MATH 308 Assignment 16: Exercises 6.4

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1

We know that $X \sim \mathcal{B}(n,p) \implies f_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$. Thus, the probability of obtaining the value X can be written as a function of the parameter p:

$$\begin{aligned} L(p) &= \binom{n}{X} p^X (1-p)^{n-X} \\ \implies L'/\binom{n}{X} &= X p^{X-1}(1-p)^{n-X} \\ &\quad - p^X(n-X)(1-p)^{n-X-1} \end{aligned}$$

We can then obtain the MLE $\hat{p}$ by setting $L' = 0$:

$$\begin{aligned} Xp^{X-1}(1-p)^{n-X} &= p^X(n-X)(1-p)^{n-X-1} \\ \implies \hat{p}(n-X) &= X(1-\hat{p}) \\ \implies \hat{p}n - \cancel{\hat{p}X} &= X - \cancel{X\hat{p}} \\ \implies \hat{p} &= X/n \quad \square \end{aligned}$$

2

$X \sim \mathcal{P}(\lambda) \implies f_X = \frac{\lambda^x e^{-\lambda}}{x!}$. So, the likelihood of obtaining the sample $\{x_1, x_2, \dots x_n\}$ is $L(\lambda) = \prod_{i=1}^n f_X(x_i)$, since the elements of the sample are i.i.d.

$$\begin{aligned} L &= \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\ &= \frac{\lambda^{\Sigma x} e^{-n\lambda}}{\prod(x_i!)} = \lambda^{\Sigma x} e^{-n\lambda}/\mathcal{C} \\ \implies L\mathcal{C} &= \lambda^{\Sigma x} e^{-n\lambda} \\ \implies \log L + \log \mathcal{C} &= \Sigma x \log \lambda - n\lambda \\ \implies L'/L &= \Sigma x/\lambda - n \\ \implies 0 &= \Sigma x/\hat{\lambda} - n \\ \implies \hat{\lambda} &= \Sigma x_i/n = \overline{x} \quad \square \end{aligned}$$

4

As in the previous problem,

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{x_i^3 e^{-x_i/\theta}}{6\theta^4} \\ &= \frac{(\Pi x)^3 e^{-\Sigma x/\theta}}{6^n \theta^{4n}} \\ &= \mathcal{C} e^{-\Sigma x/\theta} \theta^{-4n} \\ \implies \log L - \log C &= -\Sigma x/\theta - 4n \log \theta \\ \implies L'/L &= \Sigma x/\theta^2 - 4n/\theta \\ \implies 0 &= \frac{\Sigma x}{\hat{\theta}} - 4n \\ \implies \hat{\theta} &= \frac{\Sigma x}{4n} = \overline{x}/4 \end{aligned}$$

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a)

$$
\begin{aligned}
L(\mu) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} e^{-\frac{x_i^2+\mu^2-2x_i\mu}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} (e^{\frac{x_i\mu}{\sigma^2}} / e^{\frac{\mu^2}{2\sigma^2}}) \\
&= \mathcal{C} e^{\frac{\mu\Sigma x}{\sigma^2}} / e^{\frac{n\mu^2}{2\sigma^2}} \\
&= \mathcal{C} e^{\frac{2\mu\Sigma x - n\mu^2}{2\sigma^2}} \\
\implies \log L - \log \mathcal{C} &= \frac{2\mu\Sigma x - n\mu^2}{2\sigma^2} \\
\implies L'/L &= \frac{2\Sigma x - 2n\mu}{2\sigma^2} \\
\implies 0 &= \Sigma x - n\hat{\mu} \\
\implies \hat{\mu} &= \Sigma x / n = \overline{x}
\end{aligned}
$$

b)

$$
\begin{aligned}
L(\sigma) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} \prod_{i=1}^{n} e^{\frac{-x_i^2-\mu^2+2x_i\mu}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}} \\
\implies L/\mathcal{C} &= e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}} / \sigma^n \\
\implies \log L - \log \mathcal{C} &= \frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2} - n\log\sigma \\
\implies L'/L &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\sigma^3 - n/\sigma \\
\implies 0 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\hat{\sigma}^2 - n \\
\implies \hat{\sigma}^2 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/n \\
&= \frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x} \\
\implies \hat{\sigma} &= \sqrt{\frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x}}
\end{aligned}
$$

10

$$
\begin{aligned}
L(\mu) &= \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_X^2}} \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
&= \mathcal{C} e^{-\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
\implies \log L - \log \mathcal{C} &= -\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2} \\
\implies L'/L &= \frac{(x-\mu)}{\sigma_X^2} + \frac{(y-1.3\mu)}{\sigma_Y^2} \\
\implies 0 &= \frac{95-\hat{\mu}}{15^2} + \frac{130-1.3\hat{\mu}}{20^2} \\
\implies \hat{\mu} &\approx 97.1
\end{aligned}
$$

12

$$
\begin{aligned}
L(r,\lambda) &= \prod_{i=1}^{n} \frac{\lambda^r}{\Gamma(r)} x_i^{r-1} e^{-\lambda x_i} \\
&= \frac{\lambda^{nr}}{\Gamma^n(r)} (\Pi x)^{r-1} e^{-\lambda\Sigma x} \\
\log L &= nr\log\lambda + (r-1)\log(\Pi x) \\
&\quad -\lambda\Sigma x - n\log\Gamma(r) \\
\implies \frac{1}{L}\frac{\partial L}{\partial r} &= n\log\lambda + \log(\Pi x) - n\psi(r) \\
\implies n\psi(\hat{r}) &= n\log\hat{\lambda} + \log(\Pi x) \qquad (1) \\
\text{And } \frac{1}{L}\frac{\partial L}{\partial \lambda} &= nr/\lambda - \Sigma x \\
\implies \hat{\lambda} &= \frac{n\hat{r}}{\Sigma x} \qquad (2)
\end{aligned}
$$

14

$$
\begin{aligned}
X &= \{2,3,5,9,10\} \\
f(x) &= \frac{1}{\beta-\alpha+1}, \alpha \le x \le \beta \\
\implies \mathbb{E}\left(X^k\right) &= \sum_{x=\alpha}^{\beta} \frac{x^k}{\beta-\alpha+1} \\
\implies \overline{X} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x}{\hat{\beta}-\hat{\alpha}+1} \qquad (1) \\
&= (\hat{\beta}+\hat{\alpha})/2
\end{aligned}
$$

$$
\begin{aligned}
\text{And } \overline{X^2} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x^2}{\hat{\beta}-\hat{\alpha}+1} \qquad (2) \\
&= (\hat{\beta}-\hat{\alpha}+2\hat{\alpha}^2\hat{\beta}^2+2\hat{\beta}^2+2\hat{\alpha}^2)/6
\end{aligned}
$$

Where $\overline{X} = \Sigma X/n = 5.8$ and $\overline{X^2} = \Sigma X^2/n = 43.8$. Solving 1 and 2, we get:

$$
\begin{aligned}
\hat{\alpha} &= \overline{X} - \frac{\sqrt{12\overline{X}^2 + 12\overline{X^2} - 1}}{2} \approx 1 \\
\hat{\beta} &= \overline{X} + \frac{\sqrt{12\overline{X}^2 + 12\overline{X^2} - 1}}{2} \approx 11
\end{aligned}
$$

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a)

$$
\begin{aligned}
\mathbb{E}\left(X\right) &= r/\lambda \\
\implies \overline{X} &= \hat{r}/\hat{\lambda} \qquad (1) \\
\mathbb{E}\left(X^2\right) &= \mathrm{Var}(X) + \mathbb{E}\left(X\right)^2 \\
&= r/\lambda^2 + r^2/\lambda^2 \\
&= r(1+r)/\lambda \\
\implies \overline{X^2} &= \hat{r}(1+\hat{r})/\hat{\lambda}^2 \qquad (2)
\end{aligned}
$$

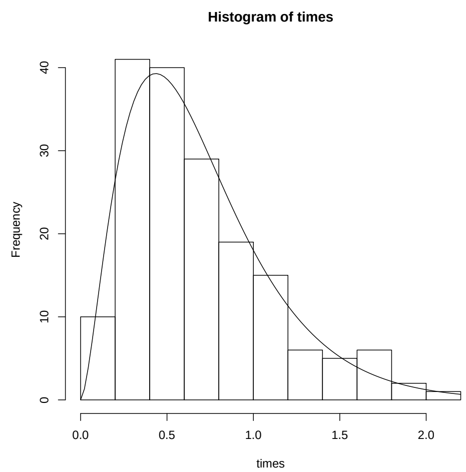
Solving 1 and 2,

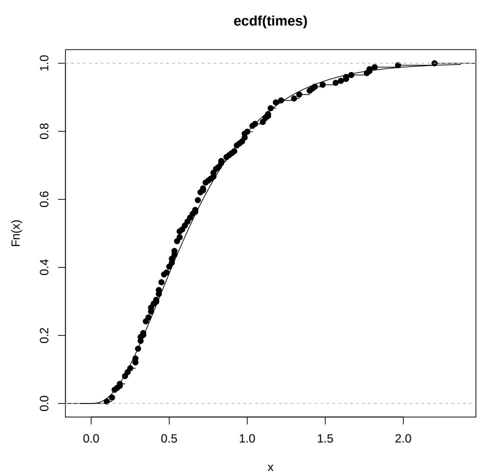
$$
\begin{aligned}
r &= \frac{\overline{X}^2}{\overline{X^2} - \overline{X}^2} \approx 2.67 \\
\lambda &= \frac{\overline{X}}{\overline{X^2} - \overline{X}^2} \approx 3.84
\end{aligned}
$$

b)

We run a test on the null hypothesis H_0 that the data are $\sim \Gamma(\hat{r}, \hat{\lambda})$, against the alternative hypothesis H_A that they are not. Running the goodness-of-fit test yields a p-value of 0.78, allowing us to strongly favour H_0.

c)





25

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right)\\ &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i)\\ &= \sum_{i=1}^{n} a_i \mu\\ &= \mu \Sigma a\end{aligned}$$

For an unbiased estimator of μ,

$$\begin{aligned}\mathbb{E}(X) &= \mu\\ \implies \cancel{\mu}\Sigma a &= \cancel{\mu}\\ \implies \Sigma a &= 1\end{aligned}$$

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a)

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{n}\Sigma(x_i - \overline{x})^2\\ &= \frac{\sigma^2}{\sigma^2}\frac{n-1}{n}\frac{1}{n-1}\Sigma(x_i - \overline{x})^2\\ &= \frac{\sigma^2}{n}\frac{(n-1)S^2}{\sigma^2}\end{aligned}$$

Where $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

$$\begin{aligned}\mathbb{E}\left(\hat{\sigma}^2\right) &= \mathbb{E}\left(\frac{\sigma^2}{n}\frac{(n-1)S^2}{\sigma^2}\right)\\ &= \frac{\sigma^2}{n}\mathbb{E}\left(\frac{(n-1)S^2}{\sigma^2}\right)\\ &= \frac{\sigma^2}{n}(n-1) = \sigma^2(1-1/n)\\ \implies \mathrm{Bias}(\hat{\sigma}^2) &= \mathbb{E}\left(\hat{\sigma}^2\right) - \sigma^2\\ &= \sigma^2(1-1/n-1) = -\sigma^2/n\end{aligned}$$

b)

$$\begin{aligned}\mathrm{Var}\ \hat{\sigma}^2 &= \frac{\sigma^4}{n^2}\mathrm{Var}\ \frac{(n-1)S^2}{\sigma^2}\\ &= \frac{\sigma^4}{n^2}2(n-1) = 2\sigma^4(n-1)/n^2\end{aligned}$$

c)

$$\begin{aligned}\mathrm{MSE} &= \mathrm{Var}(\hat{\sigma}^2) + \mathrm{Bias}^2(\hat{\sigma}^2)\\ &= \frac{2\sigma^4(n-1)}{n^2} + \frac{\sigma^4}{n^2}\\ &= \sigma^4(2n-1)/n^2\end{aligned}$$

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a)

$$\begin{aligned}f(x) &= 6x^5/\theta^6\\ \implies F(x) &= \int_0^x 6x^5/\theta^6\,\mathrm{d}t = x^6/\theta^6\\ f_{X,\max} &= f_{(n)}(x)\\ &= nf(x)F^{n-1}(x)\\ &= n\frac{6x^5}{\theta^6}\frac{x^{6n-6}}{\theta^{6n-6}}\\ &= 6nx^{6n-1}/\theta^{6n}\end{aligned}$$

b)

$$\begin{aligned}\mathbb{E}(X_{\max}) &= \int_0^\theta x\frac{6nx^{6n-1}}{\theta^{6n}}\,\mathrm{d}x\\ &= \frac{6n}{\theta^{6n}}\left.\frac{x^{6n+1}}{6n+1}\right|_0^\theta\\ &= \frac{6n}{6n+1}\theta\end{aligned}$$

c)

$$
\begin{aligned}
\text{Bias} &= \mathbb{E}\left(X_{\max}\right) - \theta \\
&= \theta\left(\frac{6n}{6n+1} - 1\right) \\
&= \frac{-\theta}{6n+1}
\end{aligned}
$$

d)

$$
\begin{aligned}
\mathbb{E}\left(X_{\max}^2\right) &= \int_0^\theta x^2 \frac{6nx^{6n-1}}{\theta^{6n}}\,\mathrm{d}x \\
&= 6n\theta^2/(6n+2) \\
\text{Var } &= EX_{\max}^2 - E^2X_{\max} \\
&= \frac{3\theta^2 n}{(3n+1)(6n+1)^2} \\
\text{MSE} = \text{Var } + \text{Bias}^2 &= \frac{\theta^2}{18n^2+9n+1}
\end{aligned}
$$

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a)

$$
\begin{aligned}
\mathbb{E}\left(W\right) &= \mathbb{E}\left(a\overline{X}\right) + \mathbb{E}\left((1-a)\overline{Y}\right) \\
&= a\mathrm{E}\overline{X} + (1-a)\mathrm{E}\overline{Y} \\
&= a\mu + (1-a)\mu = 1
\end{aligned}
$$

b)

$$
\begin{aligned}
\text{Var } W &= \text{Var }\left(a\overline{X} + (1-a)\overline{Y}\right) \\
&= a^2\text{Var }\overline{X} + (1-a)^2\text{Var }\overline{Y} \\
&= a^2\sigma_1^2/n + (1-a)^2\sigma_2^2/m \\
\implies \text{Var }'(W) &= 2a\sigma_1^2/n - 2(1-a)\sigma_2^2/m \\
\implies \hat{a}\sigma_1^2/n &= \sigma_2^2/m - \hat{a}\sigma_2^2/m \\
\implies \hat{a} &= \frac{\sigma_2^2/m}{\sigma_1^2/n + \sigma_2^2/m}
\end{aligned}
$$