MATH 308 Assignment 8:
Exercises 4.4

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March 10, 2014

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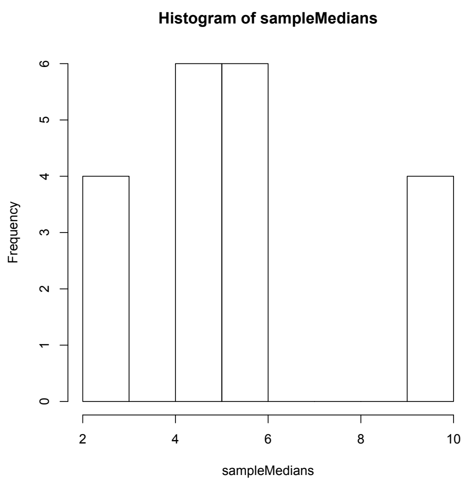

Population median: 5.5
Mean of sample medians: 5.7

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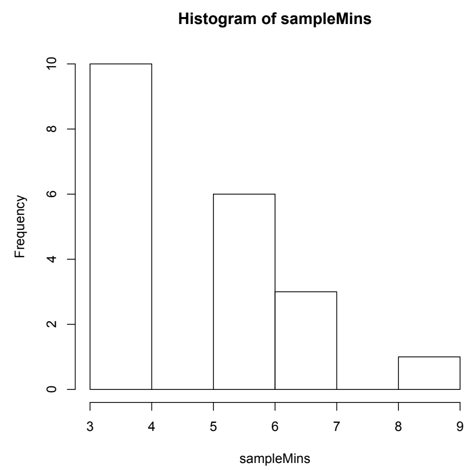

Mean of minimums: 4.8

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From the CLT, the sample mean is approximately distributed normally with $\mu_{\overline{X}} = \mu = 48$ and $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n} = 9^2/20 = 2.7$. Thus the probability of the sample mean being greater than 51 is `1-pnorm(51,mean=48,sd=sqrt(2.7))` which evaluates to $\approx 3\%$.

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The mean μ of the distribution is $\int_2^6 f(x)\,\mathrm{d}x = 4$. The variance σ^2 is $\int_2^6 (x-\mu)^2 f(x)\,\mathrm{d}x = 2.4$. By CLT, the distribution of the sample mean $\overline{X}$ is approximated by $\mathcal{N}(\mu, \frac{\sigma^2}{n}) = \mathcal{N}(4, 3/305)$. Thus, $P(\overline{X} >= 4.2)$ is `1-pnorm(4.2,mean=4,sd=sqrt(2.4))`, which evaluates to $\approx 2\%$.

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By CLT, the number of people with degrees is approximately normally dustributed with $\mu = np$ and $\sigma^2 = np(1-p)$. Thus, the probability of between 220 and 230 people in the sample having a degree is:

```
n=800
p=0.286
m=n*p
sd=sqrt(n*p*(1-p))
pn=
pnorm(230+0.5,m,sd)
-pnorm(220-0.5,m,sd)
```

Which evaluates to 0.3194833. The exact probability is given by

```
pb=pbinom(230,n,p)-pbinom(220,n,p)
```

which evaluates to 0.2959644. The error in the approximation is `(pn-pb)/pb` which evaluates to $\approx 8\%$.

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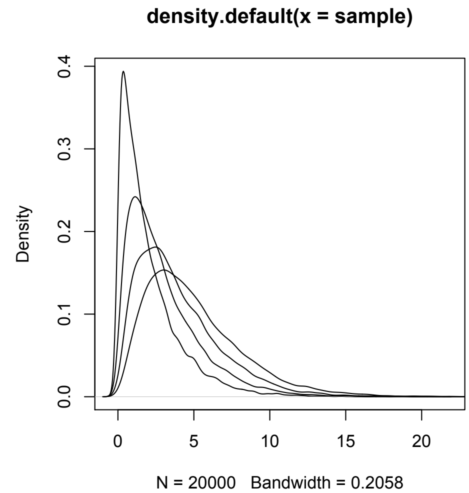
By definition, $W \sim \chi^2_n$. The sampling distributions of W_n for $n \in \{2,3,4,5\}$ are shown in the accompanying figure. The means and variances are tabulated as:

n	μ	σ^2
2	2.02	4.07
3	2.99	5.82
4	3.98	7.80
5	4.99	10.01

From the table, we can see that $\mu = n$ and $\sigma^2 = 2n$.



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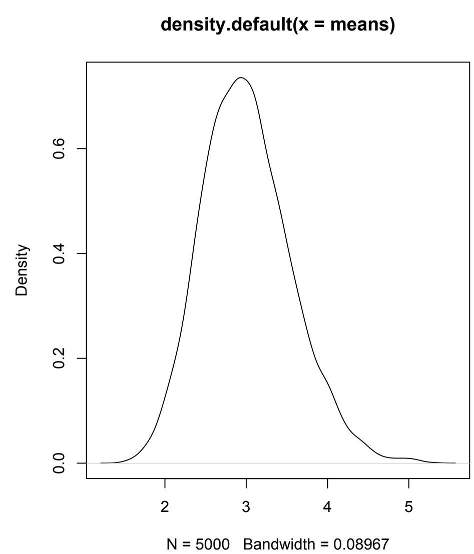
a)

The sampling distribution is shown below:



b)

The mean from sample is 3.00 with standard error 0.55. From CLT, we expect $\mu_{\overline{X}} = \mu = (1/3)^{-1} = 3$ and $\sigma_{\overline{X}} = \sigma/\sqrt{n} = (1/3)^{-1}/\sqrt{30} \approx 0.55$, giving a difference of $< 1\%$.

c)

We get $P = 0.83$.

d)

We get $P = 0.82$, which is only $\approx 1\%$ off.

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a) Distribution of Sample Minimum

$$\begin{aligned} P(X_{\min} \geq x) &= \prod_{i=1}^{n} P(X_i \geq x) \\ \implies \quad 1 - F_{\min}(x) &= (1 - F(x))^n \\ \implies \quad -f_{\min}(x) &= n(1 - F(x))^{n-1}(-f(x)) \\ \implies \quad f_{\min}(x) &= n\left(1 - F(x)\right)^{n-1} f(x) \quad \square \end{aligned}$$

b) Distribution of Sample Maximum

$$\begin{aligned} P(X_{\max} \leq x) &= \prod_{i=1}^{n} P(X_i \leq x) \\ \implies \quad F_{\max}(x) &= F^n(x) \\ \implies \quad f_{\max}(x) &= nF^{n-1}(x)f(x) \quad \square \end{aligned}$$

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a)

$F(x) = \int_1^x \frac{2}{x^2}\,\mathrm{d}x = 2 - 2/x$, so from previous result $f_{\max} = 2\left(2 - \frac{2}{x}\right)^{2-1}\frac{2}{x^2} = 2\left(2 - \frac{2}{x}\right)\frac{2}{x^2}$.

b)

$E_{X_{\max}} = \int_1^2 x f_{\max}(x)\,\mathrm{d}x \approx 1.55$.

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$$\begin{aligned} f_{X_1+X_2}(x) &= P(X_1 + X_2 = x) \\ &= \sum_{i=0}^{x}\left(\frac{\lambda_1^i}{i!}e^{-\lambda_1}\frac{\lambda_2^{x-i}}{(x-i)!}e^{-\lambda_2}\right) \\ &= e^{-(\lambda_1+\lambda_2)}\sum_{i=0}^{x}\frac{\lambda_1^i\lambda_2^{x-i}}{i!(x-i)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!}\sum_{i=0}^{x}\left(\frac{x!}{i!(x-i)!}\lambda_1^i\lambda_2^{x-i}\right) \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!}\sum_{i=0}^{x}\left(\binom{x}{i}\lambda_1^i\lambda_2^{x-i}\right) \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{x!}(\lambda_1+\lambda_2)^x \\ &= f_Z(x), \text{ where } Z \sim \mathcal{P}(\lambda_1+\lambda_2) \end{aligned}$$

$$\begin{aligned} &\implies X \sim \mathcal{P}\left(\sum_{i=1}^{10}\lambda_i\right) \\ &\implies X \sim \mathcal{P}(30) \\ &\implies f_X(x) = \frac{30^x}{x!}e^{30} \end{aligned}$$

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a)

The distribution seems quite close to normal.

b)

Theoretical mean $\mu_{\overline{X}} = \mu = 10.17$. Actual mean is 10.02. Difference is $\approx 1.5\%$.

c)

Theoretical s.e. $\sigma_{\overline{X}} = \frac{\sigma}{n}\sqrt{\frac{N-n}{N-1}} \approx 4.55$. Actual s.e. 4.56. Difference is $< 1\%$.

d)

The differences reduce with increasing n.

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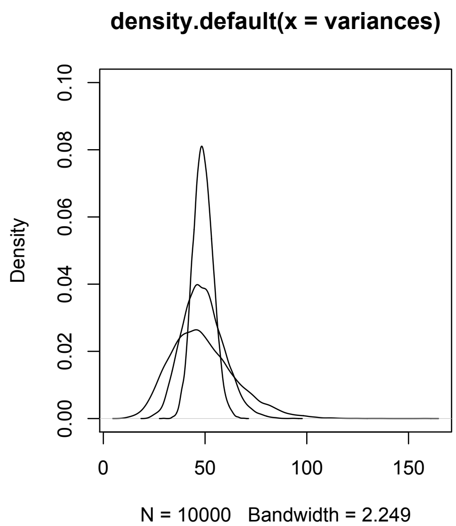
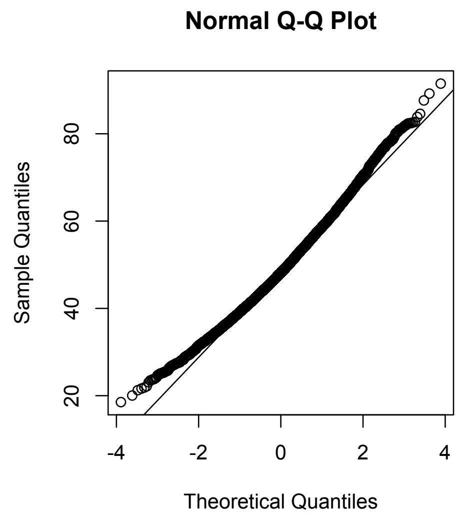
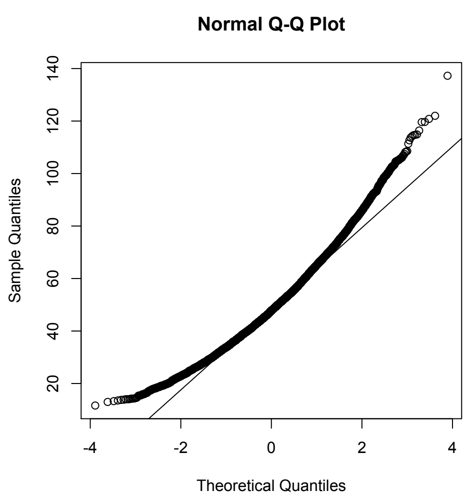
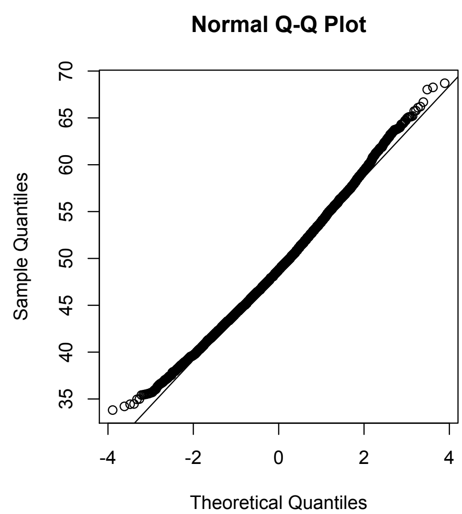
It can be seen from the plots below that the distribution of sample variances is approximately normal, and the fit to normal improves with sample size.









Math 308 Assignment 9
Moments of the Standard Normal

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March 6, 2014

1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned}
M_Z(t) &= \mathbb{E}(e^{tZ}) \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} e^{t^2/2} \, \mathrm{d}x \\
&= e^{t^2/2} \int_{-\infty}^{\infty} f_X(x) \, \mathrm{d}x
\end{aligned}$$

where $X \sim \mathcal{N}(t, 0)$

$$= e^{t^2/2}$$

2 Moments

Using the series expansion of the exponential

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

We observe that

$$\begin{aligned}
M_Z(t) &= \sum_{i=0}^{\infty} \frac{(t^2/2)^i}{i!} \\
&= 1 + \frac{t^2}{2} + \frac{t^4}{2^2 2!} + \frac{t^6}{2^3 3!} + \frac{t^8}{2^4 4!} \dots \\
M_Z'(t) &= \frac{2t}{2} + \frac{4t^3}{2^2 2!} + \frac{6t^5}{2^3 3!} + \frac{8t^7}{2^4 4!} \dots \\
M_Z''(t) &= 1 + \frac{4 \times 3t^2}{2^2 2!} + \frac{6 \times 5t^4}{2^3 3!} + \frac{8 \times 7}{2^4 4!} \dots
\end{aligned}$$

This tells us that, for odd values of n, $M_Z^{(n)}(t)$ will be an expression of the form $t \times$(a polynomial). Thus, for these values, $M_Z^{(n)}(0)$ is 0.

For the even values of n, we can see that $M_Z^{(n)}(t)$ is simply the nth derivative of the $\frac{n}{2}$th term in the series expansion of $M_Z(t)$, plus an expression of the form $t \times$(a polynomial). The latter expression always evaluates to 0 at $t = 0$, so we can drop it. To get the nth derivative of the ith term:

$$\begin{aligned}
T_i &= \frac{t^{2i}}{2^i i!} \\
\frac{\mathrm{d}^n}{\mathrm{d}t^n} T_i &= \frac{(2i)(2i-1)(2i-2)\dots(2i-n+1)}{2^i i!} t^{2i-n} \\
&= \frac{(2i)!}{(2i-n)!} \frac{1}{2^i i!} t^{2i-n}
\end{aligned}$$

Setting i to $n/2$,

$$\begin{aligned}
\mathbb{E}(Z^n) &= \frac{n!}{0!} \frac{1}{2^{(n/2)} (n/2)!} t^0 \\
&= \frac{n!}{(n/2)!} \frac{1}{2^{n/2}}
\end{aligned}$$

We can then compute and tabulate the moments:

n	0	1	2	3	4	5	6
$\mathbb{E}(Z^n)$	1	0	1	0	3	0	15

Math 308 Assignment 10
Mean and Variance of the Chi-Squared Distribution

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March 6, 2014

1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \int_0^\infty e^{tx} \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \, \mathrm{d}x \end{aligned}$$

But, $e^{-\frac{x}{2}} e^{tx} = e^{-x(t-\frac{1}{2})}$.
Substituting $u = x(1/2 - t)$, we get $\mathrm{d}u = (1/2-t)\mathrm{d}x$, and $u|_{x=0} = 0$, $u|_{x=\infty} = \infty$. So, we get:

$$\begin{aligned} M_Z(t) &= \int_0^\infty \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{u^{\frac{k}{2}-1}}{(1/2-t)^{\frac{k}{2}-1}} \frac{e^{-u}}{1/2-t} \, \mathrm{d}u \\ &= \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{1}{(1/2-t)^{\frac{k}{2}}} \int_0^\infty u^{\frac{k}{2}-1} e^{-u} \, \mathrm{d}u \\ &= \frac{1}{2^{k/2}\cancel{\Gamma\left(\frac{k}{2}\right)}} \frac{1}{(1/2-t)^{\frac{k}{2}}} \cancel{\Gamma\left(\frac{k}{2}\right)} \\ &= (1-2t)^{-\frac{k}{2}} \end{aligned}$$

2 Mean and Variance

From the above m.g.f., we can calulate the derivatives $M_X'(t) = k(1-2t)^{-\frac{k}{2}-1}$, and $M_X''(t) = k(k+2)(1-2t)^{-\frac{k}{2}-2}$.
This gives

$$\mu = \mathbb{E}(X) = M_X'(0) = k$$

.
We can also calculate

$$\begin{aligned} \sigma^2 &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= M_X''(0) - k^2 \\ &= k(k+2) - k^2 \\ &= 2k \end{aligned}$$

MATH 308 Assignment 11:
Baseball World Series Hypothesis Testing

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March 6, 2014

1 Expected Values under the Null

To win the series after playing n games, the winning team must win 3 of the first $n-1$ games, and then win the last game. Thus, for each team, the probability of winning after playing n games is:

$$P(n=k) = \binom{k-1}{3}\left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right)$$

However, the probability of n games being played is the probability of *either* team winning after n games. Thus, the p.m.f. of the number of games played, N, is twice the function above:

$$P(N=k) = \binom{k-1}{3}\left(\frac{1}{2}\right)^{k-1}$$

Thus, the expected number of series N won after k games is $P(N_k) \times$ (number of games played). We can then tabulate the expected and observed values:

k	4	5	6	7
$\mathbb{E}(N_k)$	6.25	12.5	15.625	15.625
$\mathbb{O}(N_k)$	8	8	10	24

2 Hypothesis Testing

The test statistic t is given by $\sum_{k=4}^{7} \frac{(\mathbb{O}(N_k)-\mathbb{E}(N_k))^2}{\mathbb{E}(N_k)} = 9.76$. Because k can take on only 4 values, we know that $t \sim \chi_3^2$. Hence, our p value is given by

$$p = \int_{9.76}^{\infty} \frac{e^{-x/2}\sqrt{x}}{\sqrt{2\pi}}\,\mathrm{d}x \approx 0.02$$

Thus, we can reject the null at the 5% significance level.