**227-0689-00L System Identification** **14 November 2023**

# Solution 6: Pulse Response and Persistency of excitation

We will not provide solutions for the MATLAB parts of the exercises. These will instead be discussed in the exercises sessions.

## Problem 1:

Consider the pulse response estimation problem,

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(K-1) \end{bmatrix} = \begin{bmatrix} u(0) & u(-1) & \cdots & u(-\tau_{\max}) \\ u(1) & u(0) & \cdots & u(-\tau_{\max}+1) \\ \vdots & & \ddots & \\ u(K-1) & u(K-1) & \ldots & u(K-\tau_{\max}+1) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(\tau_{\max}) \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(K-1) \end{bmatrix}. \tag{6.1}$$

and suppose that for all $\epsilon > 0$, there exists $\tau_{\max}$ such that $\sum_{i=\tau_{\max}+1}^{\infty} |g(i)| \leq \epsilon$. Show that

$$\|e\|_\infty \leq \epsilon \|u\|_\infty \tag{6.2}$$

holds in the following cases:

1. The case where $u(k) = 0$ for all $k < 0$.
2. The case where $u(k)$ is unknown for $k < 0$, but is bounded by $\|u\|_\infty$.
3. The case as in Part 2, except where one measurement $y(j)$ is corrupted and removed, where $0 < j < k - 1$.

In each case, write the relation between $K$ and $\tau_{\max}$ necessary for the bound (6.2) to hold.

### Solution

1. Since the system is at rest, we need $K > \tau_{\max}$. The truncation error at time $k$ is given by,

$$e(k) = \sum_{\tau=\tau_{\max}+1}^{\infty} g(\tau) u(k-\tau). \tag{6.3}$$

Since $u(k) = 0$ for all $k < 0$, $u(k-\tau) = 0$ whenever $k - \tau < 0$, or whenever $k < \tau$. Therefore, we know the value of any $u(k-\tau)$ appearing in (6.3), and hence we know the value of $\|u\|_\infty$. Therefore, we can write,

$$\begin{aligned}
|e(k)| &= \left| \sum_{\tau=\tau_{\max}+1}^{\infty} g(\tau) u(k-\tau) \right| && (6.4)\\
&\leq \sum_{\tau=\tau_{\max}+1}^{\infty} |g(\tau)| \cdot |u| && (6.5)\\
&\leq \sum_{\tau=\tau_{\max}+1}^{\infty} |g(\tau)| \cdot \|u\|_\infty && (6.6)\\
&= \|u\|_\infty \sum_{\tau=\tau_{\max}+1}^{\infty} |g(\tau)| && (6.7)\\
&\leq \epsilon \|u\|_\infty, && (6.8)
\end{aligned}$$

by assumption.

2. Since the inputs $u(k)$ for $k < 0$ are unknown, we need $K > 2\tau_{\max}$, since we must discard the first $\tau_{\max}$ measurements (and errors), as well as the first $\tau_{\max}$ rows of $\Phi_u$. Therefore, we only consider the errors $e(k)$ for $k > \tau_{\max}$. However, we know that all $u(k)$ for $k < 0$ satisfy $|u(k)| \leq \|u\|_\infty$ where $\|u\|_\infty$ is known, and so the calculation in the previous part holds.

3. Since the inputs $u(k)$ for $k < 0$ are unknown, we need $K > 2\tau_{\max} + 1$ since we will discard the first $\tau_{\max}$ measurements (and errors), and the first $\tau_{\max}$ rows of $\Phi_u$. We are also discarding $y(j)$ since it is corrupted, and therefore the $j$th row of $\Phi_u$, as well as $e(j)$. Since $e(k)$ for $k \neq j, k > \tau_{\max}$ does not depend on $e(j)$, the same calculation from Part 1 holds.

## Problem 2:

1. Let $u(k)$ be a PRBS signal of length $N$. Show that $u(k)$ is persistently exciting of order $N$, but not of order $N+1$.

2. Let $u(k)$ be an input signals defined as

$$u(k) = \sum_{i=1}^{S} \alpha_i \cos(\omega_i k), \tag{6.9}$$

where $\omega_1, \ldots, \omega_S \in (0, \pi)$ are different frequencies. Prove that $u(k)$ is persistently exciting of order $2S$.

3. Show that the step signal, defined as $u(k) = 1$, for all $k \neq 0$, is persistently exciting of order 1.

## Solution

1. The autocorrelation of the $N$ length PRBS, $u(k)$, is

$$R_u(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} u(k)u(k-\tau) = \begin{cases} \bar{u}^2, & \text{for } \tau = 0, \pm N, \pm 2N, \dots \\ -\frac{\bar{u}^2}{N}, & \text{else,} \end{cases}$$

where $\bar{u}$ is the amplitude of the PRBS.

The autocorrelation matrix $\bar{R}_n$ is defined as

$$\bar{R}_n := \begin{bmatrix} R_u(0) & R_u(-1) & \dots & R_u(-(n-1)) \\ R_u(1) & R_u(0) & \dots & R_u(-(n-2)) \\ \vdots & \vdots & & \vdots \\ R_u(n-1) & R_u(n-2) & \dots & R_u(0)) \end{bmatrix}.$$

The signal $u(k)$ is persistently exciting of order $n$ if and only if $\bar{R}_n$ is non-singular.

Consider order $N$:

$$\bar{R}_N = \begin{bmatrix} \bar{u}^2 & -\frac{\bar{u}^2}{N} & \dots & -\frac{\bar{u}^2}{N} \\ -\frac{\bar{u}^2}{N} & \bar{u}^2 & \dots & -\frac{\bar{u}^2}{N} \\ \vdots & \vdots & & \vdots \\ -\frac{\bar{u}^2}{N} & -\frac{\bar{u}^2}{N} & \dots & \bar{u}^2 \end{bmatrix}.$$

$\bar{R}_N$ is a strictly diagonally dominant matrix

$$|\bar{R}_{N_{ii}}| = \bar{u}^2 > \sum_{j\neq i} |\bar{R}_{N_{ij}}| = \frac{N-1}{N}\bar{u}^2 \qquad \forall i,$$

and is therefore non-singular (see Levy-Desplanques theorem).

For order $N+1$, however, we have

$$\bar{R}_{N+1} = \begin{bmatrix} \bar{u}^2 & -\frac{\bar{u}^2}{N} & \dots & -\frac{\bar{u}^2}{N} & \bar{u}^2 \\ -\frac{\bar{u}^2}{N} & \bar{u}^2 & \dots & -\frac{\bar{u}^2}{N} & -\frac{\bar{u}^2}{N} \\ \vdots & \vdots & & \vdots & \vdots \\ -\frac{\bar{u}^2}{N} & -\frac{\bar{u}^2}{N} & \dots & \bar{u}^2 & -\frac{\bar{u}^2}{N} \\ \bar{u}^2 & -\frac{\bar{u}^2}{N} & \dots & -\frac{\bar{u}^2}{N} & \bar{u}^2 \end{bmatrix}.$$

The first row of $\bar{R}_{N+1}$ is equal to the last row

$$\bar{R}_{N_{1j}} = \bar{R}_{N_{(N+1)j}} \qquad \forall j.$$

Thus, $\bar{R}_{N+1}$ is singular.

2. Define signal $u_i(k) = \alpha_i \cos(\omega_i k)$, for $i = 1, \dots, S$. From Example 2.3 in the textbook (Ljung), we know that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} u_i(k)u_i(k-\tau) = \frac{\alpha_i^2}{2}\cos(\omega_i \tau). \tag{6.10}$$

Also, for $i \neq j$, we have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}u_i(k)u_j(k-\tau) &= \alpha_i\alpha_j\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\cos(\omega_i k)\cos(\omega_j(k-\tau)) \\ &= \alpha_i\alpha_j\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2}[\cos((\omega_i-\omega_j)k+\omega_j\tau)+\cos((\omega_i+\omega_j)k-\omega_j\tau)] \\ &= \frac{\alpha_i\alpha_j}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\cos((\omega_i-\omega_j)k+\omega_j\tau)+\frac{\alpha_i\alpha_j}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\cos((\omega_i+\omega_j)k-\omega_j\tau) \\ &= 0.\end{aligned} \tag{6.11}$$

Therefore, the autocorrelation $R_u$ of the sum of sinusoids is

$$R_u(\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}u(k)u(k-\tau) = \sum_{i=1}^{S}\frac{\alpha_i^2}{2}\cos(\omega_i\tau), \tag{6.12}$$

and

$$\phi_u(e^{\mathrm{j}\omega}) = \sum_{i=1}^{S}\frac{\alpha_i^2\pi}{2}[\delta(e^{\mathrm{j}(\omega-\omega_i)})+\delta(e^{\mathrm{j}(\omega+\omega_i)})], \tag{6.13}$$

because the FT of $\cos(\omega_i\tau) = \pi(\delta(\omega-\omega_i)+\delta(\omega+\omega_i)$. Therefore, the spectrum is non-zero at $2S$ points. Using Definition 13.1 of persistent excitation in Ljung, the input is persistently exciting of order $2S$.

3. The autocorrelation of $u(k)$ is

$$R_u(\tau) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}u(k)u(k-\tau) = \lim_{N\to\infty}\frac{N-\tau}{N} = 1.$$

Therefore, we have

$$\bar{R}_n := \begin{bmatrix} R_u(0) & R_u(-1) & \dots & R_u(-(n-1)) \\ R_u(1) & R_u(0) & \dots & R_u(-(n-2)) \\ \vdots & \vdots & & \vdots \\ R_u(n-1) & R_u(n-2) & \dots & R_u(0)) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

One can easily see that $\bar{R}_n$ is positive definite only for $n = 1$.

# Matlab exercise:

Consider the ARX model

$$y(t) = a\cdot y(t-1)+b\cdot u(t-1)+c\cdot u(t-2)+v(t) \tag{6.14}$$

with $a = 5/8, b = 11/10, c = 5/6$, and $v(t)$ is a sequence of independent and identically distributed (i.i.d.) random variables with covariance 0.05.

1. The input will be a sum of sinusoids with $u(k) \in [-2, 2]$. Select an appropriate number of sinusoids using frequencies between (but not including) 0 and $\pi$. Assume that the system starts at rest. Apply two periods of this signal to the model (6.14), and using an appropriate pulse response model of the form (6.1), solve for $\hat{g}(0), \ldots, \hat{g}(80)$. Repeat the experiment 3 times (using different noise realizations), and average the resulting $\hat{g}$, i.e,

$$\hat{g} = \frac{1}{3}\sum_{i=1}^{3} \hat{g}_i. \tag{6.15}$$

2. Next, apply $N$ periods of the signal to the model (6.14) for $N \in \{2, \ldots, 30\}$, and using an appropriate pulse response model of the form (6.1), solve for $g(0), \ldots, g(80)$. Plot the error $\|\hat{g}_N - g_0\|_2$ as a function of $N$, where $g_0$ is computed with noiseless measurements. At what $N$ does the error become smaller than the averaging procedure in the previous part?

   **Hint:** Once you have code that solves parts 1 and 2 together, run it several times to see how sensitive your answer is to the random noise realization.

3. Repeat the previous part, except use a signal of length $N\tau_{\max} = 80N$ for $N \in \{2, \ldots, 30\}$ generated using the matlab command `rand`, i.e. uniform random noise. Does the error $\|\hat{g}_N - g_0\|_2$ become smaller than the averaging procedure in part 1?

## Solution hints

1. To generate the sum of sinusoids signal, you can use the function `idinput`, or you can directly compute the input $u(k)$ as follows

$$u(k) = \sum_{s=1}^{S} \alpha_s \cos(\omega_s k + \phi_s),$$

   where $\omega_s \in (0, \pi)$, and $S = 40$ to ensure that the system is persistently exiting of order $2S = 80$.

   You can use the command `toeplitz(c,r)` to get $\Phi_u$, where $c$ and $r$ are defined using the input signal and the initial conditions of the system. To make the noise for the system measurements, you can use `sqrt(sig)*randn`, which produces a sample of a Gaussian distribution with covariance `sig`. A representative solution of part 1 is shown in Figure 6.1.

2. Part 2 and 3: In the considered example, the sum of sinusoids and the random signal perform in a similar way, depending on how the parameters are chosen. In particular, the result strongly depends on how the frequencies $\omega_s$, phases $\phi_s$, and amplitudes $\alpha_s$ of the sin functions are chosen. In the Figure below, the frequencies are equally spaced between $(0, \pi)$, the phases are randomly chosen, while the amplitudes are all equal to 1. For the sum of sinusoids, and for the random signal, the input vector has been scaled so that the input takes advantage of the entire input space: $u = 2u/\max(|u|)$.

   A representative solution can be seen in Figure 6.2

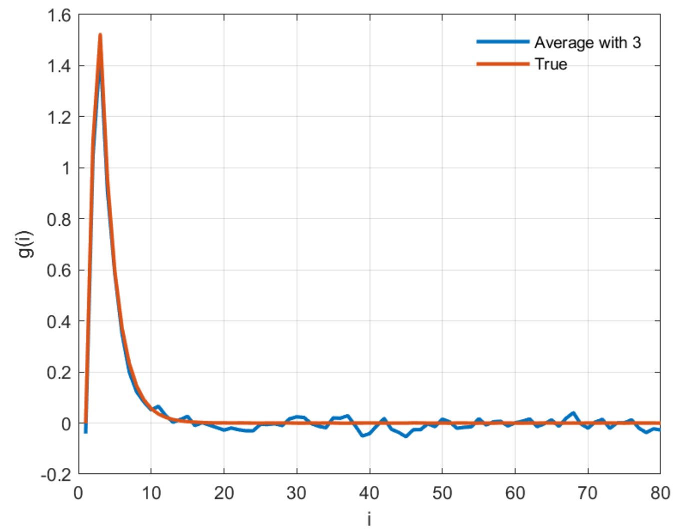


Figure 6.1: Representative results of the `matlab` simulation for Part 1

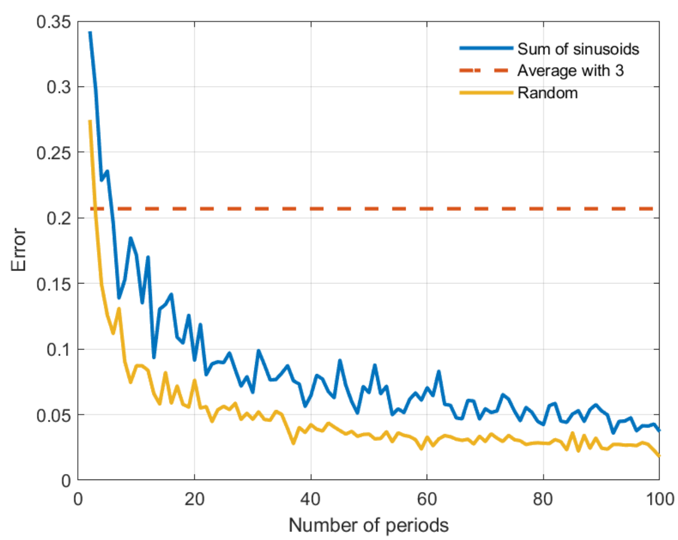


Figure 6.2: Representative results of the `matlab` simulation for Part 2 and 3