Solution 5: Estimate smoothing & experiment design

Problem 1:

Let $\omega_0 > 0$ be a base frequency and let d be a maximum index (number of frequency components). For given phase angles $\{\theta_p\}_{p=1}^d$ and nonnegative amplitudes $\{a_p\}_{p=1}^d$, consider the following multi-sine input with N samples:

$$u(k) = \sum_{p=1}^{d} a_p \cos(p\omega_0 k + \theta_p). \tag{5.1}$$

1. Derive the power spectral density of $\phi_u(\omega)$ for the following scenarios (for given amplitudes $\{a_p\}_{p=1}^d$):
   a) All phases are zero ($\forall p: \ \theta_p = 0$).
   b) The Schroeder phases are used: ($\forall p > 1: \ \theta_p = \theta_1 + \pi p(p-1)/d$).
   c) The phases θ_p are i.i.d. sampled from a uniform distribution over $[0, 2\pi]$.
2. Compare the computed power spectral densities from parts (a)-(c). Why might some choices of phase be more favorable in frequency-domain experiment design?

Hint: It may help to perform visual comparisons of plotted example $u(k)$ signals from (a)-(c).

Solution

The Fourier transform of $u(k)$ is:

$$
\begin{aligned}
U(e^{j\omega}) &= \sum_{k=0}^{N-1} u(k) e^{-jk\omega} && (5.2a)\\
&= \sum_{k=0}^{N-1} \sum_{p=1}^{d} a_p \cos(p\omega_0 k + \theta_p) e^{-j\omega k} && (5.2b)\\
&= \frac{1}{2} \sum_{k=0}^{N-1} \sum_{p=1}^{d} a_p \left(e^{j(p\omega_0 k + \theta_p)} + e^{-j(p\omega_0 k + \theta_p)} \right) e^{-j\omega k} && (5.2c)\\
&= \frac{1}{2} \sum_{k=0}^{N-1} \sum_{p=1}^{d} a_p \left(e^{j((p\omega_0 - \omega)k)} e^{j(\theta_p)} + e^{-j((p\omega_0 + \omega)k)} e^{-j(\theta_p)} \right) && (5.2d)\\
&= \frac{N}{2} \sum_{p=1}^{d} a_p \left(\delta(\omega - p\omega_0) e^{j(\theta_p)} + \delta(\omega + p\omega_0) e^{-j(\theta_p)} \right). && (5.2e)
\end{aligned}
$$

The power spectral density of $u(k)$ is

$$
\phi_u[\mathrm{e}^{j\omega}] = \frac{1}{N} U[\mathrm{e}^{j\omega}] U[\mathrm{e}^{j\omega}]^* = \frac{N}{4} \sum_{p=1}^{d} a_p^2 (\delta(\omega - k\omega_0) + \delta(\omega + k\omega_0)), \tag{5.3}
$$

given that $\delta(\omega - \omega_1)\delta(\omega - \omega_2) = 0$ whenever $\omega_1 \neq \omega_2$, and that $e^{j\theta_p} e^{-j\theta_p} = e^{j0} = 1$.

All phase choices (a)-(c) have exactly the same power spectral density, with an expression in (5.3). Parts (a)-(c) differ in their choice of phases, but the power spectral density expression in (5.3) depends only on the amplitudes a_p and the frequencies $k\omega_0$.

Even though the signals from parts (a)-(c) have the same spectral content, their temporal values may be very different. In part (a) the maximal value achieved by $u(k)$ is $u(0) = \sum_{p=1}^{d} a_p$, which may exceed the input limits/safe operation of the experimental device.

This balance between spectral content and input limits is described by the 'crest factor' C_r from (13.19) of Ljung's book:

$$
C_r^2 = \frac{\max_k u(k)^2}{\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(k)^2}. \tag{5.4}
$$

A smaller crest factor allows for lower-intensity input to be applied while still acquiring equivalent spectral information.

Figure 5.1 visualizes input cresting on a multi-sine signal (code in `crest_compare.m`) with a time horizon of $T = 2048$. Note how the zero-phase input $u(k)$ from (a) (the in-phase signal) takes on a maximal value of ≈ 40, while the other out-of-phase entries achieve maximum absolute values of ≈ 20.

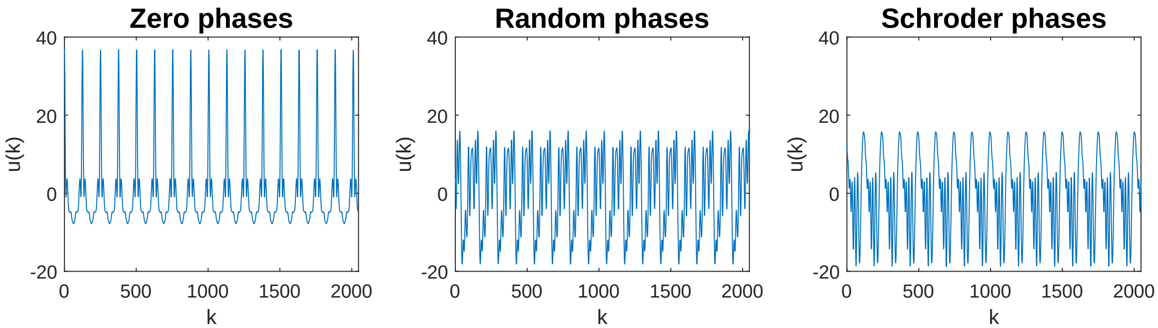

Figure 5.1: Multi-sine input cresting comparison

Problem 2:

This problem will analyze the causal linear time-invariant system g, with input u, output y, and additive noise v. All terms are collected into the expression:

$$y(k) = \sum_{\ell=0}^{\infty} g(\ell)u(k-\ell) + v(k) \qquad k = 0, 1, \ldots, \tag{5.5}$$

which can be represented in the frequency domain by

$$Y(\mathrm{e}^{j\omega}) = G(\mathrm{e}^{j\omega})U(\mathrm{e}^{j\omega}) + V(\mathrm{e}^{j\omega}),$$

where $Y(\mathrm{e}^{j\omega}), G(\mathrm{e}^{j\omega}), U(\mathrm{e}^{j\omega})$ and $V(\mathrm{e}^{j\omega})$ are the discrete time Fourier transforms of y, g, u and v, respectively. The goal of this problem is to analyze properties of the noise $v(k)$.

Let W_γ be a frequency-domain window function with $\int_{-\pi}^{\pi} W_\gamma(\xi)d\xi = 1$. For a power spectral density (such as ϕ_y of $y(k)$), let the estimate $\tilde{\phi}_y$ refer to the application of the window W_γ on the true autocorrelation ϕ_y.

Assume that we are given W_γ-smoothed power spectral densities $\tilde{\phi}_y(\omega)$ and $\tilde{\phi}_u(\omega)$, as well as a smoothed cross-spectral density $\tilde{\phi}_{yu}(\omega)$.

Let $\tilde{\kappa}_{yu}$ denote the coherency spectrum between the smoothed power spectral densities of (u, y):

$$\tilde{\kappa}_{yu}(\mathrm{e}^{j\omega}) = \sqrt{\frac{\left|\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right|^2}{\tilde{\phi}_y(\mathrm{e}^{j\omega})\tilde{\phi}_u(\mathrm{e}^{j\omega})}}. \tag{5.6}$$

1. Prove that the smoothed power spectral density approximation of the noise process $\tilde{\phi}_v$ satisfies the following estimate:

$$\tilde{\phi}_v(\mathrm{e}^{j\omega}) \approx \tilde{\phi}_y(\mathrm{e}^{j\omega}) \left(1 - \left(\tilde{\kappa}_{yu}(\mathrm{e}^{j\omega})\right)^2\right). \tag{5.7}$$

2. What is an interpretation of the coherency spectrum $\tilde{\kappa}_{yu}$? How does the quality of the transfer function estimate $G(\mathrm{e}^{j\omega}) = \frac{\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})}{\tilde{\phi}_u(e^{j\omega}}$ depend on the value of $\tilde{\kappa}_{yu}$?

Hint: Let $X_1(\mathrm{e}^{j\omega})$ and $X_2(\mathrm{e}^{j\omega})$ be a pair of spectra. When the bandwidth parameter is sufficiently small ($\gamma \to 0$ and $W_\gamma(\xi) \to \delta(\xi)$), the quantity $\int_{-\pi}^{\pi} W(\omega-\xi)[X_1(\mathrm{e}^{j\xi})X_2(\mathrm{e}^{j\xi})]d\xi$ can be approximated by $\left[\int_{-\pi}^{\pi} W(\omega-\xi)X_1(\mathrm{e}^{j\xi})d\xi\right]\left[\int_{-\pi}^{\pi} W(\omega-\xi)X_2(\mathrm{e}^{j\xi})d\xi\right]$ (warning: this approximation does not hold in general for distribution of convolution over multiplication).

Solution

The noise signal v can be isolated as

$$y(k) = \sum_{\ell=0}^{\infty} g(\ell)u(k-\ell) + v(k) \qquad k = 0, 1, \dots \tag{5.8}$$

which can be passed into the frequency domain as

$$V(\mathrm{e}^{j\omega}) = Y(\mathrm{e}^{j\omega}) - G(\mathrm{e}^{j\omega})U(\mathrm{e}^{j\omega}). \tag{5.9}$$

The W_γ-smoothed power spectral density for $v(k)$ has an expression of

$$\tilde{\phi}_v(\mathrm{e}^{j\omega}) = \int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\frac{1}{N}\left|V_N(\mathrm{e}^{j\xi})\right|^2 d\xi \tag{5.10a}$$

$$= \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|Y(\mathrm{e}^{j\xi}) - G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\right|^2 d\xi \tag{5.10b}$$

After expanding the absolute value and multiplying terms:

$$= \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left(Y(\mathrm{e}^{j\xi}) - G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\right)^*\left(Y(\mathrm{e}^{j\xi}) - G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\right) d\xi \tag{5.10c}$$

$$\begin{aligned} = \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\Big(&Y^*(\mathrm{e}^{j\xi})Y(\mathrm{e}^{j\xi}) - Y^*(\mathrm{e}^{j\xi})G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi}) \\ &- G^*(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi})Y(\mathrm{e}^{j\xi}) + G^*(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi})G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\Big)d\xi \end{aligned} \tag{5.10d}$$

Now condensing into absolute values:

$$\begin{aligned} = \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\Big(&\left|Y(\mathrm{e}^{j\xi})\right|^2 + \left|G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\right|^2 \\ &- (Y(\mathrm{e}^{j\xi})G^*(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi}) - (Y(\mathrm{e}^{j\xi})G^*(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi}))^*\Big)d\xi \end{aligned} \tag{5.10e}$$

The cross-terms can be formulated as the 2 times the real part of a complex number

$$= \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left(\left|Y(\mathrm{e}^{j\xi})\right|^2 + \left|G(\mathrm{e}^{j\xi})U(\mathrm{e}^{j\xi})\right|^2 - 2\mathrm{real}\left\{Y(\mathrm{e}^{j\xi})G^*(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi})\right\}\right) d\xi \tag{5.10f}$$

Separating the G and $\{U,Y\}$ convolutions by the hint results in

$$\begin{aligned}
&\approx \frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|Y(\mathrm{e}^{j\xi})\right|^2 d\xi && (5.10\text{g})\\
&\quad + \frac{1}{N}\left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|G(\mathrm{e}^{j\xi})\right|^2 d\xi\right)\left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|U(\mathrm{e}^{j\xi})\right|^2 d\xi\right)\\
&\quad - \frac{2}{N}\mathrm{real}\left\{\left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)G^*(\mathrm{e}^{j\xi})d\xi\right)\left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)Y(\mathrm{e}^{j\xi})U^*(\mathrm{e}^{j\xi})d\xi\right)\right\}
\end{aligned}$$

Each W_γ convolution expression may be substituted for its a-priori given spectral quantity. As an example, the term $\frac{1}{N}\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|Y(\mathrm{e}^{j\xi})\right|^2 d\xi$ is equal to $\tilde{\phi}_y(\mathrm{e}^{j\omega})$.

$$\begin{aligned}
&\approx \tilde{\phi}_y(\mathrm{e}^{j\omega}) + \left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\left|G(\mathrm{e}^{j\xi})\right|^2 d\xi\right)\tilde{\phi}_u(\mathrm{e}^{j\omega}) && (5.10\text{h})\\
&\quad - 2\mathrm{real}\left\{\left(\int_{-\pi}^{\pi} W_\gamma(\xi-\omega)G^*(\mathrm{e}^{j\xi})d\xi\right)\phi_{yu}\tilde{(\mathrm{e}^{j\omega})}\right\}
\end{aligned}$$

Properties of the gain $G(\mathrm{e}^{j\omega})$ may approximated by the ratio of windowed spectral functions $\tilde{G}(\mathrm{e}^{j\omega}) = \tilde{\phi}_{yu}(\mathrm{e}^{j\omega})/\tilde{\phi}_u(\mathrm{e}^{j\omega})$:

$$\approx \tilde{\phi}_y(\mathrm{e}^{j\omega}) + \left|\frac{\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})}{\tilde{\phi}_u(\mathrm{e}^{j\omega})}\right|^2\tilde{\phi}_u(\mathrm{e}^{j\omega}) - 2\,\mathrm{real}\left\{\frac{\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})^*}{\tilde{\phi}_u(\mathrm{e}^{j\omega})}\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right\} \qquad (5.10\text{i})$$

Note that the quantity in the real$\{\cdot\}$ expression is real on its own:

$$\begin{aligned}
&\approx \tilde{\phi}_y(\mathrm{e}^{j\omega}) + \frac{\left|\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right|^2}{\tilde{\phi}_u(\mathrm{e}^{j\omega})} - 2\frac{\left|\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right|^2}{\tilde{\phi}_u(\mathrm{e}^{j\omega})} && (5.10\text{j})\\
&\approx \tilde{\phi}_y(\mathrm{e}^{j\omega}) - \frac{\left|\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right|^2}{\tilde{\phi}_u(\mathrm{e}^{j\omega})} && (5.10\text{k})
\end{aligned}$$

The coherency expression is derived by pulling out a factor of $\tilde{\phi}_y$

$$\begin{aligned}
&\approx \tilde{\phi}_y(\mathrm{e}^{j\omega})\left(1 - \left(\sqrt{\frac{\left|\tilde{\phi}_{yu}(\mathrm{e}^{j\omega})\right|^2}{\tilde{\phi}_y(\mathrm{e}^{j\omega})\tilde{\phi}_u(\mathrm{e}^{j\omega})}}\right)^2\right) && (5.10\text{l})\\
&\approx \tilde{\phi}_y(\mathrm{e}^{j\omega})\left(1 - \left(\kappa_{yu}(\mathrm{e}^{j\omega})\right)^2\right)\,. && (5.10\text{m})
\end{aligned}$$

The coherency has an interpretation as an alignment between the input and the output. If the coherency at a frequency ω_c is large, then $\tilde{\phi}_v(e^{j\omega_c})$ is small and the noise has low power at ω_c (as compared to the true system's behavior). Conversely if the frequency is large, then most of the observational power at the output at ω_c is derived from noise. The estimated model $\tilde{G}$ therefore fits to the noise rather than to the true dynamics at ω_c.

MATLAB **Exercise:**

Swept sine identification requires long time horizons, with resets between each frequency stimulation, and therefore may be expensive in the sense of time and money. This experiment will compare the ETFE and spectral estimates produced by a single-shot experiment, for which the system is stimulated by a chirp signal. All input-output data will take place at a 1 Hz sample rate.

Download the following files from Moodle to perform this exercise:

- `HS2023_SysID_Exercise_05_GenerateData.p`
- `HS2023_SysID_Exercise_05_check.p`
- `HS2023_SysID_Exercise_05_12345678.m`

The first two files are protected files which will generate data and perform a basic check on your solution respectively. Your task is to edit the file `HS2023_SysID_Exercise_05_12345678.m` to write a MATLAB function of the following form:

```
[u, Ru, y, Ruy, Ry,etfe_raw, spec_raw, etfe_smooth, spec_smooth] = ...
                    HS2023_SysID_Exercise_05_LegiNumber(),
```

where `LegiNumber` refers to your own Legi number. Since MATLAB requires that the filename matches the function name, you must edit the filename by replacing 12345678 with your Legi Number. Then, within the file, replace 12345678 with your Legi Number in the function name. The outputs of `HS2023_SysID_Exercise_05_LegiNumber` are described in Table 5.1.

To solve this exercise, you are not allowed to use any additional files other than the three provided ones. The standard MATLAB functions along with the Control Systems Toolbox and System Identification Toolbox can be used. Use of other MATLAB toolboxes in submission is impermissible. All custom function definitions must be declared and written inside `HS2023_SysID_Exercise_05_LegiNumber.m`.

In this exercise, we will also use a sanity check script that will be used in the midterm exam. The sanity check script will test if your script can run, its dependencies on other files and toolboxes, as well as the basic format of your output. To run the sanity check script, download the file `HS2023_SysID_Exercise_05_check.p` from Moodle and put it in the same folder as your solution function. The sanity check is performed by running

```
HS2023_SysID_Exercise_05_check(LegiNumber)
```

The result of the check will be displayed in the command window. Verify that your solution satisfies the check before submission.

Table 5.1: Outputs of the homework function (each of which is an array of size 1024 × 1)

`u`:	The input (chirp) signal applied to the system $u(k)$
`Ru`:	The autocorrelation of the input $u(k)$
`y`:	Output $y(k)$ of the system
`Ruy`:	Cross correlation between $u(k)$ and $y(k)$
`Ry`:	Autocorrelation of $y(k)$
`etfe_raw`:	Empirical Transfer Function Estimate for $G(e^{j\omega})$
`spec_raw`:	Spectral estimate for $G(e^{j\omega})$
`etfe_smooth`:	Smoothed ETFE by Hann window and inverse variance weighting
`spec_smooth`:	Smoothed spectral estimate by Hann window and inverse variance weighting

This exercise comprises the following tasks:

1. Produce a chirp signal with amplitude 1, with frequency sweep from 0 Hz to 0.3 Hz in time $T = 2^{10}$ samples

   The chirp signal with $A = 1$, $f_0 = 0$, $f_1 = 0.3$, and $T = 2^{10}$ has an equation of

   $$u(k) = A\cos(2\pi(f_0 k + (f_1 - f_0)k^2)/(2T)) = \cos(\pi(0.3)k^2/1024). \tag{5.11}$$

   The signal $u(k)$ will be referred to as `u` (MATLAB array of dimensions 1024 × 1).

   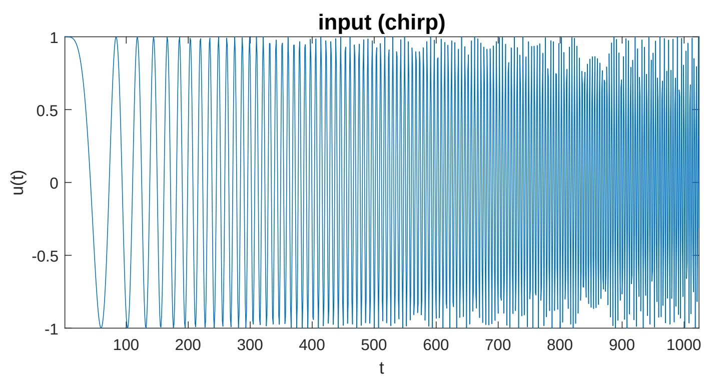

   Figure 5.2: Inputs for the experiment

2. Compute the autocorrelation of the input signal.

   The autocorrelation of the chirp can be computed as

   `Ru = ifft( fft(u) .* conj(fft(u)))/N.`

3. Probe the black-box system with your computed input to produce a noisy output response. Use the following function call to obtain the output:

   `HS2023_SysID_Exercise_05_GenerateData(LegiNumber, input)`

   The probe is obtained by executing

   `out_chirp = HS2023_SysID_Exercise_05_GenerateData(LegiNumber, u)`

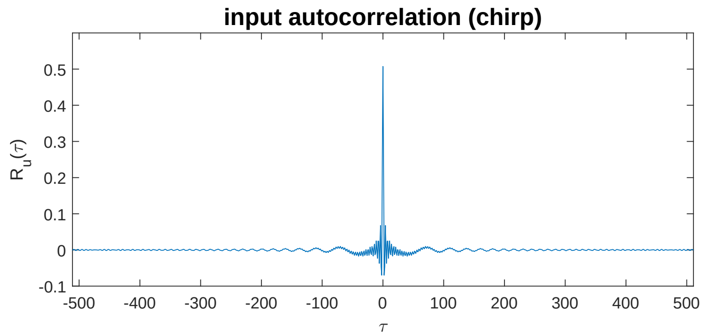

Figure 5.3: Input Autocorrelation

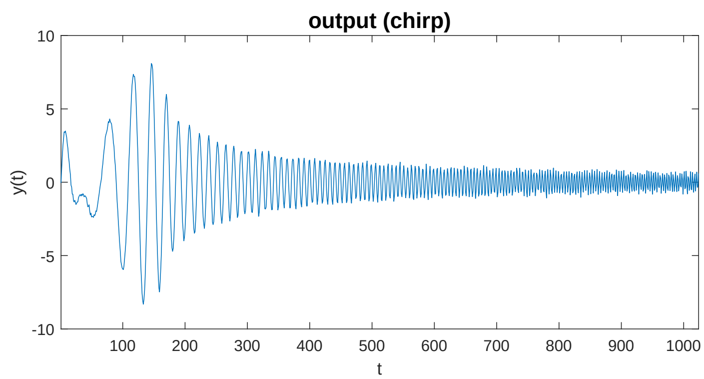

Figure 5.4: Outputs from the Experiment

4. Compute the autocorrelation of the output. Also find the cross-correlation between the input and its respective output.

   The autocorrelation of the output is `Ry = ifft( fft(y) .* conj(fft(y)) )/N`.

   The cross-correlation between the input and the output is `Ryu= ifft( fft(y) .* conj(fft(u)) )/N`.

5. Calculate the Empirical Transfer Function Estimate (ETFE) from each input-output pair. Also compute a transfer function based on spectral data (cross and auto correlations).

   The empirical transfer function may be found through the division of `etfe_raw = fft(y) ./ fft(u)`.

   Similarly, the spectral estimate is obtained by the division `spec_raw = fft(Ryu) ./ fft(Ru)`.

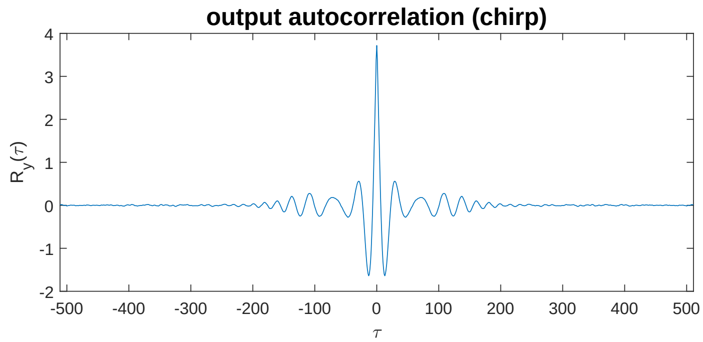

Figure 5.5: Autocorrelation of the Output

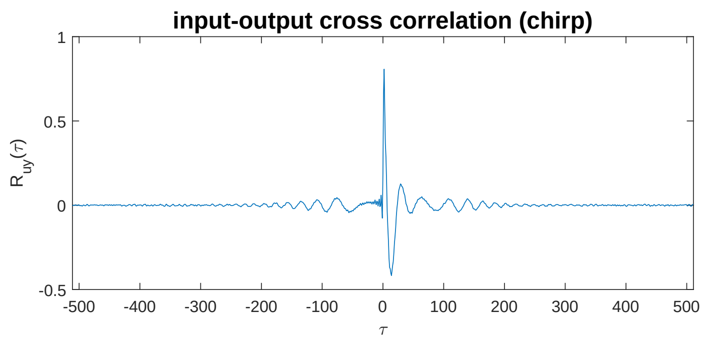

Figure 5.6: Cross-Correlation between Inputs and Outputs

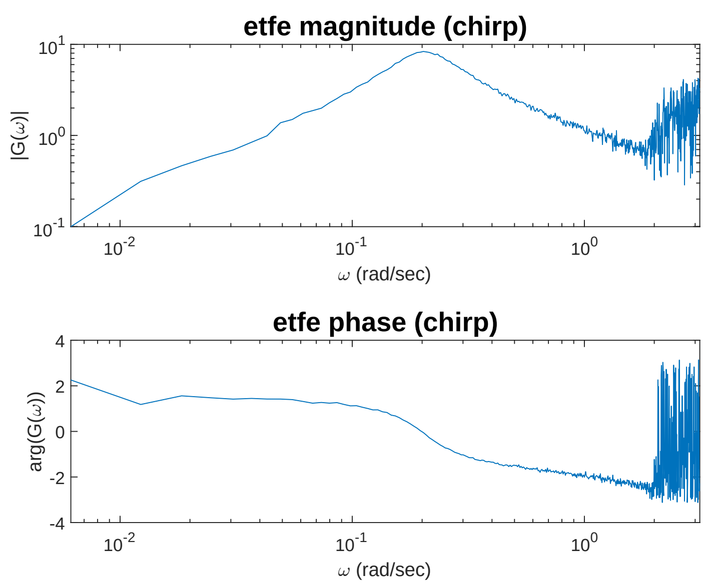

Figure 5.7: Empirical Transfer Function Estimate $\hat{G}(e^{j\omega}) = Y(e^{j\omega})/U(e^{j\omega})$

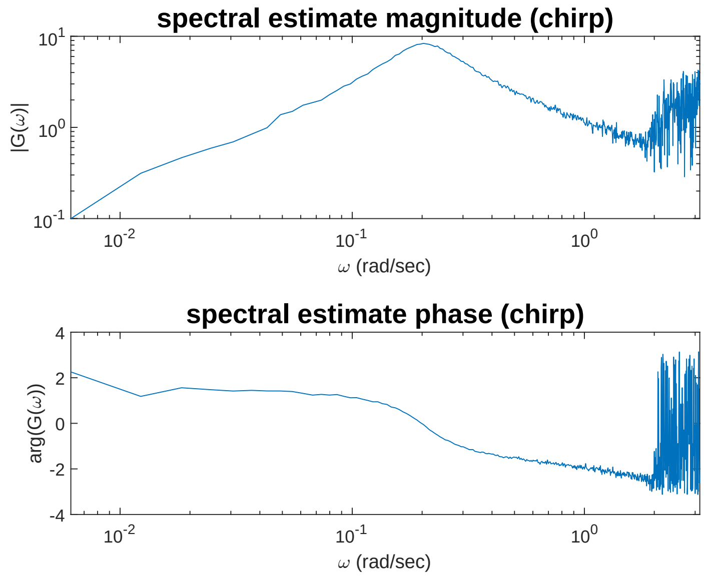

Figure 5.8: Spectral Estimate $\hat{G}(\mathrm{e}^{j\omega}) = \mathcal{F}[R_{yu}(\tau)](\mathrm{e}^{j\omega})/\mathcal{F}[R_u(\tau)](\mathrm{e}^{j\omega})$

6. Use the Hann Window with bandwidth parameter 20 in order to acquire smoothed versions of the ETFE and spectral estimates. Perform inverse variance weighting when smoothing under the assumption that the power spectral density ϕ_v is constant.

   The frequency-domain Hann window with parameter `gamma=20` can be obtained by:

   ```
   lags_w = [floor(-N/2+1):floor(N/2)]'
   wHann = zeros(1, floor(-N/2+1)+floor(N/2))
   idx = find(abs(lags_w) <= gamma)
   wHann(idx) = 0.5*(1+cos(pi*lags_w(idx)/gamma))
   ```

   Inverse variance weighting produces with flat-spectrum noise ϕ_v uses the weights

   ```
   a = fft(Ru); var_denom = ifft(fft(wh) .* fft(a));
   ```

   The smoothed ETFE is gathered by taking the frequency-domain convolution of the Hann window and the ETFE:

   ```
   etfe_smooth = ifft(fft(wh).*fft(etfe_raw .* a))./var_denom;
   ```

   A similar process occurs for the smoothed spectral estimate `spec_smooth = ifft(fft(wh).*fft(spec_raw.*a))./var_denom;`

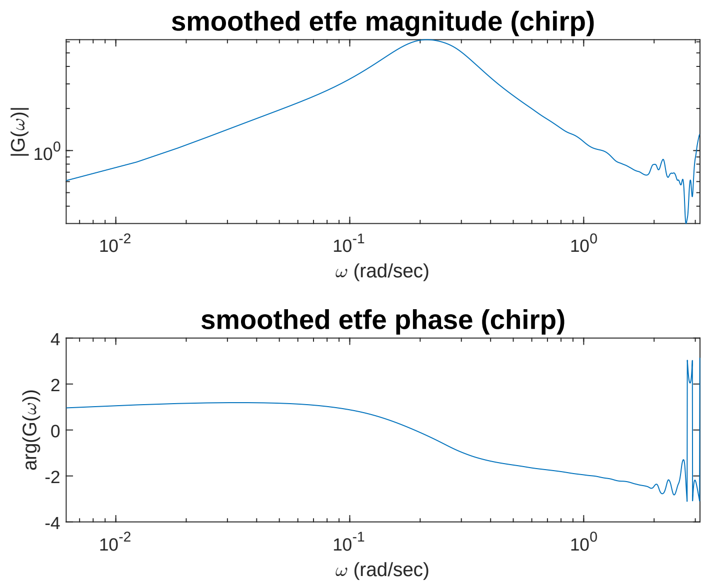

Figure 5.9: Hann-and-Inverse-Variance-Smoothed ETFE

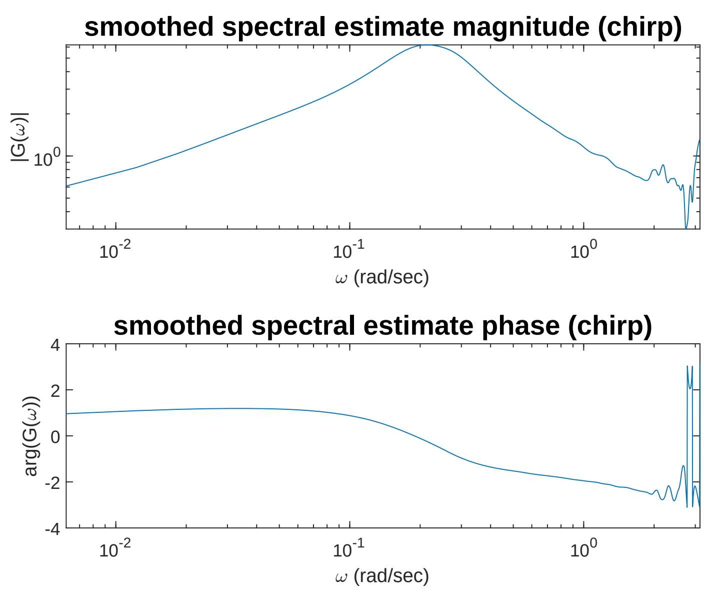

Figure 5.10: Hann-and-Inverse-Variance-Smoothed Spectral Estimate