LECTURE 2 - INTRODUCTION TO DIGITAL SIGNAL PROCESSING

ANALOG → Infinite space
↓
cannot be stored in hardware

WHAT TO CHOOSE?
Microprocessor, microcontroller, dsp.

Face recognition system for attendance example.

- 1920 X 1080 → Large information
- Floating point operations
  - Multiplication
  - Floating point not supported

multiplication Not performed by ALU of a general purpose microcontroller.

Why separate? → Not everyone uses it.

- GPU is a type of DSP.
- DSPs are generally more expensive

COMPUTER ARCHITECTURES

von Neumann

- Same memory for program and data

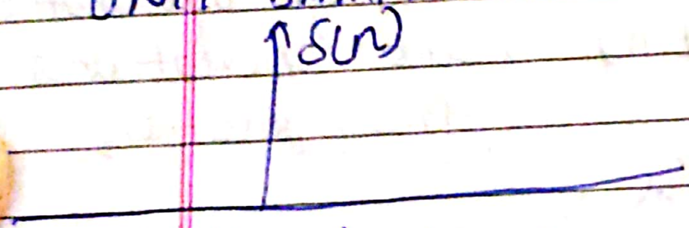
Harvard

- Faster
- DSPs use this.

Discrete time vs. **Digital**

- Discrete time
  +
- Discrete amplitude

UNIT SAMPLE



- In real life, the signal may have some width
- Area = 1.
- Ideally, height = ∞

Sine wave $\xrightarrow{FFT}$ 1 point

Square wave $\xrightarrow{FFT}$ multiple points

↓

Sudden transition from 0 to 1

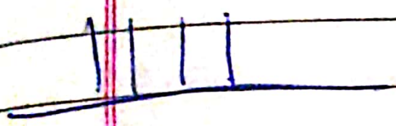
Constructed by addition of multiple ~~sin~~ sine waves

Sine wave f_1 + Sine wave f_2 $\xrightarrow{\text{FFT}}$ 2 samples

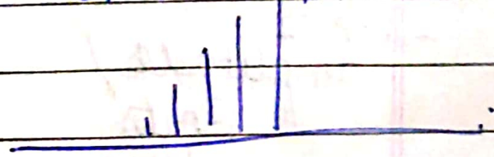
Sine wave f_1 f_2 f_N $\xrightarrow[\cancel{\text{FFT}}]{\text{Addition}}$, Triangular/Ramp

Infinite sine waves $\xrightarrow{\text{Addition}}$ Square wave.

UNIT STEP



UNIT RAMP



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BASIC SIGNALS

CLASSIFICATION:

- Deterministic & Random.
- Periodic & Non-periodic
- Energy & Power
  - Sum of squares
  - Periodic signals
- Finite amplitude / Finite time
- Limited in terms of time.

- This is not exhaustive.

- Causal & Non-Causal

↓

Realizable into the system.

- Even & Odd.
  - Half of the signal is predictable.
  - I use cosine for amplitude modulation because of its even property.

CLASSIFICATION OF SYSTEMS

- Static / Dynamic.
- Linear & Non-linear
  → Even if non-linear we break the signal into linear parts.
- Ou
- FIR & IIR
- Stable & unstable
- Time invariant & Time varying.

Scanned with CamScanner

LTI System.

Time Invariant –
Time delay in input causes the same time delay in output.

Impulse Response

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

For LTI:-

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, T[\delta(n-k)]$$

↳ Leads to convolution.

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

$$y(n) = x(n) * h(n)$$

Length of convolution result $= n+k-1$.

Recall filters where output was convolution.

Convolution with a pulse → Low pass filter

Square wave (Infinite sine waves) $\xrightarrow[\text{pulse}]{\text{conv. with}}$ Triangle (Finite sine waves)

→ Sobel edge detection.
→ Canny edge detection

TRANSFORMS

A signal finite in time domain is infinite in frequency domain, and vice versa.

LINEAR CONVOLUTION

$(n+m-1)$

$x(n) = \{1, 2, 1, 1\}$
$\uparrow$

$h(n) = \{2, 1, 2, 2\}$
$\uparrow$

	1	2	1	1
2	2	4	2	2
1	1	2	1	1
2	2	4	2	2
2	2	4	2	2
	7	14	7	7

$\uparrow$

$y(n) = \{2, 5, 6, 9, 7, 4, 2\}$

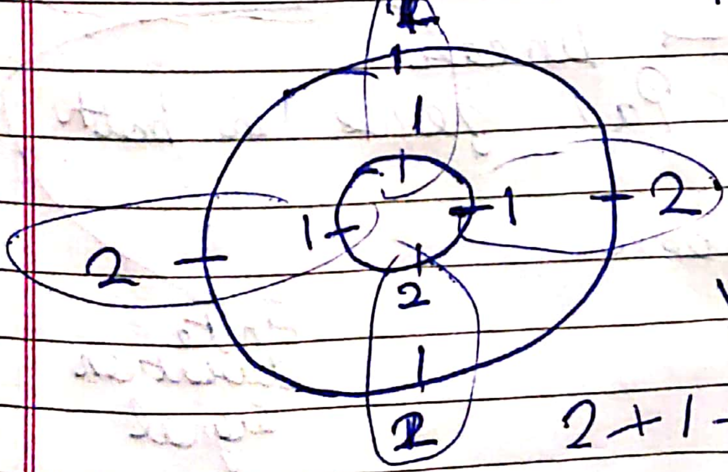
→ What if $h(n)$ & $x(n)$ are periodic?

⇓

CIRCULAR CONVOLUTION.

$x(n) = \{1, 2, 1, 1\}$

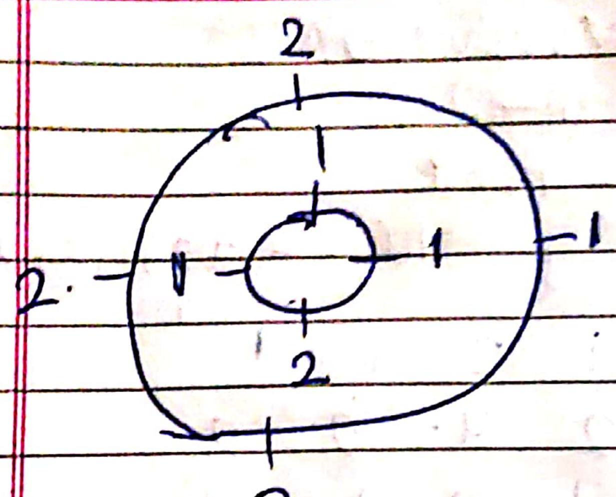
$h(n) = \{2, 1, 2, 2\}$



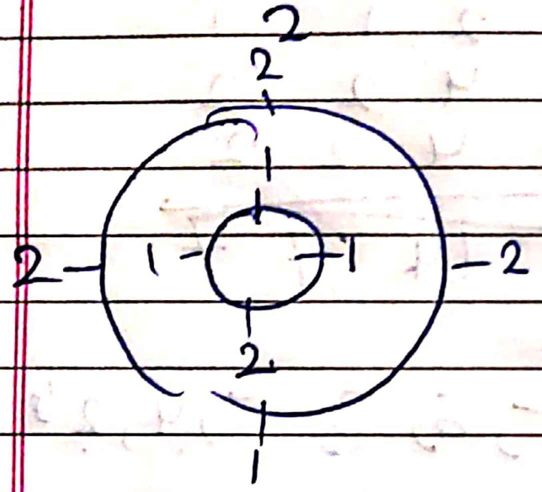
$Y(n) = \sum x(k) h(n-k)$

$Y(0) = \sum x(k) h(-k)$

$2 + 1 + 2 + 4$



$y(1) = 1+2+2+4$
$= 9$



$y(2) = 2+2+2+2$
$= 8$

$y(3) = 1+2+2+4$
$= 9$

Length
$= \max(m, n)$

$\{9, 9, 8, 9\}$

Circular → Linear

Pad zeros in both

Infinite + Periodic

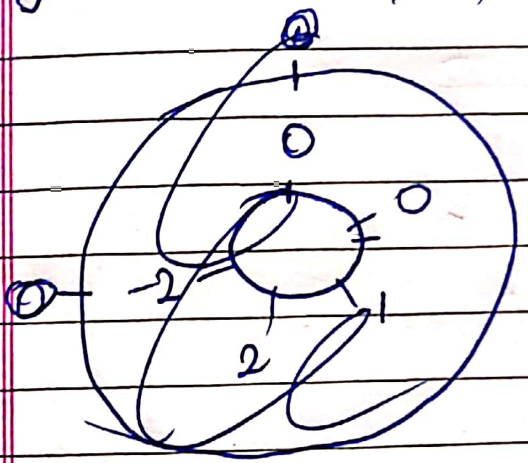
Finite duration signal

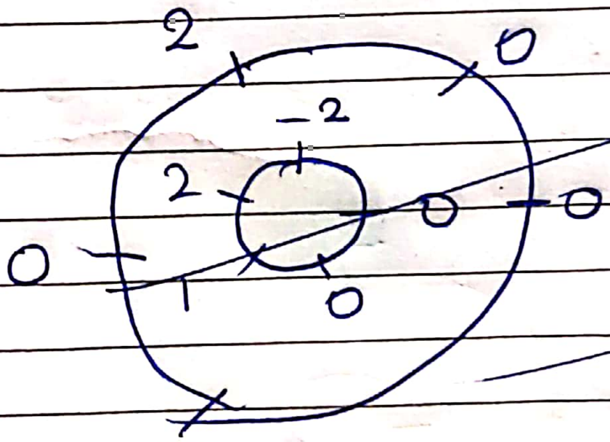
Q $x(n) = \{1, 2, -2\}$

$h(n) = \{2, 0, 1\}$

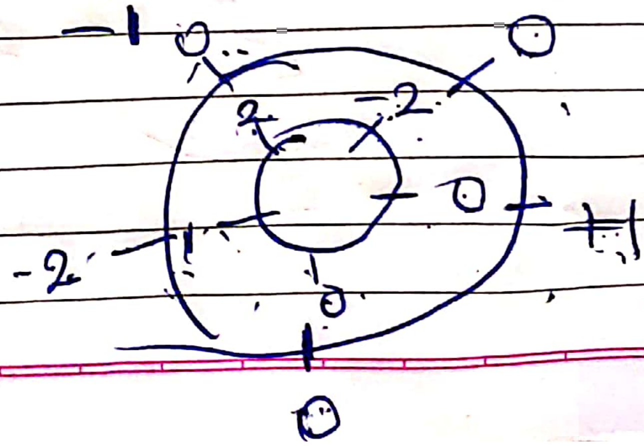
	1	2	-2
2	2	4	-4
0	0	0	0
1	1	2	-2

$y(n) = \{2, 4, -3, 2, -2\}$





$0 + 0 + 0$
$+ 4 = 4.$



$2.$

4

$+1 - 4 = -3$

2

-2

→ Linear & Circular convolution are similar provided the lengths are made same.

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THEORY OF TRANSFORMS

- Changing from one domain to another

- Any change in the original signal is some form of transformation
  - DTFT
  - DFT
  - Z-transform
  - FFT

DISCRETE TIME FOURIER TRANSFORM

CTFT should exist

↓ sampling

DTFT

HIERARCHY OF TRANSFORMS

Laplace (whole set)

↓ some values

CTFT (subset)

↓ Discretize

DTFT → DFT (Discrete time, discrete freq.) → FFT (Fast version of DFT)

Continuous — whole set — Z-transform

Discrete

Any transform should be absolutely summable.

$$|T(x(t))| < \infty$$

Time to frequency

$e^{j\theta} \rightarrow$ Basis function which helps in decomposition of sinusoids

$x(t) \leftarrow$ Input Signal.

$$T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j\theta} dt$$

change

$$T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j\theta} dt$$

$$\boxed{T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt}$$

$\rightarrow$ Continuous time fourier transform

LAPLACE TRANSFORM

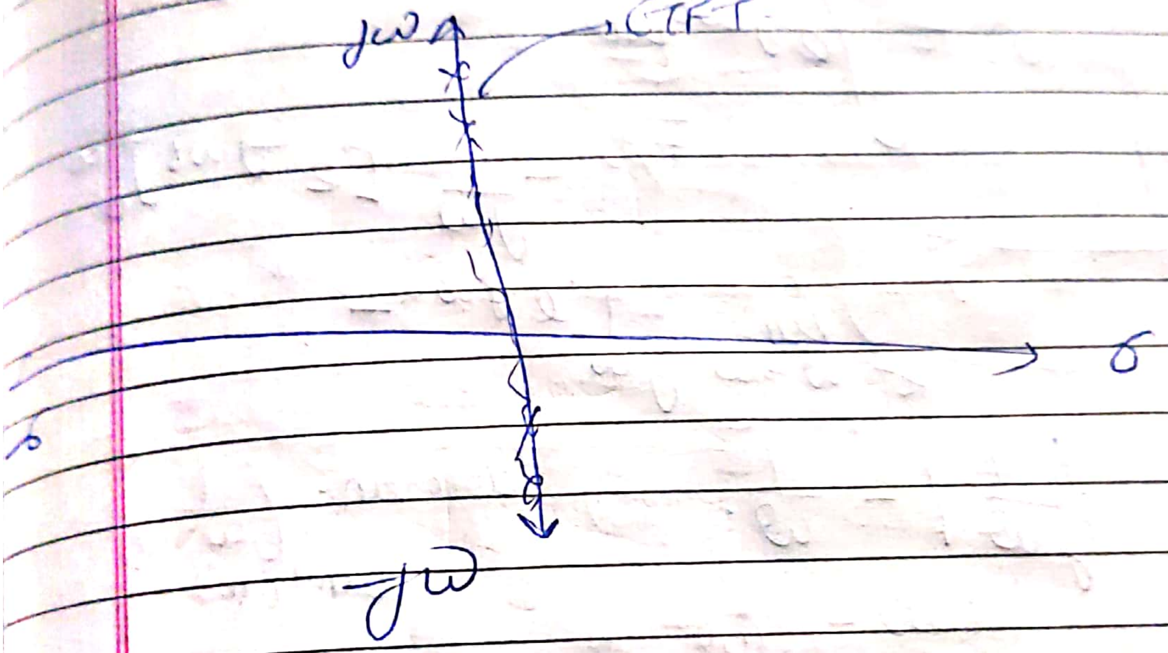
$$X[S] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

S = complex frequency

$$\boxed{s = \sigma + j\omega}$$

real freq. Imag. freq

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$



CTFT

$$T[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

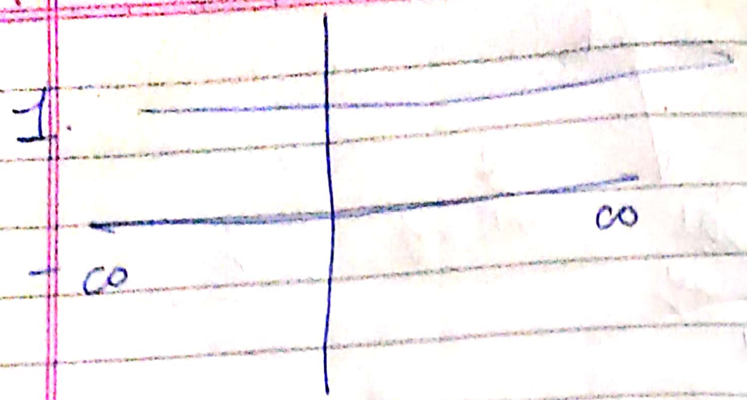
$\downarrow$

$X(\omega)$

Consider unit sample function

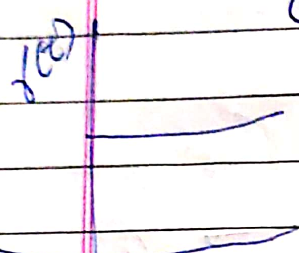
$t = 0 \quad x(t) = 1$

$$X(\omega) = x(0)\, e^{-j\omega \times 0} = 1.$$



UNIT STEP

$$X(\omega) = \int_0^\infty e^{-j\omega t}\,dt$$



$$= -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^\infty$$

$$\lim_{t\ \omega\to\infty} -\frac{1}{j\omega}e^{-j\omega t} =$$

$$\left|\frac{1}{j\omega}\right| = \frac{1}{\omega}$$

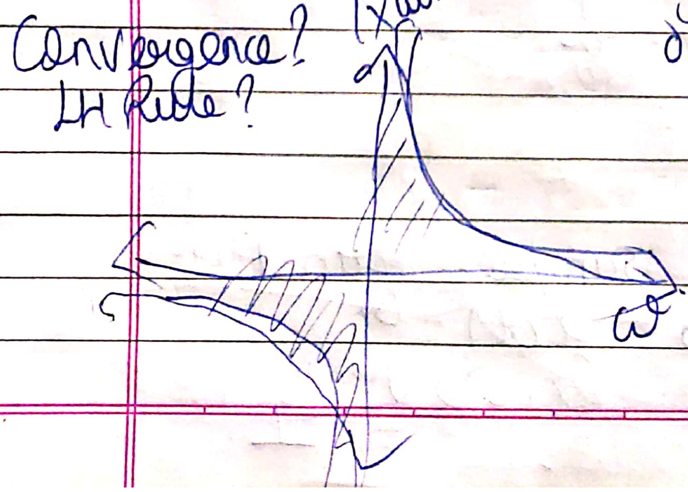
$\arctan\left(\frac{1}{\omega \times 0}\right)$

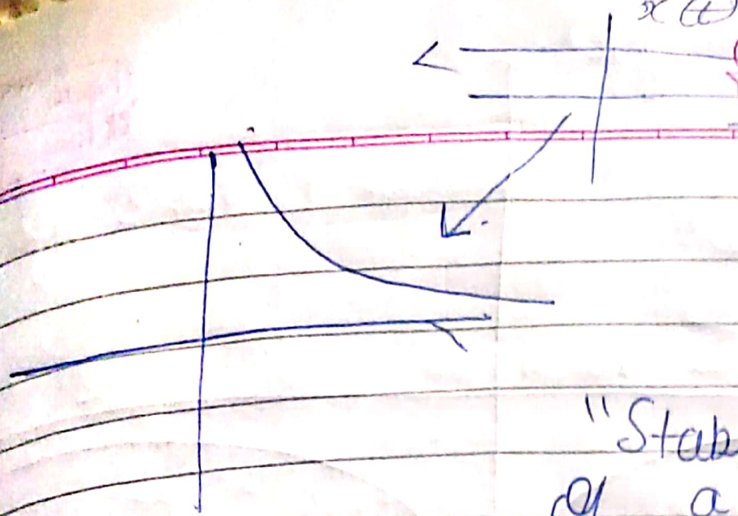
$\tan^{-1}(\infty)$

$$X(\omega) = \frac{1}{j\omega}$$

$$\int_0^\infty \frac{1}{\omega}$$

Convergence?
LH Rule?





"Stability" of a system.

CTFT

Q ① Unit Ramp.

② Sinusoidal function:

$(A\sin(\omega t+\phi))$

① Unit Ramp function.

$$r(t) = \begin{cases} 0 & t<0 \\ t & t\geq 0 \end{cases}$$

$$X(\omega) = \int_0^\infty t e^{-j\omega t}\,dt$$

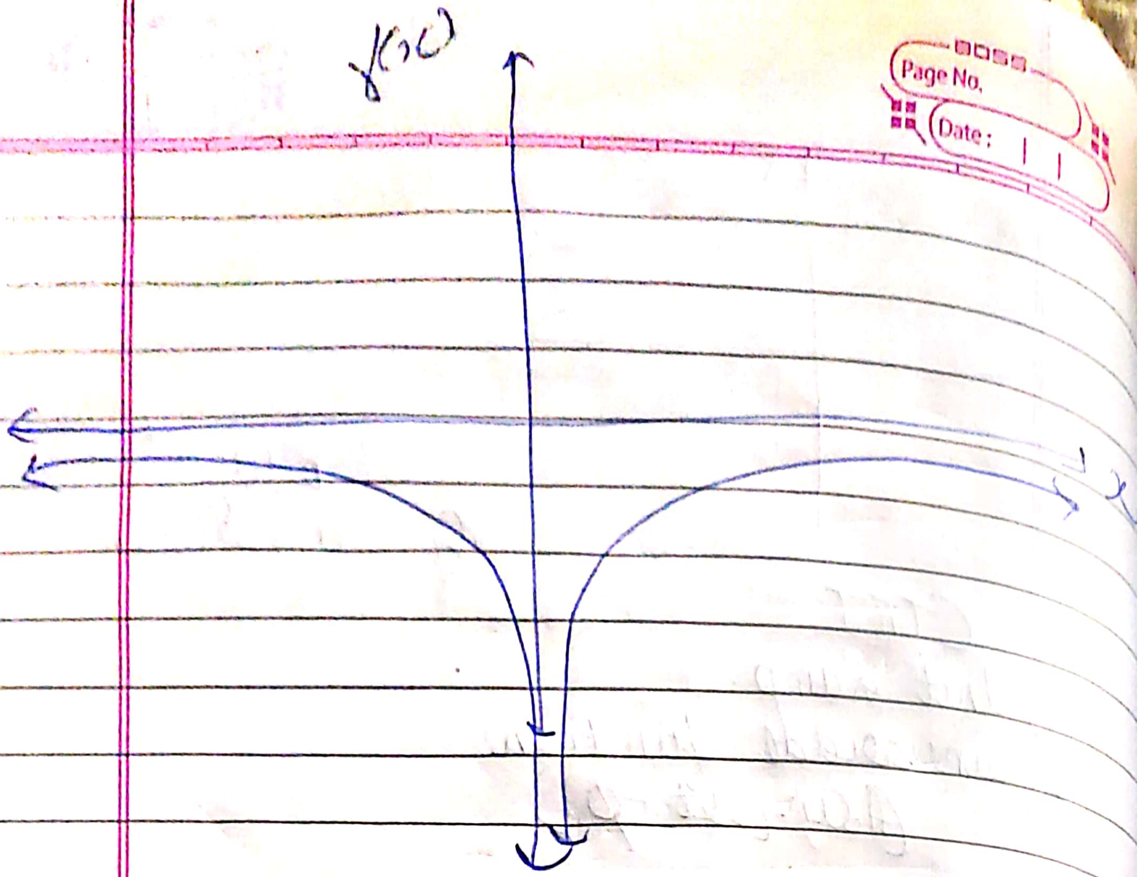
$$= \left[\frac{t}{j\omega} e^{-j\omega t}\right]_0^\infty - \left[\frac{1}{(j\omega)^2} e^{-j\omega t}\right]_0^\infty$$

$\frac{t}{e^{j\omega t}}$ $\frac{t}{e^{j\omega t}}$ $\frac{t}{e^{j\omega \infty}}$ $\frac{1}{e^{j\omega \infty}} = 0$

$\cancel{0 + \frac{1}{j\omega} + \left[\frac{1}{(j\omega)^2} \frac{1}{(j\omega)^2}\right.}$

$X(\omega) = \cancel{-1/\omega^2 - j/\omega}$

$$= -\frac{1}{(j\omega)^2} \times (-1) = \frac{1}{j^2\omega^2} = -1/\omega^2$$



(b) $x(t) = A\sin(\omega t + \phi)$

$$X(\omega) = \int A\sin(\omega t + \phi)\, e^{-j\omega t}\, dt$$

$$A\left(\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right) e^{-j\omega t}\, dt$$

$$= \frac{A}{2j} \ominus e$$

$$\int_0^{\infty} \frac{A}{2j} \left(1 - e^{-2j\omega t}\right) dt$$

$$\frac{A}{2j} \left[t + \frac{e^{-2j\omega t}}{2\omega j}\right]_0^{\infty}$$

$$\int_0^{\infty} \frac{A}{2j} e^{j\phi} - \frac{A}{2j} e^{-j\phi} e^{-2j\omega t}\, dt$$

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$$X(\omega) = \int A\sin(\omega' t + \phi)\, e^{-j\omega t}\, dt$$

$\omega \neq \omega'$

$$X(\omega) = \int_{-\infty}^{\infty} A\sin(\omega' t)$$

$$A\left(\frac{e^{j\omega' t} - e^{-j\omega' t}}{2j}\right) e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} e^{j(\omega' \pm \omega)t}\, dt$$

$$\cancel{\int_{-\infty}^{\infty} e^{j(\omega' + \omega)t}\, dt}$$

$\omega \neq \omega'$

$e^{j\omega' t} e^{-j\omega' t}$

$e^{j(\omega - \omega')t}\, dt$

$$X(\omega) = \int_{-\infty}^{\infty} \frac{A}{2j}$$

$\int e^{j\omega t}$

$$\int_{-\infty}^{\infty} [\cos(\omega t) + j\sin(\omega t)]$$

$$\frac{1}{j\omega}\left[e^{j\omega t}\right]_{-\infty}^{\infty}$$

$$\frac{e^{j\infty}}{j\omega} - 0$$

$$e^{j\omega_0 t} \to 2\pi\delta(\omega - \omega_0)$$

$$X(\omega) = \int_{-\infty}^{\infty} \frac{A}{2j}\left[e^{j(\omega' t + \phi)} - e^{-j(\omega' t + \phi)}\right] e^{-j\omega t}\, dt$$

$$\frac{A}{2j}\left[e^{j\phi} e^{j(\omega' - \omega)t} - e^{-j\phi} e^{-j(\omega' + \omega)t}\right] dt$$

$$\pi \frac{A}{j}\left[e^{j\phi}\delta(\omega - \omega') - e^{-j\phi}\delta(\omega + \omega')\right]$$

My system doesn't have infinite memory.

$\therefore$ I need to discretize

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

CTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Frequency is still continuous

discrete

Discrete Time Fourier Transform

There are cases where DTFT/CTFT doesn't exist

↳ Not absolutely summable

We're sticking to cases where DTFT ~~exists~~ exists.
/CTFT

Q. Find DTFT

(1) $x(n) = \delta(n)$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n} dn$$

$e^{-j\omega 0} = 1$

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(2) $x(n) = a^n u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$\sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$a>1$ – Exp increasing

$a<1$ – Exp. decreasing

$$\lim_{n\to\infty} \frac{a^n}{e^{j\omega n}} \quad \frac{a^n \log_e a}{j\omega e^{j\omega n}}$$

Let's have $a<1$

$b = 1/a$

$$X(\omega) = \sum_{n=0}^{\infty} b^{-n} e^{-j\omega n} \qquad a>1$$

$$= \sum_{n=0}^{\infty} (a \times e^{-j\omega})^n.$$

$$X(\omega) = \frac{1}{1-\cancel{1/b}\, e^{-j\omega}} = \frac{1}{1-ae^{-j\omega}}$$

$$X(\omega) = \cancel{\frac{a}{a-e^{j\omega}}}$$

$$\cancel{\frac{a}{a-\frac{1}{e^{j\omega}}}}$$

$$\cancel{\frac{ae^{j\omega}}{e^{j\omega}(a-1)}}$$

$\omega=0$

$\frac{1}{1-a}$

$\omega=\infty$

$$\frac{1}{1-ae^{-j\omega}} \quad 1$$

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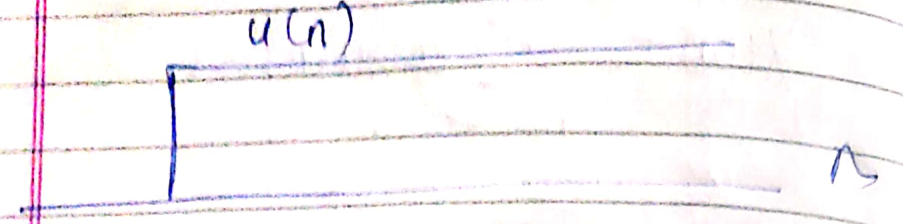
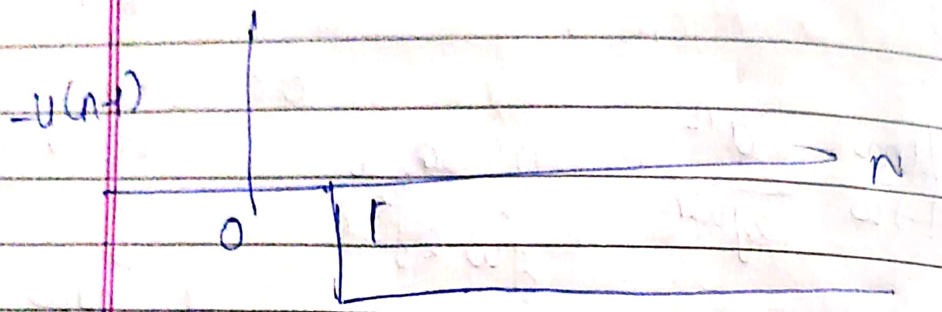
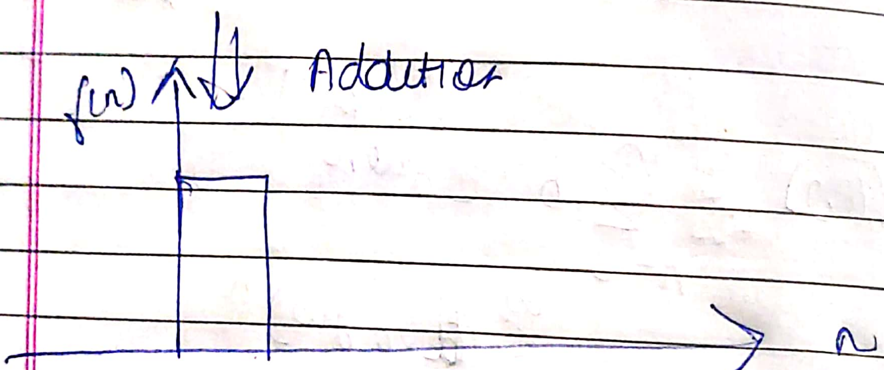
Rectangular function







$$X(\omega) = u(n) - u(n-k)$$

$$\sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=k}^{\infty} e^{-j\omega n}$$

$$= \sum_{n=0}^{k-1} e^{-j\omega n}$$

$$X(\omega) = \frac{(1 - e^{-j\omega k})}{1 - e^{-j\omega}}$$

$$\cancel{e^{j\omega}\,\frac{a\,(e^{j\omega k}-1)}{e^{j\omega k}\,(e^{j\omega}-1)}}$$

$$X(\omega) = \frac{(1-e^{-j\omega k})}{(1-e^{-j\omega})}.$$

$$= \frac{1-\cos(\omega k) - j\sin(\omega k)}{1-\cos(\omega) - j\sin(\omega)}$$

$$X(\omega) = \frac{(1-\cos(\omega k) - j\sin(\omega k))(1+\cos\omega + j\sin\omega)}{\cancel{1-(\cos(\omega) + j\sin\omega)^2}}$$

$$\hookrightarrow \cos^2\omega - \sin^2\omega + 2j\sin\omega\cos\omega$$

$$1 + \cos\omega + j\sin\omega$$

$$X(\omega) = \frac{(1-\cos(\omega k) - j\sin(\omega k))\,(1-\cos\omega + j\sin\omega)}{((1-\cos\omega) - j\sin\omega)((1-\cos\omega)+j\sin\omega)}$$

$$\hookrightarrow (1-\cos\omega)^2 + \sin^2\omega$$

$$\cos^2\omega + \sin^2\omega$$

$$2\cos\omega + 1$$

$$2-2\cos\omega$$

Try $X(\omega) = u(n+k) - u(n-k)$

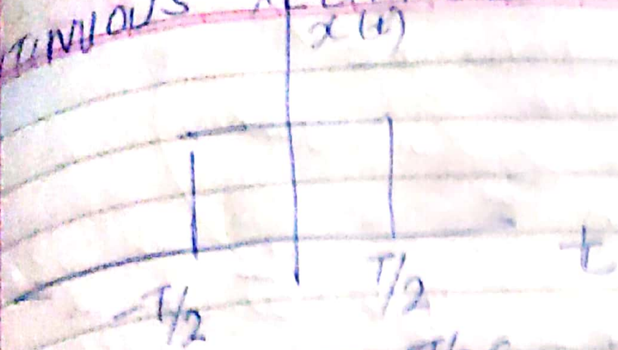
$$\sum_{n=-k}^{\infty} e^{-j\omega n} - \sum_{n=k}^{\infty} e^{-j\omega n}$$

$$\sum_{n=-k}^{k-1} e^{-j\omega n}$$

$$= \sum_{n=-k}^{k} \frac{e^{j\omega k}\left(e^{-j\omega (2k)} - 1\right)}{e^{-j\omega} - 1}$$

$$\frac{e^{j\omega k}\left(e^{-j\omega 2k} - 1\right)}{e^{-j\omega} - 1}$$

CONTINUOUS RECTANGLE



$$X(\omega) = \int_{-T/2}^{T/2} e^{-j\omega t}\, dt$$

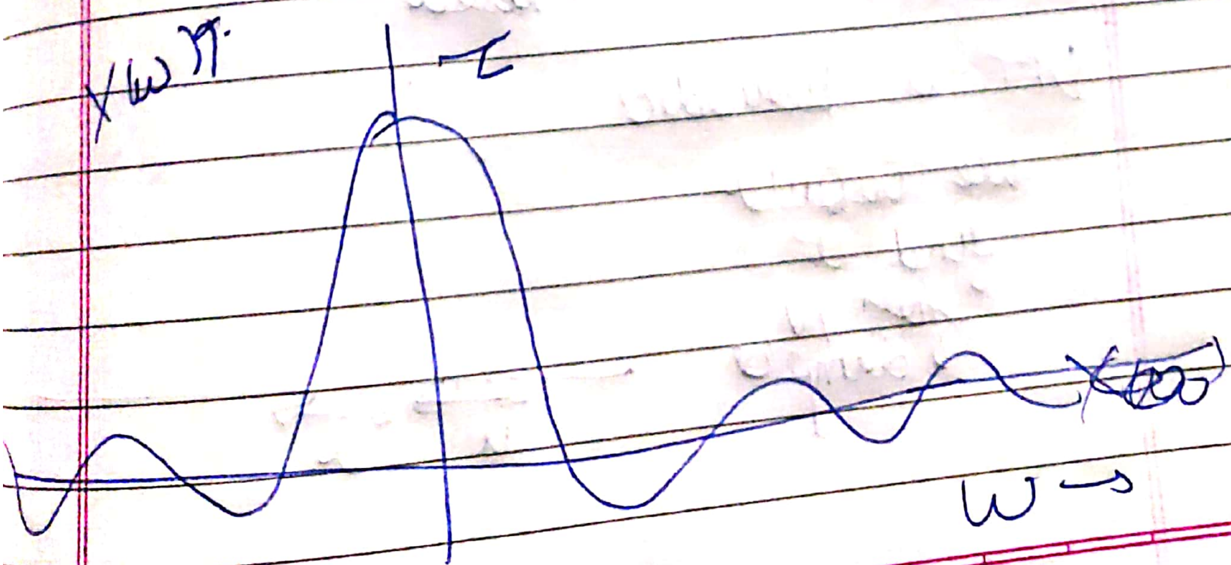
$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$= \frac{-1}{j\omega}\left[\frac{\pm 2}{\omega}\left[\sin \omega \tau/2\right]\right]$$

$$\tau/2 \;\; \frac{2}{\omega \tau/2} \sin \omega \tau/2$$

$$= \tau \operatorname{sinc}(\omega \tau/2)$$



Is DTFT Feasible?

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

No, as ω is still continuous

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi k}{N}$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}}$$

discrete

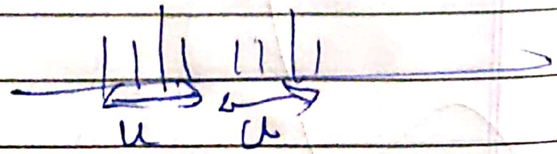
discrete

Discrete Fourier Transform.

Most practical, most widely used.

DFT is periodic

We would find it for N samples



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Q. Number of multiplications required!

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$N \times N \rightarrow N^2$

↳ complex multiplication

$N^2 \rightarrow$ complex multi → min 2 clock cycles

$N(N-1) \rightarrow$ addition

$(4096)^2 \leftarrow$ Done in min # clock cycles

DISCRETE FOURIER TRANSFORM CONTINUED

$N = 4$

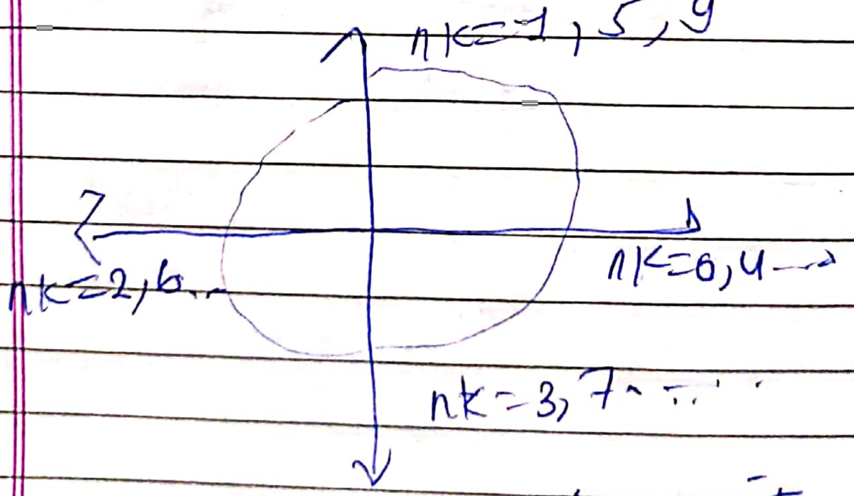
→ DFT

$$X[k] = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi nk}{4}}$$

$$X[0] = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4} 0n}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$X[1] = x(0) + x(1)e^{-j\frac{2\pi}{4}(1)(1)} + x(2)e^{-j\frac{2\pi}{4}(2)(1)} + x(3)e^{-j\frac{2\pi}{4}(3)(1)}$$



Q $x(n) = \delta(n)$ → 4 point DFT (N)

$$X[k] = \sum_{n=0}^{4} x(n)e^{-j\frac{2\pi}{4}k(n)}$$

$X[0] = 1$ $X[1] = 1$

$X[2] = 1$ $X[3] = 1$

Q. Find 4 point DFT for $x(n) = \{1, 1, 1, 1\}$

$X[0] = 1\, e^{-j\frac{2\pi}{4}} + 1 + 1 + 1$
$= 4$

~~$X[3]$~~

$X[1] = e^{-j\frac{2\pi}{4}(0)} + e^{-j\frac{2\pi}{4}} + e^{-j\frac{2\pi}{4}(2)} + e^{-j\frac{2\pi}{4}(3)}$

$= 1 + e^{-j\frac{2\pi}{2}} + e^{-j\pi} + e^{-j\frac{3\pi}{4}}$

$1 - j - 1 + j$

$= 0$

$X[2] = 1 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$

$= 0$

$X[3] = 1 + e^{-j\frac{3\pi}{2}} + e^{-j\frac{6\pi}{2}} + e^{-j\frac{9\pi}{2}} + e^{-j\frac{12\pi}{2}}$

$= 0.$

$X = \{4, 0, 0, 0\}$

ALTERNATIVE METHOD

$X[K]$, $N=4$ points

$X[0] = x(0)e^{-j\frac{2\pi}{}}$

$x(0)\cdot 1 + x(1)\cdot 1$
$+ x(2)\cdot 1 +$
$x(3)\cdot 1$

$X[1] = x(0)\cdot 1 + x(1)(-j) + x(2)(-1) + x(3)(j)$

$X[2] = x(0)\cdot 1 + x(1)(-1) + x(2)(1)$
$+ x(3)(-1)$

$X[3] = x(0)\cdot 1 + x(1)\, j + x(2)(-1) + x(3)(-j)$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Q $x(n) = \{j, 0, j, 1\}$

$$\begin{bmatrix} 2j+1 \\ j - j + j \\ j + j - 1 \\ j - j - j \end{bmatrix} = \begin{bmatrix} 2j+1 \\ j \\ 2j-1 \\ -j \end{bmatrix}$$

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Q $x(n) = \cos(0.25\pi n)$

$= \{ \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, -1 \}$

$$X = \begin{bmatrix} -1 \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - j \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 1 \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + j \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{2} - j \\ 1 \\ \sqrt{2} + j \end{bmatrix}$$

$\{1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\}$

$$X = \begin{bmatrix} 1 \\ 1 - \frac{j}{\sqrt{2}} - \frac{j}{\sqrt{2}} \\ 1 - 1/\sqrt{2} + 1/\sqrt{2} \\ 1 + j/\sqrt{2} + j/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - \sqrt{2}j \\ 1 \\ 1 + \sqrt{2}j \end{bmatrix}$$

Q $x(n) = 0.9^n$, 8 points

$x(n) = \{1, 0.9, 0.81, 0.729, 0.6561, 0.59049, 0.531441, 0.4782969\}$

$X = [5.69, 0.38 - 0.67j, 0.31 - 0.28j, 0.30 - 0.11j, 0.30, 0.30 + 0.11j,$

$0.31+0.28j, 0.38+0.67j]$

TOWARDS ~~FFT~~

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

odd even

$$\sum_{0}^{3} x(2n) e^{-j\frac{2\pi}{N}k(2n)} + \sum_{0}^{3} x(2n-1) e^{-j\frac{2\pi}{N}k(2n-1)}$$

4 points 4 points

$(N/2)^2$ $(N/2)^2$

$N^2/4$ $N^2/4 \longrightarrow N^2/2$

$$X(K) = \underbrace{\sum_{m=0}^{N/2-1} x(2m) e^{-j\frac{2\pi}{N}k(2m)}}_{\text{Even}}$$

$$\underbrace{\sum_{m=0}^{N/2-1} x(2m+1) e^{-j\frac{2\pi}{N}(2m+1)k}}_{\text{Odd}}$$

$$= \underbrace{\sum_{m=0}^{N-1} x(2m) e^{-j\frac{2\pi km}{N/2}}}_{N/2 \text{ point DFT}} + \underbrace{\sum_{m=0}^{N-1} x(2m+1) e^{-j\frac{2\pi km}{N/2}}}_{N/2 \text{ point DFT}} e^{-j\frac{2\pi}{N}}$$

extra

Twiddle factor $\leftarrow W_N^k$

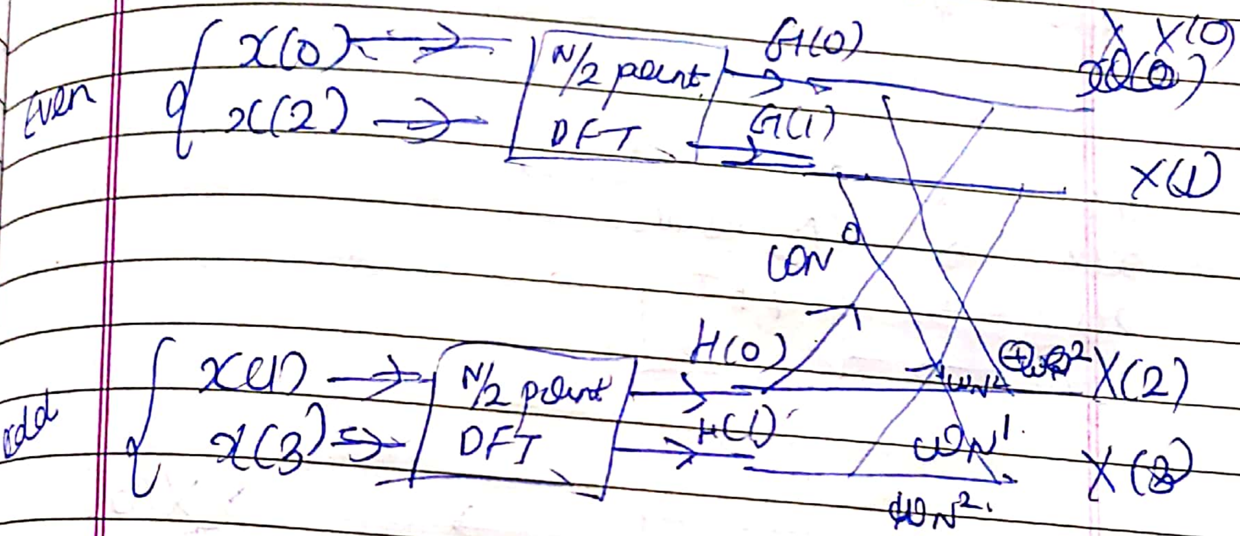
Even

odd

X

BUTTERFLY STRUCTURE (FFT)

let $x(n) = \{x(0), x(1), x(2), x(3)\}$



$X(1) = \text{DFT}[x[0], x[2]] + W_N^0 \, \text{DFT}[x(1), x(3)]$

$= ~~\text{DFT}[x[0]] + W_N^1 [\text{DFT}[x[~~$

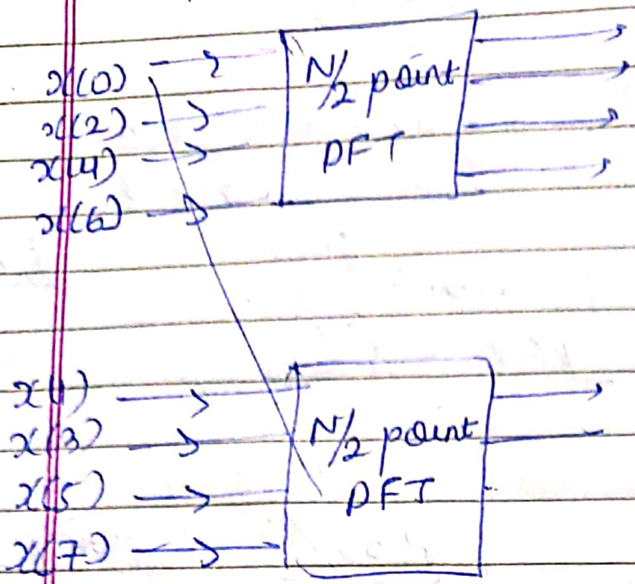
$X[2] = G(2) + W_N^2 H(2)$

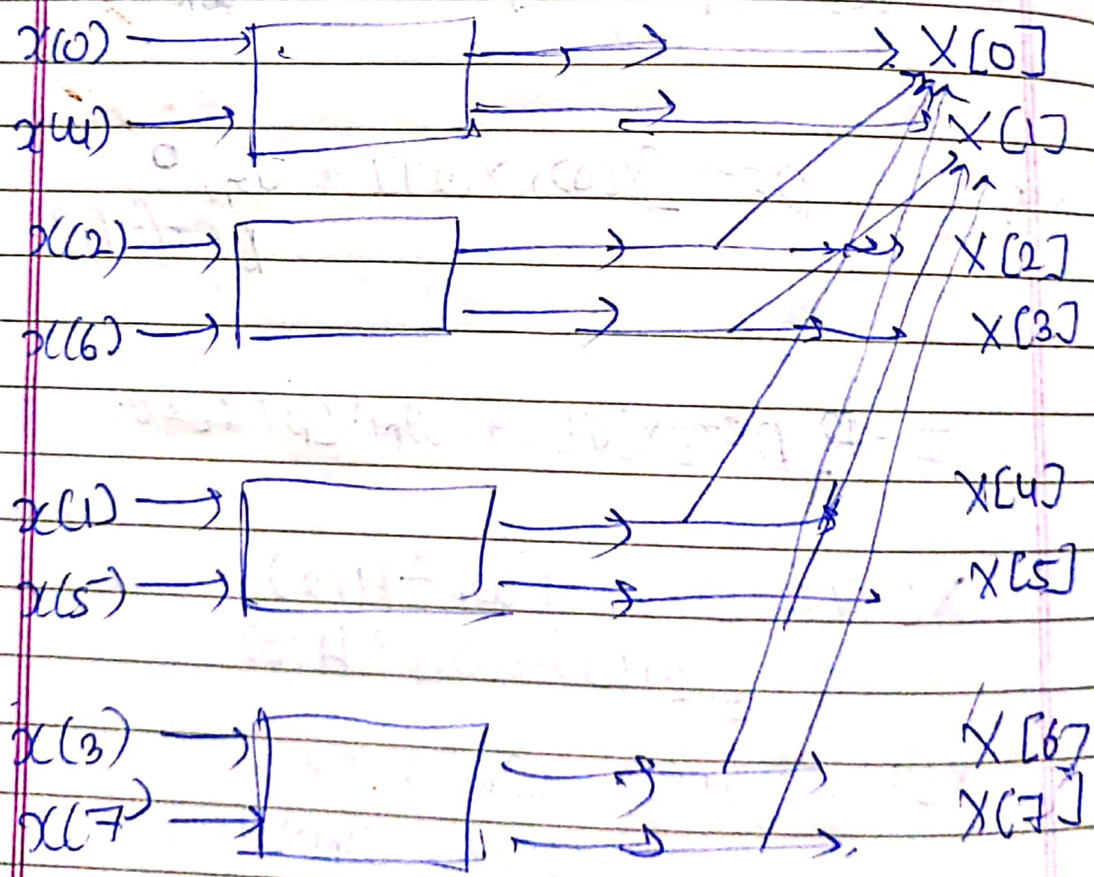
$= G(0) + W_N^0 H(0)$

$X[3] = G(3) + W_N^2 H(3)$

$= G(1) + W_N^3 H(1)$

8-point butterfly





$X(K) = DFT[x(0], x[2], x(4], x[6]]$

$\qquad + W_N^{K} DFT[x[1] x[3] x(5,7)]$

$= DFT(0,2) + W_{N/2}^{K} DFT(1,3)$

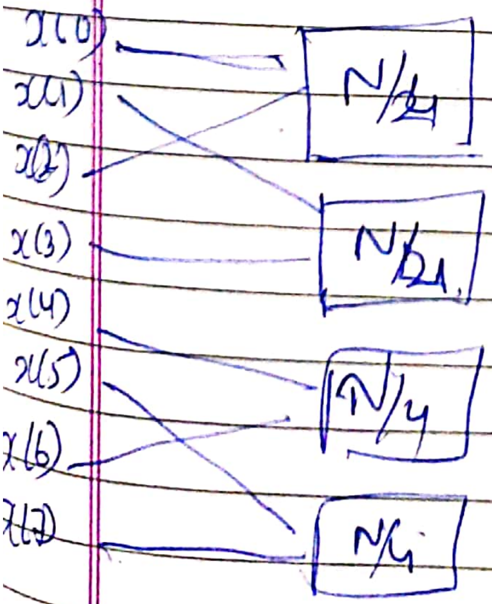
$\qquad + W_N^{K} DFT(1,5)$

$\qquad + W_N^{K} W_{N/2}^{K} DFT(3,7)$

$x(0) = G[0] + W_{\frac{N}{2}}^{K} H[0]$

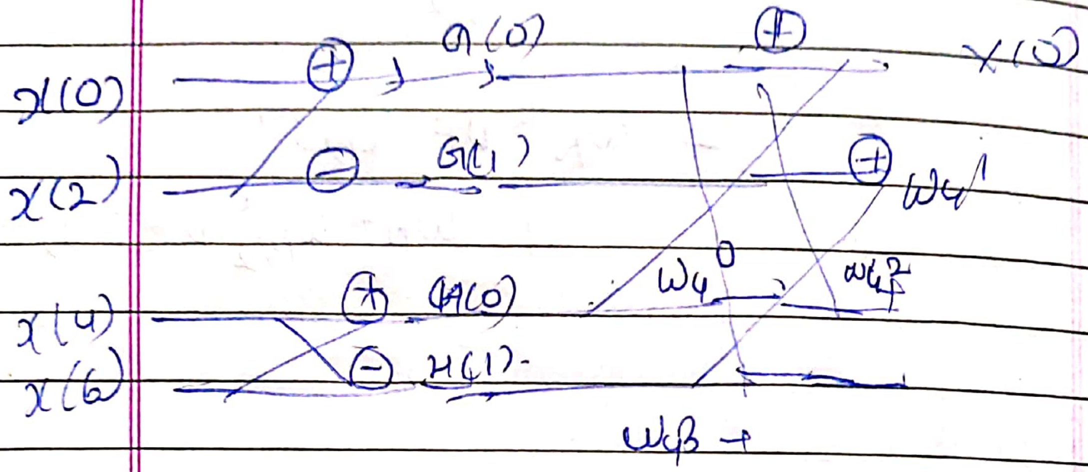
$\qquad + W_N^{K} I[0]$

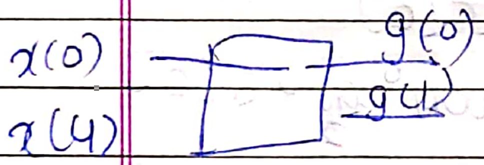
$\qquad + W_N^{0} W_{\frac{N}{2}}^{0} J[0]$

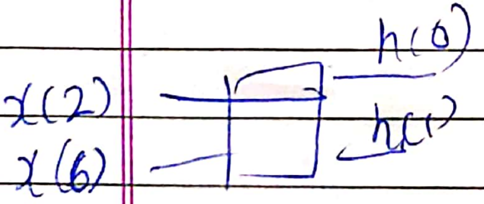


$$G(0) = x(0)\cdot 1 + x(2)\cdot 1$$
$$= x(0) + x(2)$$

$$G(1) = x(0)\cdot 1 + x(2)\cdot(-1)$$
$$x(0) - x(2)$$







x(1)
x(5)

Z-TRANSFORM

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

CTFT

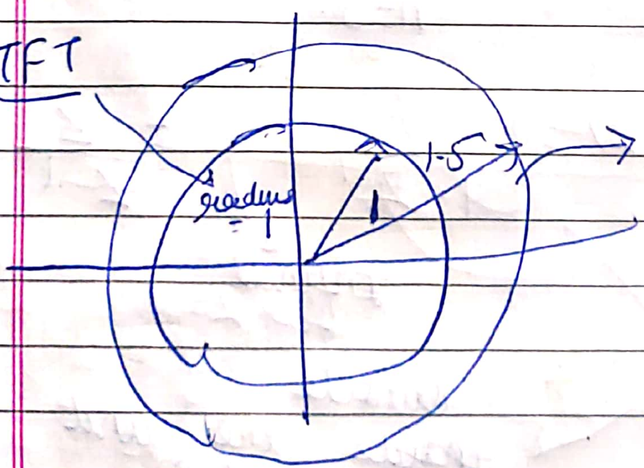
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$ (Subset of laplace)

laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$s = \sigma + j\omega$

DTFT



Can I go to the circle?

$$X(\ ) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\omega n}$$

Region of Convergence

↓

Region in which my output is finite (absolutely summable)

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$Z = re^{j\omega}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

DTFT is a special case of Z-transform at $r=1$

ROC

$x(n) = u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n)\,z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Positive for

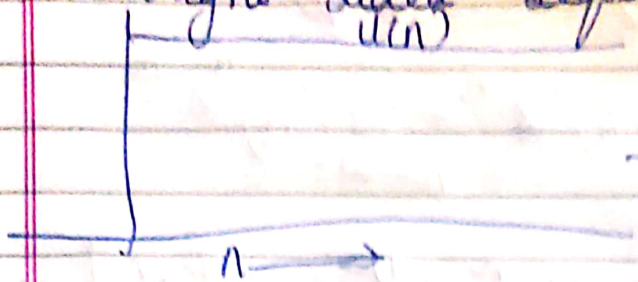
$z > 1$ → Outside the unit circle

Infinite for $z = 1$.

$$1 - 1/z > 0$$

$$\frac{z-1}{z} > 0 \qquad 1 > 1/z \qquad \boxed{z > 1}$$

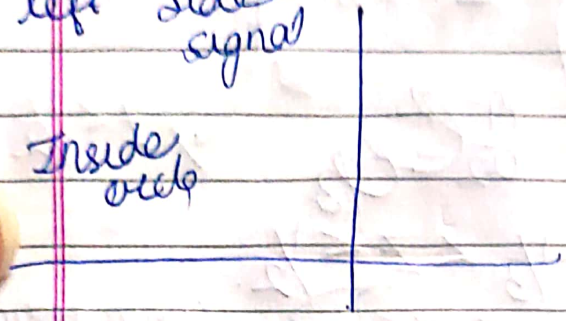
right sided sequence



→ Outside unit circle

left sided signal

Inside circle



$x(n) = a^n u(-n)$

$$X(z) = \sum_{n=-\infty}^{0} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{-n} (z/a)^n$$

$z < a$

↓

Roc inside circle with radius a

$$= \frac{1}{1 - z/a}$$

$$\frac{a}{a - z}$$

Both sided?

$x(n) = (1/2)^n u(n) + (-1/3)^n u(n)$

$$\frac{1}{1 - \frac{1}{2z}} \qquad \frac{1}{1 + \frac{1}{3z}}$$

$\boxed{z > 1/2}$ $\frac{z}{1} > \frac{1}{2}$, $z > -1/3$.

$1 + \frac{1}{3z} > 0$

$\frac{1}{3z} > -1$

$-1 < \frac{1}{3z}$ $3z > -1$

$z >$

Q $x(n) = \left(-\tfrac{1}{3}\right)^n u(n) - \left(\tfrac{1}{2}\right)^n u(-n-1)$

$$\sum_{n=0}^{\infty} \left(-\tfrac{1}{3}\right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(-\tfrac{1}{3}\right)^n z^{-n} - \left[\sum_{n=-\infty}^{0} \left(\tfrac{1}{2}\right)^n z^{-n} - 1\right]$$

$z > -1/3$

$z < 1/2$

$a_1 = 1/2$

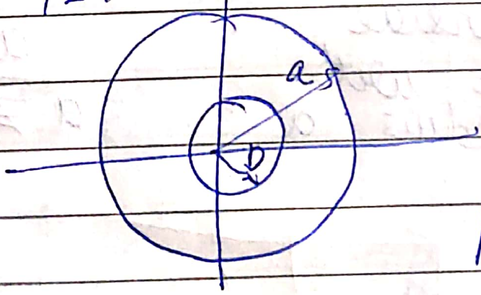
$a_2 = -1/3$

$-1/3 < z < 1/2$

$z < 1/2$ → considering positive values

For both sided sequence ROC will be a ring

$|z| < a$ | $|z| > b$



DTFT would converge if & only iff ROC includes the unit circle.

Poles & Zeros

Points at which o/p = ∞

o/p = 0

- ROC cannot contain pole

- Right sided seq, ROC will be outside the circle, vice versa for left side.

- For both sided seq, ROC will be a ring.

- For finite seq.

$$\cancel{X(z)}\ x(n) = \{1, 0, 1\}$$

$$X(z) = \sum_{n=0}^{2} x(n) z^{-n}$$

$$= 1 + z^{-2}$$

↓

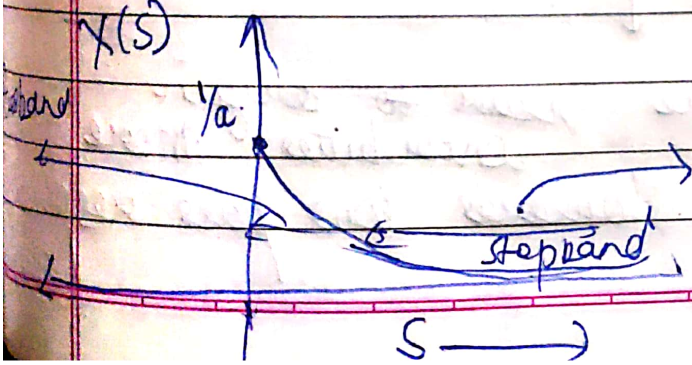
Poles at $z = 0$

FILTERS

$$X(s) = \frac{1}{s+a} \qquad s = \sigma + j\omega$$

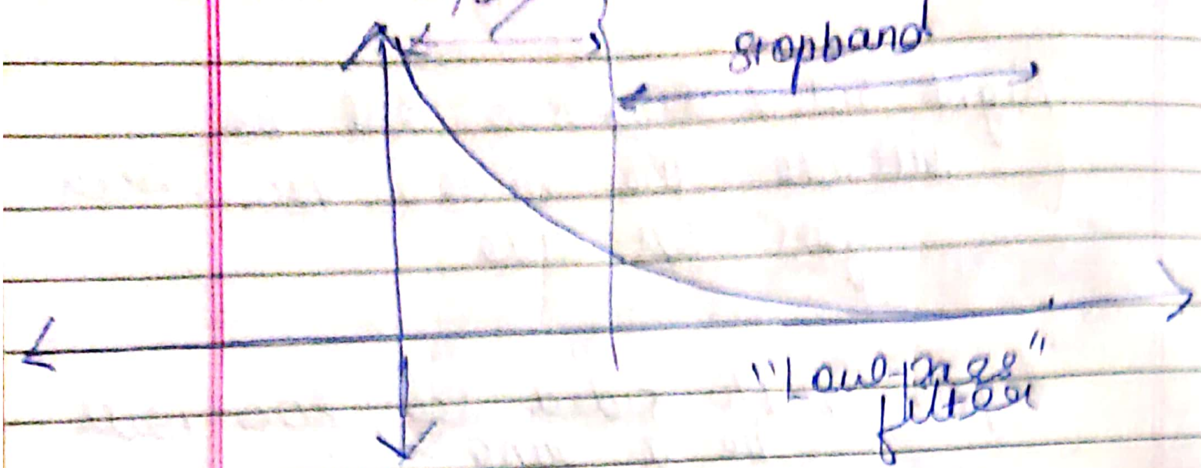
- Complex freq.

$$s = 0 \Rightarrow X(s) = \frac{1}{a}$$

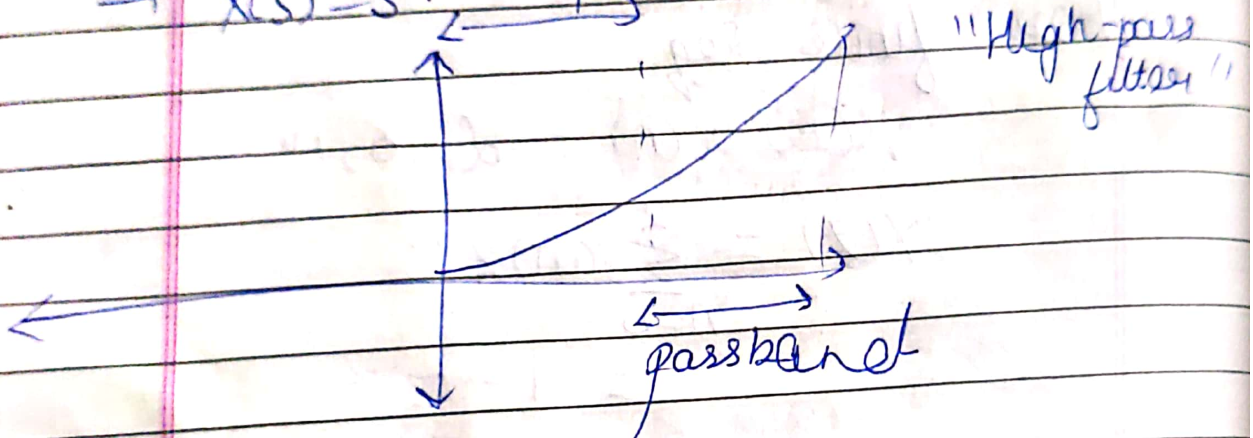


Similar to low pass filter

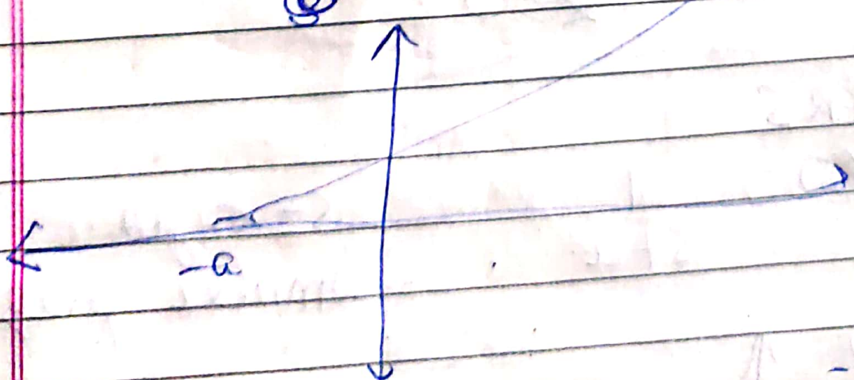
$X(s) = 1/s$



→ $X(s) = s$



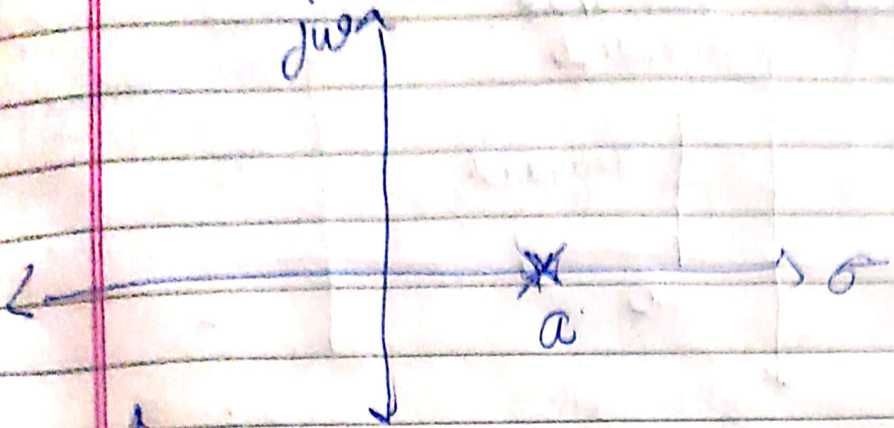
→ $\boxed{X(s) = \frac{s+a}{s}}$

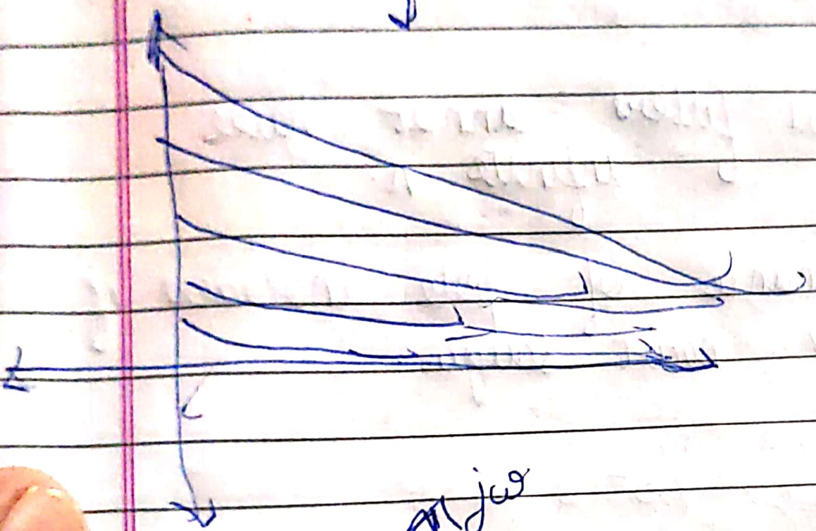


Pole ×
zero O

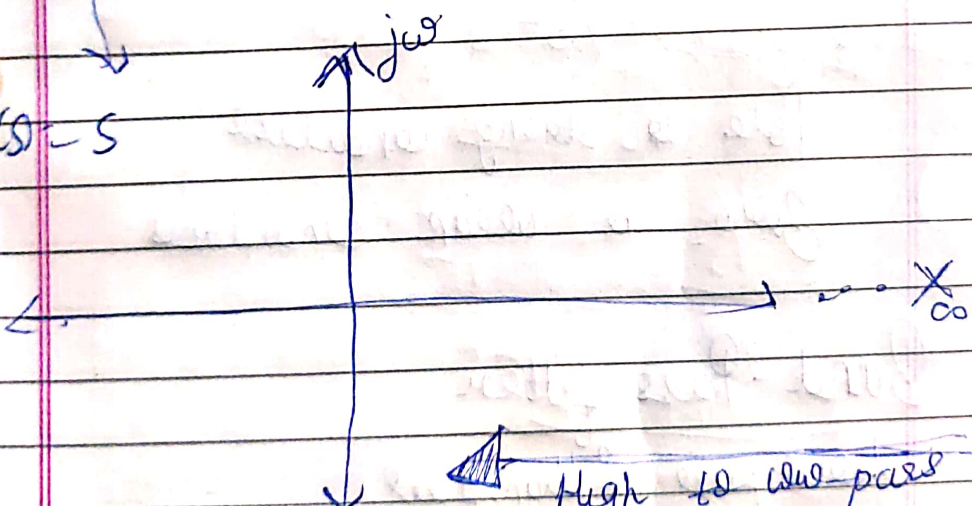
Pole near to origin contributes more towards low pass response.

$X(s) = \frac{1}{s-a}$



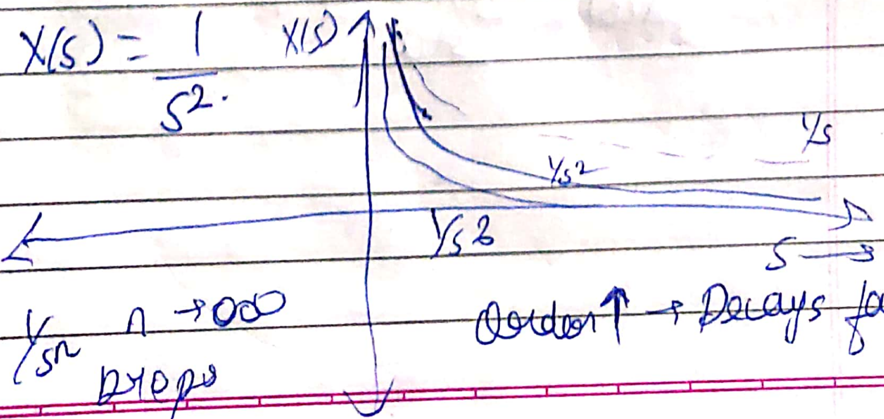


$X(s) = s$



High to low-pass

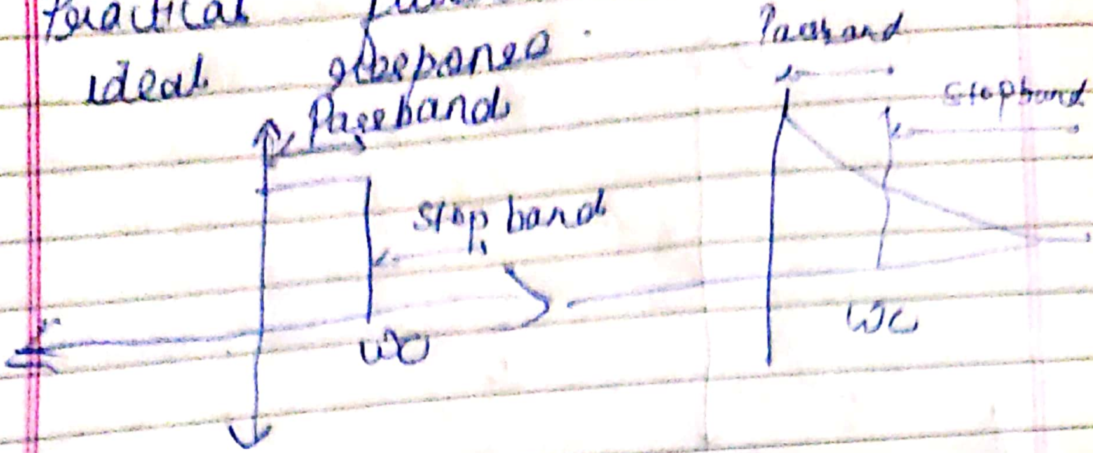
$X(s) = \frac{1}{s^2}$



$X(s) = 1/s^n \quad n \to \infty$ drops

Order ↑ → Decays faster

Practical filters cannot have ideal response.



Practical filters cannot have infinite n.

→ Dominance of pole increases if it is near origin.

$s^2, 1/s^2$

Pole is strong candidate

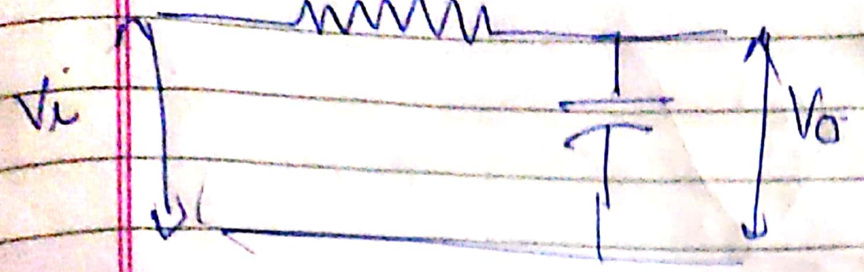
Zero is weak candidate

Band-Pass filter

Low pass & High pass

$X(s) = s/(s+a)$

LOW-PASS FILTER CIRCUIT



$X_C = \frac{1}{j\omega C}$

$R = R$

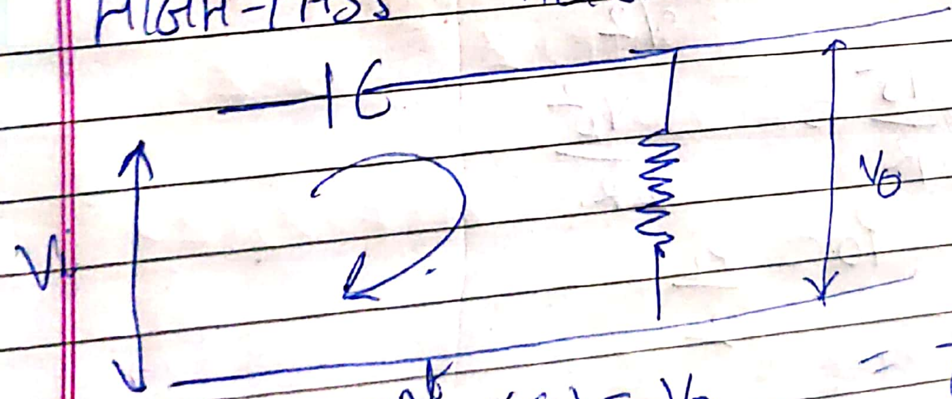
Shorts at high frequency

$$\frac{V_o}{V_i} = \frac{1/j\omega C}{R + \frac{1}{j\omega C}} = \frac{1/SC}{R + 1/SC}$$

$s = j\omega$

$$\frac{V_o}{V_i} = \frac{1}{1 + RSC} = \frac{\frac{1}{R}}{SC + 1/R}$$

HIGH-PASS FILTER CIRCUIT



$$X(s) = \frac{V_o}{V_i} = \frac{R}{R + 1/j\omega C}$$

$s = j\omega$

$RC \rightarrow$ Time constant

$$\frac{j\omega c}{\frac{1}{RC} + j\omega} \qquad \frac{S}{1/RC + S}$$

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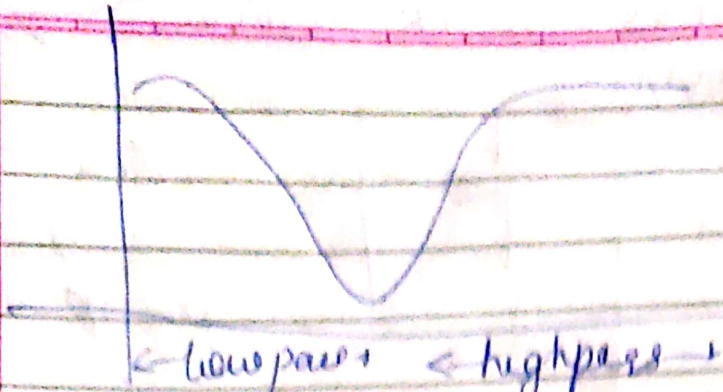

x   0   x

0   0

Low pass   High pass

$s = a$ $s = b$

$a < b$.

NEXT LECTURE:

Cutoff freq. (-3dB freq)

$$V_0 = \frac{V_i}{\sqrt{2}}$$

$$\frac{V_0^2}{R} = \frac{V_i^2}{2R}.$$

$$P_0 = \frac{P_i}{2}$$

The point at which power become half is called cutoff freq.

PROPERTIES OF Z-TRANSFORM

Linearity

$a_1 x_1(n) + a_2 x_2(n) \rightarrow a_1 X_1(z) + a_2 X_2(z)$

$$\sum_{n=-\infty}^{\infty} [a_1 x_1(n) z^{-n} + a_2 x_2(n)] z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

Time shifting

$x(n) \rightarrow X(z)$

$$x(n-k) \rightarrow \sum_{n=-\infty}^{\infty} x(n-k) \cdot z^{-n}$$

$m = n-k \qquad n = m+k$

$$\sum_{n=-\infty} x(m)\, z^{-(m+k)}$$

$$= z^{-k} \times X(z)$$

Amplitude scaling

$a^n x(n) \rightarrow X(z/a)$

$$\sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x^n (z/a)^n$$

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Convolution

$x_1(n) \quad x_2(n)$

$\downarrow \quad \downarrow$

$X_1(z) \quad X_2(z)$

$x_1(n) * x_2(n) \longrightarrow X_1(z) X_2(z)$

Z trans $\sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] z^{-n}$

$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \; z^{-n}$

$\sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$

$\sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \; X_2(z)$

$= X_1(z) X_2(z)$

INVERSE Z-TRANSFORM

$1 + z^{-1} + z^{-2} \ldots \ldots$

$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$x(n) = u(n)$

$1z + z^{1} + 4z^{-3} + 2z^{-5}$

$x\{1, 0, 1, 0, 4, 2\}$

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Finite z-transform output available
└→ Determine x(n) by looking at it.

· LONG DIVISION METHOD

$$X(z) = \frac{1}{z - 1/2}$$

$$\begin{array}{r|l} & z^{-1} + \tfrac{1}{2}z^{-2} + \tfrac{1}{4}z^{-3} \\ \hline z - 1/2 & 1 \\ & 1 - \tfrac{1}{2}z^{-1} \\ \hline & \tfrac{1}{2}z^{-1} \\ & \tfrac{1}{2}z^{-1} - \tfrac{1}{4}z^{-2} \\ \hline & \tfrac{1}{4}z^{-2} \\ & \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} \\ \hline & \tfrac{1}{8}z^{-3} \end{array}$$

$z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3} + \ldots$

$x(n) = \{0, 1, 1/2, 1/4, 1/8 \ldots\}$

Positive side, Causal nature

$|z| > 1/2$

$$X(z) = \frac{1}{z - 1/2} = \frac{1/z}{1 - 1/2\, z^{-1}} = \frac{z^{-1}}{1 - 1/2\, z^{-1}}$$

Scanned with CamScanner

$$1-\tfrac{1}{2}z^{-1} \overline{) \; z^{-1}} \quad \cancel{1/2}\ \cancel{4z}\ -2$$

$$\frac{-\left(z^{-1}-2\right)}{2}$$

$$\frac{-\left(4z-2\right)}{4z}$$

$$|z|<1/2.$$

$$\{\ldots -8, -4, -2\}$$

→ ROC exterior to the circle with radius r the long division must be performed keeping z as the term.

- ROC interior to the circle with radius r ... z^{-1} as the term.

To get accurate answer for infinite sequence becomes diff.

* Partial fraction expansion

$$X(z) = \frac{1}{z^2-2z+1} \qquad \frac{1}{(z-1)(z-1)}$$

$$\frac{1}{(z-1)(z-1)} = \frac{A}{(z-1)} + \frac{B}{(z-1)}$$

$$= \frac{A/z}{(1-z^{-1})} + \frac{B/z}{(1-z^{-1})}$$

$$Y(n) = Ax_1(n) + Bx_2(n)$$

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$$X(z) = \frac{1}{z^2-3z+2} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{(z-1)-(z-2)}{(z-1)(z-2)}$$

$$X(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$x(n) = \frac{-2}{1-z/2} + \frac{1}{1-z}$$

$$-2 \times \qquad \omega = 0,\ z = 1,\ z = 1$$

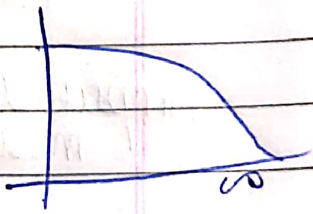
$$\omega = \infty \Rightarrow \qquad z = 0$$

$$\frac{1}{z-a} = \frac{z^{-1}}{1-az^{-1}} \qquad \omega = \infty, \qquad z = \infty$$

$$= \frac{1}{a}\left(\frac{az^{-1}}{1-az^{-1}}\right)$$

$z = 0$

$$z^{-1}\left[\frac{1}{a}\left(\frac{az^{-1}}{1-az^{-1}}\right)\right]$$



$$\frac{1}{a}\, z^{-1}\left[\frac{az^{-1}}{1-az^{-1}}\right]$$

$$= \frac{1}{z}\left(\frac{1}{1-az^{-1}}\right)$$

$1/z$

$$\frac{1}{z}\left[\frac{1}{(z-2)} - \frac{1}{(z-1)}\right] \rightarrow 2^{n-1}\, u(n-1) - u(n-1)$$

$(1-2z^{-1})$

$$\left[2^n u(n) - u(n)\right]$$

$= 2$