Lie Theory using an Intuitionistic Double Negation

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1 Introduction

In this paper we formulate and prove a version of Lie's third theorem in the language of categories and functors. Our main tool is the theory of synthetic differential geometry that provides a rigorous definition of infinitesimal objects, infinitesimal actions and infinitesimal transformations.

Recall that in classical Lie theory we have the following adjunction:

$$\begin{tikzcd} LLGGerm \arrow[r, bend left, "(-)_{int}"] \arrow[r, phantom, "\perp"] & LieGrp \arrow[l, bend left, "(-)_\infty"] \end{tikzcd} \qquad (1)$$

where *LieGrp* is the category of Lie groups and *LLGGerm* is the category of local lie group germs as defined in Definition 1.2. In this classical situation it doesn't matter if we use linear, analytic or local approximations: the categories *LieAlg* of Lie algebras, *FGLaw* of formal group laws and *LLGGerm* are all equivalent. Furthermore if one restricts to the category of simply connected Lie groups then (1) becomes an equivalence also.

In the reformulation of Lie theory presented in this paper the differences between the linear, analytic and local approximations become significant. This reformulation is closely related to those presented in [2], [3] and [4]. However the present paper differs in two respects. Firstly we prove Lie's third theorem rather than Lie's second theorem. Secondly we use an intuitionistic double negation operation to create local approximations whereas in [2] and [4] an analytic approximation involving nilpotent elements is used. Nevertheless we still use the abstract results in [3] to prove Lie's second theorem in the Dubuc topos using the infinitesimal approximation in Section 5.3. The use of the larger approximation arising from the double negation operation is a necesary one in order to prove Lie's third theorem at this level of abstraction and arises naturally from certain counterexamples that occur when using the nilpotent approximation. (For more detail please see Section 1.2.)

In this paper we make a twofold generalisation: we replace groups with categories and the category Man of smooth manifolds with a certain well adapted model of synthetic differential geometry called the Dubuc topos (see Definition 2.10). Our version of the Lie adjunction then breaks down into the following composite of a coreflection and a reflection:

$$\begin{tikzcd} Cat_\infty(\mathcal{E}) \arrow[r, tail, "i"] \arrow[r, phantom, "\perp"] & Cat(\mathcal{E}) \arrow[l, bend left, "(-)_\infty"] \arrow[r, bend left, "(-)_{int}"] \arrow[r, phantom, "\perp"] & Cat_{int}(\mathcal{E}) \arrow[l, tail, "j"] \end{tikzcd} \qquad (2)$$

1.1 Local Lie Group Germs

Definition 1.1. A *local Lie group* consists of open sets $U_0, U \subset \mathbb{R}^n$ containing $\vec{0} \in \mathbb{R}^n$, a smooth map $\mu : U \times U \to \mathbb{R}^n$ and a smooth map $i : U_0 \to \mathbb{R}^n$ such that if $X, Y, Z \in U$ then:

- $\mu(X, 0) = X = \mu(0, X)$

- $X \in U_0 \implies \mu(X, i(X)) = \mu(i(X), X) = 0$
- $\mu(X, Y), \mu(Y, Z) \in U \implies \mu(X, \mu(Y, Z)) = \mu(\mu(X, Y), Z)$

Definition 1.2. A *germ of a local Lie group* is an equivalence class of local Lie groups where $G \sim H$ iff there exists a open neighbourhoods $V_0 \subset V$ of $\vec{0} \in \mathbb{R}^n$ such that $\mu_G|V = \mu_H|V$ and $i_G|V_0 = i_H|V_0$.

1.2 Various Attempts at Lie's Third Theorem

Classical Lie algebroids.

Tseng and Zhu with Differentiable Stacks.

Groupoids.

Categories and nilpotents.

2 Infinitesimals in a Topos

2.1 The Dubuc Topos

In this paper we will work in a particular well-adapted model of synthetic differential geometry called the Dubuc topos. In this section we sketch its basic properties and refer to [5] for more details. In future work it would be interesting to see how much of the theory presented in this paper can be adapted to the theory of synthetic differential topology whose primary model is the Dubuc topos.

Notation 2.1. We use $C^\infty(\mathbb{R}^n, \mathbb{R})$ to denote the ring of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. If I is an ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$ then we use $Z(I)$ to denote the zero set $\{\vec{x} \in \mathbb{R}^n : \forall i \in I.\ i(\vec{x}) = 0\}$. We use x_i to denote the ith coordinate projection $\mathbb{R}^n \to \mathbb{R}$. We use Man to denote the category of smooth, paracompact and Hausdorff manifolds.

Definition 2.2. The *category $\mathcal{C}$ of smooth affine schemes* has as objects pairs $[n, I]$ where $n \in \mathbb{N}_{\geq 0}$ and I is an ideal in $C^\infty(\mathbb{R}^n, \mathbb{R})$. The arrows $[n, I] \to [m, J]$ in $\mathcal{C}$ are equivalence classes of smooth functions $f : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $j \in J$ the arrow $jf = 0 \mod I$ and $f \sim g$ iff $f \equiv g \mod I$.

Lemma 2.3. *There is a full and faithful embedding of Man into $\mathcal{C}$.*

Proof. Theorem 6.15 of [7] states that every smooth n-dimensional manifold M with or without boundary admits a proper smooth embedding $\iota_M : M \to \mathbb{R}^{2n+1}$. Section 1.2 of [2] shows that it is possible to choose the embeddings ι_M coherently so that the map $M \mapsto [2n + 1, I_M]$ is a full and faithful embedding $Man \to \mathcal{C}$ where I_M is the ideal of functionals vanishing on $\iota_M(M)$. $\square$

Definition 2.4. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are smooth functions and $\vec{x} \in \mathbb{R}^n$ then *f and g have the same germ at $\vec{x}$* iff there exists an open subset U containing $\vec{x}$ such that $f|_U = g|_U$.

Remark 2.5. Let $[n, I]$ be an object of $\mathcal{C}$ and $i_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that for all $\vec{x} \in Z(I)$ there exists $i_{\vec{x}} \in I$ with the same germ as i_0 at $\vec{x}$. Then it is immediate from Definition 2.2 that the presheaf $\mathcal{C}(-, [n, I])$ is isomorphic to the presheaf $\mathcal{C}(-, [n, I + (i_0)])$. By contrast the presheaf $\mathcal{C}([n, I], -)$ can be different to $\mathcal{C}([n, I + (i_0)], -)$. (To see this consider the functions $i_0 : [n, I] \to [1, (0)]$ and $i_0 : [n, I + (i_0)] \to [1, (0)]$ when $i_0 \notin I$.)

Definition 2.6. If I is an ideal of $C^\infty(\mathbb{R}^n, \mathbb{R})$ then I is *germ-determined* iff for all $i_0 : \mathbb{R}^n \to \mathbb{R}$ such that for all $\vec{x} \in Z(I)$ there exists $i_{\vec{x}} \in I$ with the same germ as i_0 then in fact $i_0 \in I$ also.

Remark 2.7. The ideals I_M defined in Lemma 2.3 are germ-determined.

Notation 2.8. We write $\mathcal{C}_{germ}$ for the full subcategory of $\mathcal{C}$ on the objects $[n, I]$ for which I is germ-determined.

Definition 2.9. If $\chi : \mathbb{R}^n \to \mathbb{R}$ then the *Dubuc open set* U_χ *associated to* χ is the arrow $(x_1, x_2, ..., x_n) : [n + 1, (\chi(x_1, x_2, ..., x_n)x_{n+1} - 1)] \to [n, (0)]$ in $\mathcal{C}$. A *Dubuc open set* is any arrow of the form U_χ or any pullback of an arrow of the form U_χ.

Definition 2.10. The *Dubuc coverage* is the Grothendieck coverage on $\mathcal{C}$ generated by the Dubuc open sets. The *Dubuc topos* $\mathcal{E}$ is the Grothendieck topos generated by the site $(\mathcal{C}_{germ}, J_{germ})$ where J_{germ} is the restriction of the Dubuc coverage on $\mathcal{C}$ to $\mathcal{C}_{germ}$.

Lemma 2.11. *There is a full and faithful embedding ι of the category Man of smooth, paracompact, Hausdorff manifolds into the Dubuc topos $\mathcal{E}$.*

Proof. Recall from [5] and Lemma 1.3 in III.1 of [9] that the site generating the Dubuc topos is subcanonical. Then the result follows from Remark 2.7. □

2.2 The Infinitesimal Part of a Category

In this section we define the local approximation to a category that we will use throughout this paper. The central feature of this definition is the use of an intuitionistic double negation operator.

Notation 2.12. We use $\mathcal{E}$ to denote an arbitrary topos. We use $\mathbb{C}$ to denote a category internal to $\mathcal{E}$ with object space M and arrow space C. We denote by s, t and e respectively the source, target and identity maps of $\mathbb{C}$.

Following Penon in [10] we make the following definition:

Definition 2.13. An arrow $c \in \mathbb{C}^{\mathbf{2}}$ is *infinitesimal* iff $\neg\neg(esc = c)$.

Remark 2.14. If c is an infinitesimal arrow then $\neg\neg(sc = tc)$.

Remark 2.15. An arrow c is infinitesimal iff $\neg\neg(etc = c)$. Also an arrow c is infinitesimal iff $\exists m \in M.\ \neg\neg(em = c)$.

Notation 2.16. We use $\iota^{\infty}_{\mathbb{C}} : \mathbb{C}^{\mathbf{2}}_{\infty} \to \mathbb{C}^{\mathbf{2}}$ to denote the subobject

$$\{c \in \mathbb{C}^{\mathbf{2}} : \neg\neg(esc = c)\} \to \mathbb{C}^{\mathbf{2}}$$

of infinitesimal arrows in $\mathbb{C}$. It is immediate that $e : M \to \mathbb{C}^{\mathbf{2}}$ factors through $\mathbb{C}^{\mathbf{2}}_{\infty}$ and we use e_{∞} to denote the unique arrow such that $\iota^{\infty}_{\mathbb{C}} e_{\infty} = e$. Similarly we write $s_{\infty} = s\iota^{\infty}_{\mathbb{C}}$ and $t_{\infty} = t\iota^{\infty}_{\mathbb{C}}$.

Lemma 2.17. *The reflexive graph*

$$\begin{tikzcd} \mathbb{C}^{\mathbf{2}}_{\infty} \arrow[r, shift left=2, "s_{\infty}"] \arrow[r, shift right=2, "t_{\infty}"'] & M \arrow[l, "e_{\infty}" description] \end{tikzcd}$$

is a category when equipped with the multiplication μ_{∞} that is the restriction of the multiplication μ of $\mathbb{C}$.

Proof. Let $c, d \in \mathbb{C}^{\mathbf{2}}_{\infty}$ such that $tc = sd$. By Definition 2.13 $\neg\neg(esc = c)$ and so $\neg\neg(es(dc) = c)$. If $d_0 \in \mathbb{C}$ then in general $(d = d_0) \implies \mu(c, d) = \mu(c, d_0)$ and so in particular $\neg\neg(esd = d) \implies \neg\neg\,(\mu(esd, c) = \mu(d, c))$. Therefore $\neg\neg(es(\mu(d, c)) = \mu(d, c))$ as required. $\square$

Definition 2.18. If $\mathbb{C}$ is a category internal to $\mathcal{E}$ then the *infinitesimal part* $\mathbb{C}_{\infty}$ *of* $\mathbb{C}$ is the subcategory of $\iota_{\mathbb{C}} : \mathbb{C}_{\infty} \to \mathbb{C}$ defined in Lemma 2.17 and Notation 2.16.

2.3 The Infinitesimal Part is a Coreflection

In this section we extend the infinitesimal part construction of Definition 2.18 to an endofunctor on the category $Cat(\mathcal{E})$ of internal categories in a topos $\mathcal{E}$. Furthermore we show that this endofunctor defines a special type of coreflective subcategory of $Cat(\mathcal{E})$.

Notation 2.19. We use $\mathcal{E}$ to denote an arbitrary topos. We use $\mathbb{C}$ to denote a category internal to $\mathcal{E}$ with object space M and arrow space C. We denote by s, t and e respectively the source, target and identity maps of $\mathbb{C}$. We use $\mathbb{C}_{\infty}$ to denote the infinitesimal part of $\mathbb{C}$ constructed in Definition 2.18.

Lemma 2.20. *The function $\mathbb{C} \mapsto \mathbb{C}_{\infty}$ extends to an endofunctor on $Cat(\mathcal{E})$.*

Proof. Let $f : \mathbb{C} \to \mathbb{D}$ be an internal functor between internal categories in $\mathcal{E}$. If $c \in C$ and $\neg\neg(esc = c)$ then $\neg\neg(f(esc) = fc)$ and so $\neg\neg(es(fc) = fc)$. Therefore there is a unique internal functor $f_{\infty} : \mathbb{C}_{\infty} \to \mathbb{D}_{\infty}$ such that

$$\begin{tikzcd} \mathbb{C}_{\infty} \arrow[r, dashed, "f_{\infty}"] \arrow[d, "\iota^{\infty}_{\mathbb{C}}"] & \mathbb{D}_{\infty} \arrow[d, "\iota^{\infty}_{\mathbb{D}}"] \\ \mathbb{C} \arrow[r, "f"] & \mathbb{D} \end{tikzcd} \tag{3}$$

commutes. $\square$

Definition 2.21. A category $\mathbb{K}$ internal to $\mathcal{E}$ is *infinitesimal* iff all of its arrows are infinitesimal.

Remark 2.22. It is immediate that the infinitesimal part $\mathbb{C}_\infty$ of a category $\mathbb{C}$ is an infinitesimal category.

Notation 2.23. We use $Cat_\infty(\mathcal{E})$ to denote the full subcategory of $Cat(\mathcal{E})$ on the infinitesimal categories as defined in Definition 2.21.

Lemma 2.24. *The subcategory $j : Cat_\infty(\mathcal{E}) \to Cat(\mathcal{E})$ is a coreflective subcategory with coreflection $(-)_\infty : Cat(\mathcal{E}) \to Cat_\infty(\mathcal{E})$. Furthermore if $\mathbb{K} \in Cat_\infty(\mathcal{E})$ and $\mathbb{D} \in Cat(\mathcal{E})$ then $(\iota^\infty_{\mathbb{D}})^{\mathbb{K}} : \mathbb{D}^{\mathbb{K}}_\infty \to \mathbb{D}^{\mathbb{K}}$ is an isomorphism in $\mathcal{E}$.*

Proof. The commutative square (3) shows that ι^∞ is a natural transformation $j \circ (-)_\infty \Rightarrow 1_{Cat(\mathcal{E})}$ where j is the inclusion $Cat_\infty(\mathcal{E}) \to Cat(\mathcal{E})$. Also it is immediate from the definition of $(-)_\infty$ that $(-)_\infty \circ j = 1_{Cat_\infty(\mathcal{E})}$. To see that $(\iota^\infty_{\mathbb{D}})^{\mathbb{K}} : \mathbb{D}^{\mathbb{K}}_\infty \to \mathbb{D}^{\mathbb{K}}$ is an isomorphism in $\mathcal{E}$ replace $\mathbb{C}$ with $\mathbb{K} \times \dot{X}$ in (3) where X is some representable object of $\mathcal{E}$ and $\dot{X}$ is the discrete category on X. $\square$

2.4 Infinitesimally Closed Arrows

In **??** we show that the infinitesimal part of a category is stable under a certain completion operation that adds macroscopic arrows. In this section we prove the primordial version of this result involving spaces rather than categories.

Notation 2.25. We use $\mathcal{E}$ to denote an arbitrary topos. We use the notation $a \in_X A$ to denote an arrow $a : X \to A$.

Definition 2.26. An arrow $\beta : A \to B$ in $\mathcal{E}$ is *infinitesimally closed* iff β is a monomorphism and the proposition

$$\forall b \in B.\ [\neg\neg\,(\exists a \in A.\ \beta(a) = b) \implies \exists a_0 \in A.\ \beta(a_0) = b] \tag{4}$$

holds in the internal logic of $\mathcal{E}$.

Lemma 2.27. *If β is infinitesimally closed and*

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & C \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \iota_C} \\ B & \xrightarrow{\iota_B} & P \end{array} \tag{5}$$

is a pushout then ι_C is infinitesimally closed.

Proof. The arrow ι_C is a monomorphism and the whole square is a pullback because $\mathcal{E}$ is a topos. Now let X be a representable object and suppose $p \in_X P$ and that (4) holds at stage of definition X. Choose a cover $(\iota_i : U_i \to X)_{i \in I}$ such that for each $i \in I$ the arrow $p_i = p\iota_i$ factors through either B or C. (Note that the principle of the excluded middle holds for the specific proposition described by the previous sentence.) We know that $c_i = c\iota_i$ factors through C and we will show that for all $i \in I$ the arrow p_i factors through C.

So suppose that p_i factors through B. Then there exists $b_i \in_{U_i} B$ such that $\iota_B(b_i) = p_i$. Now we have the following implications:

$$\begin{aligned}
&\exists b_i \in_{U_i} B.\ (\iota_B(b_i) = p_i) \wedge \neg\neg(\exists c_i \in_{U_i} C.\ \iota_C(c_i) = p_i) \\
&\implies \exists b_i \in_{U_i} B.\ (\iota_B(b_i) = p_i) \wedge \neg\neg(\exists a_i \in_{U_i} A.\ \beta(a_i) = b_i) \\
&\implies \exists b_i \in_{U_i} B.\ (\iota_B(b_i) = p_i) \wedge (\exists a_i \in_{U_i} A.\ \beta(a_i) = b_i)
\end{aligned}$$

where the first implication uses that (5) is a pullback and the second implication uses that β is infinitesimally closed. But now $p_i = \iota_B(\beta(a_i)) = \iota_C(\gamma(a_i))$ as required. $\square$

3 Integration and Order in a Topos

3.1 An Order Relation in the Dubuc Topos

Recall from Chapter 1 page 18 in [1] the definition of intuitionistic order relation.

Definition 3.1. A relation $<$ is an *(intuitionistic) order relation* iff $<$ satisfies the following propositions:

1. $(a < b) \wedge (b < c) \implies a < c$
2. $\neg(a < a)$
3. $(0 < a) \vee (a < 1)$
4. $\neg(a = b) \implies (a < b) \vee (b < a)$

The following definition is equation 2 on page 107 in Section III.1 of [9].

Definition 3.2. If $a, b : [n, I] \to R = [1, (0)]$ in the Dubuc topos then $a < b$ iff $a(\vec{x}) < b(\vec{x})$ for all $\vec{x} \in Z(I) \subset \mathbb{R}^n$.

Recall from (ii) and (iv) in Theorem 1 on page 316 in Section VI.7 of [8] the following characterisations of the join and negation operations in the sheaf semantics of a Grothendieck topos.

Theorem 3.3. *If X is an object of a Grothendieck topos $\mathcal{E}$, C is a representable object of $\mathcal{E}$, ϕ and ψ are formulas and $\alpha \in X(C)$ then:*

- *$\phi(\alpha) \vee \psi(\alpha)$ iff there is a covering family $\{f_i : C_i \to C\}$ such that for each i either $\phi(\alpha f_i)$ or $\psi(\alpha f_i)$;*
- *$\neg\phi(\alpha)$ iff for all $D \to C$ if $\phi(\alpha f)$ then the empty family is a cover of D.*

Remark 3.4. If X is an object of the Dubuc topos and the empty family covers X then $X \cong 0$.

In order to prove that $<$ defined in Definition 3.2 is an intuitionistic order relation we first record the following simple result.

Lemma 3.5. *If $a, b : \mathbb{R}^n \to \mathbb{R}$ are smooth functions and $\vec{x} \in \mathbb{R}^n$ such that $a(\vec{x}) < b(\vec{x})$ then there exists an open set $U \subset \mathbb{R}^n$ containing $\vec{x}$ such that $a(\vec{y}) < b(\vec{y})$ for all $\vec{y} \in U$.*

Lemma 3.6. *The relation $<$ defined in Definition 3.2 is an intuitionistic order relation.*

Proof. Conditions 1. and 2. are immediate from the definition of $<$. Conditions 3. and 4. follow by combining Theorem 3.3 and Lemma 3.5. □

Lemma 3.7. *The equivalence*

$$\neg\neg(a = b) \iff \neg(a < b) \wedge \neg(b < a)$$

holds in the internal logic of the Dubuc topos.

Proof. The backward implication $\Longleftarrow$ is the contrapositive of condition 4. in Definition 3.1. For the forward implication $\Longrightarrow$:

$$\begin{aligned}(a = b) &\implies \neg(a < b) \wedge \neg(b < a)\\ &\implies \neg((a < b) \vee (b < a))\end{aligned}$$

and so

$$\begin{aligned}\neg\neg(a = b) &\implies \neg\neg\neg((a < b) \vee (b < a))\\ &\implies \neg((a < b) \vee (b < a))\\ &\implies \neg(a < b) \wedge \neg(b < a)\end{aligned}$$

as required. □

3.2 The Fundamental Category on the Unit Interval

Notation 3.8. We use $\mathcal{E}$ to denote the Dubuc topos defined in Definition 2.10. We use $\mathbf{I}$ to denote the image of the unit interval under the full and faithful embedding $\iota : Man \to \mathcal{E}$ described in Lemma 2.11. Recall that this means that $\mathbf{I}$ is isomorphic to the object $[1, I_{[0,1]}]$ where $I_{[0,1]}$ is the ideal of all smooth functions vanishing on the classical unit interval $[0, 1]$. Let $<$ be the relation defined in Definition 3.2.

Definition 3.9. The *fundamental category* $\mathbb{I}$ *on the unit interval* $\mathbf{I}$ is the category internal to $\mathcal{E}$ that has underlying reflexive graph

$$\mathbb{I}^{\mathbf{2}} := \{(a, b) \in \mathbf{I}^2 : \neg(b < a)\} \overset{\xrightarrow{\pi_0}}{\underset{\xrightarrow[\pi_1]{}}{\xleftarrow{\Delta}}} \mathbf{I}$$

and the only possible composition.

Lemma 3.10. *If $\mathbb{I}$ is the fundamental category on the unit interval then the inclusion $(\iota_{\mathbb{I}}^{\infty})^{\mathbf{2}} : \mathbb{I}_{\infty}^{\mathbf{2}} \to \mathbb{I}^{\mathbf{2}}$ is infinitesimally closed.*

Proof. First

$$\begin{aligned}\mathbb{I}_{\infty}^{\mathbf{2}} &= \{(a,b) \in \mathbf{I}^2 : \neg(b < a) \wedge \neg\neg((a,a) = (a,b))\} \\ &= \{(a,b) \in \mathbf{I}^2 : \neg\neg(a = b)\}\end{aligned}$$

by Definition 2.18 and Lemma 3.7. So if $(a,b) \in \mathbb{I}^{\mathbf{2}}$ such that $\neg\neg((a,a) = (a,b))$ then $(a,b) \in \mathbb{I}_{\infty}^{\mathbf{2}}$ as required. $\square$

Lemma 3.11. *The inclusion $\iota_{\mathbb{I}}^{\infty} : \mathbb{I}_{\infty} \to \mathbb{I}$ is closed under decomposition.*

Proof. If $(a,b),(b,c) \in hom(\mathbf{2},\mathbb{I})$ such that $(a,c) \in hom(\mathbf{2},\mathbb{I}_{\infty})$ then

$$\begin{aligned}\neg(b < a) \wedge \neg(c < b) \wedge \neg\neg(a = c) &\implies \neg\neg(\neg(b < a) \wedge \neg(c < b) \wedge (a = c)) \\ &\implies \neg\neg(\neg(b < a) \wedge \neg(a < b)) \\ &\implies \neg\neg(\neg\neg(a = b)) \\ &\implies \neg\neg(a = b) \\ &\implies \neg\neg(c = b)\end{aligned}$$

where the third implication uses Lemma 3.7. $\square$

3.3 The Integral Completion of a Category

In this section we record two lemmas (Lemma 4.20) and Lemma 4.19) to help us prove our main result Theorem 4.23.

Notation 3.12. We use $\mathcal{C}$ to denote the site of the Dubuc topos and $\mathcal{E}$ to denote the Dubuc topos itself as defined in Definition 2.10.

Definition 3.13. If $\mathbb{C}$ is a category internal to $\mathcal{E}$ then *the integral completion* $\mathbb{C}_{int}$ *of* $\mathbb{C}$ is defined by the pushout

$$\begin{tikzcd} \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "\mathbb{C}^{\mathbb{I}_\infty} \times \iota_{\mathbb{I}}^{\infty}"] & \mathbb{C} \arrow[d, "\tau_{\mathbb{C}}"] \\ \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[r, "\alpha_{\mathbb{C}}"] & \mathbb{C}_{int} \end{tikzcd}$$

in $Cat(\mathcal{E})$. It is easy to see that this function extends to an endofunctor $(-)_{int}$ on $Cat(\mathcal{E})$ such that if $f : \mathbb{C} \to \mathbb{D}$ in $Cat(\mathcal{E})$ then $f_{int}\tau_{\mathbb{C}} = \tau_{\mathbb{D}} f$ and $f_{int}\alpha_{\mathbb{C}} = \alpha_{\mathbb{D}}(f^{\mathbb{I}_\infty} \times \mathbb{I})$.

4 Generalities on Closure Under Decomposition

Notation 4.1. We use $\mathbf{2}$ to denote the category defined by the preorder on two elements 0 and 1 with $0 < 1$ and $\mathbf{3}$ for the category defined by the preorder on three elements 0, 1 and 2 with $0 < 1 < 2$. We use $l : \mathbf{2} \to \mathbf{3}$ to denote the unique functor that takes $0 \mapsto 0$ and $1 \mapsto 2$.

Definition 4.2. If $\beta : \mathbb{A} \to \mathbb{B}$ is an $\mathcal{E}$-functor between $\mathcal{E}$-categories then β is *closed under decomposition* iff

$$
\begin{array}{ccc}
\mathbb{A}^{\mathbf{3}} & \xrightarrow{\mathbb{A}^l} & \mathbb{A}^{\mathbf{2}} \\
\Big\downarrow{\scriptstyle \beta^{\mathbf{3}}} & & \Big\downarrow{\scriptstyle \beta^{\mathbf{2}}} \\
\mathbb{B}^{\mathbf{3}} & \xrightarrow{\mathbb{B}^l} & \mathbb{B}^{\mathbf{2}}
\end{array}
$$

is a pullback.

4.1 The Category of Reduced Words

Notation 4.3. In this section $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are ordinary categories (i.e categories internal to Set) with arrow sets A, B and C and object sets N, N and M respectively. We write s_A, s_B and s_C for the source maps and t_A, t_B and t_C for the target maps of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ respectively. The arrow $\beta : \mathbb{A} \to \mathbb{B}$ is a functor that is the identity on objects and is closed under decomposition in Set as defined in Definition 4.2. The arrow $\gamma : \mathbb{A} \to \mathbb{C}$ is an arbitrary functor.

Notation 4.4. Since β is the identity on objects we can define an amalgamated source map

$$
\begin{aligned}
B \sqcup C &\xrightarrow{s} M \\
L &\mapsto \begin{cases} s_C(c) \text{ if } c \in C \\ \gamma(s_B(b)) \text{ if } b \in B \end{cases}
\end{aligned}
$$

and similarly an amalgamated target map

$$
\begin{aligned}
B \sqcup C &\xrightarrow{t} M \\
L &\mapsto \begin{cases} t_C(c) \text{ if } c \in C \\ \gamma(t_B(b)) \text{ if } b \in B \end{cases}
\end{aligned}
$$

and heuristically think of elements of $B \setminus \beta(A)$ as 'extra' arrows between pairs of elements of M.

Definition 4.5. of deductive system apparently in section I.1 of J. Lambek, P. J. Scott, Introduction to higher order categorical logic.

Definition 4.6. The *deductive system* $\overline{\mathbb{W}}(\mathbb{B},\mathbb{C})$ *of words in* $\mathbb{B}$ *and* $\mathbb{C}$ has as objects the set M and as arrows the set of sequences $(L_0, L_1, ..., L_n)$ of elements $L_i \in C \sqcup (B \setminus \beta(A))$ that satisfy

$$\begin{aligned} s(L_{i+1}) &= t(L_i) \\ L_i \in C &\implies L_{i+1} \in B \setminus \beta(A) \\ L_i, L_{i+1} \in B \setminus \beta(A) &\implies s_B(L_{i+1}) \neq t_B(L_i) \end{aligned}$$

for $i \in \{0, 1, ..., n-1\}$ and

$$L_i \in C \implies L_{i-1} \in B \setminus \beta(A)$$

for $i \in \{1, ..., n\}$ where s and t are the amalgamated source and target maps defined in Notation 4.4. The source map s_W of $\mathbb{W}$ is given by $s_W(L_0, L_1, .., L_n) = s(L_n)$, the target map t_W of $\mathbb{W}$ is given by $t_W(L_0, L_1, ..., L_n) = t(L_0)$ and the identity map of $\mathbb{W}$ is given by $e_W(m) = (e_C(m))$ where e_C is the identity map of $\mathbb{C}$. The composition $\circ_W$ in $\mathbb{W}$ is given by

$$(L_0, L_1, ..., L_n) \circ_W (L'_0, L'_1, ..., L'_m) =$$
$$\begin{cases} (L_0, L_1, ..., L_n \circ_B L'_0, L'_1, ..., L'_m) \text{ if } L_n, L'_0 \in B \setminus \beta(A) \text{ and } s_B(L_n) = t_B(L'_0) \\ (L_0, L_1, ..., L_n \circ_C L'_0, L'_1, ..., L'_m) \text{ if } L_n, L'_0 \in C \\ (L_0, L_1, ..., L_n, L'_0, L'_1, ..., L'_m) \text{ otherwise.} \end{cases}$$

where $\circ_B$ and $\circ_C$ denote the compositions in $\mathbb{B}$ and $\mathbb{C}$ respectively. Note that in the first of the cases $L_n \circ_B L'_0 \in B \setminus \beta(A)$ because β is closed under decomposition and that the associativity of $\circ_{\mathbb{W}}$ is inherited from the associativity of $\circ_B$, $\circ_C$ and the concatenation operation.

Definition 4.7. The *deductive system* $\mathbb{W} = \mathbb{W}(\mathbb{B},\mathbb{C})$ *of reduced words in* $\mathbb{B}$ *and* $\mathbb{C}$ has as objects the set M and as arrows the set

$$\overline{\mathbb{W}}(\mathbb{B},\mathbb{C})^{\mathbf{2}} / \sim$$

of equivalence classes of arrows in $\overline{\mathbb{W}}(\mathbb{B},\mathbb{C})$ where the equivalence relation $\sim$ is generated by equivalences of the form:

$$\begin{aligned} (L_0, ..., L_i, e_C(m), L_{i+2}, ..., L_n) &\sim (L_0, ..., L_i \circ_B L_{i+2}, ..., L_n) \text{ if } s_B(L_i) = t_B(L_{i+2}) \\ (L_0, ..., L_i, e_C(m), L_{i+2}, ..., L_n) &\sim (L_0, ..., L_i, L_{i+2}, ..., L_n) \text{ if } s_B(L_i) \neq t_B(L_{i+2}) \\ (L_0, ..., L_i, e_C(m)) &\sim (L_0, ..., L_i) \\ (e_C(m), L_{i+2}, ..., L_n) &\sim (L_{i+2}, ..., L_n) \end{aligned}$$

where $L_i, L_{i+2} \in B \setminus \beta(A)$ by construction.

Lemma 4.8. *The deductive system* $\mathbb{W}$ *is a category.*

Proof. We need to check that the identity axioms hold. If $L_i \in C$ then

$$(L_0, ..., L_i) \circ_W (e_C(m)) = (L_0, ..., L_i \circ_C e_C(m)) = (L_0, ..., L_i)$$

and if $L_i \in B \setminus \beta(A)$ then

$$(L_0, ..., L_i) \circ_W (e_C(m)) \sim (L_0, ..., L_i)$$

by definition of $\sim$ in Definition 4.7. If $L_0 \in C$ then

$$(e_C(m)) \circ_W (L_0, ..., L_n) = (e_C(m) \circ_C L_0, ..., L_n) = (L_0, ..., L_n)$$

and if $L_0 \in B \setminus \beta(A)$ then

$$(e_C(m)) \circ_W (L_0, ..., L_n) \sim (L_0, ..., L_n)$$

by definition of $\sim$ in Definition 4.7. □

Definition 4.9. The *category* $\mathbb{P}$ *of words in* $\mathbb{B}$ *and* $\mathbb{C}$ *reduced via* β *and* γ has as objects the set M and as arrows the set

$$\mathbb{W}(\mathbb{B}, \mathbb{C})^{\mathbf{2}} / \approx$$

of equivalence classes of arrows in $\mathbb{W}(\mathbb{B}, \mathbb{C})$ where the equivalence relation $\approx$ is generated by equivalences of the following form. If

$$L_i = L_i' \nu(a_0) \text{ and } L_{i+1} = \eta(a_1) L_{i+1}'$$

for some $a_0, a_1 \in A$, $L_1, L_{i+1} \in C \sqcup (B \setminus \beta(A))$ and $\nu \neq \eta \in \{\beta, \gamma\}$ then

$$(L_0, ..., L_i' \nu(a_0 a_1), L_{i+1}', ..., L_n) \approx (L_0, ..., L_i, L_{i+1}, ..., L_n)$$

and

$$(L_0, ..., L_i, L_{i+1}, ..., L_n) \approx (L_0, ..., L_i', \eta(a_0 a_1) L_{i+1}', ..., L_n)$$

which is well defined because $B \setminus \beta(A)$ is closed under decomposition.

Remark 4.10. The equivalence relation $\approx$ does not identify sequences of the different lengths.

4.2 Solving the Word Problem

Notation 4.11. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, β and γ be as in Notation 4.3 and s and t as in Notation 4.4. Let $\mathbb{P}$ be the category constructed in Definition 4.9.

Definition 4.12. The functor $\iota_C : \mathbb{C} \to \mathbb{P}$ is defined by $c \mapsto (c)$. We check that

$$c' \circ_C c \mapsto (c' \circ_C c) = (c') \circ_W (c)$$

and

$$e_C(m) \mapsto (e_C(m)) = e_W(m)$$

as required.

Definition 4.13. The functor $\iota_B : \mathbb{B} \to \mathbb{P}$ is defined by

$$b \mapsto \begin{cases} (b) \text{ if } b \in B \setminus \beta(A) \\ (\gamma(a)) \text{ if } \exists a \in A.\ b = \beta(a) \end{cases}$$

we check the following equivalences:

- If $b' = \beta(a')$ and $b = \beta(a)$ then $b' \circ_B b = \beta(a' \circ_A a)$. Therefore

$$\iota_B(b' \circ_B b) = (\gamma(a' \circ_A a)) = (\gamma(a') \circ_C \gamma(a)) = (\gamma(a')) \circ_W (\gamma(a)).$$

- If $b' \in B \setminus \beta(A)$ and $b = \beta(a)$ then $b' \circ_B b \in B \setminus \beta(A)$ because β is closed under decomposition. Therefore

$$\begin{aligned}\iota_B(b' \circ_B b) = (b' \circ_B b) = (b' \circ_B \beta(a)) &\sim (b' \circ_B \beta(a), e_C(s(b))) \\ &\approx (b', \gamma(a) \circ_C e_C(s(b)) = (b', \gamma(a)) = (b') \circ_W (\gamma(a)).\end{aligned}$$

- If $b' = \beta(a)$ and $b \in B \setminus \beta(A)$ then $b' \circ_B b \in B \setminus \beta(A)$ because β is closed under decomposition. Therefore

$$\begin{aligned}\iota_B(b' \circ_B b) = (b' \circ_B b) = (\beta(a') \circ_B b) &\sim (e_C(t(b')), \beta(a') \circ_B b) \\ &\approx (e_C(t(b')) \circ_C \gamma(a'), b) = (\gamma(a'), b) = (\gamma(a')) \circ_W (b).\end{aligned}$$

- If $b', b \in B \setminus \beta(A)$ then

$$\iota_B(b' \circ_B b) = (b' \circ_B b) = (b') \circ_W (b).$$

- Finally

$$\iota_B(e_B(n)) = \iota_B(\beta(e_A(n)) = (\gamma(e_A(n))) = (e_C(\gamma(n))) = e_W(\gamma(n)).$$

Lemma 4.14. *The square*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\gamma"] \arrow[d, "\beta"] & \mathbb{C} \arrow[d, "\iota_C"] \\ \mathbb{B} \arrow[r, "\iota_B"] & \mathbb{P} \end{tikzcd}$$

is a pushout in Cat.

Proof. Let $x : \mathbb{B} \to \mathbb{X}$ and $y : \mathbb{C} \to \mathbb{X}$ be functors such that $x\beta = y\gamma$. Then we define a functor $z : \mathbb{P} \to \mathbb{X}$ by

$$(L_0, L_1, ..., L_n) \mapsto \theta_0(L_0) \circ_X \theta_1(L_1) \circ_X ... \circ_X \theta_n(L_n)$$

where $\theta_i = x$ if $L_i \in B \setminus \beta(A)$ and $\theta_i = y$ if $L_i \in C$. First $z\iota_B = x$ because

$$z\iota_B(b) = \begin{cases} x(b) \text{ if } b \in B \setminus \beta(A) \\ y(\gamma(a)) = x(\beta(a)) = x(b) \text{ if } \exists a \in A.\ \beta(a) = b \end{cases}$$

and $z\iota_C = y$ is immediate by construction. Next we check that z respects the equivalence relation $\sim$.

$$\begin{aligned}
&z(L_0, ..., L_i, e_C(m), L_{i+1}, ..., L_N) = \\
&\theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i) \circ_X y(e_C(m)) \circ_X v_{i+2}(L_{i+2}) \circ_X ... \circ_X \theta_n(L_n) = \\
&\theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i) \circ_X v_{i+2}(L_{i+2}) \circ_X ... \circ_X \theta_n(L_n) = \\
&\begin{cases} z(L_0, .., L_i \circ_B L_{i+2}, .., L_n) \text{ if } s_B(L_i) = t_B(L_{i+2}) \\ z(L_0, .., L_i, L_{i+2}, .., L_n) \text{ if } s_B(L_i) \neq t_B(L_{i+2}) \end{cases}
\end{aligned}$$

where $L_i, L_{i+2} \in B \setminus \beta(A)$ by construction. Also

$$\begin{aligned}
z(L_0, ..., L_i, e_C(m)) &= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i) \circ_X y(e_C(m)) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i) \\
&= z(L_0, ..., L_n)
\end{aligned}$$

and

$$\begin{aligned}
z(e_C(m), L_{i+2}, ..., L_n) &= y(e_C(m)) \circ_X \theta_{i+2}(L_{i+2}) \circ_X ... \circ_X \theta_n(L_n) \\
&= \theta_{i+2}(L_{i+2}) \circ_X ... \circ_X \theta_n(L_n) \\
&= z(L_{i+2}, ..., L_n)
\end{aligned}$$

In addition z respects the equivalence relation $\approx$. Indeed if $L_i = L_i'\nu(a_0)$ and $L_{i+1} = \eta(a_1)L_{i+1}'$ for some $a_0, a_1 \in A$ and $\eta \neq \nu \in \{\beta, \gamma\}$ then

$$\begin{aligned}
&z(L_0, ..., L_i'\nu(a_0a_1), L_{i+1}', ..., L_n) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i'\nu(a_0a_1)) \circ_X \nu_{i+1}(L_{i+1}') \circ_X ... \circ_X \theta_n(L_n) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i') \circ_X \theta_i(\nu(a_0)) \circ_X \theta_i(\nu(a_1)) \circ_X \theta_{i+1}(L_{i+1}') \circ_X ... \circ_X \theta_n(L_n) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i') \circ_X \theta_i(\nu(a_0)) \circ_X \theta_{i+1}(\eta(a_1)) \circ_X \theta_{i+1}(L_{i+1}') \circ_X ... \circ_X \theta_n(L_n) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_i(L_i'\nu(a_0)) \circ_X \theta_{i+1}(\eta(a_1)L_{i+1}') \circ_X ... \circ_X \theta_n(L_n) \\
&= z(L_0, ..., L_i, L_{i+1}, ..., L_n)
\end{aligned}$$

and similarly $z(L_0, ..., L_i, L_{i+1}, ..., L_n) = z(L_0, ..., L_i', \eta(a_0a_1)L_{i+1}', ..., L_n)$. Now we show that z is the unique map $\mathbb{P} \to \mathbb{X}$ such that $z\iota_B = x$ and $z\iota_c = y$. So suppose that there were another map $w : \mathbb{P} \to \mathbb{X}$ such that $w\iota_B = x$ and $w\iota_C = y$. Then

$$\begin{aligned}
w(L_0, ..., L_n) &= w(L_0) \circ_X ... \circ_X w(L_n) \\
&= \theta_0(L_0) \circ_X ... \circ_X \theta_n(L_n) \\
&= z(L_0, ..., L_n)
\end{aligned}$$

as required. $\square$

4.3 Pushout is Closed Under Decomposition

Notation 4.15. We use the symbol $\simeq$ to denote the equivalence relation generated by both $\sim$ and $\approx$. As usual whenever we write $L = (L_0, L_1, ..., L_n)$

we assume that L is an arrow of the deductive system $\overline{\mathbb{W}}(\mathbb{B},\mathbb{C})$ defined in Definition 4.6. A, B, C etc..

Lemma 4.16. *If $c \in C$ and $(L_0, L_1, ..., L_n) \simeq (c)$ then $\forall i \in \{0, ..., n\}$. $L_i \in C$.*

Proof. First note that $\approx$ does not affect the number of L_i that are in $B \setminus \beta(A)$. So we only need to consider the case that $(L_0, L_1, ..., L_n) \sim (c)$. Now consider the four generating relations

$$
\begin{aligned}
(L_0, ..., L_i, e_C(m), L_{i+2}, ..., L_n) &\sim (L_0, ..., L_i \circ_B L_{i+2}, ..., L_n) \text{ if } s_B(L_i) = t_B(L_{i+2})\\
(L_0, ..., L_i, e_C(m), L_{i+2}, ..., L_n) &\sim (L_0, ..., L_i, L_{i+2}, ..., L_n) \text{ if } s_B(L_i) \neq t_B(L_{i+2})\\
(L_0, ..., L_i, e_C(m)) &\sim (L_0, ..., L_i)\\
(e_C(m), L_{i+2}, ..., L_n) &\sim (L_{i+2}, ..., L_n)
\end{aligned}
$$

for $\sim$ given in Definition 4.7. In the first two cases there is at least one $L_i \in B \setminus \beta(A)$ on both sides. In the final two cases there is either one $L_i \in B \setminus \beta(A)$ on each side on none on either side. The result follows immediately. $\square$

Proposition 4.17. *If β is closed under decomposition and is the identity on objects and*

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \mathbb{C} \arrow[d, "\iota_C"] \\
\mathbb{B} \arrow[r, "\iota_B"] & \mathbb{P}
\end{tikzcd}
$$

is a pushout then ι_C is closed under decomposition.

Proof. We need to prove that if $L = (L_0, L_1, ..., L_n)$ and $L' = (L'_0, ..., L'_n)$ such that $L \circ_P L' \simeq (c'')$ for some $c'' \in C$ then $L \simeq (c)$ and $L' \simeq (c')$ for some $c, c'' \in C$. Now consider the definition of $\circ_P$ given in Definition 4.6:

$$
(L_0, L_1, ..., L_n) \circ_W (L'_0, L'_1, ..., L'_m) =
$$
$$
\begin{cases}
(L_0, L_1, ..., L_n \circ_B L'_0, L'_1, ..., L'_m) \text{ if } L_n, L'_0 \in B \setminus \beta(A) \text{ and } s_B(L_n) = t_B(L'_0)\\
(L_0, L_1, ..., L_n \circ_C L'_0, L'_1, ..., L'_m) \text{ if } L_n, L'_0 \in C\\
(L_0, L_1, ..., L_n, L'_0, L'_1, ..., L'_m) \text{ otherwise.}
\end{cases}
$$

and note that by Lemma 4.16 we must be in the second case. Indeed if either of the first or third cases obtained then at least on of L_n and L'_0 would be in $B \setminus \beta(A)$ and by Lemma Lemma 4.16 this would contradict the fact that $L \simeq (c)$ and $L' \simeq (c')$.

So suppose that $L \circ_P L' = (L_0, ..., L_n \circ_C L'_0, L'_1, ..., L'_m)$. Again Lemma 4.16 implies $n, m = 0$. Therefore

$$
L \circ_P L' \simeq (c) \circ_P (c') \simeq (c \circ_C c') \in C
$$

as required. $\square$

4.4 Pushout Closed Under Decomposition in the Dubuc Topos

In order to transfer the results of Section 4 to the Dubuc topos we require the following result which is Theorem 1 from Section III.5 of [8]:

Theorem 4.18. *The inclusion functor $j : \mathcal{E} \rightarrowtail [\mathcal{C}^{op}, Set]$ has a left adjoint*

$$a : [\mathcal{C}^{op}, Set] \to \mathcal{E}$$

called the associated sheaf functor. Moreover, this functor a commutes with finite limits.

Lemma 4.19. *If $\mathbb{K}$ is an infinitesimal category in $\mathcal{E}$ and*

$$\begin{tikzcd} hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "1\times\iota"] & \mathbb{K} \arrow[d, "\tau"] \\ hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} \arrow[r, "\alpha"] & \mathbb{P} \end{tikzcd} \tag{6}$$

is a pushout in $Cat(\mathcal{E})$ then τ is closed under decomposition.

Proof. It is immediate from Lemma 3.11 that $1 \times \iota$ is closed under decomposition. Therefore $j(1 \times \iota)$ is closed under decomposition because j preserves limits. If

$$\begin{tikzcd} j\,(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty) \arrow[r] \arrow[d] & j\mathbb{K} \arrow[d, "\nu"] \\ j\,(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}) \arrow[r] & \mathbb{Q} \end{tikzcd}$$

is a pushout in $[\mathcal{C}^{op}, Cat]$ then ν is closed under decomposition by Lemma 4.19. But then

$$\begin{tikzcd} aj\,(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty) \arrow[r] \arrow[d] & aj\mathbb{K} \arrow[d, "a\nu"] \\ aj\,(hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}) \arrow[r] & a\mathbb{Q} \end{tikzcd} \tag{7}$$

is a pushout in $Cat(\mathcal{E})$ because a preserves colimits and hence the square (7) is isomorphic to the square (6). Moreover $a\nu \cong \tau$ is closed under decomposition because a preserves finite limits. $\square$

4.5 Lie's Third Theorem

In this section we prove the main result of the paper (Theorem 4.23) which will be used to deduce the appropriate version of Lie's third theorem.

Lemma 4.20. *If*

$$\begin{array}{ccc} hom(\mathbf{2}\times\mathbb{I}_\infty,\mathbb{K})\times hom(\mathbf{2},\mathbb{I}_\infty) & \xrightarrow{ev} & hom(\mathbf{2},\mathbb{K}) \\ \Big\downarrow{\scriptstyle 1\times\iota} & & \Big\downarrow{\scriptstyle v} \\ hom(\mathbf{2}\times\mathbb{I}_\infty,\mathbb{K})\times hom(\mathbf{2},\mathbb{I}) & \xrightarrow{w} & Q \end{array} \tag{8}$$

is a pushout in $\mathcal{E}$ then v_K is infinitesimally closed.

Proof. Combine Lemma 3.10 with Lemma 2.27. □

Theorem 4.23 follows quickly from Lemma 4.20 and Lemma 4.19 once we have established the legitimacy of a certain de Morgan-type implication in intuitionistic logic.

Remark 4.21. Recall that in intuitionistic logic the implications

$$\neg(A\wedge B) \implies \neg A\vee\neg B$$

and

$$\neg(\forall i\in I.\ P_i) \implies \exists i\in I.\ \neg P_i$$

are not in general valid. Nevertheless for any list P_1, P_2,..., P_n of propositions the implication

$$\neg\neg\left(\bigwedge_{i=1}^{n} P_i\right) \implies \bigwedge_{i=1}^{n}\neg\neg P_i$$

is always valid and we will use it to prove our main theorem.

Lemma 4.22. *For any finite list of propositions P_1, P_2,..., P_n the implication*

$$\neg\neg\left(\bigwedge_{i=1}^{n} P_i\right) \implies \bigwedge_{i=1}^{n}\neg\neg P_i$$

holds in intuitionistic logic.

Proof. Now

$$\begin{aligned}\neg\neg\left(\bigwedge_{i=1}^{n} P_i\right) &\implies \bigwedge_{j=1}^{n}\neg\neg\left(\bigwedge_{i=1}^{n} P_i\right)\\ &\implies \bigwedge_{j}\neg\neg P_j\end{aligned}$$

because $\bigwedge_{i=1}^{n} P_i \implies P_j$ for any $j\in\{1,...,n\}$. □

Theorem 4.23. *If $\mathbb{K}$ is an infinitesimal category and*

$$\begin{array}{ccc} hom(\mathbb{I}_\infty,\mathbb{K})\times\mathbb{I}_\infty & \xrightarrow{ev} & \mathbb{K} \\ \Big\downarrow{\scriptstyle 1\times\iota} & & \Big\downarrow{\scriptstyle \tau} \\ hom(\mathbb{I}_\infty,\mathbb{K})\times\mathbb{I} & \xrightarrow{\alpha} & \mathbb{P} \end{array}$$

is a pushout in $Cat(\mathcal{E})$ then the arrow $\tau_\infty:\mathbb{K}_\infty\to\mathbb{P}_\infty$ is an isomorphism.

Proof. Let v and Q be as in Lemma 4.20. Let $p \in P$ such that $\neg\neg(esp = p)$ and $p = L_0 \circ ... \circ L_n$ for $L_i \in Q$. First $\neg\neg(\forall i \in \{1, ..., n\}.\ L_i \in v(K))$ because $esp \in v(K)$ and (by Lemma 4.19) τ is closed under decomposition. Therefore $\forall i \in \{1, .., n\}.\ \neg\neg(L_i \in v(K))$ by Lemma 4.22. Finally $\forall i \in \{1, ..., n\}.\ L_i \in v(K)$ because v is infinitesimally closed and therefore $p \in \tau(K)$ as required. $\square$

Corollary 4.24. *(Lie's Third Theorem) The natural transformation*

$$1 \Rightarrow (-)_\infty \circ (-)_{int} : Cat_\infty(\mathcal{E}) \to Cat(\mathcal{E})$$

induced by τ_∞ is an isomorphism.

5 The Lie Adjunction

5.1 Integral Complete Categories

Notation 5.1. Let $\mathbb{C}$ be a category internal to $\mathcal{E}$ with object space M and source, target and identity maps s, t and e respectively. Let $\iota^\infty_\mathbb{C} : \mathbb{C}_\infty \to \mathbb{C}$ be the inclusion of the infinitesimal part as defined in Lemma 2.17.

Definition 5.2. A category $\mathbb{C}$ internal to $\mathcal{E}$ is *integral complete* iff the arrow

$$\mathbb{C}^{\mathbb{I}} \xrightarrow{\mathbb{C}^{\iota^\infty_\mathbb{I}}} \mathbb{C}^{\mathbb{I}_\infty}$$

is a split epimorphism in $Cat(\mathcal{E})$.

Notation 5.3. We define $\mathbb{C}_{int}$, τ and α using the pushout

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "1\times\iota^\infty_\mathbb{I}"] & \mathbb{C} \arrow[d, "\tau"] \\
\mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[r, "\alpha"] & \mathbb{C}_{int}
\end{tikzcd}$$

in $Cat(\mathcal{E})$. We use $(-)_{int}$ to denote the function that maps an internal category $\mathbb{C}$ in $\mathcal{E}$ to the internal category $\mathbb{C}_{int}$ in $\mathcal{E}$.

Lemma 5.4. *The arrow $\mathbb{C}^{\iota^\infty_\mathbb{I}}_{int} : \mathbb{C}^{\mathbb{I}}_{int} \to \mathbb{C}^{\mathbb{I}_\infty}_{int}$ is a split epimorphism. In other words, if $\mathbb{C}$ is a category internal to $\mathcal{E}$ then $\mathbb{C}_{int}$ is integral complete.*

Proof. Recall that

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "\mathbb{C}^{\mathbb{I}_\infty}\times\iota^\infty_\mathbb{I}"] & \mathbb{C} \arrow[d, "\tau_\mathbb{C}"] \\
\mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[r, "\alpha_\mathbb{C}"] & \mathbb{C}_{int}
\end{tikzcd} \tag{9}$$

is a pushout by definition of $\mathbb{C}_{int}$. Let $\hat{\alpha_{\mathbb{C}}}$: be the arrow induced by $\alpha_{\mathbb{C}}$ under the hom-tensor adjunction. If $\phi \in \mathbb{C}^{\mathbb{I}_\infty}$ and $k \in \mathbb{C}_\infty$ then

$$\begin{aligned}
\left(\mathbb{C}_{int}^{\iota_{\mathbb{I}}^{\infty}}(\hat{\alpha_{\mathbb{C}}}(\phi))\right)(k) &= \hat{\alpha_{\mathbb{C}}}(\phi)(\iota_{\mathbb{I}}^{\infty}(k)) \\
&= \alpha_{\mathbb{C}}(\phi, \iota_{\mathbb{I}}^{\infty}(k)) \\
&= \tau_{\mathbb{C}}(ev(\phi, k)) \\
&= \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}(\phi)\right)(k)
\end{aligned}$$

Now by **??** the arrow $\tau_{\mathbb{C}}^{\mathbb{I}_\infty}$ is invertible and so $\hat{\alpha_{\mathbb{C}}} \circ \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}\right)^{-1}$ splits $\mathbb{C}_{int}^{\iota_{\mathbb{I}}^{\infty}}$. □

5.2 Integration is a Reflection

Lemma 5.5. *If $\mathbb{X}$ is integral complete then $\tau_{\mathbb{X}}$ is a split monomorphism.*

Proof. Since $\mathbb{X}$ is integral complete there is a $\beta : \mathbb{X}^{\mathbb{I}_\infty} \to \mathbb{X}^{\mathbb{I}}$ such that $(\mathbb{X}^{\iota_{\mathbb{I}}^{\infty}}) \circ \beta = 1_{\mathbb{X}^{\mathbb{I}_\infty}}$. Then in

$$\begin{tikzcd}
\mathbb{X}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "1_{\mathbb{X}^{\mathbb{I}_\infty}} \times \iota_{\mathbb{I}}^{\infty}"] & \mathbb{X} \arrow[d, "\tau_{\mathbb{X}}"] \arrow[ddr, bend left, "1_{\mathbb{X}}"] & \\
\mathbb{X}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[r, "\alpha_{\mathbb{X}}"] \arrow[d, "\beta \times 1_{\mathbb{I}}"] & \mathbb{X}_{int} \arrow[dr, dashed, "p_{\mathbb{X}}"] & \\
\mathbb{X}^{\mathbb{I}} \times \mathbb{I} \arrow[rr, "ev"] & & \mathbb{X}
\end{tikzcd}$$

the top left square is the pushout defining $\mathbb{X}_{int}$, $\alpha_{\mathbb{X}}$ and $\tau_{\mathbb{X}}$ and the outer square commutes because

$$\begin{aligned}
ev\left(\beta \times \mathbb{I}\left(1 \times \iota_{\mathbb{I}}^{\infty}(\phi, k)\right)\right) &= ev\left(\beta \times \mathbb{I}\left(\phi, \iota_{\mathbb{I}}^{\infty}(k)\right)\right) \\
&= ev\left(\beta(\phi), \iota_{\mathbb{I}}^{\infty}(k)\right) \\
&= \beta(\phi)(\iota_{\mathbb{I}}^{\infty}(k)) \\
&= \mathbb{X}^{\iota_{\mathbb{I}}^{\infty}}(\beta(\phi))(k) \\
&= \phi(k)
\end{aligned}$$

for $k \in \mathbb{I}_\infty$ and $\phi \in \mathbb{X}^{\mathbb{I}_\infty}$. Then the induced arrow $p_{\mathbb{X}} : \mathbb{X}_{int} \to \mathbb{X}$ splits $\tau_{\mathbb{X}}$. □

Lemma 5.6. *The arrow $(\tau_{\mathbb{C}})_{int}$ is a monomorphism split by $p_{\mathbb{C}_{int}} : (\mathbb{C}_{int})_{int} \to \mathbb{C}_{int}$.*

Proof. By Lemma 5.4 the arrow $\beta = \hat{\alpha_{\mathbb{C}}} \circ \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}\right)^{-1}$ splits the epimorphism

$\mathbb{C}_{int}^{\iota_{\mathbb{I}^\infty}} : \mathbb{C}_{int}^{\mathbb{I}} \to \mathbb{C}_{int}^{\mathbb{I}_\infty}$. In

$$\begin{tikzcd}
\mathbb{C}_{int}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[rrr, "ev"] \arrow[dddd, "1_{\mathbb{C}_{int}^{\mathbb{I}_\infty}} \times \iota_{\mathbb{I}}^\infty"'] & & & \mathbb{C}_{int} \arrow[ddd, "\tau_{\mathbb{C}_{int}}"] \arrow[ddddr, bend left, "1_{\mathbb{C}_{int}}"] & \\
& \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[ul, "\tau_{\mathbb{C}}^{\mathbb{I}_\infty} \times 1_{\mathbb{I}_\infty}"'] \arrow[d, "1_{\mathbb{C}^{\mathbb{I}_\infty}} \times \iota_{\mathbb{I}}^\infty"] & \mathbb{C} \arrow[ur, "\tau_{\mathbb{C}}"] \arrow[d, "\tau_{\mathbb{C}}"] & & \\
& \mathbb{C}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[r, "\alpha_{\mathbb{C}}"] \arrow[dl, "\tau_{\mathbb{C}}^{\mathbb{I}_\infty} \times 1_{\mathbb{I}}"] & \mathbb{C}_{int} \arrow[dr, dashed, "(\tau_{\mathbb{C}})_{int}"] & & \\
\mathbb{C}_{int}^{\mathbb{I}_\infty} \times \mathbb{I} \arrow[rrr, "\alpha_{\mathbb{C}_{int}}"] \arrow[d, "\beta \times \mathbb{I}"] & & & (\mathbb{C}_{int})_{int} \arrow[dr, "p_{\mathbb{C}_{int}}"] & \\
\mathbb{C}_{int}^{\mathbb{I}} \times \mathbb{I} \arrow[rrrr, "ev"] & & & & \mathbb{C}_{int}
\end{tikzcd}$$

the inner square is the pushout defining $\mathbb{C}_{int}$, the middle square is the pushout defining $(\mathbb{C}_{int})_{int}$ and the rest of the diagram commutes by an argument similar to Lemma 5.4. To prove that $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} = 1_{\mathbb{C}_{int}}$ it will suffice to show that $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} \circ \tau_{\mathbb{C}} = \tau_{\mathbb{C}}$ (which is immediate by construction) and $p_{\mathbb{C}_{int}} \circ (\tau_{\mathbb{C}})_{int} \circ \alpha_{\mathbb{C}} = \alpha_{\mathbb{C}}$ which we deduce from the following equalities:

$$\begin{aligned}
p_{\mathbb{C}_{int}}\left((\tau_{\mathbb{C}})_{int}\left(\alpha_{\mathbb{C}}(\phi, k)\right)\right) &= ev(\beta \times \mathbb{I}(\tau_{\mathbb{C}}^{\mathbb{I}_\infty} \times 1_{\mathbb{I}}))(\phi, k) \\
&= ev(\beta \times \mathbb{I}(\tau_{\mathbb{C}^{\mathbb{I}_\infty}}(\phi), k)) \\
&= ev(\beta(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}(\phi)), k) \\
&= \beta(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}(\phi))(k) \\
&= \left(\hat{\alpha_{\mathbb{C}}} \circ \left(\tau_{\mathbb{C}}^{\mathbb{I}_\infty}\right)^{-1} \circ \tau_{\mathbb{C}}^{\mathbb{I}_\infty}\right)(\phi))(k) \\
&= \hat{\alpha_{\mathbb{C}}}(\phi)(k) \\
&= \alpha_{\mathbb{C}}(\phi, k)
\end{aligned}$$

where $\phi \in \mathbb{C}^{\mathbb{I}_\infty}$ and $k \in \mathbb{I}$. □

Corollary 5.7. *The functor $(-)_{int}$ is left adjoint to the inclusion $k : Cat_{int}(\mathcal{E}) \to Cat(\mathcal{E})$.*

Proof. If $\mathbb{C}$ is an arbitrary category internal to $\mathcal{E}$ then define the unit of the adjunction is $\tau_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}_{int}$. If $\mathbb{X}$ is an integral complete category internal to $\mathcal{E}$ then define the counit of the adunction is $p_{\mathbb{X}} : \mathbb{X}_{int} \to \mathbb{X}$. Then the triangle equalities are given by Lemma 5.5 and Lemma 5.6. □

5.3 Lie's Second Theorem

By composing the reflection in Corollary 5.7 with the coreflection in Lemma 2.24 we obtain the following adjunction:

$$\begin{tikzcd} Cat_\infty(\mathcal{E}) \arrow[r, tail, "i"] \arrow[r, phantom, shift right=3, "\perp"] & Cat(\mathcal{E}) \arrow[l, bend left, shift left=3, "(-)_\infty"] \arrow[r, bend left, "(-)_{int}"] \arrow[r, phantom, "\perp"] & Cat_{int}(\mathcal{E}) \arrow[l, tail, shift left=3, "j"] \end{tikzcd}$$

which generalises the classical Lie correspondence between Lie algebras and Lie groups. The appropriate formulation of Lie's second theorem in this context is that the functor $(-)_\infty \circ j$ is full and faithful when we restrict to the simply connected categories as defined in Definition 5.12. We prove this result as Theorem 5.15. To prove this we refer to results in [3] where an abstract version of Lie's second theorem is presented. In this section we give a brief sketch of the ideas in [3] and then indicate how to apply the results there to prove Theorem 5.15.

To understand the ideas in [3] it is helpful to first look at a special case involving spaces rather than groupoids or categories.

Notation 5.8. Let X and Y be topological spaces. Let $P_*(X)$ denote the space of paths in X starting at $* \in X$. Let I denote the unit interval. Let Man denote the category of smooth manifolds.

Remark 5.9. In general if we have a map $f : X \to Y$ we can construct a map $\hat{f} : P_*(X) \to Y$ by precomposing with the map $X^1 \circ \iota : P_*(X) \to X^I \to X$. However given a map $g : P_*(X) \to Y$ we may not always be able to find a map $\check{g} : X \to Y$ such that $g = \check{g} \circ X^1 \circ \iota$. Firstly for a fixed $x \in X$ there may not exist a $\gamma \in P_*(X)$ such that $\gamma(1) = x$. Secondly for a fixed $x \in X$ not all paths $\gamma, \delta : * \to x$ will necessarily have the same image under g.

Lemma 5.10. *If X is simply connected and $g : P_*(X) \to Y$ such that $g(\gamma) = g(\delta)$ for all $\gamma, \delta \in P_*(X)$ that are homotopic with fixed endpoints then there exists a unique $\check{g} : X \to Y$ such that $g = \check{g} \circ X^1 \circ \iota$.*

Proof. Since X is path connected for all $x \in X$ there exists $\gamma_x \in P_*(X)$ such that $\gamma_x(1) = x$. Now further let $\gamma, \delta \in P_*(X)$ such that $\gamma(1) = \delta(1)$. Since X is simply connected γ and δ are homotopic and hence $g(\gamma) = g(\delta)$. So we can define $\check{g}(x) = \gamma_x(1)$. $\square$

To help us generalise Lemma 5.10 we introduce the following definitions.

Notation 5.11. Let $\mathbb{I}$ be the fundamental category on the unit interval defined in Definition 3.9. Let $\mathcal{E}$ be the Dubuc topos defined in Definition 2.10.

Definition 5.12. A category $\mathbb{C}$ in $\mathcal{E}$ is *$\mathcal{E}$-path connected* iff

$$hom(\mathbb{I}, \mathbb{C}) \xrightarrow{hom((0,1),\mathbb{C})} hom(\mathbf{2}, \mathbb{C})$$

is an epimorphism in $\mathcal{E}$. A category $\mathbb{C}$ in $\mathcal{E}$ is *$\mathcal{E}$-simply connected* iff it is $\mathcal{E}$-path connected and

$$hom(\mathbb{I}^2, \mathbb{C}) \xrightarrow{hom(\iota,\mathbb{C})} hom(\partial\mathbb{I}^2, \mathbb{C})$$

is an epimorphism in $\mathcal{E}$ where $\partial\mathbb{I}^2$ is the full subcategory of $\mathbb{I}^2$ on the boundary of I^2 and $\iota : \partial\mathbb{I}^2 \to \mathbb{I}^2$ is the natural inclusion.

Definition 5.13. Recall that the 2-*truncated cube category* $\Box_2$ is the subcategory of Man generated by the following arrows:

$$\begin{tikzcd}
I^2 \arrow[r, leftarrow, shift left=5, "(1_I{,}0)"] \arrow[r, shift left=3, "\pi_1"] \arrow[r, leftarrow, shift left=1, "(1_I{,}1)"] \arrow[r, leftarrow, shift right=1, "(0{,}1_I)"] \arrow[r, shift right=3, "\pi_2"] \arrow[r, leftarrow, shift right=5, "(1{,}1_I)"] & I \arrow[r, leftarrow, shift left=2, "1"] \arrow[r, "!"] \arrow[r, leftarrow, shift right=2, "0"'] & 1
\end{tikzcd} \tag{10}$$

where I is the unit interval. Recall that if $\mathcal{E}$ is a category then *the category $c_2\mathcal{E}$ of 2-truncated cubical objects in a category $\mathcal{E}$* is the functor category $[\Box_2^{op}, \mathcal{E}]$. The arrows of $c_2\mathcal{E}$ will be called 2-cubical maps. We refer to [6] for the theory of cubical objects.

Proposition 5.14. *Let $\mathbb{C}$ and $\mathbb{X}$ be categories where $\mathbb{C}$ is simply connected and*

$$\begin{tikzcd}
\mathbb{C}^{\mathbb{I}^2} \arrow[r, shift left=5] \arrow[r, leftarrow, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, leftarrow, shift right=3] \arrow[r, shift right=5] \arrow[d, "\Psi_2"] & \mathbb{C}^{\mathbb{I}} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, "\Psi_1"] & \mathbb{C}^1 \arrow[d, "\Psi_0"] \\
\mathbb{X}^{\mathbb{I}^2} \arrow[r, shift left=5] \arrow[r, leftarrow, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, leftarrow, shift right=3] \arrow[r, shift right=5] & \mathbb{X}^{\mathbb{I}} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathbb{X}^1
\end{tikzcd}$$

is a 2-cubical map. Then there is a functor $\psi : \mathbb{C} \to \mathbb{X}$ with object map $\psi_0 = \Psi_0$ and arrow map ψ_1 satisfying $\psi_1\mathbb{C}^l = \mathbb{X}^l\Psi_1$.

Proof. Proposition 7.4 in [3]. □

Theorem 5.15 (Lie's Second Theorem)**.** *If $\mathbb{C}$ is a simply connected category in $\mathcal{E}$ such that $\mathbb{C}_\infty$ is path connected, $\mathbb{X}$ is an integral complete category and $\phi : \mathbb{C}_\infty \to \mathbb{X}$ is a functor in $Cat(\mathcal{E})$ then there exists a lift $\psi : \mathbb{C} \to \mathbb{X}$ such that $\psi \circ \iota = \phi$.*

Proof. We use Proposition 5.14. For the map on 0-cells we take $\Phi_0 = \phi$. Next we construct the map Φ_1 on 1-cells. Since $\mathbb{X}$ is integral complete the arrow $hom(\iota, \mathbb{X}) : hom(\mathbb{I}, \mathbb{X}) \to hom(\mathbb{I}_\infty, \mathbb{X})$ has a section ζ. By the second half of Lemma 2.24 the map $hom(\mathbb{I}_\infty, \iota) : hom(\mathbb{I}_\infty, \mathbb{C}_\infty) \to hom(\mathbb{I}_\infty, \mathbb{C})$ is invertible. Hence we can define Φ_1 to be the following map on 1-cells:

$$\begin{aligned} hom(\mathbb{I}, \mathbb{C}) &\xrightarrow{hom(\iota, \mathbb{C})} hom(\mathbb{I}_\infty, \mathbb{C}) \xrightarrow{hom(\mathbb{I}_\infty, \iota)^{-1}} hom(\mathbb{I}_\infty, \mathbb{C}_\infty) \\ &\xrightarrow{hom(\mathbb{I}_\infty, \phi)} hom(\mathbb{I}_\infty, \mathbb{X}) \xrightarrow{\zeta} hom(\mathbb{I}, \mathbb{X}) \end{aligned}$$

Finally we construct the map Φ_2 on 2-cells. Since $\mathbb{X}$ is integral complete the map $hom(\iota \times \iota, \mathbb{X}) = hom(\mathbb{I}_\infty \times \iota, \mathbb{X}) \times hom(\iota \times \mathbb{I}, \mathbb{X})$ has section $hom(\mathbb{I}, \zeta) \circ hom(\mathbb{I}_\infty, \zeta)$ which we call η. By the second half of Lemma 2.24 the map $hom(\mathbb{I}^2_\infty, \iota)$:

$hom(\mathbb{I}^2_\infty, \mathbb{C}_\infty) \to hom(\mathbb{I}^2_\infty, \mathbb{C})$ is invertible. Hence we define Φ_2 to be the following map on 2-cells:

$$
\begin{aligned}
hom(\mathbb{I}^2, \mathbb{C}) \xrightarrow{hom(\iota,\mathbb{C})} hom(\mathbb{I}^2_\infty, \mathbb{C}) \xrightarrow{hom(\mathbb{I}^2_\infty,\iota)^{-1}} hom(\mathbb{I}^2_\infty, \mathbb{C}_\infty) \\
\xrightarrow{hom(\mathbb{I}^2_\infty,\phi)} hom(\mathbb{I}^2_\infty, \mathbb{X}) \xrightarrow{\eta} hom(\mathbb{I}^2, \mathbb{X})
\end{aligned}
$$

It is easy to see that Φ_0, Φ_1 and Φ_2 satisfy the conditions that define a morphism of 2-cubical sets. $\square$

5.3.1 Lie's Third Theorem

Recall the main result of this paper that we proved as Theorem 4.23.

If $\mathbb{K}$ is an infinitesimal category and

$$
\begin{tikzcd}
hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I}_\infty \arrow[r, "ev"] \arrow[d, "1\times\iota"] & \mathbb{K} \arrow[d, "\tau"] \\
hom(\mathbb{I}_\infty, \mathbb{K}) \times \mathbb{I} \arrow[r, "\alpha"] & \mathbb{P}
\end{tikzcd}
$$

is a pushout in $Cat(\mathcal{E})$ then the arrow $\tau_\infty : \mathbb{K}_\infty \to \mathbb{P}_\infty$ is an isomorphism.

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