MATH 308 Assignment 18: Exercises 8.5

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p=$P(X >= 20)$ = `1-ppois(19,lambda=15)` $\approx$ 12.5%. This implies that more than 10% of the months will have a birth rate as extreme as the one observed, which means that we cannot reject the null ($\lambda = 15$) even at the 5% significance level.

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a)

Running the test gives a p-value of 10^{-4}. This lets us reject the null hypothesis (that the difference in means is zero) at the 1% significance level.

b)

The random assignment of seedlings to plots tends to remove confounding effects. Thus, the result hints at a causal relationship between presence of competition and diameter change.

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Running the chi-squared test gives a p-value of 0.012, allowing us to reject the null at the 5% significance level.

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A Type I error would be incorrectly concluding that the drug is effective when in fact it is not. This would lead to to patients wasting money on a useless drug.

A Type II error would be failing to conclude that the drug is effective, when in fact it is. This would lead to a waste of the research money put into the drug, since the pharmaceutical company would then not be able to sell it.

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Under the null, $\overline{X} \sim \mathcal{N}(25, 16/30)$. Thus $Z = \frac{\overline{X}-25}{4/\sqrt{30}}$, where Z is a standard normal random variable. Thus, the null is rejected if $Z \geq z_{1-\alpha} \implies \frac{\overline{X}-25}{4/\sqrt{30}} \geq z_{0.95} \implies \overline{X} \geq 25 + \frac{4z_{0.95}}{\sqrt{30}} \equiv C$.

But $\overline{X} \sim \mathcal{N}\left(27, 16/\sqrt{30}\right) \implies Z = \frac{\overline{X}-27}{4/\sqrt{30}}$. So,

$$
\begin{aligned}
1-\beta &= P(\overline{X} \geq C | \mu = 27) \\
&= P\left(\frac{4Z}{\sqrt{30}} + 27 \geq C\right) \\
&= P\left(Z \geq \frac{\sqrt{30}(C-27)}{4}\right) \equiv P(Z \geq C_1) \\
&= 1 - P(Z < C_1) \approx 86.3\%
\end{aligned}
$$

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The null is rejected if $\overline{X} \geq \mu_0 + \frac{z_{1-\alpha}\sigma}{\sqrt{n}} \equiv C$, where $\mu_0 = 1, \alpha = 0.01$ and $\sigma = 0.3$. So,

$$\begin{aligned}
1 - \beta &= P(\overline{X} \geq C | \mu = \mu_1) \\
&= P\left(\frac{Z\sigma}{\sqrt{n}} + \mu_1 \geq C\right) \\
&= P\left(Z \geq \frac{\sqrt{n}(C - \mu_1)}{\sigma}\right) \equiv P(Z \geq C_1) \\
&= 1 - P(Z < C_1) \\
&\implies \beta = P(Z < C_1) \\
\implies z_\beta &= C_1 = \frac{\sqrt{n}(C - \mu_1)}{\sigma} \\
\implies \frac{z_\beta \sigma}{\sqrt{n}} + \mu_1 &= C = \frac{z_{1-\alpha}\sigma}{\sqrt{n}} + \mu_0 \\
\implies n &= \left(\frac{\sigma(z_\beta - z_{1-\alpha})}{\mu_0 - \mu_1}\right)^2 = 8
\end{aligned}$$

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$$n = \left(\frac{\sigma(z_\beta - z_{1-\alpha})}{\mu_0 - \mu_1}\right)^2 = 1$$

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$f(x) = \lambda e^{-\lambda x} \implies F(x) = \int_0^x \lambda e^{-\lambda t}\,\mathrm{d}t = 1 - e^{-\lambda t}$

a)

$$\begin{aligned}
F_{X,\min}(x) &= \prod_{i=1}^{15} P(X_i \leq x) \\
&= (1 - F(x))^{15} \\
&= e^{-15\lambda x} \\
\implies \alpha &= P(X_{\min} \geq 1 | \lambda = 1/5) \\
&= 1 - F_{X,\min}(1)|_{\lambda = \frac{1}{5}} \\
&= 1 - e^{-3} \approx 95\%
\end{aligned}$$

b)

$$\begin{aligned}
1 - \beta &= P(X_{\min} \geq 1 | \lambda = 1/25) \\
&= 1 - F_{X,\min}(1)|_{\lambda = \frac{1}{25}} \\
&= 1 - e^{-3/5} \approx 45\%
\end{aligned}$$

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The most powerful test simply always rejects the null. Thus, the critical region is $X_1, X_2 \in [0, n]$.

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Under the null, the likelihood is maximised at $\mu = \mu_0$ and $\sigma^2 = \Sigma(X_i - \mu_0)^2/n \equiv \hat{\sigma}^2$. Thus, the likelihood under the null is

$$L_{H,0} = \left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^n \exp\left(\frac{-\Sigma(X_i - \mu_0)^2}{2\hat{\sigma}^2}\right) = \frac{e^{-n/2}}{(\hat{\sigma}\sqrt{2\pi})^n}$$

To maximise the likelihood, we simply use $\mu = \overline{X}$ and $\sigma = \sqrt{\Sigma(X_i - \overline{X})^2/n}$, giving

$$L = \left(\frac{1}{S\sqrt{2\pi}}\right)^n \exp\left(\frac{-\Sigma(X_i - \overline{X})^2}{2S^2}\right) = \frac{e^{-n/2}}{(S\sqrt{2\pi})^n}$$

The likelihood ratio is therefore $(S/\hat{\sigma})^n$. So, we reject if

$$\begin{aligned}
\left(\frac{\Sigma(X_i - \overline{X})^2}{\Sigma(X_i - \mu_0)^2}\right)^{n/2} \leq c &\implies \frac{\Sigma(X_i - \overline{X})^2}{\Sigma(X_i - \mu_0)^2} \leq c_1 \\
\implies c_1 \leq \frac{\Sigma(X_i - \mu_0)^2}{\Sigma(X_i - \overline{X})^2} &= \frac{\Sigma(X_i - \overline{X} + \overline{X} - \mu_0)^2}{\Sigma(X_i - \overline{X})^2} \\
&\leq 1 + \frac{n(\overline{X} - \mu_0)^2 - 2(\overline{X} - \mu_0)\cancel{\Sigma(X_i - \overline{X})}}{\Sigma(X_i - \overline{X})^2} \\
&\leq 1 + \frac{n}{\Sigma(X_i - \overline{X})^2}(\overline{X} - \mu_0)^2 \\
\implies c_2 \leq \sqrt{\frac{n}{\Sigma(X_i - \overline{X})^2}}(\overline{X} - \mu_0) &= \frac{\overline{X} - \mu_0}{S/\sqrt{n}}
\end{aligned}$$

which is a t-statistic. Thus, the test is a one-sided t-test. $\square$