Math 308 Assignment 10
Mean and Variance of the Chi-Squared Distribution

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1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \int_0^\infty e^{tx} \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \,\mathrm{d}x \end{aligned}$$

But, $e^{-\frac{x}{2}} e^{tx} = e^{-x(t-\frac{1}{2})}$.
Substituting $u = x(1/2 - t)$, we get $\mathrm{d}u = (1/2 - t)\mathrm{d}x$, and $u|_{x=0} = 0$, $u|_{x=\infty} = \infty$. So, we get:

$$\begin{aligned} M_Z(t) &= \int_0^\infty \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{u^{\frac{k}{2}-1}}{(1/2 - t)^{\frac{k}{2}-1}} \frac{e^{-u}}{1/2 - t} \,\mathrm{d}u \\ &= \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{1}{(1/2 - t)^{\frac{k}{2}}} \int_0^\infty u^{\frac{k}{2}-1} e^{-u} \,\mathrm{d}u \\ &= \frac{1}{2^{k/2}\cancel{\Gamma\left(\frac{k}{2}\right)}} \frac{1}{(1/2 - t)^{\frac{k}{2}}} \cancel{\Gamma\left(\frac{k}{2}\right)} \\ &= (1 - 2t)^{-\frac{k}{2}} \end{aligned}$$

2 Mean and Variance

From the above m.g.f., we can calulate the derivatives
$M'_X(t) = k(1 - 2t)^{-\frac{k}{2}-1}$, and
$M''_X(t) = k(k + 2)(1 - 2t)^{-\frac{k}{2}-2}$.

This gives

$$\mu = \mathbb{E}(X) = M'_X(0) = k$$

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We can also calculate

$$\begin{aligned} \sigma^2 &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= M''_X(0) - k^2 \\ &= k(k + 2) - k^2 \\ &= 2k \end{aligned}$$