Solving the problem of the firm with the EGA algorithm

João Ejarque and Nicolai Waldstrom *

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1 Introduction

This note solves a problem of the firm's optimal choice of inputs as an application of the Endogenous Grid Points Algorithm of Carroll (2006).

2 Problem of the firm

A price taking firm produces output using capital and labor. Capital has the law of motion $K_t = (1-\delta_k)\, K_{t-1} + I_t$ and the firm has the following net production function with $\alpha_k + \alpha_l < 1$, where net output N_t is given by gross output Y_t minus adjustment costs to changing the capital stock, AC_t:

$$N_t \;=\; A_t \cdot K_{t-1}^{\alpha_k} \cdot L_t^{\alpha_l} \;-\; \frac{\gamma}{2} \cdot K_{t-1} \cdot \left(\frac{I_t}{K_{t-1}} \;-\; \delta_k \right)^2 \;\equiv\; Y_t \;-\; \frac{\gamma}{2} \cdot K_{t-1} \cdot \left(\frac{K_t}{K_{t-1}} \;-\; 1 \right)^2$$

and profits are given by $\pi_t \;=\; p_t N_t \;-\; w_t L_t \;-\, p_t^I I_t$. The unconstrained firm has discount factor between time periods t and $t+1$ given by

$$\beta_{t+1} = \frac{1}{\left(1 + r_{t+1} + r_{t+1}^p\right)}$$

However, external finance is costly and therefore we add a constraint $\pi_t > 0$.

2.1 First order conditions and steady state

The first order conditions for this problem have the form

$$\frac{\partial \pi_t}{\partial K_t} + \underbrace{\beta_{t+1} \left(\frac{1+\lambda_{t+1}}{1+\lambda_t} \right)}_{\hat{\beta}_{t+1}} \frac{\partial \pi_{t+1}}{\partial K_t} \;=\; 0$$

*João Miguel Ejarque: jme@dreammodel.dk, DREAM/MAKRO, Landgreven 4, 1301 København K. The opinions contained in this paper do not represent the vews of the DREAM/MAKRO group.

where λ is the Lagrange multiplier on the positive profit constraint. This yields the expanded discount factor $\hat{\beta}$. In the case of the capital stock we can write

$$\hat{\beta}_{t+1}p_{t+1}F^K_{t+1} \;=\; p^I_t - \hat{\beta}_{t+1}p^I_{t+1}\left(1-\delta_k\right) \;+\; \frac{\partial AC_t}{\partial K_t}p_t \;+\; \hat{\beta}_{t+1}\frac{\partial AC_{t+1}}{\partial K_t}p_{t+1}$$

$$\frac{\partial AC_t}{\partial K_t}p_t \;+\; \hat{\beta}_{t+1}\frac{\partial AC_{t+1}}{\partial K_t}p_{t+1} \;\equiv\; p_t\gamma\left(\frac{K_t}{K_{t-1}}-1\right) + E_t\left\{\beta_{t+1}p_{t+1}\frac{\gamma}{2}\left[\left(\frac{K_{t+1}}{K_t}\right)^2-1\right]\right\}$$

In steady state adjustment costs are zero and the external finance constraint does not bind. In that case we obtain

$$p\alpha_k\frac{Y}{K} = p^I(1/\beta-(1-\delta_k)) \equiv U_k$$

The first order condition for employment is $p\alpha_l Y/L = w \equiv U_l$ where U_l is the user cost of labor which here equals the wage.

2.2 Applying the EGA to the problem of the firm

This is a straightforward application of Carroll's method. Define $X_{t+1} = K_{t+1}/K_t$ and start from given values of X_{t+1}, K_{t+1}, K_t, and L_{t+1}. A suitable vector of possible values for these objects can be derived from computing steady state values associated with extreme realizations of the underlying shocks. The object X would then take values in the interval K_{min}/K_{max} and K_{max}/K_{min}. Then invert the f.o.c. to obtain X_t,

$$-p^I_t + E_t\left\{\beta_{t+1}p^I_{t+1}\left(1-\delta\right)\right\} + E_t\left\{\beta_{t+1}p_{t+1}\left(\alpha_k A_{t+1}K_t^{\alpha_k-1}L^{\alpha_l}_{t+1} + \frac{\gamma}{2}\left[X^2_{t+1}-1\right]\right)\right\} = p_t\gamma\left(X_t-1\right)$$

and then use $X_t = K_t/K_{t-1}$ to get K_{t-1}, and finally the time t f.o.c. for Labor to get L_t. This is trivial if constraints do not bind.

In order to account for the profit constraint

$$\pi_t \;=\; p_tA_t\cdot K^{\alpha_k}_{t-1}\cdot L^{\alpha_l}_t \;-\; p_t\frac{\gamma}{2}\cdot K_{t-1}\cdot\left(\frac{K_t}{K_{t-1}}\;-\;1\right)^2 \;-\; w_tL_t \;-p^I_tI_t \;>\; 0$$

we note that the first order condition for labor is not affected by this constraint. In the current problem we can always write the f.o.c. for employment as the inverse function $L\left(K_{t-1},A_t,w_t,p_t\right)$ which then implies we can find the boundary value K_* through

$$\pi_t \;=\; p_tA_t\cdot K^{\alpha_k}_*\cdot L^{\alpha_l}_* \;-\; p_t\frac{\gamma}{2}\cdot K_*\cdot\left(\frac{K_t}{K_*}\;-\;1\right)^2 \;-\; w_tL_* \;-p^I_t\left(K_t-(1-\delta)K_*\right) \;=\; 0$$

This value K_* divides the state space of K_{t-1} in two sections, where only on one side is the Euler equation admissible.

2.3 The EGA in practice

This section briefly explains how the above problem is solved in the associated code. The EGA finds the optimal K_{t-1} conditional on K_t and future choices by inverting the dynamic first-order condition. In particular it solves the equation:

$$p_t^I + p_t \gamma \left(\frac{K_t}{K_{t-1}} - 1 \right) = EMCT_t \tag{1}$$

where $EMCT_t$ is the expected marginal continuation value given by:

$$EMCT_t = E_t \left\{ \beta_{t+1} p_{t+1}^I (1 - \delta) \right\} + E_t \left\{ \beta_{t+1} p_{t+1} \left(\alpha_k A_{t+1} K_t^{\alpha_k - 1} L_{t+1}^{\alpha_l} + \frac{\gamma}{2} \left[\left(\frac{K_{t+1}}{K_t} \right)^2 - 1 \right] \right) \right\}$$

The algorithm starts with an initial guess for $EMCT_t$ and solves for K_{t-1} using (1), which yields the policy function $K_{t-1}(K_t)$. It proceeds by inverting the policy function by interpolation to give the solution to the problem $K_t(K_{t-1})$. Given this one can compute a new $EMCT_t$ which can be used for further backwards iterations. This is repeated until a fixed point in K_t is reached. Algorithm 1 presents pseudocode that conducts one backwards iteration as described here.

Algorithm 1: Backwards EGA step

Calculate derivative of cost gamma function when adjust from k_j to k_i on grid (rhs of eq. (**??**));
for $j = 0 \quad nK$ **do**
  for $i = 0 \quad nK$ **do**
    $dGamma_{i,j} = \gamma \cdot (\bar{k}_i / \bar{k}_j - 1)$
Take expectation and discount future marginal continuation value ($EMCT_t$) ;
  $EMCT_t = \beta MCT_{t+1}$;
Solve dynamic FOC for k_{t-1} and interpolate policy function in one step;
  $rhs = EMCT_t$;
  $lhs = p^I + pdGamma$;
$k_t = intpolate(rhs, lhs\bar{k})$;
Calculate optimal labor input by inverting labor demand ;
  $l = (\frac{w}{alpha^L p * A * \bar{k}^{alpha^K}})^{(1/(alpha^L - 1))}$;
Calculate MCT_{t+1} for further backwards iteration ;
  $MCT_{t+1} = p_t^I (1 - delta^K) + p \left(\alpha_k A \bar{k}^{\alpha_k - 1} l^{\alpha_l} + \frac{\gamma}{2} \left[\left(\frac{k}{\bar{k}} \right)^2 - 1 \right] \right)$;

3 References

Carroll, Christopher. (2006). The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems. Economics Letters. 91. 312-320.

"This paper introduces a method for solving numerical dynamic stochastic optimization problems that avoids rootfinding operations. The idea is applicable to many microeconomic and macroeconomic problems, including life cycle, buffer-stock, and stochastic growth problems."