Math 308 Assignment 9
Moments of the Standard Normal

Nakul Joshi

March 6, 2014

1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned}
M_Z(t) &= \mathbb{E}(e^{tZ}) \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} e^{t^2/2}\,\mathrm{d}x \\
&= e^{t^2/2} \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x
\end{aligned}$$

where $X \sim \mathcal{N}(t, 0)$

$$= e^{t^2/2}$$

2 Moments

Using the series expansion of the exponential

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

We observe that

$$\begin{aligned}
M_Z(t) &= \sum_{i=0}^{\infty} \frac{(t^2/2)^i}{i!} \\
&= 1 + \frac{t^2}{2} + \frac{t^4}{2^2 2!} + \frac{t^6}{2^3 3!} + \frac{t^8}{2^4 4!} \dots \\
M_Z'(t) &= \frac{2t}{2} + \frac{4t^3}{2^2 2!} + \frac{6t^5}{2^3 3!} + \frac{8t^7}{2^4 4!} \dots \\
M_Z''(t) &= 1 + \frac{4 \times 3t^2}{2^2 2!} + \frac{6 \times 5t^4}{2^3 3!} + \frac{8 \times 7}{2^4 4!} \dots
\end{aligned}$$

This tells us that, for odd values of n, $M_Z^{(n)}(t)$ will be an expression of the form $t \times$(a polynomial). Thus, for these values, $M_Z^{(n)}(0)$ is 0.

For the even values of n, we can see that $M_Z^{(n)}(t)$ is simply the nth derivative of the $\frac{n}{2}$th term in the series expansion of $M_Z(t)$, plus an expression of the form $t \times$(a polynomial). The latter expression always evaluates to 0 at $t = 0$, so we can drop it. To get the nth derivative of the ith term:

$$\begin{aligned}
T_i &= \frac{t^{2i}}{2^i i!} \\
\frac{\mathrm{d}^n}{\mathrm{d}t^n} T_i &= \frac{(2i)(2i-1)(2i-2)\dots(2i-n+1)}{2^i i!} t^{2i-n} \\
&= \frac{(2i)!}{(2i-n)!} \frac{1}{2^i i!} t^{2i-n}
\end{aligned}$$

Setting i to $n/2$,

$$\begin{aligned}
\mathbb{E}(Z^n) &= \frac{n!}{0!} \frac{1}{2^{(n/2)}(n/2)!} t^0 \\
&= \frac{n!}{(n/2)!} \frac{1}{2^{n/2}}
\end{aligned}$$

We can then compute and tabulate the moments:

n	0	1	2	3	4	5	6
$\mathbb{E}(Z^n)$	1	0	1	0	3	0	15

Math 308 Assignment 10
Mean and Variance of the Chi-Squared Distribution

Nakul Joshi

March 6, 2014

1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \int_0^\infty e^{tx} \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \, \mathrm{d}x \end{aligned}$$

But, $e^{-\frac{x}{2}} e^{tx} = e^{-x(t-\frac{1}{2})}$.
Substituting $u = x(1/2 - t)$, we get $\mathrm{d}u = (1/2 - t)\mathrm{d}x$, and $u|_{x=0} = 0$, $u|_{x=\infty} = \infty$. So, we get:

$$\begin{aligned} M_Z(t) &= \int_0^\infty \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{u^{\frac{k}{2}-1}}{(1/2-t)^{\frac{k}{2}-1}} \frac{e^{-u}}{1/2-t} \, \mathrm{d}u \\ &= \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \frac{1}{(1/2-t)^{\frac{k}{2}}} \int_0^\infty u^{\frac{k}{2}-1} e^{-u} \, \mathrm{d}u \\ &= \frac{1}{2^{k/2}\cancel{\Gamma\left(\frac{k}{2}\right)}} \frac{1}{(1/2-t)^{\frac{k}{2}}} \cancel{\Gamma\left(\frac{k}{2}\right)} \\ &= (1-2t)^{-\frac{k}{2}} \end{aligned}$$

2 Mean and Variance

From the above m.g.f., we can calulate the derivatives $M_X'(t) = k(1-2t)^{-\frac{k}{2}-1}$, and
$M_X''(t) = k(k+2)(1-2t)^{-\frac{k}{2}-2}$.

This gives

$$\mu = \mathbb{E}(X) = M_X'(0) = k$$

.

We can also calculate

$$\begin{aligned} \sigma^2 &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= M_X''(0) - k^2 \\ &= k(k+2) - k^2 \\ &= 2k \end{aligned}$$