MATH 308 Assignment 16: Exercises 6.4

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1

We know that $X \sim \mathcal{B}(n,p) \implies f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$. Thus, the probability of obtaining the value X can be written as a function of the parameter p:

$$
\begin{aligned}
L(p) &= \binom{n}{X} p^X (1-p)^{n-X} \\
\implies L' / \binom{n}{X} &= X p^{X-1} (1-p)^{n-X} \\
&\quad - p^X (n-X)(1-p)^{n-X-1}
\end{aligned}
$$

We can then obtain the MLE $\hat{p}$ by setting $L' = 0$:

$$
\begin{aligned}
& X p^{X-1}(1-p)^{n-X} = p^X (n-X)(1-p)^{n-X-1} \\
& \implies \hat{p}(n-X) = X(1-\hat{p}) \\
& \implies \hat{p}n - \cancel{\hat{p}X} = X - \cancel{X\hat{p}} \\
& \implies \hat{p} = X/n \quad \square
\end{aligned}
$$

2

$X \sim \mathcal{P}(\lambda) \implies f_X = \frac{\lambda^x e^{-\lambda}}{x!}$. So, the likelihood of obtaining the sample $\{x_1, x_2, \dots x_n\}$ is $L(\lambda) = \prod_{i=1}^n f_X(x_i)$, since the elements of the sample are i.i.d.

$$
\begin{aligned}
L &= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\
&= \frac{\lambda^{\Sigma x} e^{-n\lambda}}{\prod (x_i!)}
\end{aligned}
$$

$$
\begin{aligned}
\implies L' / \prod (x_i!) &= \left(\lambda^{\Sigma x} e^{-n\lambda}\right)' \\
&= \Sigma x \lambda^{\Sigma x - 1} e^{-n\lambda} + \lambda^{\Sigma x} e^{-n\lambda}(-n) \\
&= e^{-n\lambda} \lambda^{\Sigma x - 1} \left(\Sigma x - n\lambda\right)
\end{aligned}
$$

Setting $L' = 0$, we get $\hat{\lambda}$:

$$
\begin{aligned}
0 &= \Sigma x_i - n\hat{\lambda} \\
\implies \hat{\lambda} &= \Sigma x_i / n = \overline{x} \quad \square
\end{aligned}
$$

4

As in the previous problem,

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{x_i^3 e^{-x_i/\theta}}{6\theta^4} \\
&= \frac{(\Pi x)^3 e^{-\Sigma x/\theta}}{6^n \theta^{4n}} \\
&= k e^{-\Sigma x/\theta} \theta^{-4n}
\end{aligned}
$$

$$
\begin{aligned}
\implies L'/k &= e^{-\Sigma x/\theta} \left(\Sigma x/\theta^2\right) \theta^{-4n} \\
&\quad + e^{-\Sigma x/\theta}(-4n)\theta^{-4n-1} \\
&= e^{-\Sigma x/\theta} \theta^{-4n-1} \left(\frac{\Sigma x}{\theta} - 4n\right)
\end{aligned}
$$

Setting $L' = 0$:

$$
\begin{aligned}
0 &= \frac{\Sigma x}{\hat{\theta}} - 4n \\
\implies \hat{\theta} &= \frac{\Sigma x}{4n} \\
&= \overline{x}/4
\end{aligned}
$$

5

a)

$$
\begin{aligned}
L(\mu) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} e^{-\frac{x_i^2+\mu^2-2x_i\mu}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^{n} (e^{\frac{x_i\mu}{\sigma^2}}/e^{\frac{\mu^2}{2\sigma^2}}) \\
&= \mathcal{C} e^{\frac{\mu\Sigma x}{\sigma^2}}/e^{\frac{n\mu^2}{2\sigma^2}} \\
&= \mathcal{C} e^{\frac{2\mu\Sigma x - n\mu^2}{2\sigma^2}} \\
\implies \log L - \log \mathcal{C} &= \frac{2\mu\Sigma x - n\mu^2}{2\sigma^2} \\
\implies L'/L &= \frac{2\Sigma x - 2n\mu}{2\sigma^2} \\
\implies 0 &= \Sigma x - n\hat{\mu} \\
\implies \hat{\mu} &= \Sigma x/n = \overline{x}
\end{aligned}
$$

b)

$$
\begin{aligned}
L(\sigma) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} \prod_{i=1}^{n} e^{\frac{-x_i^2-\mu^2+2x_i\mu}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}} \\
\implies L/\mathcal{C} &= e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}}/\sigma^n \\
\implies \log L - \log \mathcal{C} &= \frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2} - n\log\sigma \\
\implies L'/L &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\sigma^3 - n/\sigma \\
\implies 0 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\hat{\sigma}^2 - n \\
\implies \hat{\sigma}^2 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/n \\
&= \frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x} \\
\implies \hat{\sigma} &= \sqrt{\frac{\Sigma x^2}{n} + \mu^2 - 2\mu\overline{x}}
\end{aligned}
$$

10

$$
\begin{aligned}
L(\mu) &= \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_X^2}} \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
&= \mathcal{C} e^{-\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
\implies \log L - \log \mathcal{C} &= -\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2} \\
\implies L'/L &= \frac{(x-\mu)}{\sigma_X^2} + \frac{(y-1.3\mu)}{\sigma_Y^2} \\
\implies 0 &= \frac{95-\hat{\mu}}{15^2} + \frac{130-1.3\hat{\mu}}{20^2} \\
\implies \hat{\mu} &\approx 97.1
\end{aligned}
$$

12

$$
\begin{aligned}
L(r,\lambda) &= \prod_{i=1}^{n} \frac{\lambda^r}{\Gamma(r)} x_i^{r-1} e^{-\lambda x_i} \\
&= \frac{\lambda^{nr}}{\Gamma^n(r)} (\Pi x)^{r-1} e^{-\lambda\Sigma x} \\
\log L &= nr\log\lambda + (r-1)\log(\Pi x) \\
&\quad -\lambda\Sigma x - n\log\Gamma(r) \\
\implies \frac{1}{L}\frac{\partial L}{\partial r} &= n\log\lambda + \log(\Pi x) - n\psi(r) \\
\implies n\psi(\hat{r}) &= n\log\hat{\lambda} + \log(\Pi x) \qquad (1) \\
\text{And } \frac{1}{L}\frac{\partial L}{\partial \lambda} &= nr/\lambda - \Sigma x \\
\implies \hat{\lambda} &= \frac{n\hat{r}}{\Sigma x} \qquad (2)
\end{aligned}
$$

14

$$
\begin{aligned}
X &= \{2,3,5,9,10\} \\
f(x) &= \frac{1}{\beta-\alpha+1}, \alpha \leq x \leq \beta \\
\implies \mathbb{E}\left(X^k\right) &= \sum_{x=\alpha}^{\beta} \frac{x^k}{\beta-\alpha+1} \\
\implies \overline{X} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x}{\hat{\beta}-\hat{\alpha}+1} \qquad (1) \\
&= (\hat{\beta}+\hat{\alpha})/2
\end{aligned}
$$

$$
\begin{aligned}
\text{And } \overline{X^2} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x^2}{\hat{\beta}-\hat{\alpha}+1} \qquad (2) \\
&= (\hat{\beta}-\hat{\alpha}+2\hat{\alpha}^2\hat{\beta}^2+2\hat{\beta}^2+2\hat{\alpha}^2)/6
\end{aligned}
$$

Where $\overline{X} = \Sigma X/n = 5.8$ and $\overline{X^2} = \Sigma X^2/n = 43.8$. Solving 1 and 2, we get:

$$
\begin{aligned}
\hat{\alpha} &= \overline{X} - \frac{\sqrt{12\overline{X}^2 + 12\overline{X^2} - 1}}{2} \approx 1 \\
\hat{\beta} &= \overline{X} + \frac{\sqrt{12\overline{X}^2 + 12\overline{X^2} - 1}}{2} \approx 11
\end{aligned}
$$

16

a)

$$
\begin{aligned}
\mathbb{E}(X) &= r/\lambda \\
\implies \overline{X} &= \hat{r}/\hat{\lambda} \qquad (1) \\
\mathbb{E}\left(X^2\right) &= \mathrm{Var}(X) + \mathbb{E}(X)^2 \\
&= r/\lambda^2 + r^2/\lambda^2 \\
&= r(1+r)/\lambda \\
\implies \overline{X^2} &= \hat{r}(1+\hat{r})/\hat{\lambda}^2 \qquad (2)
\end{aligned}
$$

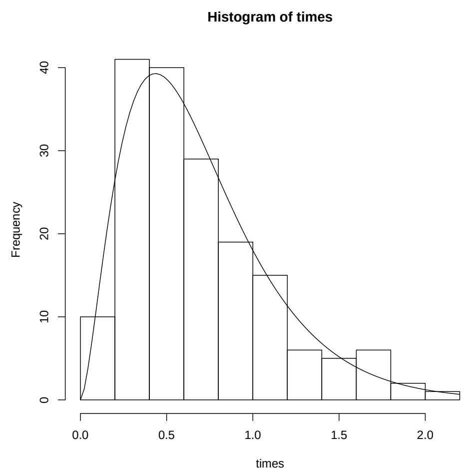
Solving 1 and 2,

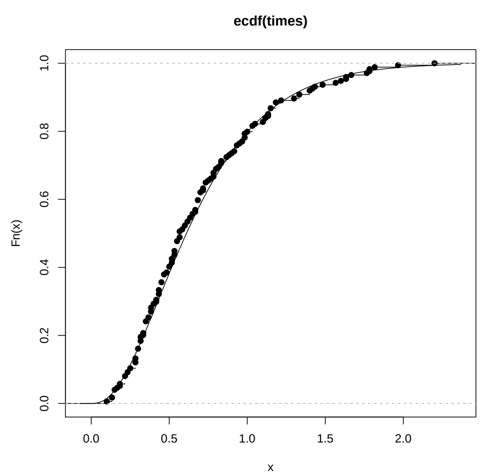
$$
\begin{aligned}
r &= \frac{\overline{X}^2}{\overline{X^2} - \overline{X}^2} \approx 2.67 \\
\lambda &= \frac{\overline{X}}{\overline{X^2} - \overline{X}^2} \approx 3.84
\end{aligned}
$$

b)

We run a test on the null hypothesis H_0 that the data are $\sim \Gamma(\hat{r}, \hat{\lambda})$, against the alternative hypothesis H_A that they are not. Running the goodness-of-fit test yields a p-value of 0.78, allowing us to strongly favour H_0.

c)





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$$
\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) \\
&= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) \\
&= \sum_{i=1}^{n} a_i \mu \\
&= \mu \Sigma a
\end{aligned}
$$

For an unbiased estimator of μ,

$$
\begin{aligned}
\mathbb{E}(X) &= \mu \\
\implies \cancel{\mu}\Sigma a &= \cancel{\mu} \\
\implies \Sigma a &= 1
\end{aligned}
$$

27

a)

$$
\begin{aligned}
&\mathbb{E}\left(\hat{\sigma}^2\right) \\
&= \frac{1}{n}\mathrm{E}\Sigma(X_i^2 + \overline{X}^2 - 2X_i\overline{X}) \\
&= \frac{1}{n}(\mathrm{E}\Sigma X_i^2 + n\mathrm{E}\overline{X}^2 - 2\mathrm{E}(\overline{X}\Sigma X_i)) \\
&= \frac{1}{n}\mathrm{E}\Sigma X_i^2 + \mathrm{E}\overline{X}^2 - 2\mathrm{E}\overline{X}^2 \\
&= \frac{1}{n}\mathrm{E}\Sigma X_i^2 - \mathrm{E}\overline{X}^2 \\
&= \cancel{\mu^2} + \sigma^2 - \cancel{\mu^2} - \sigma^2/n \\
&= \sigma^2(1 - 1/n) \\
\implies &\ \mathrm{Bias}(\hat{\sigma}^2) = \mathbb{E}\left(\hat{\sigma}^2\right) - \sigma^2 \\
&\qquad = \sigma^2(1 - 1/n - 1) = -\sigma^2/n
\end{aligned}
$$

b)

34

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