Practical Session: Bayesian reasoning

1. [Bayesian reasoning] During medical checkup, one of the tests indicates a serious disease. The test has high accuracy 99% (the probability of true positive is 99%, the probability of true negative is 99%). However, the disease is quite rare, and only one person of 10000 is affected. Calculate the probability that the examined person has the disease.

2. [Data modeling: Bayesian vs frequentist] Consider a linear regression problem. Let $\mathcal{D} = \{X, y\}$ be a training dataset of K points, where $X \in \mathbb{R}^{K \times d}$ is a matrix of feature vetors and $y \in \mathbb{R}^K$ is a vector of target variables. In order to train a linear regression model on the training data, we assume that the likelihood has the form of normal distribution:

$$p(y|X, w) = \prod_{i=1}^{K} \mathcal{N}(y_i | w^T x_i, 1) = \mathcal{N}(y | Xw, I),$$

where $w \in \mathbb{R}^d$ is a vector of trainable weights.

We would like to estimate w in a frequentist and Bayesian way. In order to perform an analytical Bayesian inference, we choose a conjugate prior. The conjugate prior distribution for the normal likelihood is a Normal distribution. We choose the following prior form:

$$p(w) = \mathcal{N}(w|0, A^{-1}), \quad A = \alpha I, \alpha \in \mathbb{R}^+$$

   (a) Compute the maximum likelihood estimate for w.

   (b) Check that the Normal distribution is indeed the conjugate distribution for the Normal likelihood.

   (c) Compute the posterior distribution $p(w|X, y)$.

   (d) Compute a maximum a posteriori estimate for w and compare it with the maximum likelihood estimate.

   (e) Compute the posterior predictive distribution $p(y_{new}|x_{new}, X, y) = \int p(y_{new}|x_{new}, w) p(w|X, y) \mathrm{d}w$.

In this task, we use the following parametrization for normal distribution:

$$\mathcal{N}(x|\mu, \Lambda^{-1}) = \sqrt{\frac{|\Lambda|}{(2\pi)^d}} \exp\left(-\frac{1}{2}(x - \mu)^T \Lambda (x - \mu)\right)$$