MATH 308 Assignment 14: Distribution of Second Largest Element

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1 Distribution of Second-Largest Element

Let the distribution being sampled have pdf $f(x)$, and let $X = \{X_1, X_2, X_3 \dots X_n\}$ be the set of random variables drawn from that distribution. Further, let Y be the second largest element of X. Then, for any value x, we have:

$$
\begin{aligned}
P(Y = x) &= P(\text{some } X = x \text{ \& } n-2 \text{ of other } X < x) \\
&= \sum_{i=1}^{n} P(X_i = x) P(n-2 \text{ of other } X < x)) \\
&= nP(X_1 = x) \\
&\binom{n-1}{n-2} P^{n-2}(X_1 < x) P(X_1 > x) \\
&= \frac{n(n-1)!}{1!(n-2)!} f(x) F^{n-2}(x)(1 - F(x)) \\
\implies f_Y(x) &= n(n-1) f(x) F^{n-2}(x)(1 - F(x))
\end{aligned}
$$

2 Expected value from $\mathcal{U}(1, 2)$

We know that $f(x) = 1$ and $F(x) = x$ in the range $[0, 1]$. Therefore,

$$
\begin{aligned}
f_Y(x) &= n(n-1) f(x) F^{n-2}(x)(1 - F(x)) \\
&= n(n-1)(1)(x^{n-2})(1 - x) \\
&= n(n-1) x^{n-1}(1 - x)
\end{aligned}
$$

$$
\begin{aligned}
\implies \mathbb{E}(Y) &= \int_0^1 n(n-1) x^{n-2}(1 - x) \, \mathrm{d}x \\
&= n(n-1) \int_0^1 x^{n-1} - x^n \, \mathrm{d}x \\
&= n(n-1) \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \cancel{n}(n-1) \frac{\cancel{n + 1 - n}}{\cancel{n}(n+1)} \\
&= \frac{n-1}{n+1}
\end{aligned}
$$