227-0689-00L System Identification **13 December 2023**

Exercise 10: Closed-loop Identification

Background reading

The background material for this exercise is the Slides of the last lecture.

Problem 1

Consider the following ARX system:

$$y(k) + a_\circ\, y(k-1) = b_\circ\, u(k-1) + v(k)\,,$$

where the system parameters $a_\circ$ and $b_\circ$ are unknown, and $v(k)$ is zero-mean white noise. The system is to be operated in closed-loop with an output feedback where

$$u(k) = r(k) - Ky(k)$$

where $K \in \mathbb{R}$ is a known feedback gain and r is an external reference signal independent of v.

a) Suppose that an input-output data record: $\{u(k), k = 0, \ldots, N-1 \text{ and } y(k), k = 0, \ldots, N\}$ is obtained in closed loop when the external reference signal is equal to zero for all times; i.e., $r(k) = 0$ for all $k = 0, \ldots, N$. Write down the cost function of the linear least-squares (LS) estimator $\begin{bmatrix} a_{\text{LS}} & b_{\text{LS}} \end{bmatrix}^T$ of the system's parameters as well as the normal equation. How many solutions does the normal equation have?

   Hint: Show that if $\begin{bmatrix} a_{\text{LS}} & b_{\text{LS}} \end{bmatrix}^T$ is a solution, then $\begin{bmatrix} a_{\text{LS}} + Kx & b_{\text{LS}} - x \end{bmatrix}^T$ for all $x \in \mathbb{R}$ is also a solution.

b) Assume now that the experiment is repeated with non-zero arbitrary reference signal $r(k)$ and the reference-output data $\{r(k), k = 0, \ldots, N-1 \text{ and } y(k), k = 0, \ldots, N\}$ is recorded. Set up the least-squares estimation problem to estimate a and b using the new data set. Can the system be estimated in this case? Is the LS estimate consistent in this case?

Solution:

a) The cost function for the LS estimator is $\|Y - \Phi\theta\|_2^2$ where

$$Y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix}, \qquad \Phi = \begin{bmatrix} -y(0) & u(0) \\ \vdots & \vdots \\ -y(N-1) & u(N-1) \end{bmatrix}, \qquad \theta = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Because the data is collected in closed loop, and $u(k) = -Ky(k)$, the regression matrix becomes

$$\Phi = \begin{bmatrix} -y(0) & -Ky(0) \\ \vdots & \vdots \\ -y(N-1) & -Ky(N-1) \end{bmatrix}$$

which is rank difficient. The normal equation $(\Phi^T\Phi)\theta = \Phi^T Y$ in this case is

$$\begin{bmatrix} \sum_{k=0}^{N-1}[y(k)]^2 & K\sum_{k=0}^{N-1}[y(k)]^2 \\ K\sum_{k=0}^{N-1}[y(k)]^2 & \sum_{k=0}^{N-1}[y(k)]^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\begin{bmatrix} \sum_{k=0}^{N-1} y(k)y(k+1) \\ K^2\sum_{k=0}^{N-1} y(k)y(k+1) \end{bmatrix}$$

The LS estimate a_{LS} and b_{LS} satisfied this equation by definition. But $a_{\text{LS}} + Kx$ and $b_{\text{LS}} - x$, for any real value x, also satisfies the normal equation. This shows that the normal equation has infinite number of solutions, and therefore the system's parameters cannot be estimated under the given experimental configurations. Notice that it did not help that we know K; the obtained data is not informative enough. Also notice that u is persistently exciting since it consists of filtered white noise. Concequently, persistency of excitation is not sufficient condition on the input in closed-loop experiments.

b) In this case one could consider the equivalent (open-loop) ARX system

$$y(k) + \underbrace{(a + Kb)}_{\bar{a}}\, y(k-1) = b\, r(k-1) + v(k)\,.$$

Therefore, a standard least squares estimate $\begin{bmatrix} \bar{a}_{\text{LS}} & b_{\text{LS}} \end{bmatrix}^T$ can be obtained from reference-output data $r(k), y(k)$. The normal equation in this case is

$$\begin{bmatrix} \sum_{k=0}^{N-1}[y(k)]^2 & \sum_{k=0}^{N-1}[y(k)r(k)] \\ \sum_{k=0}^{N-1}[y(k)r(k)] & \sum_{k=0}^{N-1}[r(k)]^2 \end{bmatrix} \begin{bmatrix} \bar{a} \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{N-1} y(k)y(k+1) \\ \sum_{k=0}^{N-1} r(k)y(k+1) \end{bmatrix}$$

Note that in order to estimate $\bar{a}$ and b, $r(k)$ should be persistently exciting of order at least 1. Then, an estimate a_{LS} of a can be obtained by the transformation $a_{\text{LS}} = \bar{a}_{\text{LS}} - Kb_{\text{LS}}$. Because $\bar{a}_{\text{LS}}, b_{\text{LS}}$ are consistent, a_{LS} will also be consistent being a linear function of $[\theta_{\text{LS}}, b_{\text{LS}}]$.

Problem 2

Consider a situation where an unknown plant G is stabilized by a known feedback controller $C(z) = \frac{X_C(z)}{Y_C(z)}$ (with $X_C(z)$ and $Y_C(z)$ stable and coprime) and the output, input and reference are related by the following equations

$$\begin{aligned} y(t) &= G(z)u(t) + H(z)e(t) \\ u(t) &= r(t) - C(z)y(t) \end{aligned}$$

Let $\frac{N_\circ}{D_\circ}$ be any system that is stabilized by C. The dual Youla parameterization set

$$\mathcal{G} = \left\{ G(z) \;:\; G(z) = \frac{N_\circ(z) + Y_C(z)R(z)}{D_\circ(z) - X_C(z)R(z)} \right\}$$

gives all transfer function models $G(z)$ that are stabilized by $C(z)$ as the Youla parameter $R(z)$ ranges over all stable transfer functions. As shown in the lecture, an equivalent open-loop rearrangement is given as

$$\alpha(t) = R(z)\beta(t) + F(z)e(t) \tag{10.1}$$

where $\alpha(t) = D_\circ(z)y(t) - N_\circ(z)u(t)$ and $\beta(t) = X_C(z)y(t) + Y_C(z)u(t)$.

Assume that an N-samples record of $y(t)$ and $r(t)$ is available.

a) Find an expression for the ETFE estimator of R, using the open-loop model (10.1), based on the DFT of the available data record.

b) Show that the obtained estimator in (a) coincides with the one obtained by applying an indirect identification method for the Youla parameterization $\mathcal{G}$

Solution:

a) Notice that the N-point DFTs of y and r can be used together with the knowledge of the controller to find that

$$U_N(e^{j\omega_n}) = R_N(e^{j\omega_n}) - \frac{X_C(e^{j\omega_n})}{Y_C(e^{j\omega_n})}Y_N(e^{j\omega_n})$$

where $\{R_N(e^{j\omega_n})\}$ denotes the DFT of $\{r(t)\}$. The ETFE is given by

$$\begin{aligned}\widehat{R(e^{j\omega_n})} &= \frac{D_\circ(e^{j\omega_n})Y_N(e^{j\omega_n}) - N_\circ(e^{j\omega_n})U_N(e^{j\omega_n})}{X_C(e^{j\omega_n})Y_N(e^{j\omega_n}) + Y_C(e^{j\omega_n})U_N(e^{j\omega_n})} \\ &= \frac{(Y_C(e^{j\omega_n})D_\circ(e^{j\omega_n}) + N_\circ(e^{j\omega_n})X_C(e^{j\omega_n}))Y_N(e^{j\omega_n}) - Y_C(e^{j\omega_n})N_\circ(e^{j\omega_n})R_N(e^{j\omega_n})}{Y_C^2(e^{j\omega_n})R_N(e^{j\omega_n})}\end{aligned}$$

where we used the above expression of $U_N(e^{j\omega_n})$

b) An indirect ETFE estimate of G is given as

$$\widehat{G(e^{j\omega_n})} = \frac{\widehat{T(e^{j\omega_n})}}{1 - C(e^{j\omega_n})\widehat{T(e^{j\omega_n})}}$$

where $\widehat{T(e^{j\omega_n})}$ is the ETFE of the complementary sensitivity function given by $\frac{Y_N(e^{j\omega_n})}{R_N(e^{j\omega_n})}$. Using Youla parameterization of G we find that

$$\frac{N_\circ(e^{j\omega_n}) + Y_C(e^{j\omega_n})\widehat{R(e^{j\omega_n})}}{D_\circ(e^{j\omega_n}) - X_C(e^{j\omega_n})\widehat{R(e^{j\omega_n})}} = \frac{\frac{Y_N(e^{j\omega_n})}{R_N(e^{j\omega_n})}}{1 - \frac{X_C(e^{j\omega_n})}{Y_C(e^{j\omega_n})}\frac{Y_N(e^{j\omega_n})}{R_N(e^{j\omega_n})}} = \frac{Y_C(e^{j\omega_n})Y_N(e^{j\omega_n})}{Y_C(e^{j\omega_n})R_N(e^{j\omega_n}) - X_C(e^{j\omega_n})Y_N(e^{j\omega_n})}$$

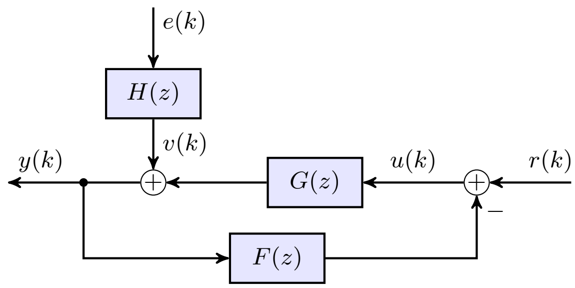
Solving for $\widehat{R(e^{j\omega_n})}$ we get the following expression

$$\widehat{R(e^{j\omega_n})} = \frac{(Y_C(e^{j\omega_n})D_\circ(e^{j\omega_n}) + N_\circ(e^{j\omega_n})X_C(e^{j\omega_n}))Y_N(e^{j\omega_n}) - Y_C(e^{j\omega_n})N_\circ(e^{j\omega_n})R_N(e^{j\omega_n})}{Y_C^2(e^{j\omega_n})R_N(e^{j\omega_n})}$$

which is the same as in (a).

MATLAB **exercise:**

Consider a closed-loop identification problem with the following configuration



where

$$
\begin{aligned}
G(z) &= \frac{b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{B(z)}{A(z)}, \\
H(z) &= \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2}} = \frac{1}{D(z)}, \\
F(z) &= z^{-1} + f_2 z^{-2}, \\
f_2 &= 0.5.
\end{aligned}
$$

An experiment was conducted, where a PRBS sequence was applied as the reference $r(k)$, and the output $y(k)$ was recorded over the time range $k = 1, \ldots, N$.

1. Use a direct method to derive a least-squares (LS) estimate of the unknown system parameters a_1, a_2 and b_1 from the reference and output sequences `p3_r1` and `p3_y1`, which you can find in the MATLAB file `p3_data.mat`.
   (a) Write the regressor matrix Φ_{LS}, the parameter vector $\hat{\theta}_{\text{LS}}$ and the regression equation.
   (b) Estimate $\hat{a}_1$, $\hat{a}_2$, and $\hat{b}_1$
   (c) Simulate the output of the plant you identified subject to the input `p3_r1`. Plot it along with `p3_y1`.
   (d) Is this an asymptotically unbiased estimation? Why?
2. Assume now that $H(z) = \frac{1}{1+a_1 z^{-1} + a_2 z^{-2}} = \frac{1}{A(z)}$. Implement a least-squares method for the identification of $A(z)$ and $B(z)$ in the equation error framework.
   (a) Write the regressor matrix Φ_{LS}, the parameter vector $\hat{\theta}_{\text{LS}}$ and the regression equation.
   (b) Is this an asymptotically unbiased estimation? Why?

Solution hints

1. The direct method requires the knowledge of the output y and the input u. However the given data set contains only y and r. To find the values of u, we use the knowledge of the controller

F and compute $u = r - Fy$. Once u is found, we can formulate a linear regression problem by fitting the data to the ARX model

$$A(z)y(k) = B(z)u(k) + w(k),$$

where $w(k)$ is unknown noise. Using the definitions of $A(z)$ and $B(z)$ given in the question, we find that the value of the output at time k is given as

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_1 u(k-1) + w(k)$$

We can then define a linear regression porblem with a regression matrix

$$\Phi_{\text{LS}} = \begin{bmatrix} -y(2) & -y(1) & u(2) \\ \vdots & \vdots & \vdots \\ -y(254) & -y(253) & u(254) \end{bmatrix}$$

where we removed the first three rows because they contain unknown initial conditions. The output vector $Y = \begin{bmatrix} y(3) & \dots & y(255) \end{bmatrix}^T$.

The least squares estimate is then computed using the backslash command, and is found to be

$$\hat{a}_1 = 0.2980, \qquad \hat{a}_2 = 0.2631 \qquad \hat{b}_1 = 0.275$$

We then create a transfer function using the estimated parameters, which can then be used to simulate the closed-loop output. Figure 10.1 shows a comparison between the simulated output and the noisy output from the given data set. consistent

The LS estimate obtained here is asymptotically biased in general, for the same reasons an output-error model estimated using a LS estimate is biased in open-loop. wrong noise model as noise is not white = correlated

2. In this case we have $D(z) = A(z)$. We then notice that, using the given block diagram,

$$\begin{aligned} y(k) =& \frac{B(z)}{A(z)} u(k) + \frac{1}{A(z)} e(k) \\ u(k) =& r(k) - F(z)y(k) \end{aligned}$$

Rearranging, we see that

$$(A(z) + B(z)F(z))y(k) = B(z)r(k) + e(k)$$

and the value of the output at time k is given as

$$\begin{aligned} y(k) &= -a_1 y(k-1) - a_2 y(k-2) - b_1 y(k-2) - b_1 f_1 y(k-3) + b_1 r(k-1) + e(k). \\ &= -a_1 y(k-1) - a_2 y(k-2) + b_1 (r(k-1) - y(k-2) - f_1 y(k-3)) + e(k) \end{aligned}$$

But we know that $u(k-1) = r(k-1) - y(k-2) - f_1 y(k-3)$. This means that

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_1 u(k-1) + e(k)$$

Observe that this is a very similar equation to the one we found in the first part above, but the noise term is not exactly the same. Therefore, the linear regression matrix and the output vector are the same, and we get the same LS estimate. However here, with $D(z) = A(z)$, the LS estimate is asymptotically unbiased.

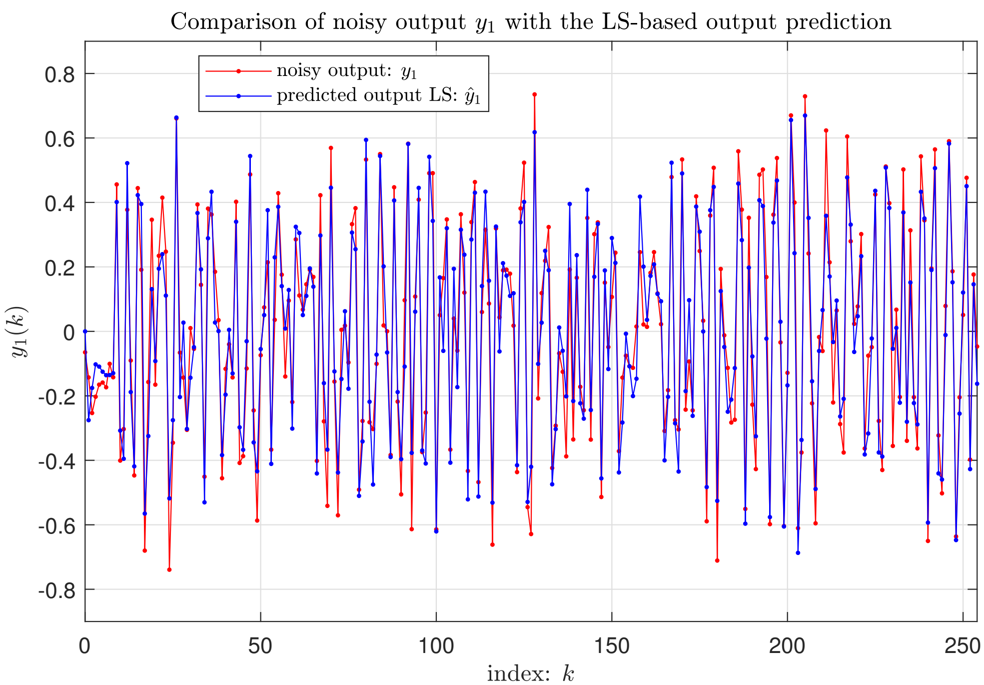


Figure 10.1: Noisy output versus the one obtained via simulating the estimated model.