8. Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\overline{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

   (a) Give your posterior distribution for θ. (Your answer will be a function of n.)

Observe $y = (y_1, y_2 \dots y_n)$

$$y_i|\theta \sim N(\theta, \sigma^2) = N(\theta, 20^2)$$
$$\theta \sim N(\mu_0, \sigma_0^2) = N(180, 40^2)$$
$$\bar{y} = 150 \Rightarrow \frac{1}{n}\sum_{i=1}^{n} y_i = 150$$

Posterior:

$$p(\theta|y) \propto p(y|\theta)\,p(\theta)$$
$$= \left[\prod_{i=1}^{n} p(y_i|\theta)\right] p(\theta)$$
$$\boxed{\propto} \exp\left(\frac{\sum(y_i-\theta)^2}{2\sigma^2}\right)\exp\left(\frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right)$$
$$= \exp\left(\frac{1}{2}\left[\frac{\sigma_0^2\sum(y_i-\theta)^2 + \sigma^2(\theta-\mu_0)^2}{\sigma^2\sigma_0^2}\right]\right)$$

– complete square –

$$= \exp\left(\frac{1}{2}\left[\underline{(\theta - \mu_1)}\ \boxed{\frac{(\sigma^2+\sigma_0^2 n)}{\sigma^2\sigma_0^2}}\right]\right) \quad \Rightarrow = \frac{1}{\sigma_1^2}$$

$$\mu_1 = \frac{\sigma^2\mu_0 + \sigma_0^2 \overbrace{\sum y_i}^{=n\bar{y}}}{\sigma^2 + n\sigma_0^2} = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}}{\underbrace{\left(\frac{\sigma^2+\sigma_0^2 n}{\sigma^2\sigma_0^2}\right)}_{\Rightarrow = \left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right) = \sigma_1^2}}$$

$$\frac{\frac{\mu}{\sigma_0^2} + \frac{\bar{y}n}{\sigma^2}}{\left(\frac{\sigma^2+\sigma_0^2 n}{\sigma^2\sigma_0^2}\right.}$$

Thus, $\theta|y \sim N(\mu_1, \sigma_1^2)$

(b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$. (Your answer will still be a function of n.)

$$p(\tilde{y}|y) \propto \int_{-\infty}^{\infty} p(\tilde{y},\theta|y)\, d\theta$$

$$= \int p(\tilde{y}|\theta, y)\, p(\theta|y)\, d\theta$$

$$= \int p(\tilde{y}|\theta)\, p(\theta|y)\, d\theta$$

$$\boxed{\propto} \int \exp\left(\frac{(\tilde{y}-\theta)^2}{2\underbrace{\sigma^2}_{\text{known}}}\right) \exp\left(\frac{(\theta - \overbrace{\mu_1}^{\text{known}})^2}{2\underbrace{\sigma_1^2}_{\text{known}}}\right) d\theta$$

$$= \int \exp\left(\frac{(\tilde{y}-\mu_1)^2}{2(\sigma^2+\sigma_1^2)}\right)$$

$$\Rightarrow \quad \tilde{y}|y \sim N(\mu_1, \sigma^2+\sigma_1^2)$$

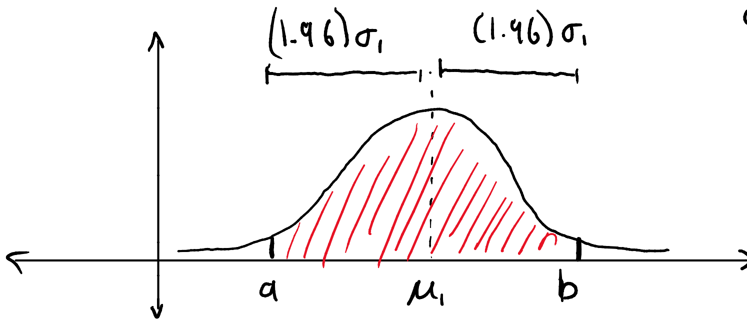
(c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

$n=10$ means:

$\mu_1 = 150.7$

$\sigma_1 = 6.25$

$$\theta|y \sim N(\mu_1, \sigma_1^2)$$



$$= 0.95 = \int_a^b p(\theta|y)\, d\theta$$

Thus, interval is $150.7 \pm 1.96(6.25)$

For $\tilde{y}|y_1$ interval is $150.7 \pm 1.96\,(20.95)$

9. Setting parameters for a beta prior distribution: suppose your prior distribution for θ, the proportion of Californians who support the death penalty, is beta with mean 0.6 and standard deviation 0.3.

(a) Determine the parameters α and β of your prior distribution. Sketch the prior density function.

If $x \sim \text{Beta}(\alpha, \beta)$, $\quad p(x) \propto x^{\alpha-1}(1-x)^{\beta-1}$

$$E[x] = \frac{\alpha}{\alpha+\beta} \Rightarrow 1 - E[x] = \frac{\alpha+\beta-\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$$

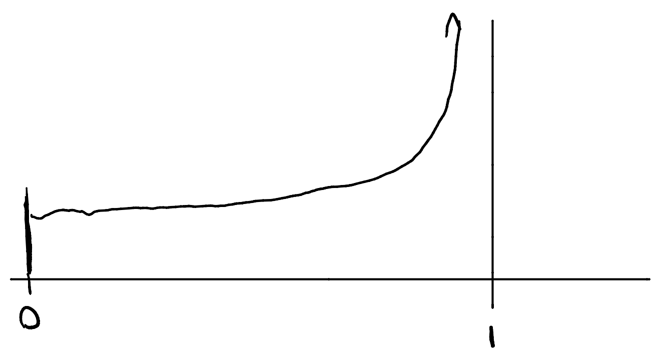
$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{E(x)\,(1-E(x))}{(\alpha+\beta+1)}$$

$$= \frac{E(x)\,(1-E(x))}{E[x]\alpha+1}$$

$\Rightarrow \alpha =$ <algebra> $= 1$

$\beta =$ <algebra> $= 0.67$



(b) A random sample of 1000 Californians is taken, and 65% support the death penalty. What are your posterior mean and variance for θ? Draw the posterior density function.

Binomial distribution, $n = 1000$, $\quad y_i = \begin{cases} 0 & \text{against} \\ 1 & \text{for penalty} \end{cases}$

$$\sum y_i = 650$$

$$p(\theta|y) = \text{Bin}(0.65, 1000)\,\text{Beta}(\alpha, \beta)$$

$$p(\theta|y) = \mathrm{Bin}(0.65, 1000)\,\mathrm{Beta}(\alpha, \beta)$$
$$= \mathrm{Beta}(650+\alpha, 350+\beta)$$
$$= \mathrm{Beta}(651, 350.67)$$
$$\Rightarrow E[\theta|y] = 0.6499 \qquad SD(\theta|y) = 0.015$$

(c) Examine the sensitivity of the posterior distribution to different prior means and widths including a non-informative prior.

Data Dominates.