Solution 4: Empirical Transfer Function Estimation

Problem 1:

Consider the ETFE of the LTI system $G\left(e^{j\omega}\right)$ given by

$$\hat{G}\left(e^{j\omega_n}\right) = \frac{Y_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)} = G\left(e^{j\omega_n}\right) + \frac{R_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)} + \frac{V_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}.$$

where $U_N\left(e^{j\omega_n}\right), Y_N\left(e^{j\omega_n}\right), R_N\left(e^{j\omega_n}\right), V_N\left(e^{j\omega_n}\right)$ are the discrete Fourier transform of the input, output, transient, and noise respectively.

a) If the transient is neglected and the noise is zero-mean, the ETFE $\hat{G}\left(e^{j\omega_n}\right)$ is unbiased. However, this does not imply that $\left|\hat{G}\left(e^{j\omega_n}\right)\right|$ is an unbiased estimate of $\left|G\left(e^{j\omega_n}\right)\right|$. Show that

$$\mathbb{E}\left\{\left|\hat{G}\left(e^{j\omega_n}\right)\right|^2\right\} = \left|G\left(e^{j\omega_n}\right)\right|^2 + \frac{\phi_v\left(e^{j\omega_n}\right)}{\frac{1}{N}|U_N\left(e^{j\omega_n}\right)|^2}$$

asymptotically for large N with $\phi_v\left(e^{j\omega_n}\right)$ defined as the noise spectrum.

b) If the transient is neglected and the noise is zero-mean, show that the estimates at different frequencies satisfy

$$\mathbb{E}\left\{\left(\hat{G}\left(e^{j\omega_n}\right) - G\left(e^{j\omega_n}\right)\right)\left(\hat{G}\left(e^{j\omega_i}\right) - G\left(e^{j\omega_i}\right)\right)^*\right\} = 0, \quad \omega_n \neq \omega_i.$$

c) If the input is random, show that the expected transient bias error $\mathbb{E}\left\{\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2\right\} \to 0$, as $N \to \infty$, with a convergence rate of $\frac{1}{N}$.

d) If the input is periodic, show that the transient bias error $\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2 \to 0$, as $N \to \infty$, with a convergence rate of $\frac{1}{N^2}$.

Hint: First show that for a periodic signal $u(k)$ of period M, $U_{mM}\left(\mathrm{e}^{j\omega_n}\right) = mU_M\left(\mathrm{e}^{j\omega_n}\right)$.

Solution

a) Neglecting transients we have $\hat{G}\left(e^{j\omega_n}\right) = G\left(e^{j\omega_n}\right) + \dfrac{V_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}$, such that

$$\begin{aligned}
\mathbb{E}\left\{\left|\hat{G}\left(e^{j\omega_n}\right)\right|^2\right\} &= \mathbb{E}\left\{\hat{G}\left(e^{j\omega_n}\right)\overline{\hat{G}\left(e^{j\omega_n}\right)}\right\} \\
&= \mathbb{E}\left\{\hat{G}\left(e^{j\omega_n}\right)\hat{G}\left(e^{-j\omega_n}\right)\right\} \\
&= \mathbb{E}\left\{\left(G\left(e^{j\omega_n}\right) + \frac{V_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}\right)\left(G\left(e^{-j\omega_n}\right) + \frac{V_N\left(e^{-j\omega_n}\right)}{U_N\left(e^{-j\omega_n}\right)}\right)\right\} \\
&= \left|G\left(e^{j\omega_n}\right)\right|^2 + \frac{G\left(e^{j\omega_n}\right)}{U_N\left(e^{-j\omega_n}\right)}\mathbb{E}\left\{V_N\left(e^{-j\omega_n}\right)\right\} + \\
&\qquad \frac{G\left(e^{-j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}\mathbb{E}\left\{V_N\left(e^{j\omega_n}\right)\right\} + \frac{\mathbb{E}\left\{\left|V_N\left(e^{j\omega_n}\right)\right|^2\right\}}{\left|U_N\left(e^{j\omega_n}\right)\right|^2}.
\end{aligned}$$

Since the noise is zero-mean, it follows that

$$\mathbb{E}\left\{V_N\left(e^{-j\omega_n}\right)\right\} = \mathbb{E}\left\{V_N\left(e^{j\omega_n}\right)\right\} = 0.$$

Note that the periodogram of the noise is an asymptotically unbiased estimator of the spectrum:

$$\lim_{N\to\infty}\mathbb{E}\left\{\frac{1}{N}\left|V_N\left(e^{j\omega_n}\right)\right|^2\right\} = \phi_v\left(e^{j\omega_n}\right).$$

We have

$$\mathbb{E}\left\{\left|\hat{G}\left(e^{j\omega_n}\right)\right|^2\right\} = \left|G\left(e^{j\omega_n}\right)\right|^2 + \frac{\phi_v\left(e^{j\omega_n}\right)}{\frac{1}{N}\left|U_N\left(e^{j\omega_n}\right)\right|^2}$$

asymptotically for large N.

b)

$$\begin{aligned}
\mathbb{E}\left\{V_N\left(e^{j\omega_n}\right)V_N^*\left(e^{j\omega_i}\right)\right\} &= \sum_{r=0}^{N-1}\sum_{s=0}^{N-1}\mathbb{E}\left\{v(r)\mathrm{e}^{-j\omega_n r}v(s)\mathrm{e}^{j\omega_i s}\right\} \\
&= \sum_{r=0}^{N-1}\sum_{s=0}^{N-1}R_v(r-s)\mathrm{e}^{-j(\omega_n r-\omega_i s)} \\
&= \sum_{r=0}^{N-1}\mathrm{e}^{-j(\omega_n-\omega_i)r}\sum_{\tau=r-N+1}^{r}R_v(\tau)\mathrm{e}^{-j\omega_i\tau},
\end{aligned}$$

where the last equality uses variable transformation $\tau = r - s$. Since $\omega_n - \omega_i = \dfrac{2\pi(n-i)}{N}$, $n - i \neq 0$,

$$\sum_{r=0}^{N-1}\mathrm{e}^{-j(\omega_n-\omega_i)r} = \frac{1-\mathrm{e}^{-j(\omega_n-\omega_i)N}}{1-\mathrm{e}^{-j(\omega_n-\omega_i)}} = 0. \quad \text{(sum of geometric series)}$$

So,

$$\mathbb{E}\left\{V_N\left(e^{j\omega_n}\right)V_N^*\left(e^{j\omega_i}\right)\right\}=0,$$

and

$$\mathbb{E}\left\{\left(\hat{G}\left(e^{j\omega_n}\right)-G\left(e^{j\omega_n}\right)\right)\left(\hat{G}\left(e^{j\omega_i}\right)-G\left(e^{j\omega_i}\right)\right)^*\right\}=\frac{\mathbb{E}\left\{V_N\left(e^{j\omega_n}\right)V_N^*\left(e^{j\omega_i}\right)\right\}}{U_N\left(e^{j\omega_n}\right)U_N^*\left(e^{j\omega_i}\right)}=0.$$

c) For random inputs, according to the property of the periodogram,

$$\lim_{N\to\infty}\mathbb{E}\left\{\frac{1}{N}\left|U_N\left(e^{j\omega_n}\right)\right|^2\right\}=\phi_u\left(e^{j\omega_n}\right).$$

Since the transient response is decaying exponentially, it is finitely summable. So the DFT of the transient response $R_N\left(\mathrm{e}^{j\omega_n}\right)$ converges to the DTFT as $N\to\infty$. So,

$$\lim_{N\to\infty}\mathbb{E}\left\{N\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2\right\}=\frac{|R\left(\mathrm{e}^{j\omega_n}\right)|^2}{\phi_u\left(e^{j\omega_n}\right)},$$

i.e., $\mathbb{E}\left\{\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2\right\}$ converges to zero at a rate of $\frac{1}{N}$ for random inputs.

d) The DFT for a single period of the signal is

$$U_M(e^{j\omega_n})=\sum_{k=0}^{M-1}u(k)e^{-jk\omega_n}.$$

Now consider increasing the experiment length to $N=mM$ where m is an integer. Then,

$$\begin{aligned}U_{mM}(e^{j\omega_n})&=\sum_{k=0}^{mM-1}u(k)e^{-jk\omega_n}=\sum_{r=0}^{m-1}\sum_{s=0}^{M-1}u(s+rM)e^{-j(s+rM)\omega_n}\\&=\sum_{r=0}^{m-1}\sum_{s=0}^{M-1}u(s)e^{-js\omega_n}e^{-jrM\omega_n}=m\sum_{s=0}^{M-1}u(s)e^{-js\omega_n}=mU_M(e^{j\omega_n})\end{aligned}$$

since $M\omega_n=2\pi n$. So for periodic inputs, $\left|U_N\left(\mathrm{e}^{j\omega_n}\right)\right|\to N\left|U_M\left(\mathrm{e}^{j\omega_n}\right)\right|/M$ as $N\to\infty$.

According to part c), $\left|R_N\left(\mathrm{e}^{j\omega_n}\right)\right|$ converges as $N\to\infty$. So $\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2$ converges to zero at a rate of $\frac{1}{N^2}$ for periodic inputs.

MATLAB **Exercise:**

Consider the discrete-time system

$$G(z) = \frac{0.1z}{z^4 - 2.2z^3 + 2.42z^2 - 1.87z + 0.7225}$$

and noise model

$$H(z) = \frac{0.5(z - 0.9)}{(z - 0.25)}.$$

The measured output $y(k)$ is defined as the sum of the outputs of $G(z)$ and $H(z)$, such that

$$y(k) = Gu(k) + He(k),$$

where the noise signal $v(k) = He(k)$ is driven by Gaussian white noise $e \sim \mathcal{N}(0, 0.01)$. The input $u(k)$ is bounded between -1 and 1. The sample time can be taken as 1 s.

a) Generate a 1024-point MATLAB simulation as an identification 'experiment'.

b) Estimate $G(z)$ by ETFE and compare it to the true system - particularly in the frequency range around the resonant peak. Plot both the transfer functions and the magnitude of the errors.

c) Split the data record into 4 parts and calculate an averaged ETFE from the 4 parts. Compare this to the true system as well as the original 1024-point ETFE.

d) Repeat the first three parts with different 'experiments'. How can you achieve the best estimate?

Solution hints

a) You should first generate the noise (e) and input (u) signals of the appropriate length. The noise signal can be generated by randomly distributed vectors using the `randn` command. You can freely design the input signal as long as it is bounded between -1 and 1.

   Transfer functions $G(z)$ and $H(z)$ can be defined using the `tf` function. The noise-free signal y' and the colored noise v can then be obtained using the `lsim` function with the transfer functions $G(z)$ and $H(z)$ and the vectors u and e as inputs. Finally, the measured output can be generated by adding together y' and v.

b) You can calculate the DFT of the input and output signals $U_N\left(e^{j\omega_n}\right)$ and $Y_N\left(e^{j\omega_n}\right)$ by applying the `fft` function to the vectors u and y. The ETFE is then obtained by point-wise division of $Y_N\left(e^{j\omega_n}\right)$ by $U_N\left(e^{j\omega_n}\right)$ (./ in MATLAB). Note that due to the symmetry of the DFT, you actually need to take care of just half of the points in $U_N(\omega)$ and $Y_N(\omega)$ corresponding to non-negative frequencies.

   You can calculate the frequency response of the true systems $G(z)$ and $H(z)$ for the same frequencies at which the DFT of u and y are calculated using the `freqresp` function. With the `loglog` command, you can plot the amplitude of the estimate and the actual frequency responses in logarithmic scale for both axes.

c) You can form 4 new input and output signals (denoted as u_i and y_i, $i = 1, 2, 3, 4$) by splitting the initial vectors u and y into 4 vectors of same length. For each i you can then repeat the steps from part b) in order to calculate the ETFE $\hat{G}_i, i = 1, 2, 3, 4$. The averaged estimate can then be calculated by taking the average over these 4 estimates using the `mean` function. Note that the ETFE in this case is calculated for different frequency values to part b) since different data lengths are used.

   Compared to the results obtained in part b), it should be observed that the estimate is less noisy but the resolution is poorer.

d) There are mainly two things you can do to enhance the quality of your estimate:

   1. Increase the power of your input signal. For systems with constraints on the magnitude of inputs, a good choice of the input is random binary signals, which can be generated by the `idinput` command. You will learn more about input design in the next lecture.
   2. Use periodic input signals. To reduce the transient bias error, you can choose a periodic input signal such that the 4 parts in part c) are obtained with the same input signal of length 256. In this case, you can discard the first part and obtain the averaged estimate with the last three parts. In this way, the transient bias error will almost vanish.

You can find the sample plots as follows with a random binary input signal.

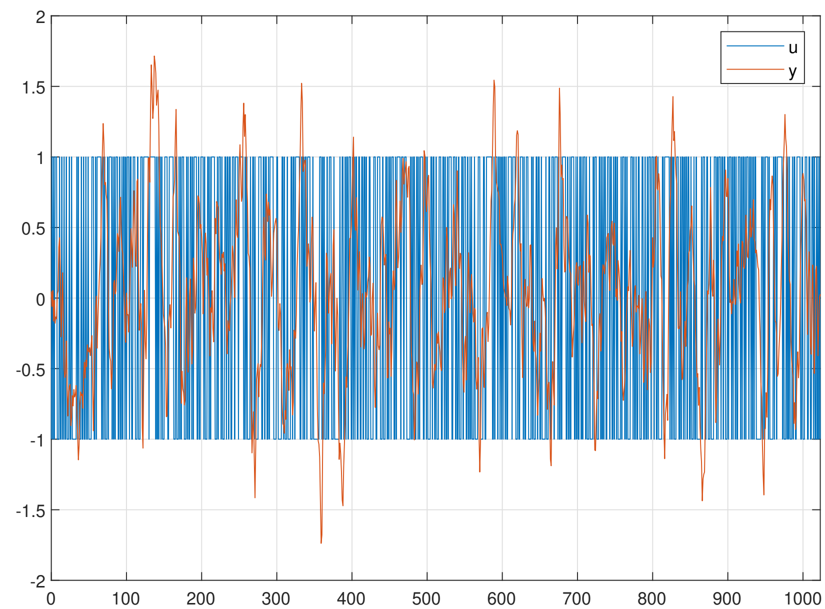

Figure 4.1: ‘Experiment’ data with a random binary input signal.

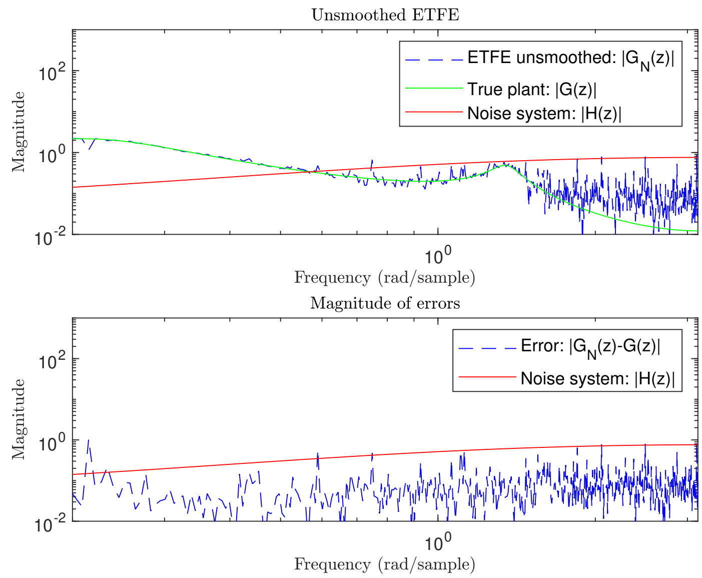

Figure 4.2: Comparison of the ETFE and the true plant.

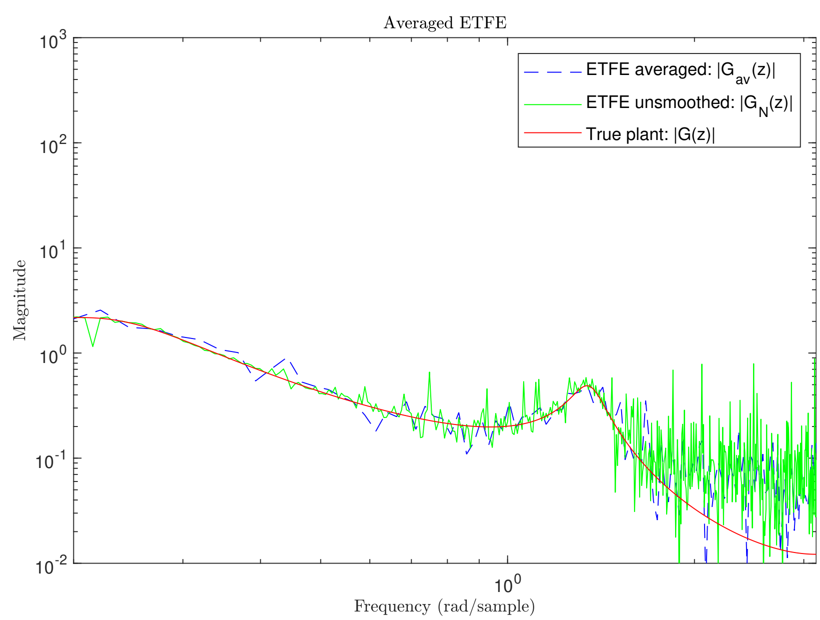

Figure 4.3: Comparison of the averaged ETFE and the full-length ETFE.