50.002 COMPUTATIONAL STRUCTURES

INFORMATION SYSTEMS TECHNOLOGY AND DESIGN

Problem Set 1

1 Warm Up

Suppose that you are to guess the value of a 16-bit number: 0x$Z_1Z_2Z_3Z_4$ You are told that the value of Z_1 is B. Thus you have been given [N] bits of information. What is the value of [N]?

2 ISTD Prize

Your cohort contains 100 students, 51 of whom are male and 49 are female. There are 31 male students who are above 19 years old. On the other hand, there are 19 female students who are above 19 years old. There are one male student and three female students who like to have a final exam. You can assume that students either like or hate a final exam and no indifference. Two students like exam and is above 19 years old.
Now someone in your class won ”the first to join ISTD” prize. Answer the following questions:

1. If you are told the student ID of this winner, how much information did you receive in bits?

2. If you are told the student ID of the last 33 students who joined ISTD, how much information did you receive in bits?

3. If you are told that the student who won the ”first to join ISTD prize” is a male, how much information did you receive in bits?

4. If you are told that the student who won the ”first to join ISTD prize” is above 19 years old instead, how much information did you receive in bits?

5. If you are told that the student who won the ”first to join ISTD prize” hated a final exam and is below 19 years old, how much information did you receive in bits?

3 Keyboard Presses

1. Bob used an enhanced keyboard that was made up of 101 keys. He told Alice that he pressed one of the letter keys. How much information did Bob give to Alice? Hint: There are 26 letters in an alphabet.

2. Bob used an enhanced keyboard that was made up of 101 keys. He told Alice that he pressed two of the letter keys consecutively. Bob did not mention whether the two keys are the same or not. How much information did Bob give to Alice? Hint: There are 26 letters in an alphabet.

4 Deck of Cards

1. Someone picks a name out of a hat known to contain the names of 5 women and 3 men, and tells you a man has been selected. How much information have they given you about the selection?

2. You're given a standard deck of 52 playing cards that you start to turn face up, card by card. So far as you know, they're in completely random order. How many new bits of information do you get when the first card is flipped over? The fifth card? The last card?

3. X is an unknown N-bit binary number ($N > 3$). You are told that the first three bits of X are 011. How many bits of information have you been given?

4. X is an unknown 8-bit binary number. You are given another 8-bit binary number, Y, and told that the Hamming distance (number of different bits) between X and Y is one. How many bits of information about X have you been given when Y is presented to you?

5 Measuring Information

After spending the afternoon in the dentist's chair, Ben has invented a new language called DDS made up entirely of vowels (the only sounds he could make with someone's hand in his mouth). The DDS alphabet consists of the five letters: A, E, I, U, O, which occur with the following probabilities,

Letter	Prob. of occurrence
A	$p(A) = 0.15$
E	$p(B) = 0.4$
I	$p(C) = 0.15$
O	$p(D) = 0.15$
U	$p(E) = 0.15$

Table 1

If you're told that the first letter of the message is "A", give an expression for the number of bits of information you have received.

6 Modular arithmetic and 2's complement representation

Most computers choose a particular word length (measured in bits) for representing integers and provide hardware that performs various arithmetic operations on word-size operands. The current generation of processors have word lengths of 32 bits; restricting the size of the operands and the result to a single word means that the arithmetic operations are actually performing arithmetic modulo 2^{32}.

Almost all computers use a 2's complement representation for integers since the 2's complement addition operation is the same for both positive and negative numbers. In 2's complement notation, one negates a number by forming the 1's complement (i.e: for each bit, changing 0 to a 1 and vice versa) representation of the number and then adding 1. **By convention**, we write 2's complement integers with the most-significant bit (MSB) on the left and the least-significant bit (LSB) on the right. Also, **by convention**, if the MSB is 1, the number is negative, otherwise it's non-negative.

1. How many different values can be encoded in a 32-bit word?
2. Please use a 32-bit 2's complement representation to answer the following questions. What are the representations for:
   (a) Zero
   (b) The most positive integer that can be represented
   (c) The most negative integer that can be represented

   What are the decimal values for the most positive and the most negative number?
3. Since writing a string of 32 bits gets tedious, it's often convenient to use hexadecimal representation where a single digit in the range of 0-9 or A-F is used to represent groups of 4 bits. Give the 8-digit hexadecimal equivalent of the following decimal and binary numbers:
   (a) 37_{10}
   (b) -32768_{10}
   (c) `1101 1110 1010 1101 1011 1110 1110 1111`
4. Calculate the following using 6-bit 2's complement arithmetic (which is just a fancy way of saying to do ordinary addition in base 2, keeping only 6 bits of

your answer). Show your work using binary notation. Remember that subtraction can be performed by negating the second operand and then adding to the first operand.

(a) 13 + 10

(b) 15 - 18

(c) 27 - 6

7 Dice Throwing Game

A group of five friends are playing a game that requires them to generate random numbers using 10 fair dice in the beginning before proceeding with the game. They each will throw the 10 dice and sum up all the outcomes of the dice to get the random number. Answer the following questions:

1. How many bits at the minimum (so round up your answer to the nearest integer) are required to encode all distinct numeric outcomes of 10
2. Someone in the group suggests that they can just use a die and throw it 10 times to get the random number required for the game. This way, they don't have to deal with carrying so many dice. The game began and then he proceeded with throwing the die. His first 3 throws are: 1, 3, and 4. How many bits of information has been given so far? Give your answer in 3 decimal places.
3. After throwing the die 9 times in total, how many new bits of information did he get from making the last (the 10th) throw? Give your answer in 3 decimal places.
4. Finally, he found that the number he got in total from all 10 throws is 53. Express this number in 3-digit hex, formatted as 0xZZZ where Z is your answer.
5. The second person in the team proceeded to make his 10 throws and found that the number he got is 31. Given that there are five people in the team, how many bits of information are revealed when it is known that the first person's random number is 53 and the second person's random number is 31? Recall that the goal is for everybody in the team to each have a random number before proceeding with the game. Give your answer in 3 decimal places.

8 Another Base Conversion

Consider an 8-bit number systems. Do the following base conversion, and indicate with a `0b` prefix for binary systems and `0x` prefix for hexadecimal systems. Octal and decimal systems do not have prefixes.

1. 76 (decimal) to binary

2. 0b10000001 (binary) to decimal
3. 0b10011101 (binary) to hexadecimal
4. 0xBF (hexadecimal) to binary
5. 0xC6 (hexadecimal) to octal

9 Representing -32 on different number systems

Which of the following signed numbers is equivalent to the number -32 for either an 8-bit or 16-bit system?

a. `0b1010 0000`

b. `0b1110 0000`

c. `0b0001 0000`

d. `0xE0`

e. `0x80E0`

f. `0xFFE0`

g. `0x10E0`

h. `0x800000E0`

i. `0xFFFFFFE0`

10 Challenge – Proof of 2's Complement

At first blush, "Complement and add 1" doesn't seem like an obvious way to negate a two's complement number. By manipulating the expression $A + (-A) = 0$, show that "complement and add 1" does produce correct representation for the negative of a two's complement number.
Hint: express 0 as (-1 + 1) and rearrange the terms to get -A on one side and ZZZ+1 on the other and then think about how the expression ZZZ is related to A using only logical operations (AND, OR, NOT).