

MATH 308 Assignment 16:

Exercises 6.4

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We know that $X \sim \mathcal{B}(n, p) \implies f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$. Thus, the probability of obtaining the value X can be written as a function of the parameter p :

$$\begin{aligned} L(p) &= \binom{n}{X} p^X (1-p)^{n-X} \\ \implies L'/\binom{n}{X} &= X p^{X-1} (1-p)^{n-X} \\ &\quad - p^X (n-X) (1-p)^{n-X-1} \end{aligned}$$

We can then obtain the MLE \hat{p} by setting $L' = 0$:

$$\begin{aligned} X p^{X-1} (1-p)^{n-X} &= p^X (n-X) (1-p)^{n-X-1} \\ \implies \hat{p}(n-X) &= X(1-\hat{p}) \\ \implies \hat{p}n - \hat{p}X &= X - X\hat{p} \\ \implies \hat{p} &= X/n \quad \square \end{aligned}$$

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$X \sim \mathcal{P}(\lambda) \implies f_X = \frac{\lambda^x e^{-\lambda}}{x!}$. So, the likelihood of obtaining the sample $\{x_1, x_2, \dots, x_n\}$ is $L(\lambda) = \prod_{i=1}^n f_X(x_i)$, since the elements of the sample are i.i.d.

$$\begin{aligned} L &= \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\ &= \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod (x_i!)} = \lambda^{\sum x_i} e^{-n\lambda} / \mathcal{C} \end{aligned}$$

$$\begin{aligned} \implies LC &= \lambda^{\sum x_i} e^{-n\lambda} \\ \implies \log L + \log \mathcal{C} &= \sum x_i \log \lambda - n\lambda \\ \implies L'/L &= \sum x_i / \lambda - n \\ \implies 0 &= \sum x_i / \hat{\lambda} - n \\ \implies \hat{\lambda} &= \sum x_i / n = \bar{x} \quad \square \end{aligned}$$

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As in the previous problem,

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n \frac{x_i^3 e^{-x_i/\theta}}{6\theta^4} \\ &= \frac{(\prod x_i)^3 e^{-\sum x_i/\theta}}{6^n \theta^{4n}} \\ &= \mathcal{C} e^{-\sum x_i/\theta} \theta^{-4n} \\ \implies \log L - \log \mathcal{C} &= -\sum x_i / \theta - 4n \log \theta \\ \implies L'/L &= \sum x_i / \theta^2 - 4n / \theta \\ \implies 0 &= \frac{\sum x_i}{\hat{\theta}} - 4n \\ \implies \hat{\theta} &= \frac{\sum x_i}{4n} = \bar{x} / 4 \end{aligned}$$

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a)

$$\begin{aligned}
L(\mu) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^n e^{-\frac{x_i^2 + \mu^2 - 2x_i\mu}{2\sigma^2}} \\
&= \mathcal{C} \prod_{i=1}^n (e^{\frac{x_i\mu}{\sigma^2}} / e^{\frac{\mu^2}{2\sigma^2}}) \\
&= \mathcal{C} e^{\frac{\mu\Sigma x}{\sigma^2}} / e^{\frac{n\mu^2}{2\sigma^2}} \\
&= \mathcal{C} e^{\frac{2\mu\Sigma x - n\mu^2}{2\sigma^2}} \\
\Rightarrow \log L - \log \mathcal{C} &= \frac{2\mu\Sigma x - n\mu^2}{2\sigma^2} \\
\Rightarrow L'/L &= \frac{2\Sigma x - 2n\mu}{2\sigma^2} \\
\Rightarrow 0 &= \Sigma x - n\hat{\mu} \\
\Rightarrow \hat{\mu} &= \Sigma x/n = \bar{x}
\end{aligned}$$

b)

$$\begin{aligned}
L(\sigma) &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} \prod_{i=1}^n e^{-\frac{-x_i^2 - \mu^2 + 2x_i\mu}{2\sigma^2}} \\
&= \frac{\mathcal{C}}{\sigma^n} e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}} \\
\Rightarrow L/\mathcal{C} &= e^{\frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2}} / \sigma^n \\
\Rightarrow \log L - \log \mathcal{C} &= \frac{-\Sigma x^2 - n\mu^2 + 2\mu\Sigma x}{2\sigma^2} - n \log \sigma \\
\Rightarrow L'/L &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\sigma^3 - n/\sigma \\
\Rightarrow 0 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/\hat{\sigma}^2 - n
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \hat{\sigma}^2 &= (\Sigma x^2 + n\mu^2 - 2\mu\Sigma x)/n \\
&= \frac{\Sigma x^2}{n} + \mu^2 - 2\mu\bar{x} \\
\Rightarrow \hat{\sigma} &= \sqrt{\frac{\Sigma x^2}{n} + \mu^2 - 2\mu\bar{x}}
\end{aligned}$$

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$$\begin{aligned}
L(\mu) &= \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_X^2}} \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
&= \mathcal{C} e^{-\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2}} \\
\Rightarrow \log L - \log \mathcal{C} &= -\frac{(x-\mu)^2}{2\sigma_X^2} - \frac{(y-1.3\mu)^2}{2\sigma_Y^2} \\
\Rightarrow L'/L &= \frac{(x-\mu)}{\sigma_X^2} + \frac{(y-1.3\mu)}{\sigma_Y^2} \\
\Rightarrow 0 &= \frac{95 - \hat{\mu}}{15^2} + \frac{130 - 1.3\hat{\mu}}{20^2} \\
\Rightarrow \hat{\mu} &\approx 97.1
\end{aligned}$$

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$$\begin{aligned}
L(r, \lambda) &= \prod_{i=1}^n \frac{\lambda^r}{\Gamma(r)} x_i^{r-1} e^{-\lambda x_i} \\
&= \frac{\lambda^{nr}}{\Gamma^n(r)} (\Pi x)^{r-1} e^{-\lambda \Sigma x} \\
\log L &= nr \log \lambda + (r-1) \log(\Pi x) \\
&\quad - \lambda \Sigma x - n \log \Gamma(r) \\
\Rightarrow \frac{1}{L} \frac{\partial L}{\partial r} &= n \log \lambda + \log(\Pi x) - n\psi(r) \\
\Rightarrow n\psi(\hat{r}) &= n \log \hat{\lambda} + \log(\Pi x) \quad (1) \\
\text{And } \frac{1}{L} \frac{\partial L}{\partial \lambda} &= nr/\lambda - \Sigma x \\
\Rightarrow \hat{\lambda} &= \frac{n\hat{r}}{\Sigma x} \quad (2)
\end{aligned}$$

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$$\begin{aligned}
X &= \{2, 3, 5, 9, 10\} \\
f(x) &= \frac{1}{\beta - \alpha + 1}, \alpha \leq x \leq \beta \\
\Rightarrow \mathbb{E}(X^k) &= \sum_{x=\alpha}^{\beta} \frac{x^k}{\beta - \alpha + 1} \\
\Rightarrow \bar{X} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x}{\hat{\beta} - \hat{\alpha} + 1} \\
&= (\hat{\beta} + \hat{\alpha})/2
\end{aligned} \tag{1}$$

$$\begin{aligned}
\text{And } \bar{X^2} &= \frac{\sum_{x=\hat{\alpha}}^{\hat{\beta}} x^2}{\hat{\beta} - \hat{\alpha} + 1} \\
&= (\hat{\beta} - \hat{\alpha} + 2\hat{\alpha}^2\hat{\beta}^2 + 2\hat{\beta}^2 + 2\hat{\alpha}^2)/6
\end{aligned} \tag{2}$$

Where $\bar{X} = \Sigma X/n = 5.8$ and $\bar{X^2} = \Sigma X^2/n = 43.8$. Solving 1 and 2, we get:

$$\begin{aligned}
\hat{\alpha} &= \bar{X} - \frac{\sqrt{12\bar{X^2} + 12\bar{X}^2} - 1}{2} \approx 1 \\
\hat{\beta} &= \bar{X} + \frac{\sqrt{12\bar{X^2} + 12\bar{X}^2} - 1}{2} \approx 11
\end{aligned}$$

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a)

$$\begin{aligned}
\mathbb{E}(X) &= r/\lambda \\
\Rightarrow \bar{X} &= \hat{r}/\hat{\lambda} \\
\mathbb{E}(X^2) &= \text{Var}(X) + \mathbb{E}(X)^2 \\
&= r/\lambda^2 + r^2/\lambda^2 \\
&= r(1+r)/\lambda \\
\Rightarrow \bar{X^2} &= \hat{r}(1+\hat{r})/\hat{\lambda}^2
\end{aligned} \tag{1}$$

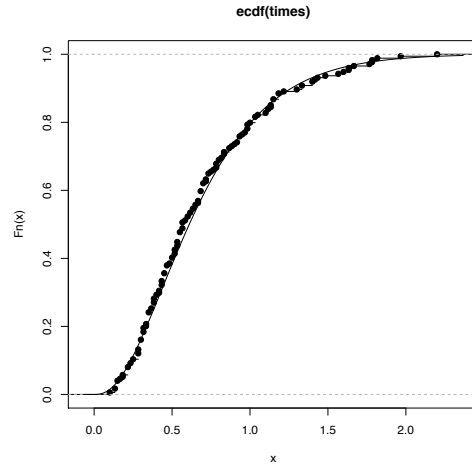
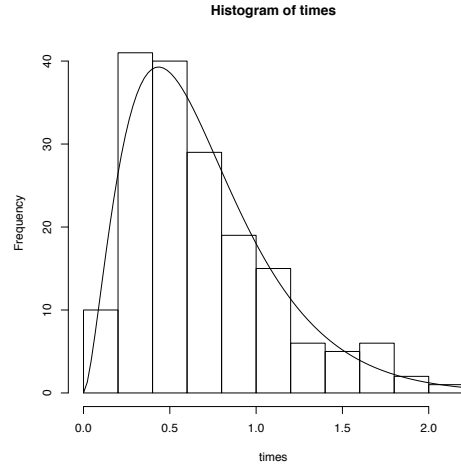
Solving 1 and 2,

$$\begin{aligned}
r &= \frac{\bar{X^2}}{\bar{X^2} - \bar{X}^2} \approx 2.67 \\
\lambda &= \frac{\bar{X}}{\bar{X^2} - \bar{X}^2} \approx 3.84
\end{aligned}$$

b)

We run a test on the null hypothesis H_0 that the data are $\sim \Gamma(\hat{r}, \hat{\lambda})$, against the alternative hypothesis H_A that they are not. Running the goodness-of-fit test yields a p -value of 0.78, allowing us to strongly favour H_0 .

c)



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$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^n a_i X_i\right) \\ &= \sum_{i=1}^n a_i \mathbb{E}(X_i) \\ &= \sum_{i=1}^n a_i \mu \\ &= \mu \Sigma a\end{aligned}$$

For an unbiased estimator of μ ,

$$\begin{aligned}\mathbb{E}(X) &= \mu \\ \implies \mu \Sigma a &= \mu' \\ \implies \Sigma a &= 1\end{aligned}$$

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a)

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{n} \Sigma (x_i - \bar{x})^2 \\ &= \frac{\sigma^2}{\sigma^2} \frac{n-1}{n} \frac{1}{n-1} \Sigma (x_i - \bar{x})^2 \\ &= \frac{\sigma^2}{n} \frac{(n-1)S^2}{\sigma^2}\end{aligned}$$

Where $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$.

$$\begin{aligned}\mathbb{E}(\hat{\sigma}^2) &= \mathbb{E}\left(\frac{\sigma^2}{n} \frac{(n-1)S^2}{\sigma^2}\right) \\ &= \frac{\sigma^2}{n} \mathbb{E}\left(\frac{(n-1)S^2}{\sigma^2}\right) \\ &= \frac{\sigma^2}{n} (n-1) = \sigma^2(1 - 1/n) \\ \implies \text{Bias}(\hat{\sigma}^2) &= \mathbb{E}(\hat{\sigma}^2) - \sigma^2 \\ &= \sigma^2(1 - 1/n - 1) = -\sigma^2/n\end{aligned}$$

b)

$$\begin{aligned}\text{Var } \hat{\sigma}^2 &= \frac{\sigma^4}{n^2} \text{Var } \frac{(n-1)S^2}{\sigma^2} \\ &= \frac{\sigma^4}{n^2} 2(n-1) = 2\sigma^4(n-1)/n^2\end{aligned}$$

c)

$$\begin{aligned}\text{MSE} &= \text{Var}(\hat{\sigma}^2) + \text{Bias}^2(\hat{\sigma}^2) \\ &= \frac{2\sigma^4(n-1)}{n^2} + \frac{\sigma^4}{n^2} \\ &= \sigma^4(2n-1)/n^2\end{aligned}$$

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a)

$$\begin{aligned}f(x) &= 6x^5/\theta^6 \\ \implies F(x) &= \int_0^x 6x^5/\theta^6 dt = x^6/\theta^6 \\ f_{X,\max} &= f_{(n)}(x) \\ &= nf(x)F^{n-1}(x) \\ &= n \frac{6x^5}{\theta^6} \frac{x^{6n-6}}{\theta^{6n-6}} \\ &= 6nx^{6n-1}/\theta^{6n}\end{aligned}$$

b)

$$\begin{aligned}\mathbb{E}(X_{\max}) &= \int_0^\theta x \frac{6nx^{6n-1}}{\theta^{6n}} dx \\ &= \frac{6n}{\theta^{6n}} \frac{x^{6n+1}}{6n+1} \Big|_0^\theta \\ &= \frac{6n}{6n+1} \theta\end{aligned}$$

c)

$$\begin{aligned}\text{Bias} &= \mathbb{E}(X_{\max}) - \theta \\ &= \theta \left(\frac{6n}{6n+1} - 1 \right) \\ &= \frac{-\theta}{6n+1}\end{aligned}$$

d)

$$\begin{aligned}\mathbb{E}(X_{\max}^2) &= \int_0^\theta x^2 \frac{6nx^{6n-1}}{\theta^{6n}} dx \\ &= 6n\theta^2/(6n+2) \\ \text{Var} &= EX_{\max}^2 - E^2 X_{\max} \\ &= \frac{3\theta^2 n}{(3n+1)(6n+1)^2} \\ \text{MSE} = \text{Var} + \text{Bias}^2 &= \frac{\theta^2}{18n^2 + 9n + 1}\end{aligned}$$

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a)

$$\begin{aligned}\mathbb{E}(W) &= \mathbb{E}(a\bar{X}) + \mathbb{E}((1-a)\bar{Y}) \\ &= aE\bar{X} + (1-a)E\bar{Y} \\ &= a\mu + (1-a)\mu = 1\end{aligned}$$

b)

$$\begin{aligned}\text{Var } W &= \text{Var}(a\bar{X} + (1-a)\bar{Y}) \\ &= a^2 \text{Var } \bar{X} + (1-a)^2 \text{Var } \bar{Y} \\ &= a^2 \sigma_1^2/n + (1-a)^2 \sigma_2^2/m \\ \implies \text{Var}'(W) &= 2a\sigma_1^2/n - 2(1-a)\sigma_2^2/m \\ \implies \hat{a}\sigma_1^2/n &= \sigma_2^2/m - \hat{a}\sigma_2^2/m \\ \implies \hat{a} &= \frac{\sigma_2^2/m}{\sigma_1^2/n + \sigma_2^2/m}\end{aligned}$$