Math 308 Assignment 9
Moments of the Standard Normal

Nakul Joshi

March 6, 2014

1 Moment Generating Function

From the definition of the m.g.f.,

$$\begin{aligned} M_Z(t) &= \mathbb{E}(e^{tZ}) \\ &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} e^{t^2/2} \, \mathrm{d}x \\ &= e^{t^2/2} \int_{-\infty}^{\infty} f_X(x) \, \mathrm{d}x \end{aligned}$$

where $X \sim \mathcal{N}(t, 0)$

$$= e^{t^2/2}$$

2 Moments

Using the series expansion of the exponential

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

We observe that

$$\begin{aligned} M_Z(t) &= \sum_{i=0}^{\infty} \frac{(t^2/2)^i}{i!} \\ &= 1 + \frac{t^2}{2} + \frac{t^4}{2^2 2!} + \frac{t^6}{2^3 3!} + \frac{t^8}{2^4 4!} \dots \\ M'_Z(t) &= \frac{2t}{2} + \frac{4t^3}{2^2 2!} + \frac{6t^5}{2^3 3!} + \frac{8t^7}{2^4 4!} \dots \\ M''_Z(t) &= 1 + \frac{4 \times 3t^2}{2^2 2!} + \frac{6 \times 5t^4}{2^3 3!} + \frac{8 \times 7}{2^4 4!} \dots \end{aligned}$$

This tells us that, for odd values of n, $M_Z^{(n)}(t)$ will be an expression of the form $t \times$(a polynomial). Thus, for these values, $M_Z^{(n)}(0)$ is 0.

For the even values of n, we can see that $M_Z^{(n)}(t)$ is simply the nth derivative of the $\frac{n}{2}$th term in the series expansion of $M_Z(t)$, plus an expression of the form $t \times$(a polynomial). The latter expression always evaluates to 0 at $t = 0$, so we can drop it. To get the nth derivative of the ith term:

$$\begin{aligned} T_i &= \frac{t^{2i}}{2^i i!} \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} T_i &= \frac{(2i)(2i-1)(2i-2)\dots(2i-n+1)}{2^i i!} t^{2i-n} \\ &= \frac{(2i)!}{(2i-n)!} \frac{1}{2^i i!} t^{2i-n} \end{aligned}$$

Setting i to $n/2$,

$$\begin{aligned} \mathbb{E}(Z^n) &= \frac{n!}{0!} \frac{1}{2^{(n/2)}(n/2)!} t^0 \\ &= \frac{n!}{(n/2)!} \frac{1}{2^{n/2}} \end{aligned}$$

We can then compute and tabulate the moments:

n	0	1	2	3	4	5	6
$\mathbb{E}(Z^n)$	1	0	1	0	3	0	15