

The Algebroid of a Groupoid in a Tangent Category

Work in Progress

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Abstract

We generalise the construction of the Lie algebroid of a Lie groupoid so that it can be carried out in any tangent category. First we reconstruct the bijection between left invariant vector fields and source constant tangent vectors based at an identity element for a groupoid in a category equipped with an endofunctor that has a retraction onto the identity functor. Second we use the full structure of a tangent category to construct the algebroid of a groupoid. Finally we show how the classical result concerning the splitting of the tangent bundle of a Lie group can be carried out for any pregroupoid.

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1 Introduction

Recall that in classical Lie theory we approximate a Lie group using an algebraic structure called a Lie algebra. To construct the Lie algebra that approximates a particular Lie group $\mathbb{G} = (G, \mu, e)$ we first consider the tangent space of the underlying smooth manifold G at the identity element e of the underlying group. To define the Lie bracket $[v, w]$ of two tangent vectors v and w based at the identity element we use the group multiplication μ to extend v and w to (left

invariant) vector fields X and Y on G then define $[v, w]$ to be the evaluation of $[X, Y]$ at the identity element. The Lie bracket defined in this way is well defined due to two facts. Firstly there is a bijection between tangent vectors at the identity and left invariant vector fields. Secondly if X and Y are left invariant vector fields then $[X, Y]$ is left invariant also. For more details concerning the classical theory of Lie groups and Lie algebras please see [3]. A more conceptual way of phrasing the correspondence between tangent vectors at the identity element and left invariant vector fields is to say that the tangent space TG splits into the product $G \times T_e G$ where TG denotes the tangent space of G and $T_e G$ the subspace of TG consisting of vectors based at the identity element.

In the established generalisation of Lie theory involving Lie groupoids and Lie algebroids (see for instance [2]) we can also construct a Lie algebroid that approximates a Lie groupoid. Recall that a *Lie groupoid* is a groupoid in the category *Man* of smooth manifolds such that the source and target maps are submersions. Recall further that a *Lie algebroid* is a smooth vector bundle $A \rightarrow M$ together with a bundle homomorphism $\rho: A \rightarrow TM$ such that the space of sections $\Gamma(A)$ is a Lie algebra satisfying the following Leibniz law: for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$ the equality

$$[X, fY] = \rho(X)(f) \cdot Y + f[X, Y]$$

holds. The construction of the Lie algebroid that is the linear approximation to a Lie groupoid is analogous to the construction of the Lie algebra that approximates a Lie group described above. First we consider the space $T_M^s G$ of tangent vectors that are based at an identity element of \mathbb{G} and are parallel to the source fibres of \mathbb{G} . The reason we insist on vectors that are parallel to the source fibres of \mathbb{G} is that we can then use the composition of the Lie groupoid to extend a section of the natural projection $\pi_M^s: T_M^s G \rightarrow M$ to a vector field on G . Then we can use the standard Lie bracket of vector fields to induce a Lie bracket on sections of π_M^s . For more details on the approximation of a Lie groupoid by a Lie algebroid please see Section 3.5 of [5].

In this paper we show how to generalise this technique of approximating a groupoid by an algebroid to any category \mathbb{X} equipped with a tangent structure (or *tangent category*) as defined in Section 2 of [1]. In fact some of the theory can be carried out at a higher level of generality. In Section 2 we show that the bijection between left invariant vector fields and sections of π_M^s holds in any category equipped with an endofunctor that has a retraction onto the identity functor. In Section 3 we assume the full structure of a tangent category to construct the algebroid that is the linear approximation to a groupoid in \mathbb{X} . In Section 4 we show that the splitting of the tangent bundle holds not only for groupoids but also for pregroupoids (as defined in [4]) equipped with an extra arrow satisfying a certain equation. In this section also we do not need the full structure provided by a tangent category and work in a category \mathbb{X} equipped with a functor that has a retraction onto the identity functor.

2 Invariant Vector Fields

In this section we construct an abstract bijection which generalises the classical bijection between left invariant vector fields on the arrow space of a Lie groupoid and sections of the bundle of source constant tangent vectors based at an identity element of the groupoid. These constructions only require part of the structure of a tangent category. Therefore in this section we assume only the existence of: a category \mathbb{X} , an endofunctor $T : \mathbb{X} \rightarrow \mathbb{X}$, a natural transformation $p : T \Rightarrow 1_{\mathbb{X}}$ and a natural transformation $0 : 1_{\mathbb{X}} \Rightarrow T$. We further assume that 0 is a section of p . In addition the definitions we make in this section require that certain pullbacks exist in \mathbb{X} . Therefore we assume without further comment that these pullbacks exist and moreover that they are preserved by T .

2.1 Basic Definitions

We define the arrows π^s and π_M^s of \mathbb{X} that generalise the two classical bundles that we are interested in. The arrow π^s corresponds to the bundle of source constant tangent vectors. The arrow π_M^s corresponds to the bundle of source constant tangent vectors based at an identity element.

Notation 2.1. Let \mathbb{X} be a category and $T : \mathbb{X} \rightarrow \mathbb{X}$ a functor. Let $p : T \Rightarrow 1_{\mathbb{X}}$ and $0 : 1_{\mathbb{X}} \Rightarrow T$ be a natural transformations such that 0 is a section of p . Let $\mathbb{G} = G \rightrightarrows M$ be a groupoid in \mathbb{X} with source, target, inverse and composition denoted by $s, t, (-)^{-1}$ and μ respectively. This means that the pullback $G_{t \times s} G$ exists and further we assume that T preserves this pullback.

Definition 2.2. We denote by $\pi^s : T^s G \rightarrow M$ the pullback

$$\begin{array}{ccc} T^s G & \xrightarrow{\iota^s} & TG \\ \downarrow \pi^s & & \downarrow Ts \\ M & \xrightarrow{0} & TM \end{array} \quad (1)$$

consisting of s -constant tangent vectors of G and write $p^s = \iota^s p$. Finally we denote by $\pi_M^s : T_M^s G \rightarrow M$ the pullback

$$\begin{array}{ccc} T_M^s G & \xrightarrow{\iota_M} & T^s G \\ \downarrow \pi_M^s & & \downarrow p^s \\ M & \xrightarrow{e} & G \end{array} \quad (2)$$

and write $\iota_M^s = \iota_M \iota^s$ and $p_M^s = \iota_M p^s$.

Remark 2.3. The arrow $\pi_M^s : T_M^s G \rightarrow M$ is equivalently defined by the double

pullback

$$\begin{array}{ccc}
T_M^s G & \xrightarrow{q} & M \\
\downarrow \pi_M^s & \searrow \iota_M^s & \downarrow 0 \\
& TG & \xrightarrow{Ts} TM \\
& \downarrow p & \\
M & \xrightarrow{e} & G
\end{array} \tag{3}$$

Remark 2.4. In fact $q = \pi_M^s$. Indeed if

$$\begin{array}{ccc}
Z & \xrightarrow{z_2} & M \\
\downarrow z_0 & \searrow z_1 & \downarrow 0 \\
& TG & \xrightarrow{Ts} TM \\
& \downarrow p & \\
M & \xrightarrow{e} & G
\end{array} \tag{4}$$

is an arbitrary cone over the diagram defining p_M^s then we can paste the commutative square

$$\begin{array}{ccc}
TG & \xrightarrow{Ts} & TM \\
\downarrow p & & \downarrow p \\
G & \xrightarrow{s} & M
\end{array} \tag{5}$$

onto the bottom right of (4) and conclude that $z_2 = z_2 0 p = z_0 e s = z_0$.

In fact we can make a further simplification in the definition of $T_M^s G$. Since the components z_0 and z_2 of any cone over the limit defining $T_M^s G$ are equal we can conflate the two copies of M in (4) and conclude that $T_M^s G$ is the limit of the diagram

$$\begin{array}{ccc}
TG & \xrightarrow{Ts} & TM \\
\downarrow p & & \uparrow 0 \\
G & \xleftarrow{e} & M
\end{array} \tag{6}$$

and so in fact π_M^s is equivalently defined using the pullback

$$\begin{array}{ccc}
T_M^s G & \xrightarrow{s} & TG \\
\downarrow \pi_M^s & & \downarrow (Ts, p) \\
M & \xrightarrow{(0, e)} & TM \times G
\end{array} \tag{7}$$

2.2 Extending and Restricting Vector Fields

In this section we recall how to postcompose an element of a groupoid with a source constant tangent vector and then how to define a left invariant vector field. With these definitions in place we show how to extend a section of π_M^s to a left invariant vector field and restrict a left invariant vector field to a section of π_M^s . Finally we show that these extension and restriction functions are inverses.

Notation 2.5. The arrow $(\pi_0 0, \pi_1 \iota^s) T\mu : G \times_{t \times \pi^s} T^s G \rightarrow TG$ factors through $T^s G$ because

$$(\pi_0 0, \pi_1 \iota^s) T\mu Ts = (\pi_0 0, \pi_1 \iota^s) \pi_0 Ts \quad (8)$$

$$= \pi_0 0 Ts \quad (9)$$

$$= \pi_0 s 0 \quad (10)$$

We call this unique factorisation $\mu^s : G \times_{t \times \pi^s} T^s G \rightarrow T^s G$. Let $\Gamma(M, T_M^s G)$ be the set of sections of π_M^s .

Definition 2.6. A *left invariant vector field* $X : G \rightarrow T^s G$ is a section of $p^s : T^s G \rightarrow G$ such that

$$\begin{array}{ccc} G \times T^s G & \xrightarrow{\mu^s} & T^s G \\ (\pi_0, \pi_1 X) \uparrow & & \uparrow X \\ G \times G & \xrightarrow{\mu} & G \end{array} \quad (11)$$

commutes. We write $LeftInv(G, T^s G)$ for the set of sections of p^s that are left invariant.

Lemma 2.7.

$$\mu^s p^s = \mu^s \iota^s p \quad (12)$$

$$= (\pi_0 0, \pi_1 \iota^s) T\mu p \quad (13)$$

$$= (\pi_0 0, \pi_1 \iota^s) p \mu \quad (14)$$

$$= (\pi_0, \pi_1 p^s) \mu \quad (15)$$

Lemma 2.8. Every section $v : M \rightarrow T_M^s G$ of $\pi_M^s : T_M^s G \rightarrow M$ can be extended to a left invariant vector field $\hat{v} : G \rightarrow T^s G$ of $p^s : T^s G \rightarrow G$.

Proof. Define \hat{v} as the composite

$$G \xrightarrow{(1_G, t)} G \times_{t \times 1_M} M \xrightarrow{(\pi_0, \pi_1 v \iota_M)} G \times_{t \times \pi^s} T^s G \xrightarrow{\mu^s} T^s G \quad (16)$$

Then \hat{v} is a section of p^s because

$$\hat{v} p^s = (1_G, tv \iota_M) \mu^s p^s \quad (17)$$

$$= (1_G, tv \iota_M) (\pi_0, \pi_1 p^s) \mu \quad (18)$$

$$= (1_G, tv \iota_M p^s) \mu \quad (19)$$

$$= (1_G, tv \pi_M^s e) \mu \quad (20)$$

$$= (1_G, te) \mu = 1_G \quad (21)$$

Finally \hat{v} is left invariant because

$$(\pi_0, \pi_1 \hat{v}) \mu^s \iota^s = (\pi_0, \pi_1 (1_G, tv\iota_M) \mu^s) (\pi_0 0, \pi_1 \iota^s) T\mu \quad (22)$$

$$= (\pi_0 0, \pi_1 (1_G, tv\iota_M) \mu^s \iota^s) T\mu \quad (23)$$

$$= (\pi_0 0, \pi_1 (1_G, tv\iota_M) (\pi_0 0, \pi_1 \iota^s) T\mu) T\mu \quad (24)$$

$$= (\pi_0 0, \pi_1 (1_G 0, tv\iota_M \iota^s) T\mu) T\mu \quad (25)$$

$$= (\pi_0 0, (\pi_1 0, \pi_1 tv\iota_M \iota^s) T\mu) T\mu \quad (26)$$

$$= ((\pi_0 0, \pi_1 0) T\mu, \pi_1 tv\iota_M \iota^s) T\mu \quad (27)$$

$$= ((\pi_0, \pi_1) \mu 0, \pi_1 tv\iota_M \iota^s) T\mu \quad (28)$$

$$= ((\pi_0, \pi_1) \mu, \pi_1 tv\iota_M) (\pi_0 0, \pi_1 \iota^s) T\mu \quad (29)$$

$$= ((\pi_0, \pi_1) \mu, \pi_1 tv\iota_M) \mu^s \iota^s \quad (30)$$

$$= ((\pi_0, \pi_1) \mu, (\pi_0, \pi_1) tv\iota_M) \mu^s \iota^s \quad (31)$$

$$= (\pi_0, \pi_1) \mu (1_G, tv\iota_M) \mu^s \iota^s \quad (32)$$

$$= \mu \hat{v} \iota^s \quad (33)$$

and so $(\pi_0, \pi_1 \hat{v}) \mu^s = \mu \hat{v}$ because ι^s is a monomorphism. \square

Lemma 2.9. *Every section $X : G \rightarrow T^s G$ of $p^s : T^s G \rightarrow G$ restricts to a section $\check{X} : M \rightarrow T_M^s G$ of $\pi_M^s : T_M^s G \rightarrow M$.*

Proof. The arrow $eX : M \rightarrow T^s G$ factors through $T_M^s G$ because $eX p^s = e$. Let \check{X} be this unique factorisation induced by $(1_M, eX)$. This means that $\check{X} \pi_M^s = 1_M$ and so \check{X} is a section of π_M^s . \square

Proposition 2.10. *The function $(-)^{\sim} : \Gamma(M, T_M^s G) \rightarrow \text{LeftInv}(G, T^s G)$ has inverse $(-)^{\check{}}$.*

Proof. Let $v : M \rightarrow T_M^s G$ be a section of $\pi_M^s : T_M^s G \rightarrow G$. Then

$$(\hat{v})^{\sim} = ((1_G, tv\iota_M) \mu^s)^{\sim} \quad (34)$$

$$= (1_M, e(1_G, tv\iota_M) \mu^s) \quad (35)$$

$$= (1_M, (e, v\iota_M) \mu^s) \quad (36)$$

$$= (1_M, v\iota_M) \quad (37)$$

$$= v \quad (38)$$

Let $X : G \rightarrow T^s G$ be a left invariant vector field. Then

$$(\check{X})^{\sim} = (1_M, eX)^{\sim} \quad (39)$$

$$= (1_G, t(1_M, eX) \iota_M) \mu^s \quad (40)$$

$$= (1_G, teX) \mu^s \quad (41)$$

$$= (1_G, te) (\pi_0, \pi_1 X) \mu^s \quad (42)$$

$$= (1_G, te) \mu X = X \quad (43)$$

where the penultimate equality is due to the fact that X is left invariant. \square

3 The Algebroid of a Groupoid in a Tangent Category

In this section we generalise the classical construction that assigns a Lie algebroid to each Lie groupoid. To carry out this work we must use the full structure of a tangent category. Therefore for this section only we assume that the category \mathbb{X} that we are working in is a tangent category. If $\mathbb{G} = G \rightrightarrows M$ is a groupoid in \mathbb{X} then the underlying bundle of the algebroid will be the bundle π_M^s defined in Definition 4.7. We define the Lie bracket of two sections of π_M^s in two steps. First we use the bijection of Proposition 2.10 to obtain two left invariant vector fields on G . Second we use the description of the Lie bracket of vector fields in [1] to define a Lie bracket. Then in order to prove that this bracket is well defined we check that the vector field thus obtained is source constant and left invariant.

Notation 3.1. In this section we make further assumptions about \mathbb{X} , T , p and 0 so that they form part of a tangent structure as defined in [1]. This means that we have extra natural transformations $c : T^2 \rightarrow T^2$ etc..

The following result is Lemma 2.13 in [1].

Lemma 3.2. *In any tangent category the following is a (triple) equaliser diagram:*

$$TG \xrightarrow{l} T^2G \begin{array}{c} \xrightarrow{Tp} \\ \xrightarrow{-p} \\ \xrightarrow{pp0} \end{array} TG \quad (44)$$

The following is Definition 3.14 in [1].

Definition 3.3. In a group tangent category with vector fields $X, Y : G \rightarrow TG$ the *Lie bracket* of X and Y is defined to be the morphism

$$[X, Y] := \{XTY - YTXc\} : G \rightarrow TG \quad (45)$$

where c is the canonical flip and $\{f\}$ denotes the factorisation of f through l for any arrow f that equalises diagram (44).

Lemma 3.4. *If $X, Y \in \text{LeftInv}(G, T^sG)$ then $[X\iota^s, Y\iota^s]$ factors through T^sG . (This means that there is an arrow $[X, Y] : G \rightarrow T^sG$ such that $[X, Y]\iota^s = [X\iota^s, Y\iota^s]$.)*

Proof. First

$$\begin{array}{ccccccc} G & \xrightarrow{X} & T^sG & \xrightarrow{\iota^s} & TG & \xrightarrow{TY} & T(T^sG) \xrightarrow{T(\iota^s)} T^2G \\ \downarrow s & & \downarrow \pi^s & & \downarrow Ts & & \downarrow T(\pi^s) \\ M & \xrightarrow{=} & M & \xrightarrow{0} & TM & \xrightarrow{=} & TM \xrightarrow{T0} T^2M \end{array} \quad (46)$$

where the first and third squares commute because $\pi^s = \pi^s 0p = \iota^s Tsp = \iota^s ps = p^s s$. Similarly $Y\iota^s T(X\iota^s)T^2s = s0T0$. Now we show that the arrow $[X\iota^s, Y\iota^s] : G \rightarrow TG$ factors through T^sG :

$$[X\iota^s, Y\iota^s]Tsl = \{X\iota^s T(Y\iota^s) - Y\iota^s T(X\iota^s)c\}lT^2s \quad (47)$$

$$= (X\iota^s T(Y\iota^s) - Y\iota^s T(X\iota^s)c)T^2s \quad (48)$$

$$= X\iota^s T(Y\iota^s)T^2s - Y\iota^s T(X\iota^s)cT^2s \quad (49)$$

$$= X\iota^s T(Y\iota^s)T^2s - Y\iota^s T(X\iota^s)T^2sc \quad (50)$$

$$= s0T0 - s0T0c \quad (51)$$

$$= s0T0 \quad (52)$$

$$= s0l \quad (53)$$

and so $[X\iota^s, Y\iota^s]Ts = (s - s)0$ because l is a monomorphism. \square

Lemma 3.5. *If $X, Y \in \text{LeftInv}(G, T^sG)$ then the arrow $[X, Y]$ defined in Lemma 3.4 is in $\Gamma(G, T^sG)$.*

Proof. We show that $[X\iota^s, Y\iota^s]$ is a section of p :

$$[X\iota^s, Y\iota^s]p = \{X\iota^s T(Y\iota^s) - Y\iota^s T(X\iota^s)c\}p \quad (54)$$

$$= (X\iota^s T(Y\iota^s) - Y\iota^s T(X\iota^s)c)pp \quad (55)$$

$$= X\iota^s T(Y\iota^s)pp \quad (56)$$

$$= (X\iota^s p)(Y\iota^s p) \quad (57)$$

$$= 1_G 1_G = 1_G \quad (58)$$

as required. \square

Lemma 3.6. *If $X, Y \in \text{LeftInv}(G, T^sG)$ then the arrow $[X, Y]$ defined in Lemma 3.4 is in $\text{LeftInv}(G, T^sG)$.*

Proof.

$$\begin{aligned}
(\pi_0, \pi_1[X, Y])\mu^s\iota^sl &= (\pi_0, \pi_1[X, Y])(\pi_0 0, \pi_1\iota^s)T\mu l \\
&= (\pi_0 0, \pi_1[X, Y]\iota^s)T\mu l \\
&= (\pi_0 0, \pi_1[X\iota^s, Y\iota^s])T\mu l \\
&= (\pi_0 0, \pi_1[X\iota^s, Y\iota^s])lT^2\mu \\
&= (\pi_0 0l, \pi_1[X\iota^s, Y\iota^s]l)T^2\mu \\
&= (\pi_0 0l, \pi_1(X\iota^sT(Y\iota^s) - Y\iota^sT(X\iota^s)c))T^2\mu \\
&= (\pi_0 0l, \pi_1X\iota^sT(Y\iota^s))T^2\mu - (\pi_0 0l, \pi_1Y\iota^sT(X\iota^s)c)T^2\mu \\
&= (\pi_0 0T0, \pi_1X\iota^sT(Y\iota^s))T^2\mu - (\pi_0 0T0, \pi_1Y\iota^sT(X\iota^s)c)T^2\mu c \\
&= (\pi_0 0, \pi_1X\iota^s)(T\pi_0T0, T\pi_1T(Y\iota^s))T^2\mu - (\pi_0 0, \pi_1Y\iota^s)(T\pi_0T0, T\pi_1T(X\iota^s))T^2\mu c \\
&= (\pi_0 0, \pi_1X\iota^s)T(\pi_0 0, \pi_1Y\iota^s)T^2\mu - (\pi_0 0, \pi_1Y\iota^s)T(\pi_0 0, \pi_1X\iota^s)T^2\mu c \\
&= (\pi_0 0, \pi_1X\iota^s)T((\pi_0, \pi_1Y)(\pi_0 0, \pi_1\iota^s)T\mu) - (\pi_0 0, \pi_1Y\iota^s)T((\pi_0, \pi_1X)(\pi_0 0, \pi_1\iota^s)T\mu)c \\
&= (\pi_0 0, \pi_1X\iota^s)T((\pi_0, \pi_1Y)\mu^s\iota^s) - (\pi_0 0, \pi_1Y\iota^s)T((\pi_0, \pi_1X)\mu^s\iota^s)c \\
&= (\pi_0 0, \pi_1X\iota^s)T(\mu Y\iota^s) - (\pi_0 0, \pi_1Y\iota^s)T(\mu X\iota^s)c \\
&= (\pi_0, \pi_1X)(\pi_0 0, \pi_1\iota^s)T\mu T(Y\iota^s) - (\pi_0, \pi_1Y)(\pi_0 0, \pi_1\iota^s)T\mu T(X\iota^s)c \\
&= (\pi_0, \pi_1X)\mu^s\iota^sT(Y\iota^s) - (\pi_0, \pi_1Y)\mu^s\iota^sT(X\iota^s)c \\
&= \mu X\iota^sT(Y\iota^s) - \mu Y\iota^sT(X\iota^s)c \\
&= \mu(X\iota^sT(Y\iota^s) - Y\iota^sT(X\iota^s)c) \\
&= \mu[X\iota^s, Y\iota^s]l \\
&= \mu[X, Y]\iota^sl
\end{aligned}$$

and so $(\pi_0, \pi_1[X, Y])\mu^s = \mu[X, Y]$ because ι^sl is a monomorphism. \square

4 Splitting the Tangent Bundle of a Pregroupoid

In this section we generalise the classical result that the bundle of source constant tangent vectors of a Lie groupoid splits as the product of the target bundle of the Lie groupoid and the bundle of source constant tangent vectors to the Lie groupoid based at an identity element. In fact we generalise this result in two directions. Firstly we replace the category of smooth manifolds with an arbitrary category equipped with a endofunctor which has a retraction onto the identity functor. Secondly we prove our results for the generalisation of groupoids called pregroupoids defined in [4].

4.1 Pregroupoids and an Involution

In this section we recall the definition of a pregroupoid. In addition we construct from this structure an involution that will be important in the sequel. Explicitly in Section 4.2 we assume the existence of an extra arrow in addition to the pregroupoid structure; in the presence of this extra arrow the involution we

define in this section will pullback in a natural way to give the splitting of the tangent bundle.

Notation 4.1. Let \mathbb{X} be a category and $T : \mathbb{X} \rightarrow \mathbb{X}$ a functor. Let $p : T \Rightarrow 1_{\mathbb{X}}$ and $0 : 1_{\mathbb{X}} \Rightarrow T$ be natural transformations such that 0 is a section of p .

The following definition is in Section 1 of [4].

Definition 4.2. A *pregroupoid* in \mathbb{X} consists of a span

$$\begin{array}{ccc} & X & \\ \alpha \swarrow & & \searrow \beta \\ A & & B \end{array} \quad (59)$$

in \mathbb{X} and a pre-composition $\mu = - \circ_- - : X_{\beta \times \beta} X_{\alpha \times \alpha} X \rightarrow X$ satisfying the following equations:-

- $\alpha(x \circ_y z) = \alpha(x)$;
- $\beta(x \circ_y z) = \beta(z)$;
- if $\beta(x) = \beta(y)$ then $x \circ_y y = x$;
- if $\alpha(x) = \alpha(y)$ then $x \circ_x y = y$;
- if $\beta(x) = \beta(y) = \beta(z)$ and $\alpha(z) = \alpha(w)$ then $x \circ_y (y \circ_z w) = x \circ_z w$;
- if $\beta(x) = \beta(y)$ and $\alpha(y) = \alpha(z) = \alpha(w)$ then $(x \circ_y z) \circ_z w = x \circ_y w$.

Lemma 4.3. *The arrow*

$$X_{\beta \times \beta} X_{\alpha \times \alpha} X \xrightarrow{\check{\xi}} X_{\beta \times \beta} X_{\alpha \times \alpha} X$$

defined by $(\pi_1, \pi_0, \pi_0 \circ_{\pi_1} \pi_2)$ is an involution.

Proof.

$$\begin{aligned} \check{\xi}\check{\xi} &= (\pi_1, \pi_0, \pi_0 \circ_{\pi_1} \pi_2)(\pi_1, \pi_0, \pi_0 \circ_{\pi_1} \pi_2) \\ &= (\pi_0, \pi_1, \pi_1 \circ_{\pi_0} (\pi_0 \circ_{\pi_1} \pi_2)) \\ &= (\pi_0, \pi_1, \pi_2) = 1 \end{aligned}$$

□

Definition 4.4. The arrow π^α is defined as the pullback

$$\begin{array}{ccc} T^\alpha X & \xrightarrow{\iota^\alpha} & TX \\ \downarrow \pi^\alpha & & \downarrow T\alpha \\ A & \xrightarrow{0} & TA \end{array} \quad (60)$$

and we write $p^\alpha = \iota^\alpha p$.

Lemma 4.5. *The arrow*

$$X_{\beta \times \beta} X_{\alpha \times \pi^\alpha} T^\alpha X t \xrightarrow{\xi} X_{\beta \times \beta} X_{\alpha \times \pi^\alpha} T^\alpha X$$

defined by $(\pi_1, \pi_0, (\pi_0 0, \pi_1 0, \pi_2 \iota^\alpha) T \mu)$ is an involution (i.e. $\xi \xi = 1_C$).

Proof.

$$\xi \xi = (\pi_1, \pi_0, (T \mu)(\pi_0 0, \pi_1 0, \pi_2))(\pi_1, \pi_0, (T \mu)(\pi_0 0, \pi_1 0, \pi_2)) \quad (61)$$

$$= (\pi_0, \pi_1, (T \mu)(\pi_1 0, \pi_0 0, (T \mu)(\pi_0 0, \pi_1 0, \pi_2))) \quad (62)$$

$$= (\pi_0, \pi_1, (T \mu)(\pi_1 0, \pi_1 0, \pi_2)) \quad (63)$$

$$= (\pi_0, \pi_1, \pi_2) \quad (64)$$

□

4.2 Splitting of the Vertical Bundle

In this section we prove that when we assume the existence of an arrow $e : B \rightarrow X$ such that $\beta e \beta = \beta$ that we can split the bundle $T_B^\alpha X$ as $X_{\beta \times \pi_B^\alpha} T_M^\alpha X$. The isomorphism exhibiting the splitting is the pullback of the involution ξ defined in Lemma 4.5 along the arrow $(\pi_0, \pi_1 p_B^\alpha, \pi_1 \iota_B) : X_{\beta \times \pi_M^\alpha} T_M^\alpha X \rightarrow X_{\beta \times \beta} X_{\alpha \times \pi^\alpha} T^\alpha X$.

Notation 4.6. Let $e : B \rightarrow X$ be an arrow in \mathbb{X} satisfying $\beta e \beta = \beta$. Note that if β is an epimorphism then e is a section of β .

Definition 4.7. The arrow $\pi_B^\alpha : T_B^\alpha X \rightarrow B$ is defined by the pullback

$$\begin{array}{ccc} T_B^\alpha X & \xrightarrow{\iota_B} & T^\alpha X \\ \downarrow \pi_B^\alpha & & \downarrow \iota^\alpha p \\ B & \xrightarrow{e} & X \end{array} \quad (65)$$

and we write $p_B^\alpha = \iota_B p^\alpha$.

Lemma 4.8. *The square*

$$\begin{array}{ccc} X_{\beta \times \pi_B^\alpha} T_B^\alpha X & \xrightarrow{(\pi_0, \pi_1 p_B^\alpha, \pi_1 \iota_B)} & X_{\beta \times \beta} X_{\alpha \times \pi^\alpha} T^\alpha X \\ \downarrow \pi_0 & & \downarrow (\pi_0, \pi_1, \pi_2 p^\alpha) \\ X & \xrightarrow{(1_X, \beta e, \beta e)} & X_{\beta \times \beta} X_{\alpha \times \alpha} X \end{array} \quad (66)$$

is a pullback in \mathbb{X} .

Proof. First the right square of

$$\begin{array}{ccccc} X_{\beta \times \pi_B^\alpha} T_B^\alpha X & \xrightarrow{\pi_1} & T_B^\alpha X & \xrightarrow{\iota_B} & T^\alpha X \\ \downarrow \pi_0 & & \downarrow \pi_B^\alpha & & \downarrow p^\alpha \\ X & \xrightarrow{\beta} & B & \xrightarrow{e} & X \end{array}$$

is a pullback by definition of π_B^α and the left square is immediately seen to be a pullback. Therefore the whole square is a pullback. Therefore the outer square of

$$\begin{array}{ccccc} X_{\beta} \times_{\pi_B^\alpha} T_B^\alpha X & \xrightarrow{(\pi_0, \pi_1 p_B^\alpha p, \pi_1 \iota_B)} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\pi_\alpha} T^\alpha X & \xrightarrow{\pi_2} & T^\alpha X \\ \downarrow \pi_0 & & \downarrow (\pi_0, \pi_1, \pi_2 \iota^\alpha p) & & \downarrow p^\alpha \\ X & \xrightarrow{(1_X, \beta e, \beta e)} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X & \xrightarrow{\pi_2} & X \end{array} \quad (67)$$

is a pullback by the definition of $T_B^\alpha X$ and the right hand square is immediately seen to be a pullback also. Therefore by the pullback pasting lemma the left hand square is a pullback as required. \square

Lemma 4.9. *The square*

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \downarrow (\beta e, 1_X, 1_X) & & \downarrow (1_X, \beta e, \beta e) \\ X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X & \xrightarrow{\check{\xi}} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X \end{array} \quad (68)$$

is a pullback.

Proof. First the square commutes because

$$(\beta e, 1_X, 1_X) \check{\xi} = (\beta e, 1_X, 1_X) (\pi_1, \pi_0, \pi_0 \circ \pi_1 \pi_2) \quad (69)$$

$$= (1_X, \beta e, \beta e \circ 1_X) = (1_X, \beta e, \beta e) \quad (70)$$

Second the diagram is a pullback because 1_x and $\check{\xi}$ are isomorphisms (and the diagram commutes). \square

Lemma 4.10. *The square*

$$\begin{array}{ccc} T^\alpha X & \xrightarrow{(p^\alpha \beta e, p^\alpha, 1_{T^\alpha X})} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\pi_\alpha} T^\alpha X \\ \downarrow p^\alpha & & \downarrow (\pi_0, \pi_1, \pi_2 p^\alpha) \\ X & \xrightarrow{(\beta e, 1_X, 1_X)} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X \end{array} \quad (71)$$

is a pullback.

Proof. In the diagram

$$\begin{array}{ccccc} T^\alpha X & \xrightarrow{(p^\alpha \beta e, p^\alpha, 1_{T^\alpha X})} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\pi_\alpha} T^\alpha X & \xrightarrow{\pi_2} & T^\alpha X \\ \downarrow \iota^\alpha p & & \downarrow (\pi_0, \pi_1, \pi_2 p^\alpha) & & \downarrow p^\alpha \\ X & \xrightarrow{(\beta e, 1_X, 1_X)} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X & \xrightarrow{\pi_2} & X \end{array} \quad (72)$$

the right hand square is immediately seen to be a pullback and the whole square is the trivial pullback. Therefore by the pasting lemma the left square is a pullback as required. \square

Proposition 4.11. *The objects $X_{\beta} \times_{\pi_B^{\alpha}} T_B^{\alpha} X$ and $T^{\alpha} X$ are isomorphic.*

Proof. In the cube

$$\begin{array}{ccccc}
 T^{\alpha} X & \xrightarrow{(\iota^{\alpha} p \beta e, \iota^{\alpha} p, 1_{T^{\alpha} X})} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\pi^{\alpha}} T^{\alpha} X & & \\
 \downarrow p^{\alpha} & \searrow & \downarrow \xi & & \downarrow \\
 & X_{\beta} \times_{\pi_B^{\alpha}} T_B^{\alpha} X \xrightarrow{(\pi_0, \pi_1 p_B^{\alpha}, \pi_1 \iota_B)} X_{\beta} \times_{\beta} X_{\alpha} \times_{\pi^{\alpha}} T^{\alpha} X & & & \\
 & \downarrow \pi_0 & \downarrow (\pi_0, \pi_1, \pi_2 p^{\alpha}) & & \downarrow (\pi_0, \pi_1, \pi_2 p^{\alpha}) \\
 & X \xrightarrow{(1_X, \beta e, \beta e)} X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X & & & \\
 \downarrow 1_X & \nearrow & \downarrow \tilde{\xi} & & \\
 X & \xrightarrow{(\beta e, 1_X, 1_X)} & X_{\beta} \times_{\beta} X_{\alpha} \times_{\alpha} X & &
 \end{array} \tag{73}$$

the middle square, the lower square and the outer square are pullbacks by Lemma 4.8, Lemma 4.9 and Lemma 4.10 respectively. \square

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