 2.18.

Lemma 2.20. *The function $\mathbb{C} \mapsto \mathbb{C}_{\infty}$ extends to an endofunctor on $Cat(\mathcal{E})$.*

Proof. Let $f : \mathbb{C} \rightarrow \mathbb{D}$ be an internal functor between internal categories in \mathcal{E} . If $c \in C$ and $\neg\neg(esc = c)$ then $\neg\neg(f(esc) = fc)$ and so $\neg\neg(es(fc) = fc)$. Therefore there is a unique internal functor $f_\infty : \mathbb{C}_\infty \rightarrow \mathbb{D}_\infty$ such that

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{f_\infty} & \mathbb{D}_\infty \\ \downarrow \iota_{\mathbb{C}}^\infty & & \downarrow \iota_{\mathbb{D}}^\infty \\ \mathbb{C} & \xrightarrow{f} & \mathbb{D} \end{array} \quad (3)$$

commutes. \square

Definition 2.21. A category \mathbb{K} internal to \mathcal{E} is *infinitesimal* iff all of its arrows are infinitesimal.

Remark 2.22. It is immediate that the infinitesimal part \mathbb{C}_∞ of a category \mathbb{C} is an infinitesimal category.

Notation 2.23. We use $Cat_\infty(\mathcal{E})$ to denote the full subcategory of $Cat(\mathcal{E})$ on the infinitesimal categories as defined in Definition 2.21.

Lemma 2.24. The subcategory $j : Cat_\infty(\mathcal{E}) \rightarrow Cat(\mathcal{E})$ is a coreflective subcategory with coreflection $(-)_\infty : Cat(\mathcal{E}) \rightarrow Cat_\infty(\mathcal{E})$. Furthermore if $\mathbb{K} \in Cat_\infty(\mathcal{E})$ and $\mathbb{D} \in Cat(\mathcal{E})$ then $(\iota_{\mathbb{D}}^\infty)^\mathbb{K} : \mathbb{D}_\infty^\mathbb{K} \rightarrow \mathbb{D}^\mathbb{K}$ is an isomorphism in \mathcal{E} .

Proof. The commutative square (3) shows that ι^∞ is a natural transformation $j \circ (-)_\infty \Rightarrow 1_{Cat(\mathcal{E})}$ where j is the inclusion $Cat_\infty(\mathcal{E}) \rightarrow Cat(\mathcal{E})$. Also it is immediate from the definition of $(-)_\infty$ that $(-)_\infty \circ j = 1_{Cat_\infty(\mathcal{E})}$. To see that $(\iota_{\mathbb{D}}^\infty)^\mathbb{K} : \mathbb{D}_\infty^\mathbb{K} \rightarrow \mathbb{D}^\mathbb{K}$ is an isomorphism in \mathcal{E} replace \mathbb{C} with $\mathbb{K} \times \dot{X}$ in (3) where X is some representable object of \mathcal{E} and \dot{X} is the discrete category on X . \square

2.4 Infinitesimally Closed Arrows

In ?? we show that the infinitesimal part of a category is stable under a certain completion operation that adds macroscopic arrows. In this section we prove the primordial version of this result involving spaces rather than categories.

Notation 2.25. We use \mathcal{E} to denote an arbitrary topos. We use the notation $a \in_X A$ to denote an arrow $a : X \rightarrow A$.

Definition 2.26. An arrow $\beta : A \rightarrow B$ in \mathcal{E} is *infinitesimally closed* iff β is a monomorphism and the proposition

$$\forall b \in B. [\neg\neg(\exists a \in A. \beta(a) = b) \implies \exists a_0 \in A. \beta(a_0) = b] \quad (4)$$

holds in the internal logic of \mathcal{E} .

Lemma 2.27. If β is infinitesimally closed and

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & C \\ \downarrow \beta & & \downarrow v \\ B & \xrightarrow{w} & Q \end{array} \quad (5)$$

is a pushout then v is infinitesimally closed.

Proof. The arrow v is a monomorphism and the whole square is a pullback because \mathcal{E} is a topos. Now suppose that $q \in Q$ and $\neg\neg(\exists c \in C. v(c) = q)$. Then

$$\begin{aligned}
& (q \in Q) \wedge \neg\neg(\exists c \in C. v(c) = q) \\
& \implies [(\exists b \in B. w(b) = q) \vee (\exists c \in C. v(c) = q)] \wedge \neg\neg(\exists c \in C. v(c) = q) \\
& \implies (\exists b \in B. w(b) = q) \wedge \neg\neg(\exists c \in C. v(c) = q) \\
& \implies \neg\neg(\exists b \in B, \exists c \in C. (w(b) = q) \wedge (v(c) = q)) \wedge (\exists b \in B. w(b) = q) \\
& \implies \neg\neg(\exists a \in A. \beta(a) = b) \wedge (\exists b \in B. w(b) = q) \\
& \implies (\exists a \in A. \beta(a) = b) \wedge (\exists b \in B. w(b) = q) \\
& \implies \exists c' \in C. v(c') = q
\end{aligned}$$

where the first implication is an elementary property of pushouts in a topos (see for instance [8]) and the penultimate implication is due to the fact that β is infinitesimally closed. \square

3 Integration and Order in a Topos

3.1 An Order Relation in the Dubuc Topos

Recall from Chapter 1 page 18 in [1] the definition of intuitionistic order relation.

Definition 3.1. A relation $<$ is an (*intuitionistic*) *order relation* iff $<$ satisfies the following propositions:

1. $(a < b) \wedge (b < c) \implies a < c$
2. $\neg(a < a)$
3. $(0 < a) \vee (a < 1)$
4. $\neg(a = b) \implies (a < b) \vee (b < a)$

The following definition is equation 2 on page 107 in Section III.1 of [12].

Definition 3.2. If $a, b : [n, I] \rightarrow R = [1, (0)]$ in the Dubuc topos then $a < b$ iff $a(\vec{x}) < b(\vec{x})$ for all $\vec{x} \in Z(I) \subset \mathbb{R}^n$.

Recall from (ii) and (iv) in Theorem 1 on page 316 in Section VI.7 of [11] the following characterisations of the join and negation operations in the sheaf semantics of a Grothendieck topos.

Theorem 3.3. If X is an object of a Grothendieck topos \mathcal{E} , C is a representable object of \mathcal{E} , ϕ and ψ are formulas and $\alpha \in X(C)$ then:

- $\phi(\alpha) \vee \psi(\alpha)$ iff there is a covering family $\{f_i : C_i \rightarrow C\}$ such that for each i either $\phi(\alpha f_i)$ or $\psi(\alpha f_i)$;

- $\neg\phi(\alpha)$ iff for all $D \rightarrow C$ if $\phi(\alpha f)$ then the empty family is a cover of D .

Remark 3.4. If X is an object of the Dubuc topos and the empty family covers X then $X \cong 0$.

In order to prove that $<$ defined in Definition 3.2 is an intuitionistic order relation we first record the following simple result.

Lemma 3.5. *If $a, b : \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth functions and $\vec{x} \in \mathbb{R}^n$ such that $a(\vec{x}) < b(\vec{x})$ then there exists an open set $U \subset \mathbb{R}^n$ containing \vec{x} such that $a(\vec{y}) < b(\vec{y})$ for all $\vec{y} \in U$.*

Lemma 3.6. *The relation $<$ defined in Definition 3.2 is an intuitionistic order relation.*

Proof. Conditions 1. and 2. are immediate from the definition of $<$. Conditions 3. and 4. follow by combining Theorem 3.3 and Lemma 3.5. \square

Lemma 3.7. *The equivalence*

$$\neg\neg(a = b) \iff \neg(a < b) \wedge \neg(b < a)$$

holds in the internal logic of the Dubuc topos.

Proof. The backward implication \Leftarrow is the contrapositive of condition 4. in Definition 3.1. For the forward implication \Rightarrow :

$$\begin{aligned} (a = b) &\implies \neg(a < b) \wedge \neg(b < a) \\ &\implies \neg((a < b) \vee (b < a)) \end{aligned}$$

and so

$$\begin{aligned} \neg\neg(a = b) &\implies \neg\neg\neg((a < b) \vee (b < a)) \\ &\implies \neg((a < b) \vee (b < a)) \\ &\implies \neg(a < b) \wedge \neg(b < a) \end{aligned}$$

as required. \square

3.2 The Fundamental Category on the Unit Interval

Notation 3.8. We use \mathcal{E} to denote the Dubuc topos defined in Definition 2.10. We use \mathbf{I} to denote the image of the unit interval under the full and faithful embedding $\iota : \mathbf{Man} \rightarrow \mathcal{E}$ described in Lemma 2.11. Recall that this means that \mathbf{I} is isomorphic to the object $[1, I_{[0,1]}]$ where $I_{[0,1]}$ is the ideal of all smooth functions vanishing on the classical unit interval $[0, 1]$. Let $<$ be the relation defined in Definition 3.2.

Definition 3.9. The *fundamental category* \mathbb{I} on the unit interval \mathbf{I} is the category internal to \mathcal{E} that has underlying reflexive graph

$$\mathbb{I}^2 := \{(a, b) \in \mathbf{I}^2 : \neg(b < a)\} \xrightleftharpoons[\pi_1]{\pi_0} \mathbf{I}$$

and the only possible composition.

Lemma 3.10. *If \mathbb{I} is the fundamental category on the unit interval then the inclusion $(\iota_{\mathbb{I}}^\infty)^2 : \mathbb{I}_\infty^2 \rightarrow \mathbb{I}^2$ is infinitesimally closed.*

Proof. First

$$\begin{aligned} \mathbb{I}_\infty^2 &= \{(a, b) \in \mathbf{I}^2 : \neg(b < a) \wedge \neg\neg((a, a) = (a, b))\} \\ &= \{(a, b) \in \mathbf{I}^2 : \neg\neg(a = b)\} \end{aligned}$$

by Definition 2.18 and Lemma 3.7. So if $(a, b) \in \mathbb{I}^2$ such that $\neg\neg((a, a) = (a, b))$ then $(a, b) \in \mathbb{I}_\infty^2$ as required. \square

Lemma 3.11. *The inclusion $\iota_{\mathbb{I}}^\infty : \mathbb{I}_\infty \rightarrow \mathbb{I}$ is closed under decomposition.*

Proof. If $(a, b), (b, c) \in \text{hom}(\mathbf{2}, \mathbb{I})$ such that $(a, c) \in \text{hom}(\mathbf{2}, \mathbb{I}_\infty)$ then

$$\begin{aligned} \neg(b < a) \wedge \neg(c < b) \wedge \neg\neg(a = c) &\implies \neg\neg(\neg(b < a) \wedge \neg(c < b) \wedge (a = c)) \\ &\implies \neg\neg(\neg(b < a) \wedge \neg(a < b)) \\ &\implies \neg\neg(\neg\neg(a = b)) \\ &\implies \neg\neg(a = b) \\ &\implies \neg\neg(c = b) \end{aligned}$$

where the third implication uses Lemma 3.7. \square

3.3 The Integral Completion of a Category

In this section we record two lemmas (Lemma 4.3) and Lemma 4.2) to help us prove our main result Theorem 4.6.

Notation 3.12. We use \mathcal{C} to denote the site of the Dubuc topos and \mathcal{E} to denote the Dubuc topos itself as defined in Definition 2.10.

Definition 3.13. If \mathbb{C} is a category internal to \mathcal{E} then the *integral completion* \mathbb{C}_{int} of \mathbb{C} is defined by the pushout

$$\begin{array}{ccc} \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{C} \\ \downarrow \mathbb{C}^{\mathbb{I}_\infty} \times \iota_{\mathbb{I}}^\infty & & \downarrow \tau_{\mathbb{C}} \\ \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} & \xrightarrow{\alpha_{\mathbb{C}}} & \mathbb{C}_{int} \end{array}$$

in $\text{Cat}(\mathcal{E})$. It is easy to see that this function extends to an endofunctor $(-)_{int}$ on $\text{Cat}(\mathcal{E})$ such that if $f : \mathbb{C} \rightarrow \mathbb{D}$ in $\text{Cat}(\mathcal{E})$ then $f_{int}\tau_{\mathbb{C}} = \tau_{\mathbb{D}}f$ and $f_{int}\alpha_{\mathbb{C}} = \alpha_{\mathbb{D}}(f^{\mathbb{I}_\infty} \times \mathbb{I})$.

4 Pushout Closed Under Decomposition in the Dubuc Topos

In order to transfer the results of [?] to the Dubuc topos we require the following result which is Theorem 1 from Section III.5 of [11]:

Theorem 4.1. *The inclusion functor $j : \mathcal{E} \hookrightarrow [\mathcal{C}^{op}, Set]$ has a left adjoint*

$$a : [\mathcal{C}^{op}, Set] \rightarrow \mathcal{E}$$

called the associated sheaf functor. Moreover, this functor a commutes with finite limits.

Lemma 4.2. *If \mathbb{K} is an infinitesimal category in \mathcal{E} and*

$$\begin{array}{ccc} hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \downarrow 1 \times \iota & & \downarrow \tau \\ hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{P} \end{array} \quad (6)$$

is a pushout in $Cat(\mathcal{E})$ then τ is closed under decomposition.

Proof. It is immediate from Lemma 3.11 that $1 \times \iota$ is closed under decomposition. Therefore $j(1 \times \iota)$ is closed under decomposition because j preserves limits. If

$$\begin{array}{ccc} j(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty) & \longrightarrow & j\mathbb{K} \\ \downarrow & & \downarrow \nu \\ j(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}) & \longrightarrow & \mathbb{Q} \end{array}$$

is a pushout in $[\mathcal{C}^{op}, Cat]$ then ν is closed under decomposition by Lemma 4.2. But then

$$\begin{array}{ccc} aj(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty) & \longrightarrow & aj\mathbb{K} \\ \downarrow & & \downarrow a\nu \\ aj(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}) & \longrightarrow & a\mathbb{Q} \end{array} \quad (7)$$

is a pushout in $Cat(\mathcal{E})$ because a preserves colimits and hence the square (7) is isomorphic to the square (6). Moreover $a\nu \cong \tau$ is closed under decomposition because a preserves finite limits. \square

4.1 Lie's Third Theorem

In this section we prove the main result of the paper (Theorem 4.6) which will be used to deduce the appropriate version of Lie's third theorem.

Lemma 4.3. *If*

$$\begin{array}{ccc}
\text{hom}(\mathbf{2} \times \mathbb{I}_\infty, \mathbb{K}) \times \text{hom}(\mathbf{2}, \mathbb{I}_\infty) & \xrightarrow{ev} & \text{hom}(\mathbf{2}, \mathbb{K}) \\
\downarrow 1 \times \iota & & \downarrow v \\
\text{hom}(\mathbf{2} \times \mathbb{I}_\infty, \mathbb{K}) \times \text{hom}(\mathbf{2}, \mathbb{I}) & \xrightarrow{w} & Q
\end{array} \tag{8}$$

is a pushout in \mathcal{E} then v_K is infinitesimally closed.

Proof. Combine Lemma 3.10 with Lemma 2.27. \square

Theorem 4.6 follows quickly from Lemma 4.3 and Lemma 4.2 once we have established the legitimacy of a certain de Morgan-type implication in intuitionistic logic.

Remark 4.4. Recall that in intuitionistic logic the implications

$$\neg(A \wedge B) \implies \neg A \vee \neg B$$

and

$$\neg(\forall i \in I. P_i) \implies \exists i \in I. \neg P_i$$

are not in general valid. Nevertheless for any list P_1, P_2, \dots, P_n of propositions the implication

$$\neg \neg \left(\bigwedge_{i=1}^n P_i \right) \implies \bigwedge_{i=1}^n \neg \neg P_i$$

is always valid and we will use it to prove our main theorem. Johnstone Stone Spaves.

Lemma 4.5. *For any finite list of propositions P_1, P_2, \dots, P_n the implication*

$$\neg \neg \left(\bigwedge_{i=1}^n P_i \right) \implies \bigwedge_{i=1}^n \neg \neg P_i$$

holds in intuitionistic logic.

Proof. Now

$$\begin{aligned}
\neg \neg \left(\bigwedge_{i=1}^n P_i \right) &\implies \bigwedge_{j=1}^n \neg \neg \left(\bigwedge_{i=1}^n P_i \right) \\
&\implies \bigwedge_j \neg \neg P_j
\end{aligned}$$

because $\bigwedge_{i=1}^n P_i \implies P_j$ for any $j \in \{1, \dots, n\}$. \square

Theorem 4.6. *If \mathbb{K} is an infinitesimal category and*

$$\begin{array}{ccc} \text{hom}(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \downarrow 1 \times \iota & & \downarrow \tau \\ \text{hom}(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{P} \end{array}$$

is a pushout in $\text{Cat}(\mathcal{E})$ then the arrow $\tau_\infty : \mathbb{K}_\infty \rightarrow \mathbb{P}_\infty$ is an isomorphism.

Proof. Let v and Q be as in Lemma 4.3. Let $p \in P$ such that $\neg\neg(esp = p)$ and $p = L_0 \circ \dots \circ L_n$ for $L_i \in Q$. First $\neg\neg(\forall i \in \{1, \dots, n\}. L_i \in v(K))$ because $esp \in v(K)$ and (by Lemma 4.2) τ is closed under decomposition. Therefore $\forall i \in \{1, \dots, n\}. \neg\neg(L_i \in v(K))$ by Lemma 4.5. Finally $\forall i \in \{1, \dots, n\}. L_i \in v(K)$ because v is infinitesimally closed and therefore $p \in \tau(K)$ as required. \square

Corollary 4.7. *(Lie's Third Theorem) The natural transformation*

$$1 \Rightarrow (-)_\infty \circ (-)_{int} : \text{Cat}_\infty(\mathcal{E}) \rightarrow \text{Cat}(\mathcal{E})$$

induced by τ_∞ is an isomorphism.

5 The Lie Adjunction

5.1 Integral Complete Categories

Notation 5.1. Let \mathbb{C} be a category internal to \mathcal{E} with object space M and source, target and identity maps s, t and e respectively. Let $\iota_\mathbb{C}^\infty : \mathbb{C}_\infty \rightarrow \mathbb{C}$ be the inclusion of the infinitesimal part as defined in Lemma 2.17.

Definition 5.2. A category \mathbb{C} internal to \mathcal{E} is *integral complete* iff the arrow

$$\mathbb{C}^\mathbb{I} \xrightarrow{\mathbb{C}^{\iota_\mathbb{I}^\infty}} \mathbb{C}^{\mathbb{I}_\infty}$$

is a split epimorphism in $\text{Cat}(\mathcal{E})$.

Notation 5.3. We define \mathbb{C}_{int} , τ and α using the pushout

$$\begin{array}{ccc} \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{C} \\ \downarrow 1 \times \iota_\mathbb{I}^\infty & & \downarrow \tau \\ \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{C}_{int} \end{array}$$

in $\text{Cat}(\mathcal{E})$. We use $(-)_{int}$ to denote the function that maps an internal category \mathbb{C} in \mathcal{E} to the internal category \mathbb{C}_{int} in \mathcal{E} .

Lemma 5.4. *The arrow $\mathbb{C}_{int}^{\iota_\mathbb{I}^\infty} : \mathbb{C}_{int}^\mathbb{I} \rightarrow \mathbb{C}_{int}^{\mathbb{I}_\infty}$ is a split epimorphism. In other words, if \mathbb{C} is a category internal to \mathcal{E} then \mathbb{C}_{int} is integral complete.*

Proof. Recall that

$$\begin{array}{ccc} \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{C} \\ \downarrow \mathbb{C}^{\mathbb{I}_\infty} \times \iota_{\mathbb{I}}^\infty & & \downarrow \tau_{\mathbb{C}} \\ \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} & \xrightarrow{\alpha_{\mathbb{C}}} & \mathbb{C}_{int} \end{array} \quad (9)$$

is a pushout by definition of \mathbb{C}_{int} . Let $\hat{\alpha}_{\mathbb{C}} :$ be the arrow induced by $\alpha_{\mathbb{C}}$ under the hom-tensor adjunction. If $\phi \in \mathbb{C}^{\mathbb{I}_\infty}$ and $k \in \mathbb{C}_\infty$ then

$$\begin{aligned} \left(\mathbb{C}_{int}^{\iota_{\mathbb{I}}^\infty} (\hat{\alpha}_{\mathbb{C}}(\phi)) \right) (k) &= \hat{\alpha}_{\mathbb{C}}(\phi)(\iota_{\mathbb{I}}^\infty(k)) \\ &= \alpha_{\mathbb{C}}(\phi, \iota_{\mathbb{I}}^\infty(k)) \\ &= \tau_{\mathbb{C}}(ev(\phi, k)) \\ &= \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}(\phi) \right) (k) \end{aligned}$$

Now by ?? the arrow $\tau_{\mathbb{C}}^{\mathbb{I}_\infty}$ is invertible and so $\hat{\alpha}_{\mathbb{C}} \circ \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty} \right)^{-1}$ splits $\mathbb{C}_{int}^{\iota_{\mathbb{I}}^\infty}$. \square

5.2 Integration is a Reflection

Lemma 5.5. *If \mathbb{X} is integral complete then $\tau_{\mathbb{X}}$ is a split monomorphism.*

Proof. Since \mathbb{X} is integral complete there is a $\beta : \mathbb{X}^{\mathbb{I}_\infty} \rightarrow \mathbb{X}^{\mathbb{I}}$ such that $(\mathbb{X}^{\iota_{\mathbb{I}}^\infty}) \circ \beta = 1_{\mathbb{X}^{\mathbb{I}_\infty}}$. Then in

$$\begin{array}{ccccc} \mathbb{X}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{X} & & \\ \downarrow 1_{\mathbb{X}^{\mathbb{I}_\infty}} \times \iota_{\mathbb{I}}^\infty & & \downarrow \tau_{\mathbb{X}} & \searrow 1_{\mathbb{X}} & \\ \mathbb{X}^{\mathbb{I}_\infty} \times \mathbb{I} & \xrightarrow{\alpha_{\mathbb{X}}} & \mathbb{X}_{int} & & \\ \downarrow \beta \times 1_{\mathbb{I}} & & \downarrow p_{\mathbb{X}} & \searrow & \\ \mathbb{X}^{\mathbb{I}} \times \mathbb{I} & \xrightarrow{ev} & \mathbb{X} & & \end{array}$$

the top left square is the pushout defining \mathbb{X}_{int} , $\alpha_{\mathbb{X}}$ and $\tau_{\mathbb{X}}$ and the outer square commutes because

$$\begin{aligned} ev(\beta \times \mathbb{I}(1 \times \iota_{\mathbb{I}}^\infty(\phi, k))) &= ev(\beta \times \mathbb{I}(\phi, \iota_{\mathbb{I}}^\infty(k))) \\ &= ev(\beta(\phi), \iota_{\mathbb{I}}^\infty(k)) \\ &= \beta(\phi)(\iota_{\mathbb{I}}^\infty(k)) \\ &= \mathbb{X}^{\iota_{\mathbb{I}}^\infty}(\beta(\phi))(k) \\ &= \phi(k) \end{aligned}$$

for $k \in \mathbb{I}_\infty$ and $\phi \in \mathbb{X}^{\mathbb{I}_\infty}$. Then the induced arrow $p_{\mathbb{X}} : \mathbb{X}_{int} \rightarrow \mathbb{X}$ splits $\tau_{\mathbb{X}}$. \square

Lemma 5.6. *The arrow $(\tau_{\mathbb{C}})_{int}$ is a monomorphism split by $p_{\mathbb{C}_{int}} : (\mathbb{C}_{int})_{int} \rightarrow \mathbb{C}_{int}$.*

Proof. By Lemma 5.4 the arrow $\beta = \hat{\alpha}_{\mathbb{C}} \circ (\tau_{\mathbb{C}}^{\mathbb{I}_{\infty}})^{-1}$ splits the epimorphism $\mathbb{C}_{int}^{\mathbb{I}_{\infty}} : \mathbb{C}_{int}^{\mathbb{I}} \rightarrow \mathbb{C}_{int}^{\mathbb{I}_{\infty}}$. In

$$\begin{array}{ccccc}
\mathbb{C}_{int}^{\mathbb{I}_{\infty}} \times \mathbb{I}_{\infty} & \xrightarrow{ev} & \mathbb{C}_{int} & & \\
\downarrow \tau_{\mathbb{C}}^{\mathbb{I}_{\infty}} \times 1_{\mathbb{I}_{\infty}} & \swarrow & \uparrow \tau_{\mathbb{C}} & & \downarrow \tau_{\mathbb{C}_{int}} \\
& \mathbb{C}^{\mathbb{I}_{\infty}} \times \mathbb{I}_{\infty} \xrightarrow{ev} \mathbb{C} & & & \\
& \downarrow 1_{\mathbb{C}^{\mathbb{I}_{\infty}}} \times \iota_{\mathbb{I}^{\infty}} & \downarrow \tau_{\mathbb{C}} & & \\
& \mathbb{C}^{\mathbb{I}_{\infty}} \times \mathbb{I} \xrightarrow{\alpha_{\mathbb{C}}} \mathbb{C}_{int} & & & \\
& \swarrow \tau_{\mathbb{C}}^{\mathbb{I}_{\infty}} \times 1_{\mathbb{I}} & \searrow (\tau_{\mathbb{C}})_{int} & & \\
\mathbb{C}_{int}^{\mathbb{I}_{\infty}} \times \mathbb{I} & \xrightarrow{\alpha_{\mathbb{C}_{int}}} & (\mathbb{C}_{int})_{int} & & \\
\downarrow \beta \times \mathbb{I} & & \downarrow p_{\mathbb{C}_{int}} & & \downarrow 1_{\mathbb{C}_{int}} \\
\mathbb{C}_{int}^{\mathbb{I}} \times \mathbb{I} & \xrightarrow{ev} & \mathbb{C}_{int} & &
\end{array}$$

the inner square is the pushout defining \mathbb{C}_{int} , the middle square is the pushout defining $(\mathbb{C}_{int})_{int}$ and the rest of the diagram commutes by an argument similar to Lemma 5.4. To prove that $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} = 1_{\mathbb{C}_{int}}$ it will suffice to show that $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} \circ \tau_{\mathbb{C}} = \tau_{\mathbb{C}}$ (which is immediate by construction) and $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} \circ \alpha_{\mathbb{C}} = \alpha_{\mathbb{C}}$ which we deduce from the following equalities:

$$\begin{aligned}
p_{\mathbb{C}_{int}}((\tau_{\mathbb{C}})_{int}(\alpha_{\mathbb{C}}(\phi, k))) &= ev(\beta \times \mathbb{I}(\tau_{\mathbb{C}}^{\mathbb{I}_{\infty}} \times 1_{\mathbb{I}}))(\phi, k) \\
&= ev(\beta \times \mathbb{I}(\tau_{\mathbb{C}^{\mathbb{I}_{\infty}}}(\phi), k)) \\
&= ev(\beta(\tau_{\mathbb{C}}^{\mathbb{I}_{\infty}}(\phi)), k) \\
&= \beta(\tau_{\mathbb{C}}^{\mathbb{I}_{\infty}}(\phi))(k) \\
&= \left(\hat{\alpha}_{\mathbb{C}} \circ (\tau_{\mathbb{C}}^{\mathbb{I}_{\infty}})^{-1} \circ \tau_{\mathbb{C}}^{\mathbb{I}_{\infty}} \right)(\phi)(k) \\
&= \hat{\alpha}_{\mathbb{C}}(\phi)(k) \\
&= \alpha_{\mathbb{C}}(\phi, k)
\end{aligned}$$

where $\phi \in \mathbb{C}^{\mathbb{I}_{\infty}}$ and $k \in \mathbb{I}$. □

Corollary 5.7. *The functor $(-)_{int}$ is left adjoint to the inclusion $k : Cat_{int}(\mathcal{E}) \rightarrow Cat(\mathcal{E})$.*

Proof. If \mathbb{C} is an arbitrary category internal to \mathcal{E} then define the unit of the adjunction is $\tau_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}_{int}$. If \mathbb{X} is an integral complete category internal to \mathcal{E} then define the counit of the adjunction is $p_{\mathbb{X}} : \mathbb{X}_{int} \rightarrow \mathbb{X}$. Then the triangle equalities are given by Lemma 5.5 and Lemma 5.6. □

5.3 Lie's Second Theorem

By composing the reflection in Corollary 5.7 with the coreflection in Lemma 2.24 we obtain the following adjunction:

$$\begin{array}{ccccc}
 Cat_\infty(\mathcal{E}) & \xrightarrow{i} & Cat(\mathcal{E}) & \xrightarrow{(-)_{int}} & Cat_{int}(\mathcal{E}) \\
 & \perp & & \perp & \\
 & \xleftarrow{(-)_\infty} & & \xleftarrow{j} &
 \end{array}$$

which generalises the classical Lie correspondence between Lie algebras and Lie groups. The appropriate formulation of Lie's second theorem in this context is that the functor $(-)_\infty \circ j$ is full and faithful when we restrict to the simply connected categories as defined in Definition 5.12. We prove this result as Theorem 5.15. To prove this we refer to results in [3] where an abstract version of Lie's second theorem is presented. In this section we give a brief sketch of the ideas in [3] and then indicate how to apply the results there to prove Theorem 5.15.

To understand the ideas in [3] it is helpful to first look at a special case involving spaces rather than groupoids or categories.

Notation 5.8. Let X and Y be topological spaces. Let $P_*(X)$ denote the space of paths in X starting at $*$ in X . Let I denote the unit interval. Let Man denote the category of smooth manifolds.

Remark 5.9. In general if we have a map $f : X \rightarrow Y$ we can construct a map $\hat{f} : P_*(X) \rightarrow Y$ by precomposing with the map $X^1 \circ \iota : P_*(X) \rightarrow X^I \rightarrow X$. However given a map $g : P_*(X) \rightarrow Y$ we may not always be able to find a map $\check{g} : X \rightarrow Y$ such that $g = \check{g} \circ X^1 \circ \iota$. Firstly for a fixed $x \in X$ there may not exist a $\gamma \in P_*(X)$ such that $\gamma(1) = x$. Secondly for a fixed $x \in X$ not all paths $\gamma, \delta : * \rightarrow x$ will necessarily have the same image under g .

Lemma 5.10. *If X is simply connected and $g : P_*(X) \rightarrow Y$ such that $g(\gamma) = g(\delta)$ for all $\gamma, \delta \in P_*(X)$ that are homotopic with fixed endpoints then there exists a unique $\check{g} : X \rightarrow Y$ such that $g = \check{g} \circ X^1 \circ \iota$.*

Proof. Since X is path connected for all $x \in X$ there exists $\gamma_x \in P_*(X)$ such that $\gamma_x(1) = x$. Now further let $\gamma, \delta \in P_*(X)$ such that $\gamma(1) = \delta(1)$. Since X is simply connected γ and δ are homotopic and hence $g(\gamma) = g(\delta)$. So we can define $\check{g}(x) = g(\gamma_x)$. \square

To help us generalise Lemma 5.10 we introduce the following definitions.

Notation 5.11. Let \mathbb{I} be the fundamental category on the unit interval defined in Definition 3.9. Let \mathcal{E} be the Dubuc topos defined in Definition 2.10.

Definition 5.12. A category \mathbb{C} in \mathcal{E} is \mathcal{E} -path connected iff

$$hom(\mathbb{I}, \mathbb{C}) \xrightarrow{hom((0,1), \mathbb{C})} hom(\mathbf{2}, \mathbb{C})$$

is an epimorphism in \mathcal{E} . A category \mathbb{C} in \mathcal{E} is \mathcal{E} -*simply connected* iff it is \mathcal{E} -path connected and

$$\text{hom}(\mathbb{I}^2, \mathbb{C}) \xrightarrow{\text{hom}(\iota, \mathbb{C})} \text{hom}(\partial\mathbb{I}^2, \mathbb{C})$$

is an epimorphism in \mathcal{E} where $\partial\mathbb{I}^2$ is the full subcategory of \mathbb{I}^2 on the boundary of I^2 and $\iota : \partial\mathbb{I}^2 \rightarrow \mathbb{I}^2$ is the natural inclusion.

Definition 5.13. Recall that the 2-truncated cube category \square_2 is the subcategory of Man generated by the following arrows:

$$\begin{array}{ccc} \leftarrow (1_I, 0) \longrightarrow & & \\ \xrightarrow{\pi_1} & & \\ I^2 \leftarrow (1_I, 1) \longrightarrow & I \xrightleftharpoons[0]{1} 1 & \\ \xrightarrow{\pi_2} & & \\ \leftarrow (1, 1_I) \longrightarrow & & \end{array} \quad (10)$$

where I is the unit interval. Recall that if \mathcal{E} is a category then the category $c_2\mathcal{E}$ of 2-truncated cubical objects in a category \mathcal{E} is the functor category $[\square_2^{op}, \mathcal{E}]$. The arrows of $c_2\mathcal{E}$ will be called 2-cubical maps. We refer to [6] for the theory of cubical objects.

Proposition 5.14. Let \mathbb{C} and \mathbb{X} be categories where \mathbb{C} is simply connected and

$$\begin{array}{ccccc} \mathbb{C}^{\mathbb{I}^2} & \xrightleftharpoons{\quad} & \mathbb{C}^{\mathbb{I}} & \xrightleftharpoons{\quad} & \mathbb{C}^1 \\ \downarrow \Psi_2 & & \downarrow \Psi_1 & & \downarrow \Psi_0 \\ \mathbb{X}^{\mathbb{I}^2} & \xrightleftharpoons{\quad} & \mathbb{X}^{\mathbb{I}} & \xrightleftharpoons{\quad} & \mathbb{X}^1 \end{array}$$

is a 2-cubical map. Then there is a functor $\psi : \mathbb{C} \rightarrow \mathbb{X}$ with object map $\psi_0 = \Psi_0$ and arrow map ψ_1 satisfying $\psi_1 \mathbb{C}^I = \mathbb{X}^I \Psi_1$.

Proof. Proposition 7.4 in [3]. \square

Theorem 5.15 (Lie's Second Theorem). *If \mathbb{C} is a simply connected category in \mathcal{E} such that \mathbb{C}_∞ is path connected, \mathbb{X} is an integral complete category and $\phi : \mathbb{C}_\infty \rightarrow \mathbb{X}$ is a functor in $\text{Cat}(\mathcal{E})$ then there exists a lift $\psi : \mathbb{C} \rightarrow \mathbb{X}$ such that $\psi \circ \iota = \phi$.*

Proof. We use Proposition 5.14. For the map on 0-cells we take $\Phi_0 = \phi$. Next we construct the map Φ_1 on 1-cells. Since \mathbb{X} is integral complete the arrow $\text{hom}(\iota, \mathbb{X}) : \text{hom}(\mathbb{I}, \mathbb{X}) \rightarrow \text{hom}(\mathbb{I}_\infty, \mathbb{X})$ has a section ζ . By the second half of Lemma 2.24 the map $\text{hom}(\mathbb{I}_\infty, \iota) : \text{hom}(\mathbb{I}_\infty, \mathbb{C}_\infty) \rightarrow \text{hom}(\mathbb{I}_\infty, \mathbb{C})$ is invertible. Hence we can define Φ_1 to be the following map on 1-cells:

$$\begin{aligned} \text{hom}(\mathbb{I}, \mathbb{C}) &\xrightarrow{\text{hom}(\iota, \mathbb{C})} \text{hom}(\mathbb{I}_\infty, \mathbb{C}) \xrightarrow{\text{hom}(\mathbb{I}_\infty, \iota)^{-1}} \text{hom}(\mathbb{I}_\infty, \mathbb{C}_\infty) \\ &\xrightarrow{\text{hom}(\mathbb{I}_\infty, \phi)} \text{hom}(\mathbb{I}_\infty, \mathbb{X}) \xrightarrow{\zeta} \text{hom}(\mathbb{I}, \mathbb{X}) \end{aligned}$$

Finally we construct the map Φ_2 on 2-cells. Since \mathbb{X} is integral complete the map $\text{hom}(\iota \times \iota, \mathbb{X}) = \text{hom}(\mathbb{I}_\infty \times \iota, \mathbb{X}) \times \text{hom}(\iota \times \mathbb{I}, \mathbb{X})$ has section $\text{hom}(\mathbb{I}, \zeta) \circ \text{hom}(\mathbb{I}_\infty, \zeta)$

which we call η . By the second half of Lemma 2.24 the map $hom(\mathbb{I}_\infty^2, \iota) : hom(\mathbb{I}_\infty^2, \mathbb{C}_\infty) \rightarrow hom(\mathbb{I}_\infty^2, \mathbb{C})$ is invertible. Hence we define Φ_2 to be the following map on 2-cells:

$$\begin{aligned} hom(\mathbb{I}^2, \mathbb{C}) &\xrightarrow{hom(\iota, \mathbb{C})} hom(\mathbb{I}_\infty^2, \mathbb{C}) \xrightarrow{hom(\mathbb{I}_\infty^2, \iota)^{-1}} hom(\mathbb{I}_\infty^2, \mathbb{C}_\infty) \\ &\xrightarrow{hom(\mathbb{I}_\infty^2, \phi)} hom(\mathbb{I}_\infty^2, \mathbb{X}) \xrightarrow{\eta} hom(\mathbb{I}^2, \mathbb{X}) \end{aligned}$$

It is easy to see that Φ_0 , Φ_1 and Φ_2 satisfy the conditions that define a morphism of 2-cubical sets. \square

5.3.1 Lie's Third Theorem

Recall the main result of this paper that we proved as Theorem 4.6.

If \mathbb{K} is an infinitesimal category and

$$\begin{array}{ccc} hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \downarrow 1 \times \iota & & \downarrow \tau \\ hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} & \xrightarrow{\alpha} & \mathbb{P} \end{array}$$

is a pushout in $Cat(\mathcal{E})$ then the arrow $\tau_\infty : \mathbb{K}_\infty \rightarrow \mathbb{P}_\infty$ is an isomorphism.

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