50.002 COMPUTATIONAL STRUCTURES

INFORMATION SYSTEMS TECHNOLOGY AND DESIGN

Problem Set 1

1 Warm Up

Suppose that you are to guess the value of a 16-bit number: $0xZ_1Z_2Z_3Z_4$ You are told that the value of Z_1 is B. Thus you have been given [N] bits of information. What is the value of [N]?

Solution:
Obviously Z_x represents 4 bits since these are in hexadecimal number system (indicated with the prefix of 0x.) We are literally told that the first hex digit is $B = 1011$. Hence we are given **4 bits of information**. There are still other 12 bits that we do not know of its value.

2 ISTD Prize

Your cohort contains 100 students, 51 of whom are male and 49 are female. There are 31 male students who are above 19 years old. On the other hand, there are 19 female students who are above 19 years old. There are one male student and three female students who like to have a final exam. You can assume that students either like or hate a final exam and no indifference. Two students like exam and is above 19 years old.
Now someone in your class won "the first to join ISTD" prize. Answer the following questions:

1. If you are told the student ID of this winner, how much information did you receive in bits?

   Solution:
   If we are given the student ID, we effectively know who the winner is. We instantly narrow down our pool of winner candidates from 100 students into just 1 student, hence the bits of information given is $\log_2(100)$.

2. If you are told the student ID of the last 33 students who joined ISTD, how much information did you receive in bits?
   Solution:
   If we know the last 33 students who joined ISTD, we know that *these students cannot be the winner*. Hence our pool of candidates are narrowed down into 67 students. We are then given $\log_2(100) - \log_2(67)$ bits of information.

3. If you are told that the student who won the "first to join ISTD prize" is a male, how much information did you receive in bits?

   Solution:
   Similarly, there are 51 males in ISTD. Hence, we are given $\log_2(100) - \log_2(51)$ bits of information.

4. If you are told that the student who won the "first to join ISTD prize" is above 19 years old instead, how much information did you receive in bits?
   Solution:
   Our candidates are narrowed down into 31 + 19 = 50 students instead since there are 50 students who are above 19 years old. The amount of information given is $\log_2(100) - \log_2(50)$ bits.

5. If you are told that the student who won the "first to join ISTD prize" hated a final exam and is below 19 years old, how much information did you receive in bits?
   Solution:
   There are $51 - 31 = 20$ male students that are below 19 years old. There are also 30 female students that are below 19 years old. Since there are only 4 people who like exam, and 2 of them are above 19 years old, there's only 2 students who are below 19 years old and like exam as well.

   Hence our candidates are further narrowed into 48 students after knowing that the person does not like exam. This gives us $\log_2(100) - \log_2(48)$ bits of information.

3 Keyboard Presses

1. Bob used an enhanced keyboard that was made up of 101 keys. He told Alice that he pressed one of the letter keys. How much information did Bob give to Alice? Hint: There are 26 letters in an alphabet.

   Solution:
   Initially, there's 101 choices. The information that Bob gave Alice narrows down the choices into 26. The information given is therefore $\log_2(101) - \log_2(26) = 1.958$.

2. Bob used an enhanced keyboard that was made up of 101 keys. He told Alice that he pressed two of the letter keys consecutively. Bob did not mention whether the two keys are the same or not. How much information did Bob give to Alice? Hint: There are 26 letters in an alphabet.

   Solution:
   Initially, there are $101 * 101$ choices. Pressing two letter keys consecutively (might be repeated) narrows down the choices onto $26 * 26$. Hence the information given is $\log_2(101^2) - \log_2(26^2) = 3.916$.

4 Deck of Cards

1. Someone picks a name out of a hat known to contain the names of 5 women and 3 men, and tells you a man has been selected. How much information have they given you about the selection?

   Solution:
   Originally, we have 8 options: $M = 8$. The option is narrowed down into 3, hence $N = 3$, we have:

   $$I = \log_2\left(\frac{1}{N/M}\right) = \log_2\left(\frac{1}{3/8}\right) \text{ bits of information.} \tag{1}$$

2. You're given a standard deck of 52 playing cards that you start to turn face up, card by card. So far as you know, they're in completely random order. How many new bits of information do you get when the first card is flipped over? The fifth card? The last card?

   Solution:
   The first card has $\log_2 52$ bits of information. The fifth card has $\log_2 48$ bits of information. The last card has 0 bits of information.

3. X is an unknown N-bit binary number ($N > 3$). You are told that the first three bits of X are 011. How many bits of information have you been given?

   Solution:
   The bits of information that has been given is 3.

4. X is an unknown 8-bit binary number. You are given another 8-bit binary number, Y, and told that the Hamming distance (number of different bits) between X and Y is one. How many bits of information about X have you been given when Y is presented to you?

   Solution:

Originally, we have 8 unknown bits, that is $M = 2^8$ choices. After given Y, we are down to $N = 8$ choices, each having 1 bitt different for X. Therefore, we have been given,

$$I = \log_2\left(\frac{1}{N/M}\right) = \log_2\left(\frac{1}{8/2^8}\right) = 5 \text{ bits of information.} \tag{2}$$

5 Measuring Information

After spending the afternoon in the dentist's chair, Ben has invented a new language called DDS made up entirely of vowels (the only sounds he could make with someone's hand in his mouth). The DDS alphabet consists of the five letters: A, E, I, U, O, which occur with the following probabilities,

Letter	Prob. of occurrence
A	$p(A) = 0.15$
E	$p(B) = 0.4$
I	$p(C) = 0.15$
O	$p(D) = 0.15$
U	$p(E) = 0.15$

Table 1

If you're told that the first letter of the message is ”A”, give an expression for the number of bits of information you have received.

Solution:
Recall that the information received is inversely proportional to the probability of that choice occurring, and we take $\log_2$ of the probability of that choice occurring to quantify the information in terms of bits. Hence the expression is,

$$I = \log_2 \frac{1}{p(A)} \tag{3}$$

6 Modular arithmetic and 2's complement representation

Most computers choose a particular word length (measured in bits) for representing integers and provide hardware that performs various arithmetic operations on word-size operands. The current generation of processors have word lengths of 32 bits; restricting the size of the operands and the result to a single word means that the arithmetic operations are actually performing arithmetic modulo 2^{32}.

Almost all computers use a 2's complement representation for integers since the 2's complement addition operation is the same for both positive and negative numbers. In 2's complement notation, one negates a number by forming the 1's complement (i.e: for each bit, changing 0 to a 1 and vice versa) representation of the number and then adding 1. **By convention**, we write 2's complement integers with the most-significant bit (MSB) on the left and the least-significant bit (LSB) on the right. Also, **by convention**, if the MSB is 1, the number is negative, otherwise it's non-negative.

1. How many different values can be encoded in a 32-bit word?
   Solution:
   2^{32}.

2. Please use a 32-bit 2's complement representation to answer the following questions. What are the representations for:

   (a) Zero
   Solution:
   `0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0`

   (b) The most positive integer that can be represented
   Solution:
   `0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1`

   (c) The most negative integer that can be represented
   Solution:
   `1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0`

   What are the decimal values for the most positive and the most negative number?
   Solution:
   The most positive value in decimal: 2147483647. The most negative value in decimal: -2147483648

3. Since writing a string of 32 bits gets tedious, it's often convenient to use hexadecimal representation where a single digit in the range of 0-9 or A-F is used to represent groups of 4 bits. Give the 8-digit hexadecimal equivalent of the following decimal and binary numbers:

   (a) 37_{10}
   Solution:
   `0x 0000 0025`

   (b) -32768_{10}
   Solution:
   We begin by converting 32768 (positive) number to Hex: `0x 0000 8000`. Then we take the 1's complement[1]: `0x FFFF 7FFF`, and finally $+1$: `0x FFFF 8000`.

[1]Note: transform the hex to binary first and flip the bits, then transform back to hex

(c) `1101 1110 1010 1101 1011 1110 1110 1111`
Solution:
`0x DEAD BEEF`

4. Calculate the following using 6-bit 2's complement arithmetic (which is just a fancy way of saying to do ordinary addition in base 2, keeping only 6 bits of your answer). Show your work using binary notation. Remember that subtraction can be performed by negating the second operand and then adding to the first operand.

   (a) 13 + 10
   Solution:
   `001101 + 001010 = 010111`

   (b) 15 - 18
   Solution:
   18 is `010010`. -18 is `101110`. Hence, `001111 + 101110 = 111101`

   (c) 27 - 6
   Solution:
   `011011 + 111010 = 010101`

7 Dice Throwing Game

A group of five friends are playing a game that requires them to generate random numbers using 10 fair dice in the beginning before proceeding with the game. They each will throw the 10 dice and sum up all the outcomes of the dice to get the random number. Answer the following questions:

1. How many bits at the minimum (so round up your answer to the nearest integer) are required to encode all distinct numeric outcomes of 10

   Solution:
   The maximum number you can get from summing 10 dice throws is 60, and the minimum is 10. Therefore you have 51 possible combinations, which require $\log_2(51)$ bits to represent. Rounding up, this results in **6 bits**.

2. Someone in the group suggests that they can just use a die and throw it 10 times to get the random number required for the game. This way, they don't have to deal with carrying so many dice. The game began and then he proceeded with throwing the die. His first 3 throws are: 1, 3, and 4. How many bits of information has been given so far? Give your answer in 3 decimal places.

   Solution:
   Each dice throw can result in any number between 1 to 6, which requires $\log_2(6)$ bits to encode. Three dice throws give you $3 * \log_2(6) = 7.755$ bits.

3. After throwing the die 9 times in total, how many new bits of information did he get from making the last (the 10th) throw? Give your answer in 3 decimal places.

   Solution:
   With the same idea as the previous part, the last dice throws solve the mystery on whether we will get number 1, 2, 3, 4, 5, or 6. Hence we are given $\log_2(6) = 2.585$ bits from the last throw.

   Note that this is significantly different from the bits of information that has been given to us so far. The nine dice throws have given us $9 * log2(6)$ bits of information. The last nice throws give us another $\log_2(6)$ bits of information. Please be careful with the wording

4. Finally, he found that the number he got in total from all 10 throws is 53. Express this number in 3-digit hex, formatted as 0xZZZ where Z is your answer.

   Solution:
   `0x035`

5. The second person in the team proceeded to make his 10 throws and found that the number he got is 31. Given that there are five people in the team, how many bits of information are revealed when it is known that the first person's random number is 53 and the second person's random number is 31? Recall that the goal is for everybody in the team to each have a random number before proceeding with the game. Give your answer in 3 decimal places.

   Solution:
   From part (a), we know that we need $\log_2(51)$ bits of information to encode the possible random number of a person. We are given the random numbers of two people, hence we are given $2 * \log_2(51) = 11.345$ bits of information.

8 Another Base Conversion

Consider an 8-bit number systems. Do the following base conversion, and indicate with a `0b` prefix for binary systems and `0x` prefix for hexadecimal systems. Octal and decimal systems do not have prefixes.

1. 76 (decimal) to binary

   Solution:
   `0b01001100`

2. 0b10000001 (binary) to decimal

Solution:
`-127`

3. 0b10011101 (binary) to hexadecimal

Solution:
`0x9D`

4. 0xBF (hexadecimal) to binary

Solution:
`0b10111111`

5. 0xC6 (hexadecimal) to octal

Solution:
`0306`

9 Representing -32 on different number systems

Which of the following signed numbers is equivalent to the number -32 for either an 8-bit or 16-bit system?

a. `0b1010 0000`

b. `0b1110 0000`

c. `0b0001 0000`

d. `0xE0`

e. `0x80E0`

f. `0xFFE0`

g. `0x10E0`

h. `0x800000E0`

i. `0xFFFFFFE0`

Solution:
(b), (d), (f)

10 Challenge – Proof of 2's Complement

At first blush, "Complement and add 1" doesn't seem like an obvious way to negate a two's complement number. By manipulating the expression $A + (-A) = 0$, show that "complement and add 1" does produce correct representation for the negative of a two's complement number.
Hint: express 0 as (-1 + 1) and rearrange the terms to get -A on one side and ZZZ+1 on the other and then think about how the expression ZZZ is related to A using only logical operations (AND, OR, NOT).

Solution:

$$(-A) = -A - 1 + 1 \tag{4}$$
$$= (-1 - A) + 1 \tag{5}$$

In this case, ZZZ is (-1 - A). Let's say we have 8 bit number. -1 is represented by all 1's : `1111 1111`. Let's represent A arbitrarily as `a7 a6 a5 a4 a3 a2 a1 a0`, where a_i can be 0 or 1.

Subtracting -1 with A will flip the bits of A, such that if $a_i = 0$, then $1 - a_i = 1$, and if $a_i = 1$ then $1 - a_i = 0$ (see how binary subtraction 'borrow' method works at `https://www.wikihow.com/Subtract-Binary-Numbers` if you dont know how it works). Hence, we can rewrite the above into,

$$(-A) = \textit{bitwise} \text{ NOT } A + 1 \tag{6}$$