MATH 308 Assignment 21: Least-squares estimates for Linear Models

Nakul Joshi

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1 Least Squares Estimate

$$
\begin{aligned}
L(\beta, \sigma) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(Y_i - \beta x_i)^2}{2\sigma^2}} \\
&= (\sigma\sqrt{2\pi})^n e^{-\frac{\Sigma(Y_i - \beta x_i)^2}{2\sigma^2}} \\
\implies \log L &= -n \log \sigma - n \log \sqrt{2\pi} - \frac{\Sigma(Y_i - \beta x_i)^2}{2\sigma^2} \\
\implies \frac{\partial L}{\partial \beta} &= -\frac{1}{2\sigma^2} \Sigma(2(Y_i - \beta x_i)(-x_i)) \\
\implies 0 &= \Sigma(\hat{\beta} x_i^2 - x_i Y_i) = \hat{\beta}\Sigma x_i^2 - \Sigma x_i Y_i \\
\implies \hat{\beta} &= \frac{\Sigma x_i Y_i}{\Sigma x_i^2}
\end{aligned}
$$

2 Confidence Interval

$$
\begin{aligned}
\text{Var } [\hat{\beta}] &= \text{Var } \left[\frac{\Sigma x_i Y_i}{\Sigma x_i^2}\right] \\
&= \frac{1}{(\Sigma x_i^2)^2} \Sigma(x_i^2 \text{Var } [Y_i]) \\
&= \frac{\Sigma(x_i^2 \sigma^2)}{(\Sigma x_i^2)^2} \\
&= \frac{\sigma^2}{\Sigma x_i^2}
\end{aligned}
$$

σ^2 is unknown, so we use the estimate $S^2 = \frac{\Sigma Y_i^2}{n-1}$, where the $n-1$ denominator is chosen because only one parameter is being estimated.

The statistic $T \equiv \frac{\hat{\beta}-\beta}{S/\sqrt{\Sigma x_i^2}}$ then has a t-distribution on $n-1$ degrees of freedom.

Thus, the $1-\alpha$ confidence interval is given by

$$
\hat{\beta} \pm \frac{S}{\sqrt{\Sigma x_i^2}} t_{n-1,1-\alpha/2}
$$