227-0689-00L System Identification **18 October 2023**

Exercise 4: Empirical Transfer Function Estimation

Background reading

The background material for this exercise is Chapter 6 of Ljung's book (*System Identification: Theory for the User*, 2nd Ed., Prentice-Hall, 1998).

Problem 1:

Consider the ETFE of the LTI system $G\left(e^{j\omega}\right)$ given by

$$\hat{G}\left(e^{j\omega_n}\right)=\frac{Y_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}=G\left(e^{j\omega_n}\right)+\frac{R_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}+\frac{V_N\left(e^{j\omega_n}\right)}{U_N\left(e^{j\omega_n}\right)}.$$

where $U_N\left(e^{j\omega_n}\right), Y_N\left(e^{j\omega_n}\right), R_N\left(e^{j\omega_n}\right), V_N\left(e^{j\omega_n}\right)$ are the discrete Fourier transform of the input, output, transient, and noise respectively.

a) If the transient is neglected and the noise is zero-mean, the ETFE $\hat{G}\left(e^{j\omega_n}\right)$ is unbiased. However, this does not imply that $\left|\hat{G}\left(e^{j\omega_n}\right)\right|$ is an unbiased estimate of $\left|G\left(e^{j\omega_n}\right)\right|$. Show that

$$\mathbb{E}\left\{\left|\hat{G}\left(e^{j\omega_n}\right)\right|^2\right\}=\left|G\left(e^{j\omega_n}\right)\right|^2+\frac{\phi_v\left(e^{j\omega_n}\right)}{\frac{1}{N}\left|U_N\left(e^{j\omega_n}\right)\right|^2}$$

asymptotically for large N with $\phi_v\left(e^{j\omega_n}\right)$ defined as the noise spectrum.

b) If the transient is neglected and the noise is zero-mean, show that the estimates at different frequencies are uncorrelated, i.e.,

$$\mathbb{E}\left\{\left(\hat{G}\left(e^{j\omega_n}\right)-G\left(e^{j\omega_n}\right)\right)\left(\hat{G}\left(e^{j\omega_i}\right)-G\left(e^{j\omega_i}\right)\right)^*\right\}=0, \quad \omega_n \neq \omega_i.$$

c) If the input is random, show that the expected transient bias error $\mathbb{E}\left\{\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2\right\} \to 0$, as $N \to \infty$, with a convergence rate of $\frac{1}{N}$.

d) If the input is periodic, show that the transient bias error $\left|\frac{R_N\left(\mathrm{e}^{j\omega_n}\right)}{U_N\left(\mathrm{e}^{j\omega_n}\right)}\right|^2 \to 0$, as $N \to \infty$, with a convergence rate of $\frac{1}{N^2}$.
Hint: First show that for a periodic signal $u(k)$ of period M, $U_{mM}\left(\mathrm{e}^{j\omega_n}\right)=mU_M\left(\mathrm{e}^{j\omega_n}\right)$.

MATLAB **Exercise:**

Consider the discrete-time system

$$G(z) = \frac{0.1z}{z^4 - 2.2z^3 + 2.42z^2 - 1.87z + 0.7225}$$

and noise model

$$H(z) = \frac{0.5(z - 0.9)}{(z - 0.25)}.$$

The measured output $y(k)$ is defined as the sum of the outputs of $G(z)$ and $H(z)$, such that

$$y(k) = Gu(k) + He(k),$$

where the noise signal $v(k) = He(k)$ is driven by Gaussian white noise $e \sim \mathcal{N}(0, 0.01)$. The input $u(k)$ is bounded between -1 and 1. The sample time can be taken as 1 s.

a) Generate a 1024-point MATLAB simulation as an identification 'experiment'.

b) Estimate $G(z)$ by ETFE and compare it to the true system - particularly in the frequency range around the resonant peak. Plot both the transfer functions and the magnitude of the errors.

c) Split the data record into 4 parts and calculate an averaged ETFE from the 4 parts. Compare this to the true system as well as the original 1024-point ETFE.

d) Repeat the first three parts with different 'experiments'. How can you achieve the best estimate?