Solving the Shallow Water Equations

CFD Final Project

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Noah Brenowitz

Outline

- Finite Volume Methods for Conservation Laws
  - 1st Order
  - High Resolution
- Shallow Water Equations
  - With Rotation
- Results

Conservation Laws

$$q_t + \nabla \cdot f(q) = 0$$

$$q_t + A(u) \cdot q_x + B(u) \cdot q_y = 0$$

- Called hyperbolic if A and B are diagonalizable with real eigenvalues
- Examples include (Burger's Equation, Traffic Flow, Shallow water, Gas Dynamics, etc.)

Finite Volume Methods for Conservation laws

- Nodes represent averages over a cell
- Solve a Riemann problem on each interface
- Upwinding is by characteristic decomposition
- Second order accuracy achieved using fully discrete (Lax-Wendroff) scheme
  - Regularity is enforced using flux-limiters
  - This is Leveque's approach
- Higher order accuracy can be obtained using WENO
  - Very costly for higher dimensions. Easier to use finite differencing

First Order Scheme

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}\left[f_{i+1/2} - f_{i-1/2}\right] \\
&= Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+_{i-1/2}\Delta Q_{i-1/2} + A^-_{1+1/2}\Delta Q_{i+1/2}\right]
\end{aligned}
$$

- Diagonalize A

$$A = LSR$$
$$A^{\pm} = LS^{\pm}R$$

- Where

$$x^+ = max(0, x)$$
$$x^- = min(0, x)$$

How to find $A_{i-1/2}$

- For linear problems this is easy
- For nonlinear problems, we need to solve a Riemann problem on each interface
  - Impossible analytically and/or computationally costly
- Solution: Use an *approximate* Riemann solver

Roe Averaging

Need to find a diagonalizable matrix with

$$\hat{A}(Q_\ell, Q_r) \to A(q)$$

- Simple averaging does not guarantee that A is diagonalizeable
- Roe averaging assumption:

Q_i and Q_{i+1} are connected by one characteristic.

$$f(Q_{i+1}) - f(Q_i) = s(Q_{i+1} - Q_i)$$

Roe Averaging Continued

- This implies that

$$f(Q_{i+1}) - f(Qi) = \hat{A}(Q_{i+1} - Q_i)$$

- Roe devised some tricks to help find such an A_hat.
- Diagonalize A_hat and form the right going and left going fluctuations at each interface
- This can violate entropy condition!
  – Use “entropy” fix

Second Order (1D)

- Use fully discrete Lax-Wendroff style scheme with flux limiter

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[A^+_{i-1/2}\Delta Q_{i-1/2} + A^-_{1+1/2}\Delta Q_{i+1/2}\right]$$
$$- \frac{\Delta t}{\Delta x}\left[\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right]$$

- The higher order correction terms are flux limited and depend only on the solution to the Riemann problem at the two neighboring interfaces

2D Higher Order Scheme

- Similar flux correction terms
- Tricky part is to upwind properly.
  - Characteristics can be diagonal!
- The solution is to split the left and right fluctuations into left-up/left-down and right-up/right-down fluctuations.
- "Right-up" from interface (i-1/2,j) is (i,j+1/2)
  - Draw picture

Shallow Water Equations

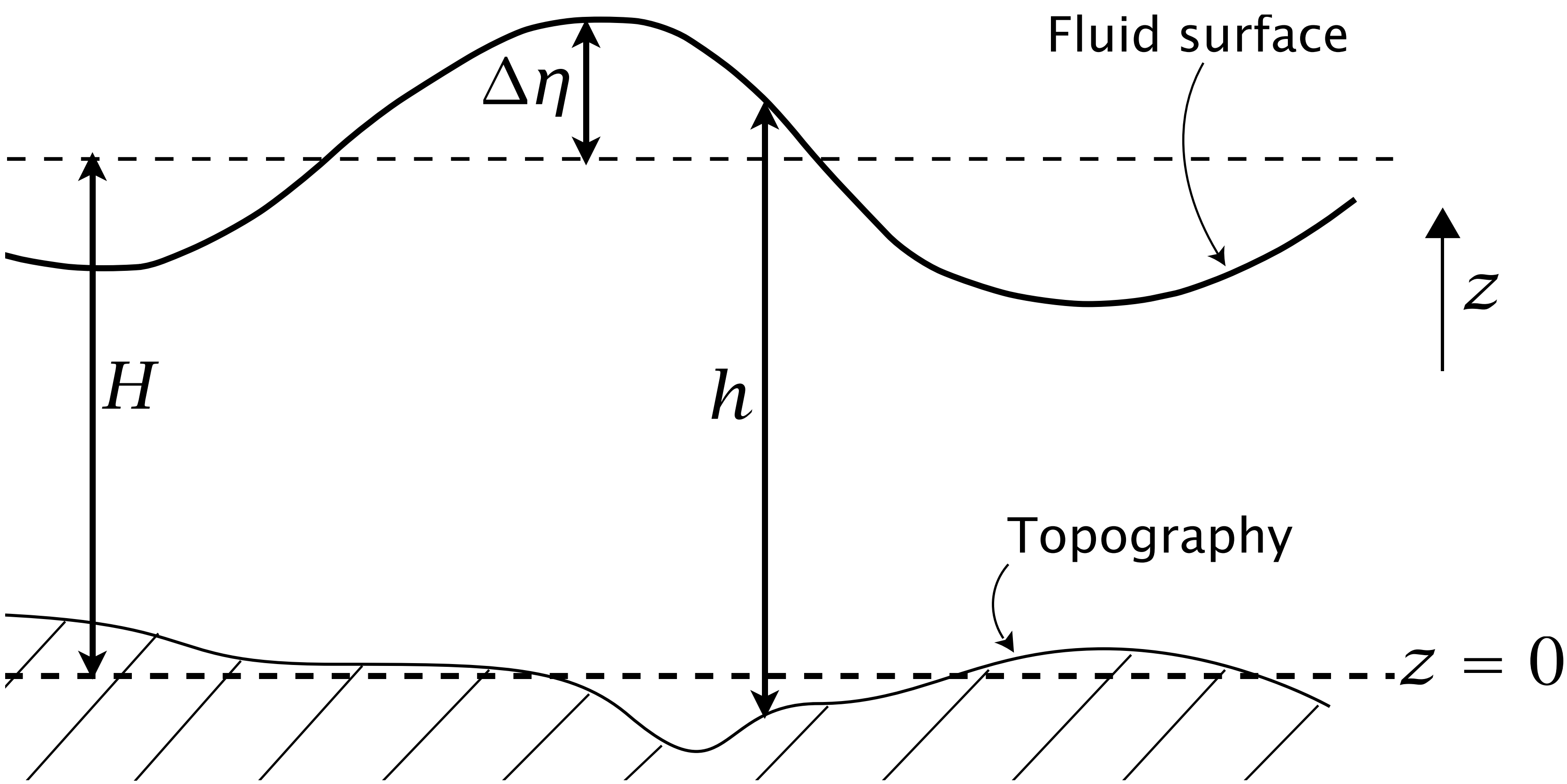


Fig 3.1 in Vallis

Shallow Water Equations

Without Rotation

$$\frac{Du}{Dt} = -g\nabla h$$

$$\frac{Dh}{Dt} + h\nabla \cdot u = 0$$

With Rotation

$$\frac{Du}{Dt} + f \times u = -g\nabla h$$

$$\frac{Dh}{Dt} + h\nabla \cdot u = 0$$

Derivation (on board)

SWE Conservation Form

Without Rotation

$$\begin{pmatrix} h \\ hu \\ hv \end{pmatrix}_t + \begin{pmatrix} hu \\ \frac{1}{2}gh^2 + hu^2 \\ huv \end{pmatrix}_x + \begin{pmatrix} hv \\ huv \\ \frac{1}{2}gh^2 + hv^2 \end{pmatrix}_y = 0$$

With Rotation

$$\begin{pmatrix} h \\ hu \\ hv \end{pmatrix}_t + \begin{pmatrix} hu \\ \frac{1}{2}gh^2 + hu^2 \\ huv \end{pmatrix}_x + \begin{pmatrix} hv \\ huv \\ \frac{1}{2}gh^2 + hv^2 \end{pmatrix}_y = f \begin{pmatrix} 0 \\ hv \\ -hu \end{pmatrix}$$

Roe Averaging for SWE

$$\hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}} \qquad \bar{h} = \frac{h_l + h_r}{2}$$

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0 & 1 & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \\ -\hat{v}^2 + g\bar{h} & 0 & 2\hat{v} \end{pmatrix}$$

Roe Averaging for SWE

- Eigenvalues and Eigenvectors of the above are easy to calculate
- A and B are very similar
- Easy to implement and cheap computationally

What about this $f \times u$ term?

- Just use a Godunov splitting approach

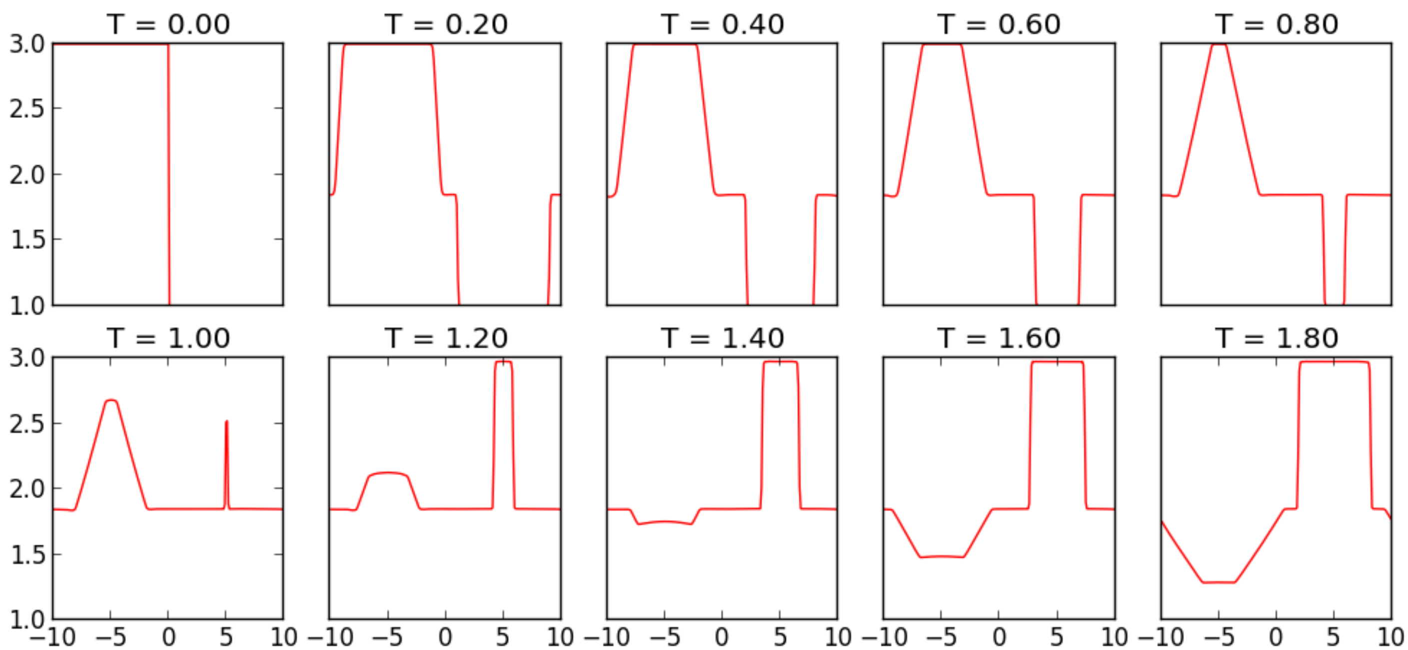
$$q^* = q^n + \Delta t \cdot (\text{flux terms})$$
$$q^{n+1} = q^* + \Delta t \cdot (-hf \times u)$$

- Or Strang Splitting (2nd order in time)
  - Advance homogenous problem dt/2
  - Advance source problem dt
  - Advance homogenous problem dt/2

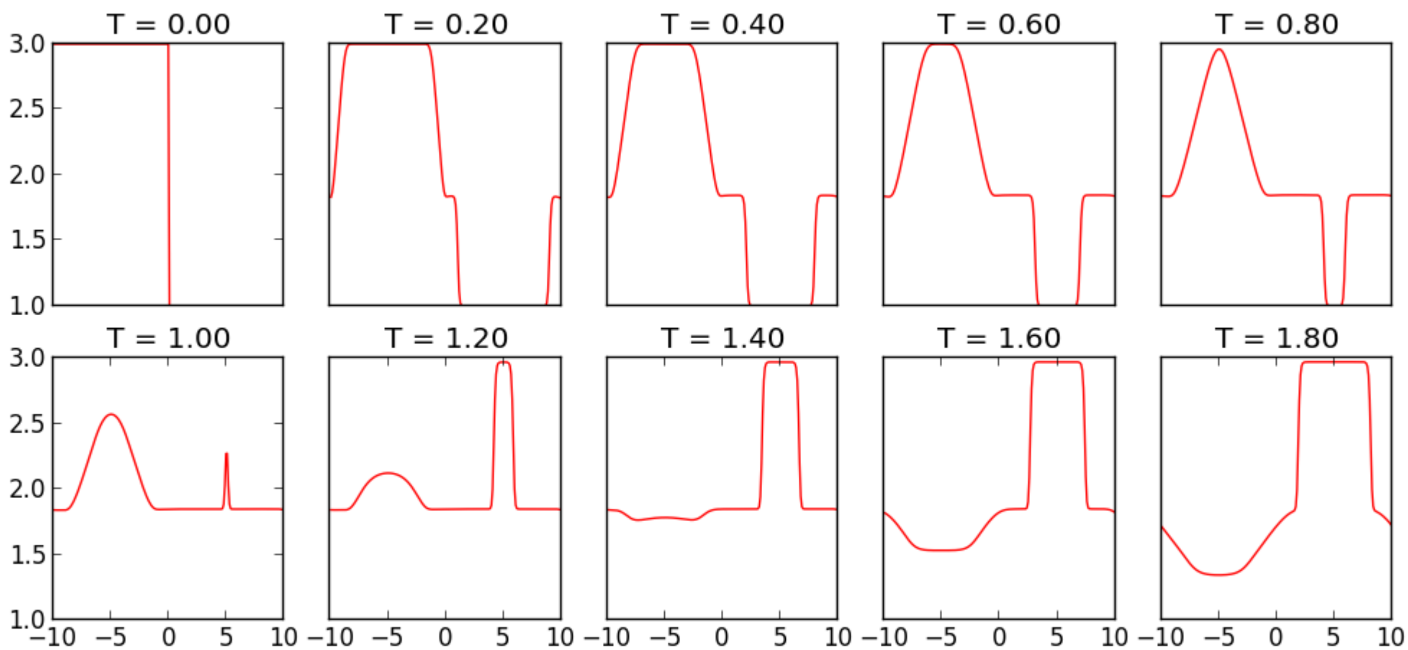
Results

- 1D Dam Break
- 2D Radial Dam Break
- 2D Smooth Radial Dam Break

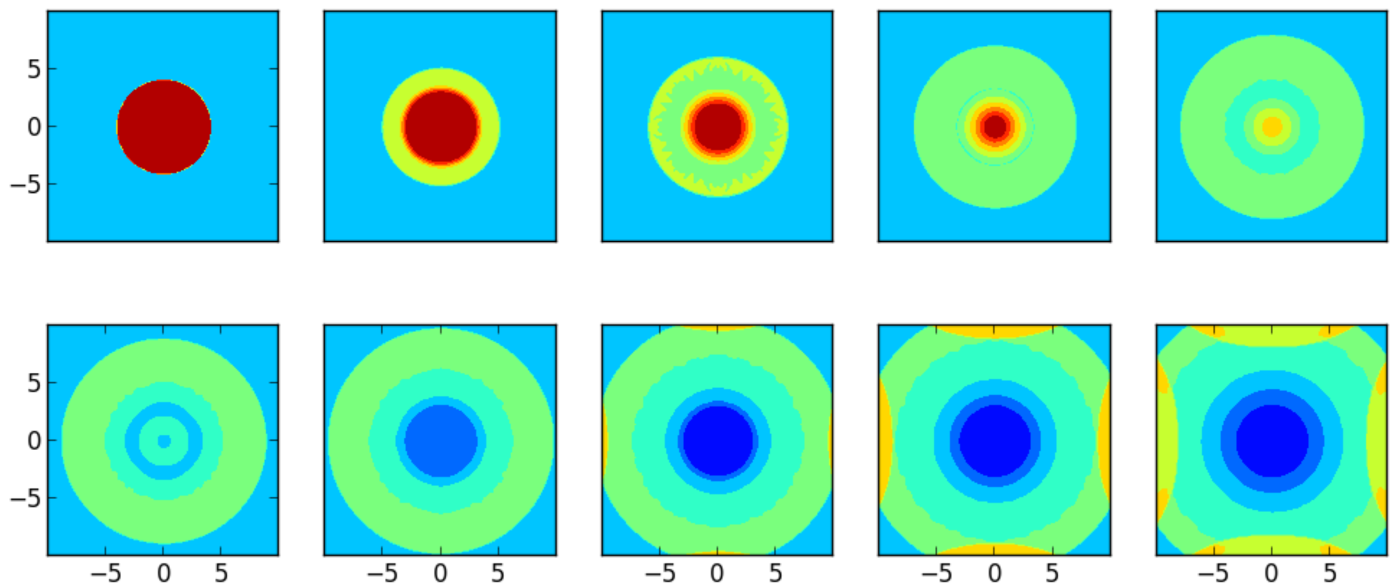
Dam Break Problem in 1D



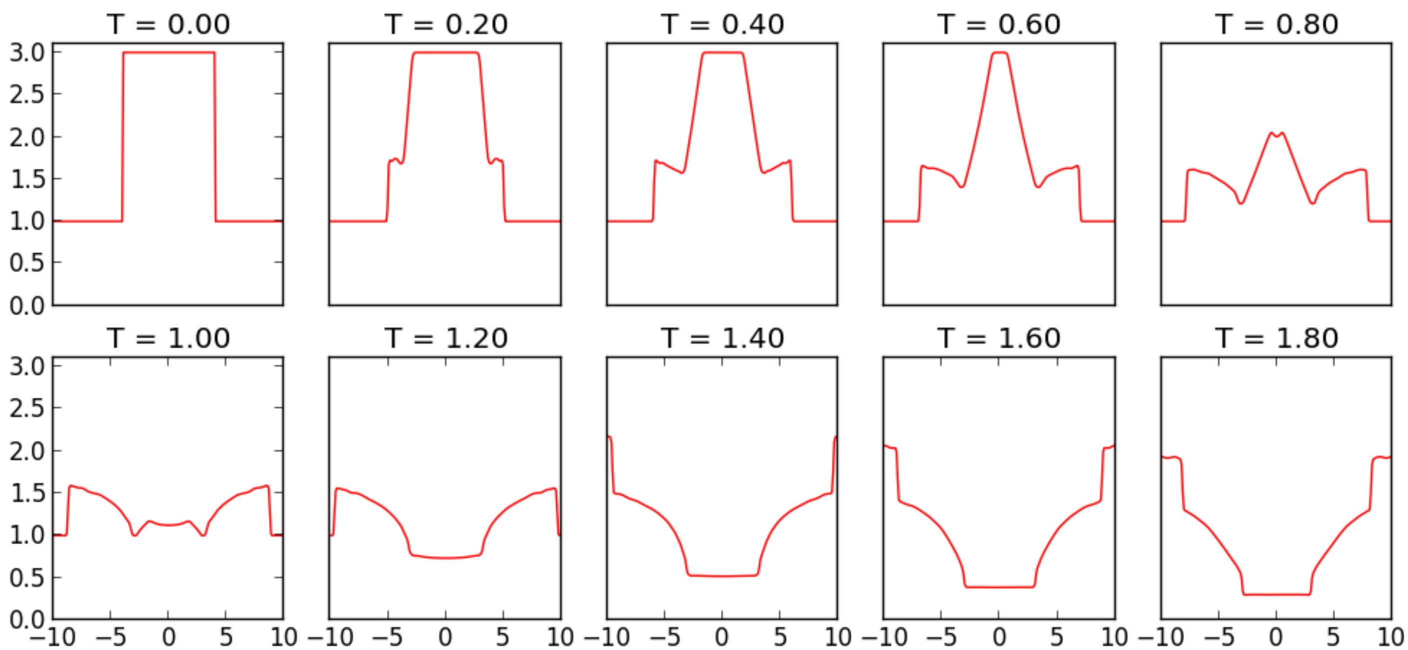
Radial Dam Break Problem



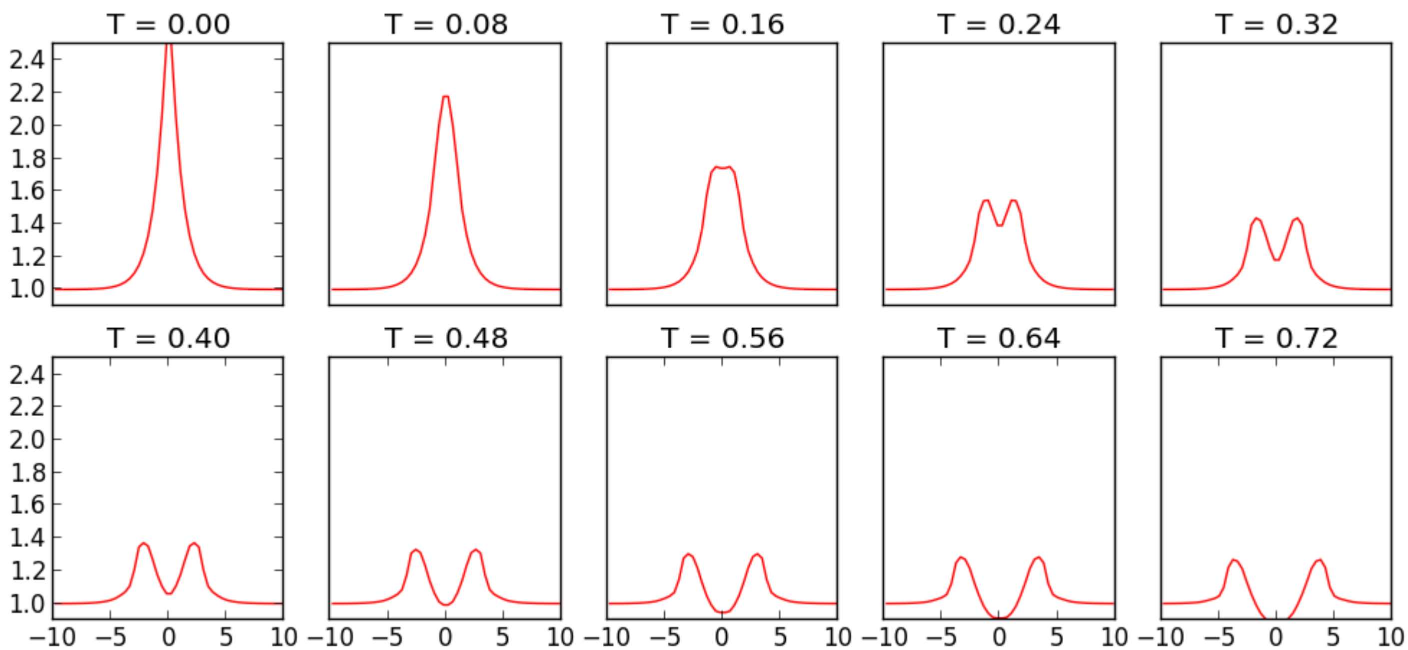
Radial Dam Break Problem



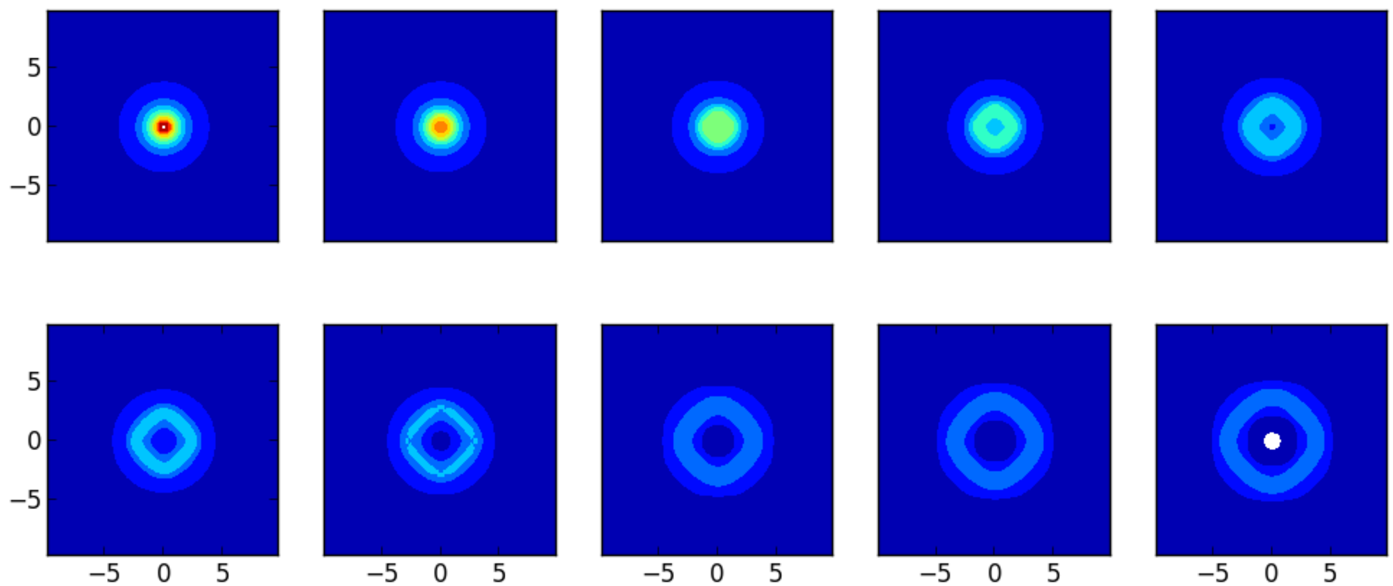
Radial Dam Break Problem



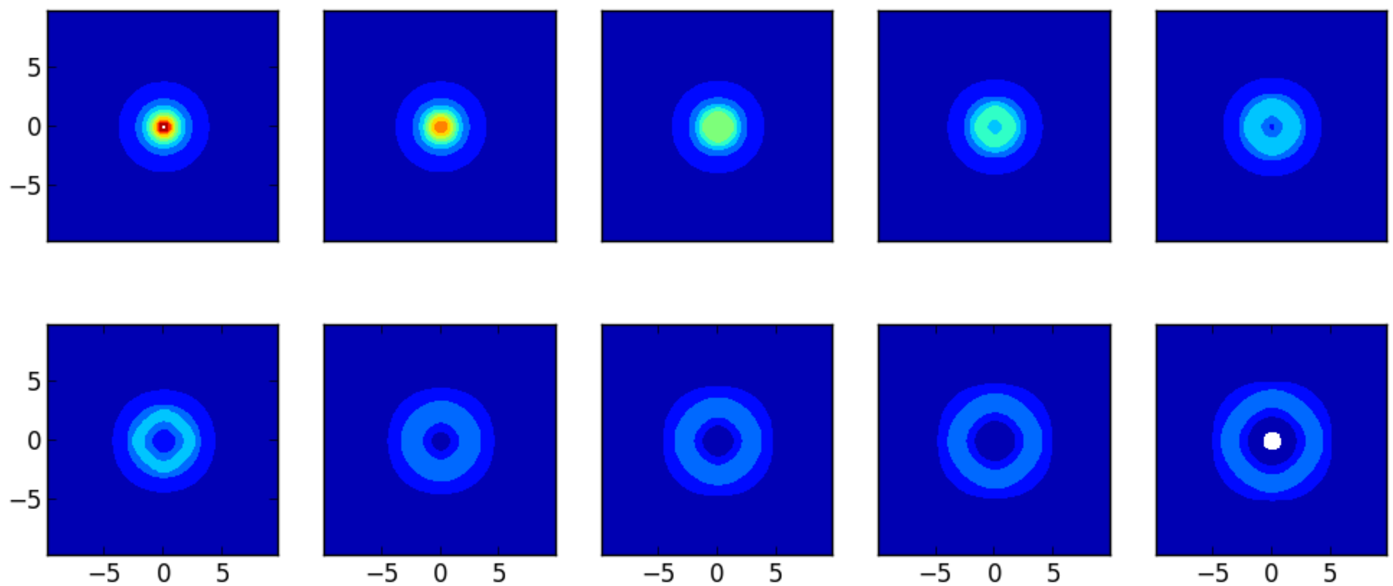
Corner Fixing Helps



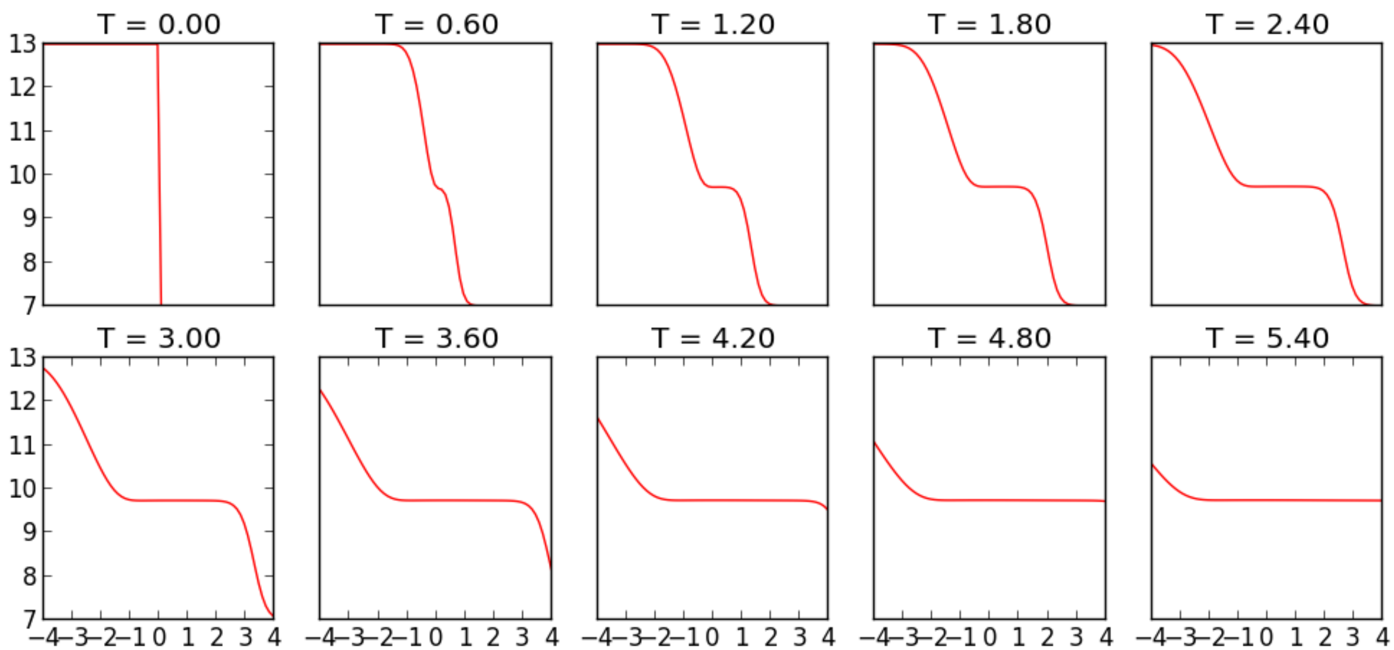
Corner Fixing Helps



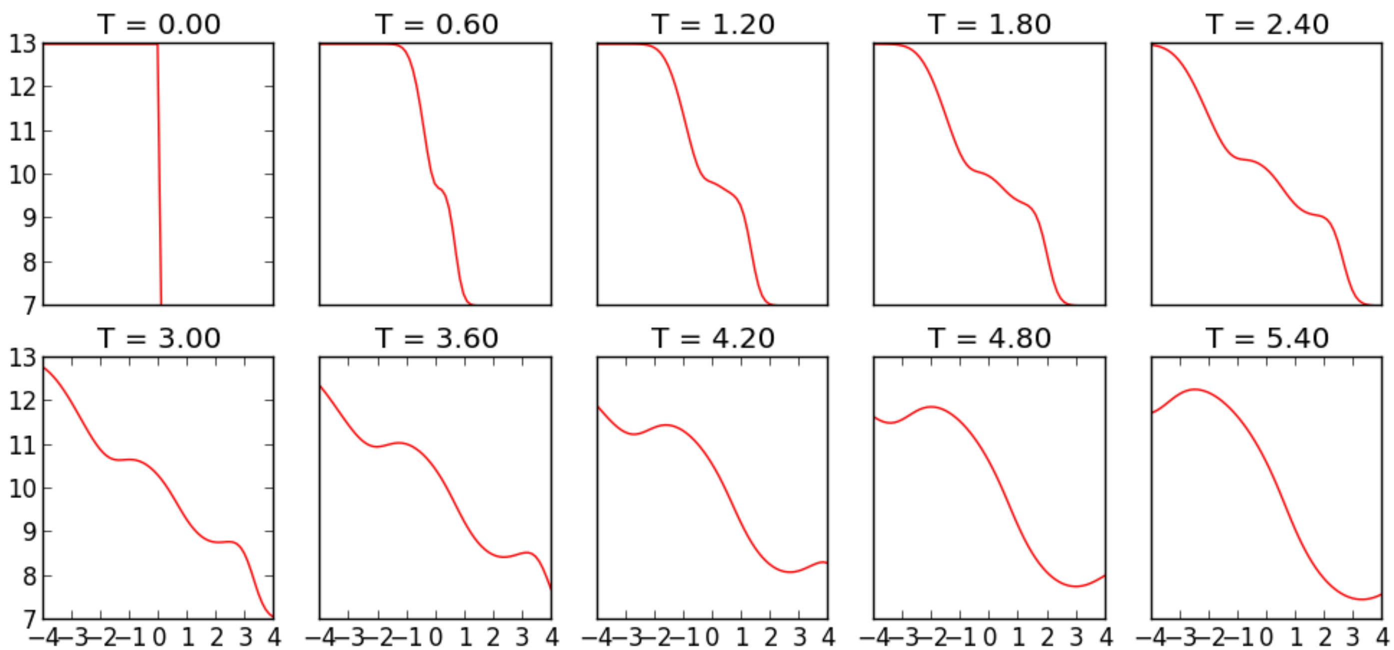
Corner Fixing Helps



Geostrophic Adjustment: Dam break with f=.01

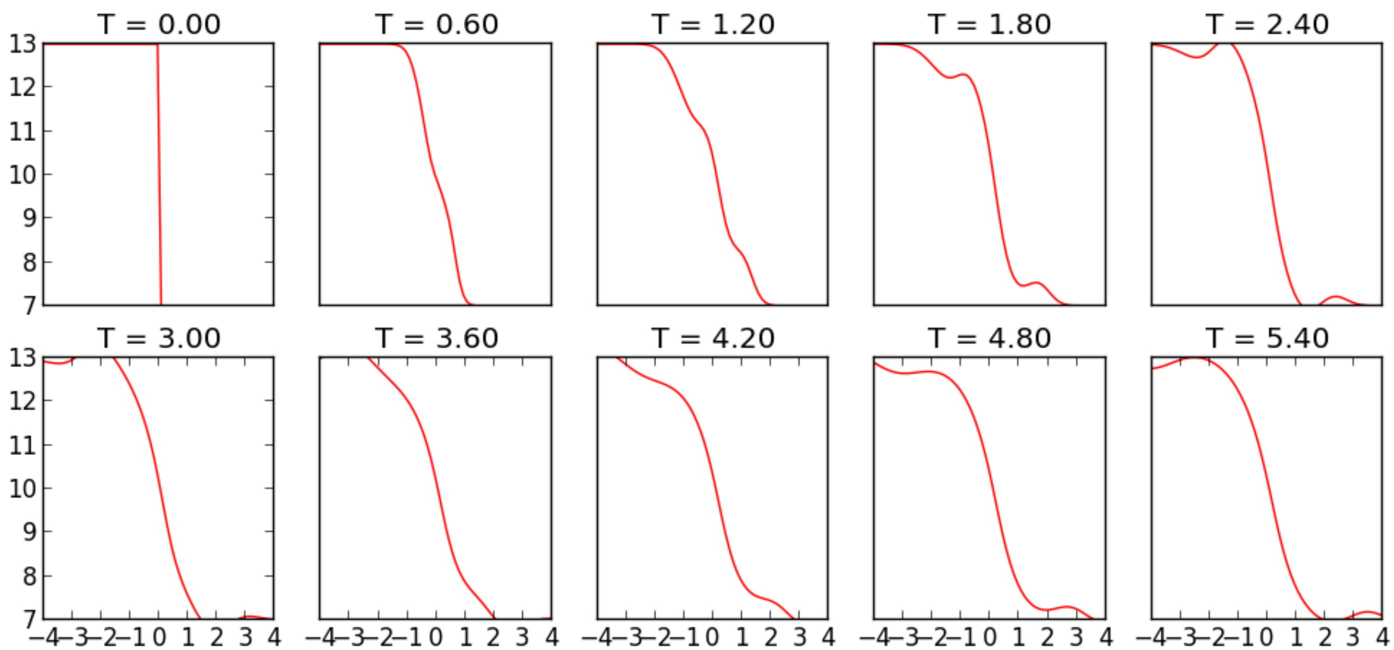


F=.3



T = 0.00
T = 0.60
T = 1.20
T = 1.80
T = 2.40
T = 3.00
T = 3.60
T = 4.20
T = 4.80
T = 5.40
13
12
11
10
9
8
7
−4 −3 −2 −1 0 1 2 3 4

F=1



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13
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11
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9
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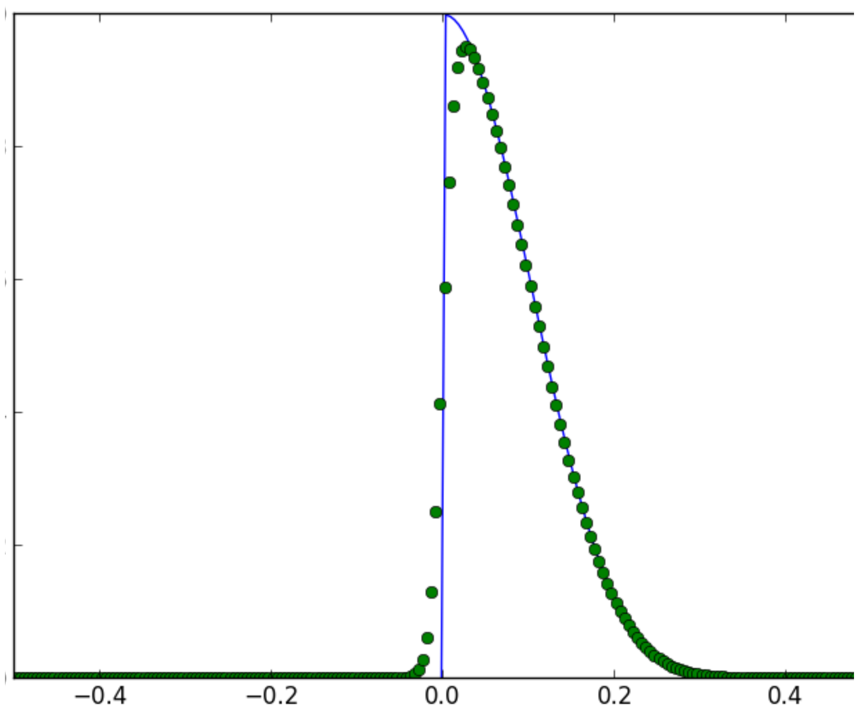
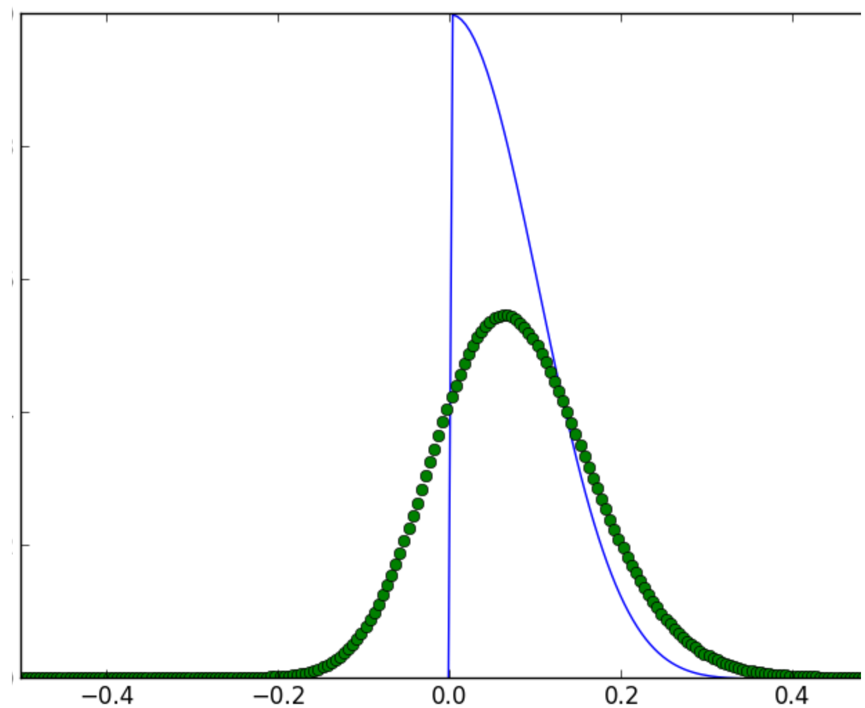
7
−4−3−2−1 0 1 2 3 4

WENO on uniform advection with n=100

WENO r=3 (order 5)



1st order upwind



References

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Repository

- https://github.com/nbren12/cfd-final