MATH 308 Assignment 16: Exercises 6.4

Nakul Joshi

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We know that $X \sim \mathcal{B}(n, p) \implies f_X(x) = \binom{n}{x} p^x (1 - p)^{n-x}$. Thus, the probability of obtaining the value X can be written as a function of the parameter p:

$$\begin{aligned} L(p) &= \binom{n}{X} p^X (1-p)^{n-X} \\ \implies L' / \binom{n}{X} &= X p^{X-1} (1-p)^{n-X} \\ &\quad - p^X (n - X)(1-p)^{n-X-1} \end{aligned}$$

We can then obtain the MLE $\hat{p}$ by setting $L' = 0$:

$$\begin{aligned} &X p^{X-1} (1-p)^{n-X} = p^X (n-X)(1-p)^{n-X-1} \\ &\implies \hat{p}(n - X) = X(1 - \hat{p}) \\ &\implies \hat{p}n - \cancel{\hat{p}X} = X - \cancel{X\hat{p}} \\ &\implies \hat{p} = X/n \quad \square \end{aligned}$$

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$X \sim \mathcal{P}(\lambda) \implies f_X = \frac{\lambda^x e^{-\lambda}}{x!}$. So, the likelihood of obtaining the sample $\{x_1, x_2, \ldots x_n\}$ is $L(\lambda) = \prod_{i=1}^n f_X(x_i)$, since the elements of the sample are i.i.d.

$$\begin{aligned} L &= \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\ &= \frac{\lambda^{\Sigma x} e^{-n\lambda}}{\prod(x_i!)} \end{aligned}$$

$$\begin{aligned} \implies L' / \prod(x_i!) &= \left(\lambda^{\Sigma x} e^{-n\lambda}\right)' \\ &= \Sigma x \lambda^{\Sigma x - 1} e^{-n\lambda} + \lambda^{\Sigma x} e^{-n\lambda}(-n) \\ &= e^{-n\lambda} \lambda^{\Sigma x - 1} \left(\Sigma x - n\lambda\right) \end{aligned}$$

Setting $L' = 0$, we get $\hat{\lambda}$:

$$\begin{aligned} 0 &= \Sigma x_i - n\hat{\lambda} \\ \implies \hat{\lambda} &= \Sigma x_i / n = \overline{x} \quad \square \end{aligned}$$

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As in the previous problem,

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{x_i^3 e^{-x_i/\theta}}{6\theta^4} \\ &= \frac{(\Pi x)^3 e^{-\Sigma x/\theta}}{6^n \theta^{4n}} \\ &= k e^{-\Sigma x/\theta} \theta^{-4n} \end{aligned}$$

$$\begin{aligned} \implies L'/k &= e^{-\Sigma x/\theta} \left(\Sigma x / \theta^2\right) \theta^{-4n} \\ &\quad + e^{-\Sigma x/\theta} (-4n) \theta^{-4n-1} \\ &= e^{-\Sigma x/\theta} \theta^{-4n-1} \left(\frac{\Sigma x}{\theta} - 4n\right) \end{aligned}$$

Setting $L' = 0$:

$$\begin{aligned} 0 &= \frac{\Sigma x}{\hat{\theta}} - 4n \\ \implies \hat{\theta} &= \frac{\Sigma x}{4n} \\ &= \overline{x}/4 \end{aligned}$$

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a)

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\begin{aligned}
L(\mu) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} e^{-\frac{x_i^2+\mu^2-2x_i\mu}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} (e^{\frac{x_i\mu}{\sigma^2}}/e^{\frac{\mu^2}{2\sigma^2}}) \\
&= \mathcal{C} e^{\frac{\mu\Sigma x}{\sigma^2}}/e^{\frac{n\mu^2}{2\sigma^2}} \\
&= \mathcal{C} e^{\frac{2\mu\Sigma x-n\mu^2}{2\sigma^2}} \\
\implies \log L - \log\mathcal{C} &= \frac{2\mu\Sigma x - n\mu^2}{2\sigma^2} \\
\implies L'/L &= \frac{2\Sigma x - 2n\mu}{2\sigma^2} \\
\implies 0 &= \Sigma x - n\hat{\mu} \\
\implies \hat{\mu} &= \Sigma x/n = \overline{x}
\end{aligned}
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b)

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\begin{aligned}
L(\sigma) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} \prod_{i=1}^{n} e^{\frac{-x_i^2-\mu^2+2x_i\mu}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} e^{\frac{-\Sigma x^2-n\mu^2+2\mu\Sigma x}{2\sigma^2}} \\
\implies L/\mathcal{C} &= e^{\frac{-\Sigma x^2-n\mu^2+2\mu\Sigma x}{2\sigma^2}}/\sigma^n \\
\implies \log L - \log\mathcal{C} &= \frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2} - n\log\sigma \\
\implies L'/L &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\sigma^3 - n/\sigma \\
\implies 0 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\hat{\sigma}^2 - n \\
\implies \hat{\sigma}^2 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/n \\
&= \frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x} \\
\implies \hat{\sigma} &= \sqrt{\frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x}}
\end{aligned}
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\begin{aligned}
L(\mu) &= \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_X^2}} \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
&= \mathcal{C} e^{-\frac{(x-\mu)^2}{2\sigma_X^2}-\frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
\implies \log L - \log\mathcal{C} &= -\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2} \\
\implies L'/L &= \frac{(x-\mu)}{\sigma_X^2} + \frac{(y-1.3\mu)}{\sigma_Y^2} \\
\implies 0 &= \frac{95-\hat{\mu}}{15^2} + \frac{130-1.3\hat{\mu}}{20^2} \\
\implies \hat{\mu} &\approx 97.1
\end{aligned}
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\begin{aligned}
L(r,\lambda) &= \prod_{i=1}^{n} \frac{\lambda^r}{\Gamma(r)} x_i^{r-1} e^{-\lambda x_i} \\
&= \frac{\lambda^{nr}}{\Gamma^n(r)} (\Pi x)^{r-1} e^{-\lambda\Sigma x} \\
\log L &= nr\log\lambda + (r-1)\log(\Pi x) \\
&\quad -\lambda\Sigma x - n\log\Gamma(r) \\
\implies \frac{\partial \log L}{\partial r} &= n\log\lambda + \log(\Pi x)
\end{aligned}
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