MATH 308 Assignment 8: Exercises 4.4

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March 10, 2014

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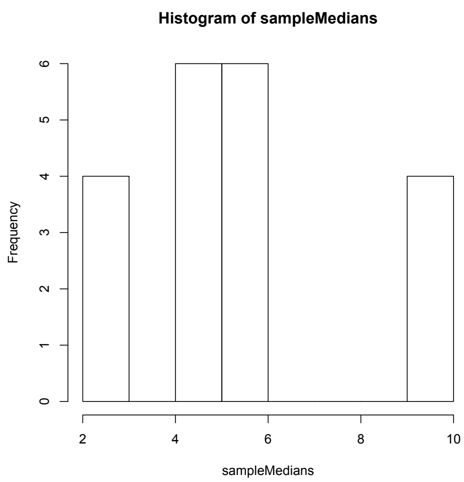

Population median: 5.5
Mean of sample medians: 5.7

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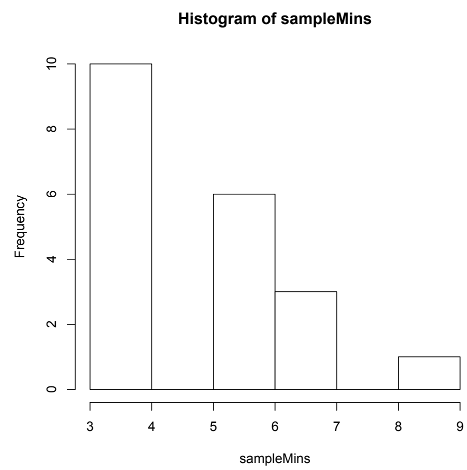

Mean of minimums: 4.8

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From the CLT, the sample mean is approximately distributed normally with $\mu_{\overline{X}} = \mu = 48$ and $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n} = 9^2/20 = 2.7$. Thus the probability of the sample mean being greater than 51 is `1-pnorm(51,mean=48,sd=sqrt(2.7))` which evaluates to $\approx 3\%$.

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The mean μ of the distribution is $\int_2^6 f(x)\,\mathrm{d}x = 4$. The variance σ^2 is $\int_2^6 (x-\mu)^2 f(x)\,\mathrm{d}x = 2.4$. By CLT, the distribution of the sample mean $\overline{X}$ is approximated by $\mathcal{N}(\mu, \frac{\sigma^2}{n}) = \mathcal{N}(4, 3/305)$. Thus, $P(\overline{X} >= 4.2)$ is `1-pnorm(4.2,mean=4,sd=sqrt(2.4))`, which evaluates to $\approx 2\%$.

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By CLT, the number of people with degrees is approximately normally dustributed with $\mu = np$ and $\sigma^2 = np(1-p)$. Thus, the probability of between 220 and 230 people in the sample having a degree is:

```
n=800
p=0.286
m=n*p
sd=sqrt(n*p*(1-p))
pn=
pnorm(230+0.5,m,sd)
-pnorm(220-0.5,m,sd)
```

Which evaluates to 0.3194833. The exact probability is given by

```
pb=pbinom(230,n,p)-pbinom(220,n,p)
```

which evaluates to 0.2959644. The error in the approximation is `(pn-pb)/pb` which evaluates to $\approx 8\%$.

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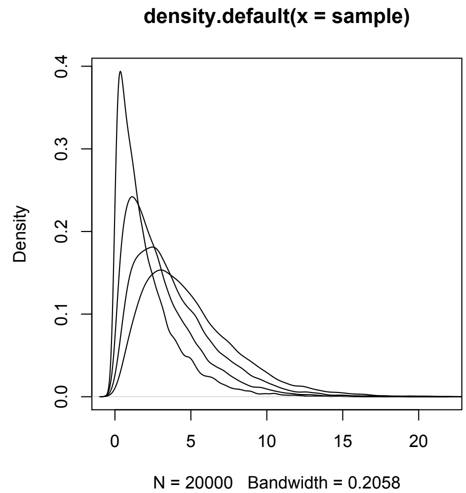
By definition, $W \sim \chi^2_n$. The sampling distributions of W_n for $n \in \{2, 3, 4, 5\}$ are shown in the accompanying figure. The means and variances are tabulated as:

n	μ	σ^2
2	2.02	4.07
3	2.99	5.82
4	3.98	7.80
5	4.99	10.01

From the table, we can see that $\mu = n$ and $\sigma^2 = 2n$.



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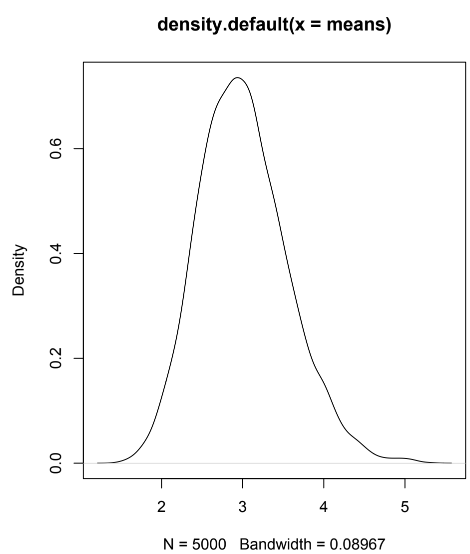
a)

The sampling distribution is shown below:



b)

The mean from sample is 3.00 with standard error 0.55. From CLT, we expect $\mu_{\overline{X}} = \mu = (1/3)^{-1} = 3$ and $\sigma_{\overline{X}} = \sigma/\sqrt{n} = (1/3)^{-1}/\sqrt{30} \approx 0.55$, giving a difference of $< 1\%$.

c)

We get $P = 0.83$.

d)

We get $P = 0.82$, which is only $\approx 1\%$ off.

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a) Distribution of Sample Minimum

$$\begin{aligned} P(X_{\min} \geq x) &= \prod_{i=1}^{n} P(X_i \geq x) \\ \implies \quad 1 - F_{\min}(x) &= (1 - F(x))^n \\ \implies \quad -f_{\min}(x) &= n(1 - F(x))^{n-1}(-f(x)) \\ \implies \quad f_{\min}(x) &= n\,(1 - F(x))^{n-1}\, f(x) \quad \square \end{aligned}$$

b) Distribution of Sample Maximum

$$\begin{aligned} P(X_{\max} \leq x) &= \prod_{i=1}^{n} P(X_i \leq x) \\ \implies \quad F_{\max}(x) &= F^n(x) \\ \implies \quad f_{\max}(x) &= nF^{n-1}(x)f(x) \quad \square \end{aligned}$$

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a)

$F(x) = \int_1^x \frac{2}{x^2}\,\mathrm{d}x = 2 - 2/x$, so from previous result $f_{\max} = 2\left(2 - \frac{2}{x}\right)^{2-1} \frac{2}{x^2} = 2\left(2 - \frac{2}{x}\right)\frac{2}{x^2}$.

b)

$E_{X_{\max}} = \int_1^2 x f_{\max}(x)\,\mathrm{d}x \approx 1.55$.

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$$\begin{aligned} f_{X_1+X_2}(x) &= P(X_1 + X_2 = x) \\ &= \sum_{i=0}^{x} \left(\frac{\lambda_1^i}{i!} e^{-\lambda_1} \frac{\lambda_2^{x-i}}{(x-i)!} e^{-\lambda_2} \right) \\ &= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{x} \frac{\lambda_1^i \lambda_2^{x-i}}{i!(x-i)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!} \sum_{i=0}^{x} \left(\frac{x!}{i!(x-i)!} \lambda_1^i \lambda_2^{x-i} \right) \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!} \sum_{i=0}^{x} \left(\binom{x}{i} \lambda_1^i \lambda_2^{x-i} \right) \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!} (\lambda_1 + \lambda_2)^x \\ &= f_Z(x), \text{ where } Z \sim \mathcal{P}(\lambda_1 + \lambda_2) \end{aligned}$$

$$\begin{aligned} &\implies X \sim \mathcal{P}\left(\sum_{i=1}^{10} \lambda_i \right) \\ &\implies X \sim \mathcal{P}(30) \\ &\implies f_X(x) = \frac{30^x}{x!} e^{30} \end{aligned}$$

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a)

The distribution seems quite close to normal.

b)

Theoretical mean $\mu_{\overline{X}} = \mu = 10.17$. Actual mean is 10.02. Difference is $\approx 1.5\%$.

c)

Theoretical s.e. $\sigma_{\overline{X}} = \frac{\sigma}{n}\sqrt{\frac{N-n}{N-1}} \approx 4.55$. Actual s.e. 4.56. Difference is $< 1\%$.

d)

The differences reduce with increasing n.

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It can be seen from the plots below that the distribution of sample variances is approximately normal, and the fit to normal improves with sample size.