Choice with risk

Chapter 3

Most of our choices are choices under uncertainty:

- A person buys groceries without knowing for sure how tasty they will be.

How people make choices under uncertainty? We need a theory.

What is uncertainty and what is risk?

We define **uncertainty** as a *situation where one does not know the consequences of their choice*.

One might not know:

- the possible outcomes

Or

- they might know the outcomes but they do not know the likelihood that one of these outcomes realizes (i.e. probabilities associated with each outcome).

A situation of risk

Choice under risk is a subfield of choice under uncertainty.

In a situation of risk the outcome is unknown, but the individual does know

- What outcomes are possible
- The probability of each outcome.

A **prospect** is a list of probabilities and outcomes

$$(p_1, x_1; p_2, x_2; \ldots; p_n, x_n).$$

where p_i is the probability of getting monetary payoff x_i .

Example: Toss a coin; there are 2 possible outcomes 'tails'and 'heads' with the same probability. Suppose that if it is 'heads' you win 100, while if it is 'tails' you win 0\$. The prospect offered to you is then $(p_1 = 0.5, x_1 = 100; p_2 = 0.5, x_2 = 0)$

Expected Utility Theory

Let u(*x*) be the standard utility function that transforms money x into utility u(x)

The expected utility of prospect $A = (p_1, x_1; p_2, x_2; \ldots; p_n, x_n)$ is given by

$$EU(A) = \sum_{i=1}^{n} p_i u(x_i) = p_1 u(x_1) + p_2 u(x_2) + \cdots + p_n u(x_n)$$

Expected value vs Expected utility

People confer a subjective value (utility) to objective monetary outcome.

In the expected utility theory there is the utility function u(x) that transforms objective monetary outcomes into subjective values.

It follows that the prospect will have a subjective value for the consumer: the expected utility that it gives to the consumer.

$$EU(A) = \sum_{i=1}^{n} p_i u(x_i) = p_1 u(x_1) + p_2 u(x_2) + \cdots + p_n u(x_n)$$

In general, **a consumer will choose the prospect that has the highest expected utility.**

The shape of the utility function *u(x)* is able to capture the different attitudes that the subjects can have toward the **variability of the returns**, that is their **attitude toward risk**.

The Expected Value vs Expected Utility of a Prospect

The expected value (EV) is a weighted averege of the outcomes of a prospect $x_1, x_2, \dots, x_n$, where these are indeed weighted by the associated probabilities, $p_1, p_2, \dots, p_n$:

$$EV = \sum_{i=1}^{n} x_i * p_i = x_1 * p_1 + x_2 * p_2 + \cdots + x_n * p_n$$

This tells us how much one can earn on average from playing this prospect.

The only difference between equations Expected Value and Expected utility is that we do not transform monetary payoffs into utility payoffs. This means we obtain expected monetary payoff rather than expected utility payoff.

<u>Why does this matter? Because the utility function u(x) captures attitudes toward risk.</u>

Consider the following example:
Mr Rossi wants to invest 10000 euros. He has to choose between Government Bonds (G) and private equities (E). The government bonds give a sure return, while the return on equities depends on the market trend: by the end of the year Mr Rossi will surely have 11000 euros if he invests in Government Bonds ($x_1 = 11000, p_1 = 1$), while he might end up with 16000 euros or 6000 euros with the same probability if he invests in equities ($x_1 = 16000, p_1 = 0.5;\ x_2 = 6000,\ p_2 = 0.5$).

	Expansion	Recession
Government Bonds	11000	11000
Private Equities	16000	6000

$$EV^E = \sum_{i=1}^{2} x_i * p_i = x_1 * p_1 + x_2 * p_2 = 16000 * 0.5 + 6000 * 0.5 = 11000$$

$$EV^G = \sum_{i=1}^{1} x_i * p_i = x_1 * p_1 = 11000 * 1 = 11000$$

The expected value is the same in the two prospects. But it is unlikely that Mr Rossi would be indifferent between the two. Indeed, the crucial point is that the expected value does not consider the variability of the returns, i.e., the risk.

But if I say that Mr Rossi has utility function: $u(x) = \sqrt{x}$

$$EU(G) = \sum_{i=1}^{N} p_i u(x_i) = p_1\sqrt{x} + p_2\sqrt{x} = 1 * \sqrt{11000} = 104.88$$

$$EU(E) = \sum_{i=1}^{N} p_i u(x_i) = p_1\sqrt{x} + p_2\sqrt{x} = 0.5 * \sqrt{16000} + 0.5 * \sqrt{6000} = 63.24 + 38.72 = 101.96$$

$EU^G > EU^E$. Indeed, in this case the consumer is **risk averse**

But if I say that Mr Rossi has utility function: $\boldsymbol{u(x) = x^2}$

$$EU(G) = \sum_{i=1}^{N} p_i u(x_i) = p_1 x^2 + p_2 x^2 = 1 * 11000^2 = 121M$$

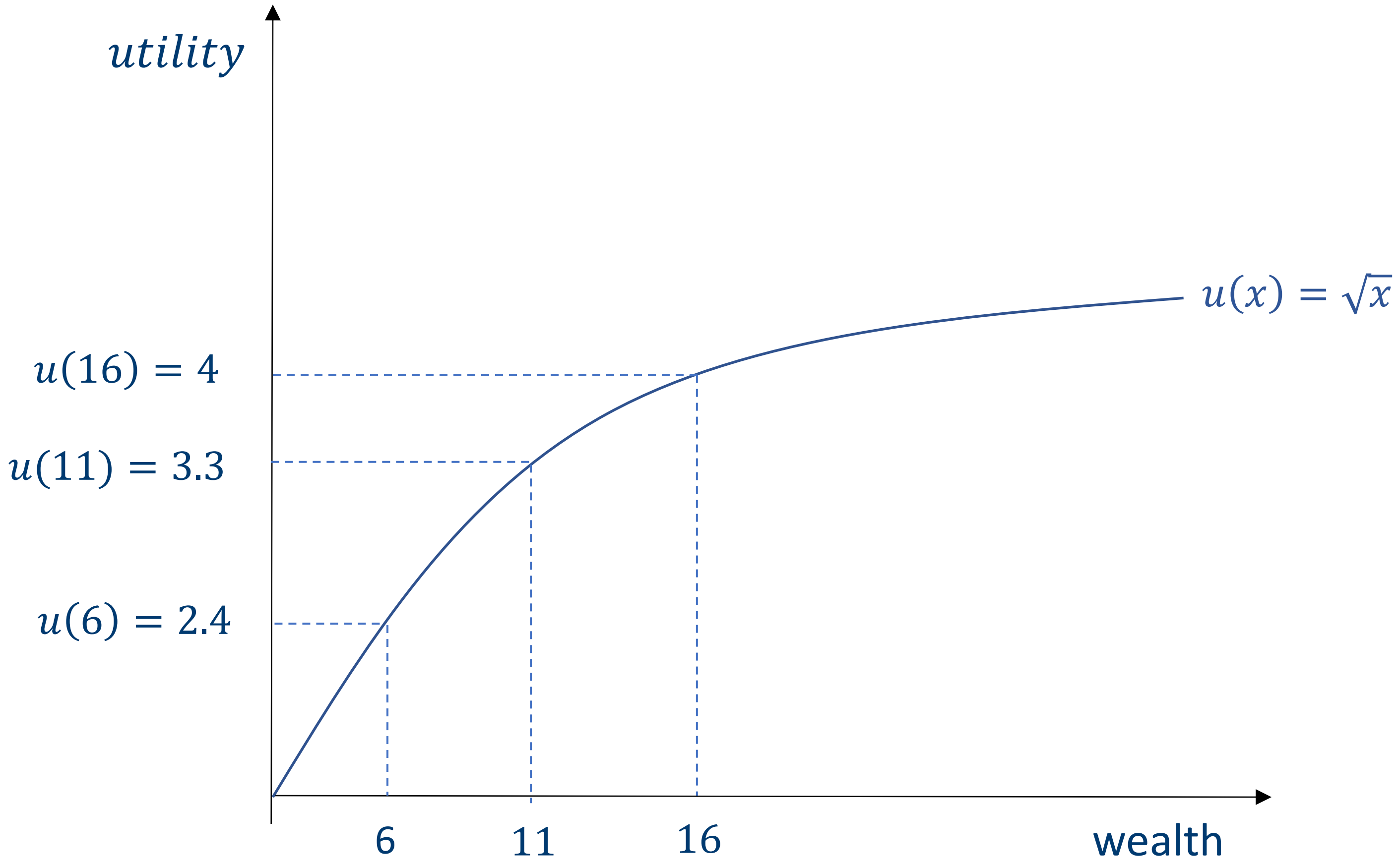
$$EU(E) = \sum_{i=1}^{N} p_i u(x_i) = p_1 x^2 + p_2 x^2 = 0.5 * 16000^2 + 0.5 * 6000^2 = \\ = 128M + 18M = 146M$$

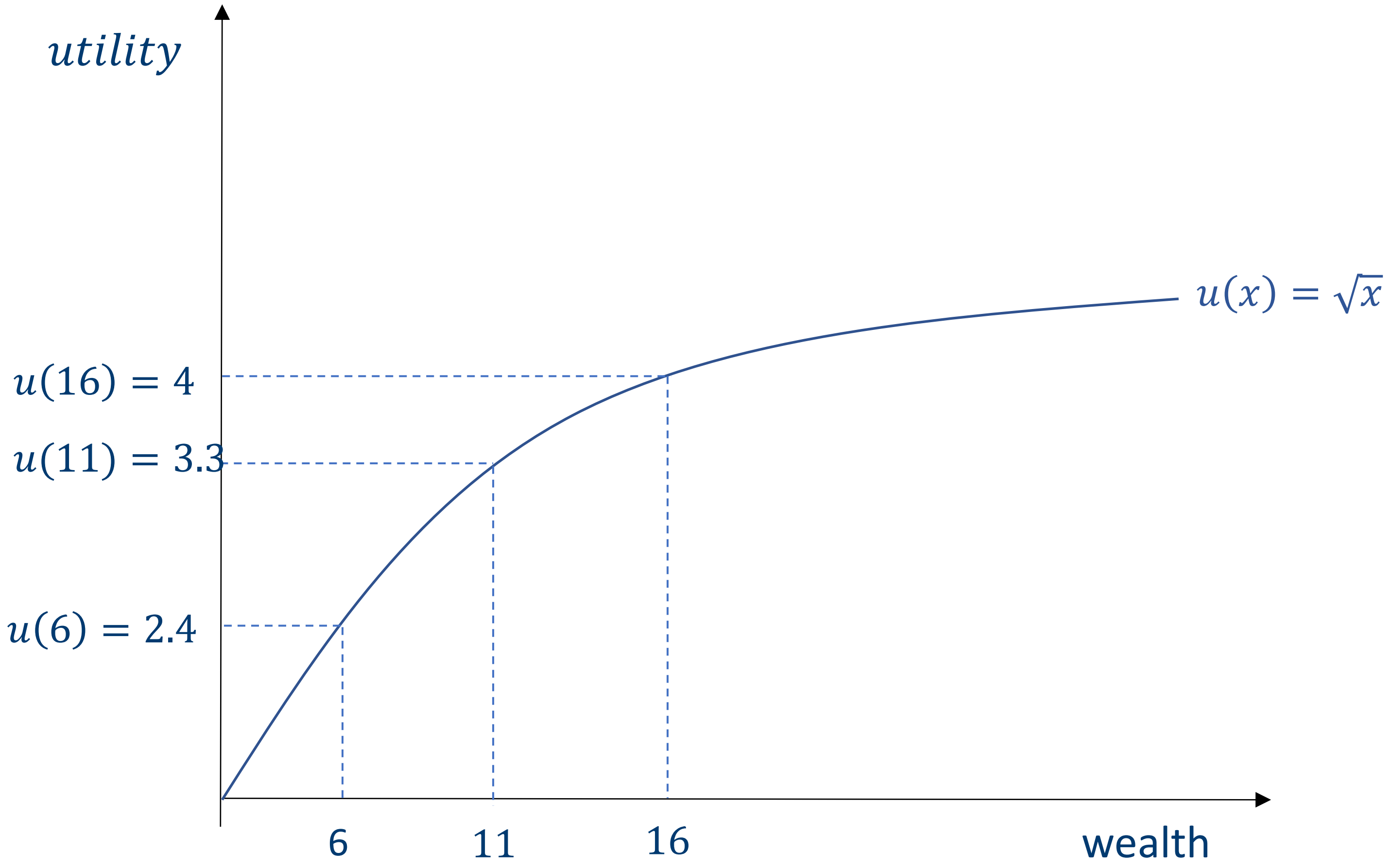
$EU^E > EU^G$. Indeed, in this case the consumer is **risk-seeking**.

Risk aversion

Concave utility function as for example $u(x) = \sqrt{x}$ is used to represent the risk attitudes of a risk-averse consumer.

Concave u(x) implies diminishing marginal utility: additional units of wealth will increase the utility at a decreasing rate.





If this utility function is concave, Mr Rossi loses relatively more if his wealth goes down than he gains if his wealth goes up. Indeed, the increase in utility if the high outcome (4-3.3=0.7) is lower than the decrease in utility (3.3-2.4=0.9) if there is low outcome.

So, he would rather not risk a loss for a gain, this is why we define him as risk averse.

In general, the prospect with the highest expected utility will be chosen.

In this case ($u(x) = \sqrt{x}$ the highest expected utility is given by the Government Bonds prospect:

$$EU^E = p_1^E\sqrt{x_1^E} + p_2^E\sqrt{x_2^E} = 0.50 * \sqrt{16} + 0.50 * \sqrt{6} = 3.22$$

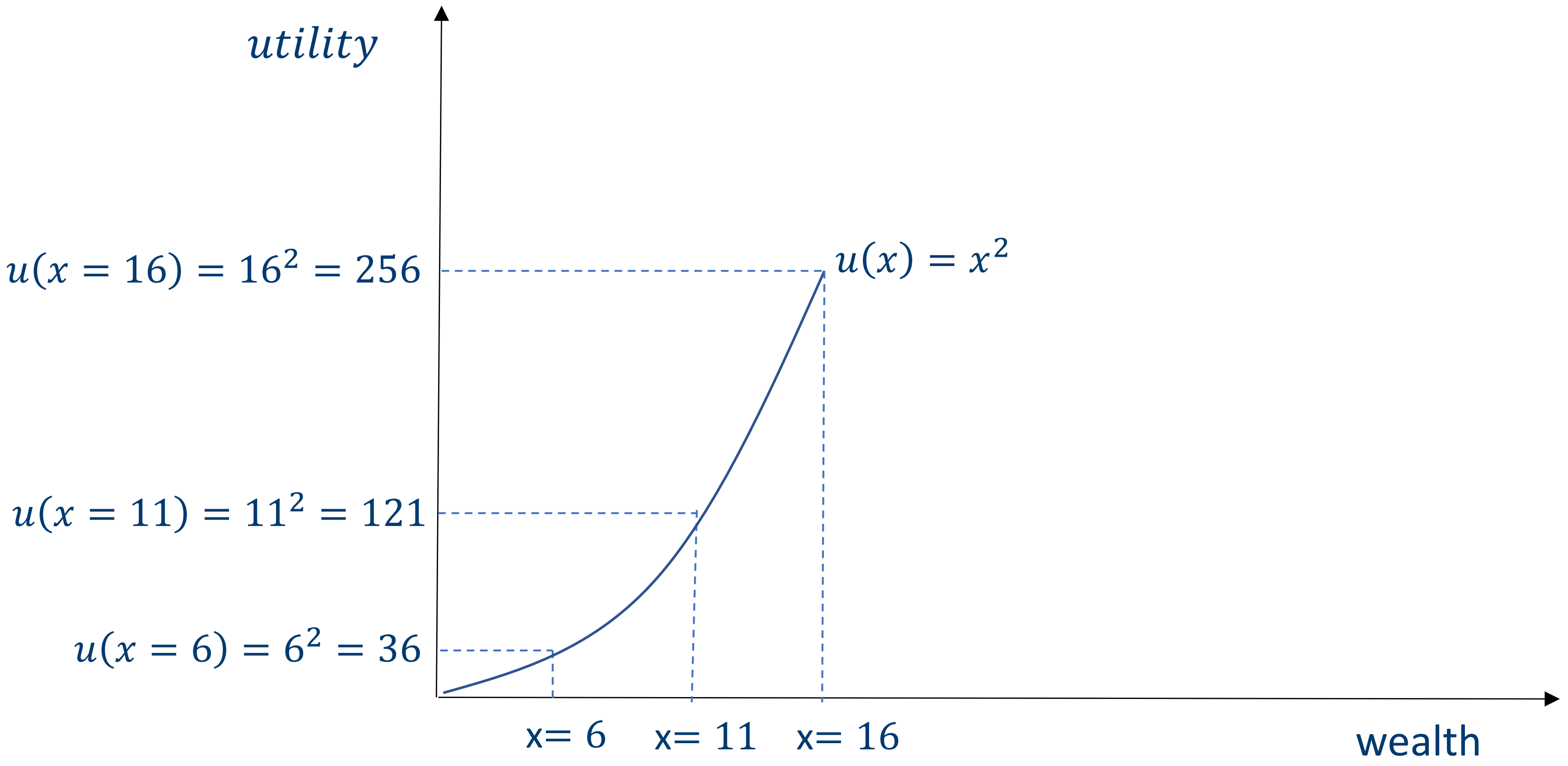
$$EU^G = p_1^G\sqrt{x_1^G} = 1 * \sqrt{11} = 3.31$$

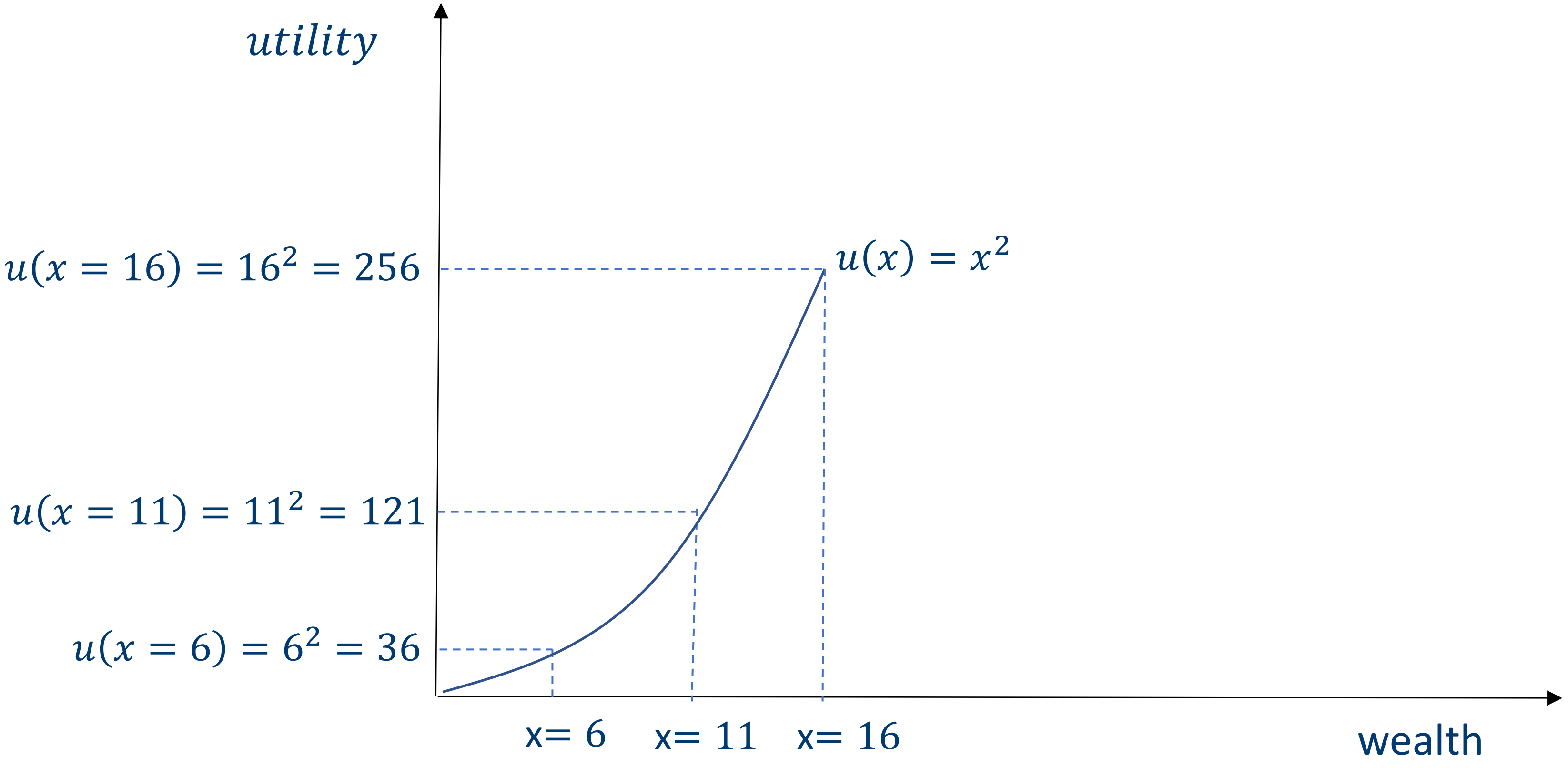
$EU^G > EU^E$. Indeed, in this case the consumer is risk averse.

Risk-seeking

Convex utility function as for example $u(x) = x^2$ is used to represent the risk attitudes of a risk-seeking consumer.

The marginal utility is increasing in this case: additional units of wealth will increase the utility at an increasing rate.





If his utility function is convex, Mr Rossi loses relatively less if his wealth goes down than he gains if his wealth goes up. Indeed, the increase in utility if the high outcome (256-121=135) is higher than the decrease in utility (121-36=85) if there is low outcome.
He would risk a loss for a gain , this is why we define him as risk loving.

In general, the prospect with the highest expected utility will be chosen.

In this case ($u(x) = x^2$) the highest expected utility is given by the private equities prospect:

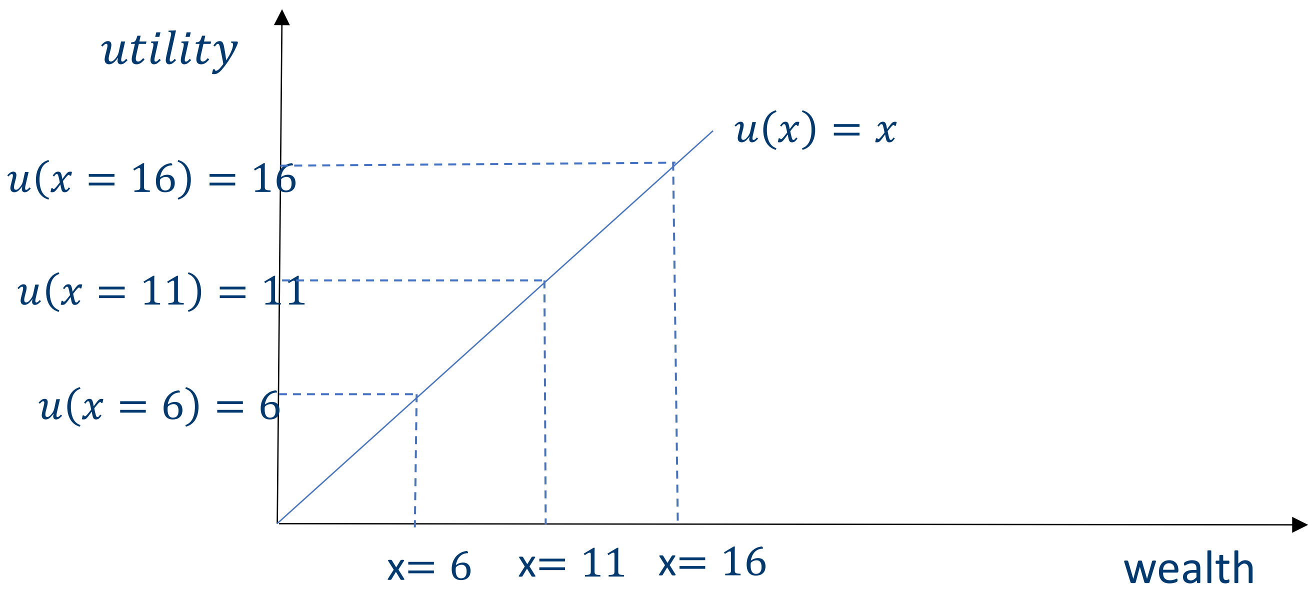
$$EU^E = p_1^P(x_1^P)^2 + p_2^P(x_2^P)^2 = 0.50 * (16)^2 + 0.50 * (6)^2 = 146$$

$$EU^G = p_1^S(x_1^S)^2 + p_2^S(x_2^S)^2 = 0.50 * (11)^2 + 0.50 * (11)^2 = 121$$

$EU^E > EU^G$. Indeed, in this case the consumer is risk-seeking.

Risk Neutrality

If the utility function is linear $u(x) = \mathrm{x}$ the consumer is risk-neutral.
The marginal utility is constant: additional units of wealth will increase the utility at a constant rate.

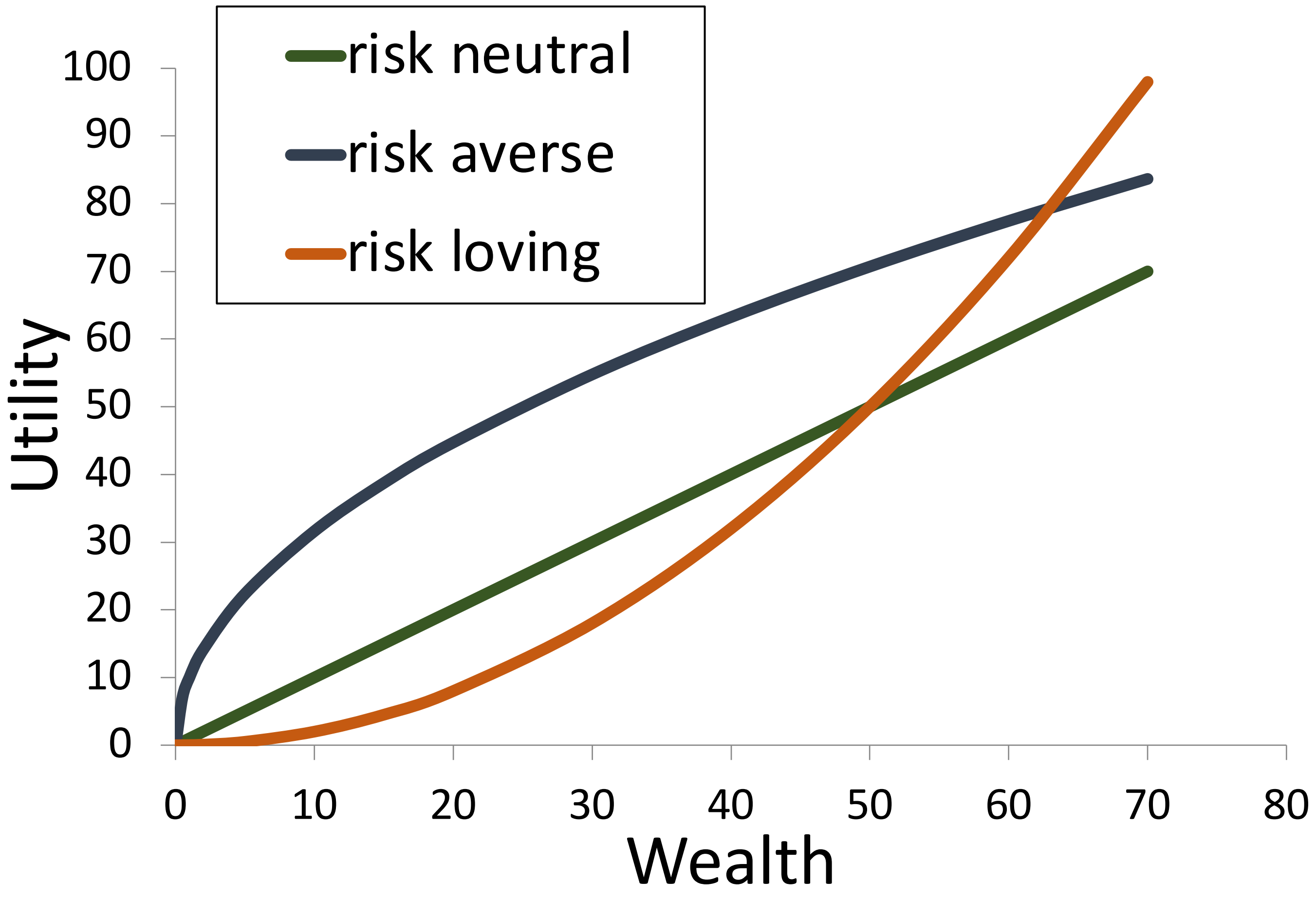


The two prospects have the same expected utility since they have the same expected value.

$$EU^G = EU^E = 11 = EV$$

Mr Rossi will be indifferent between the two prospects.

Attitudes to risk



How to measure risk attitudes?

We can estimate risk attitudes observing choices over prospects:

- **In the lab:** the experimental laboratory is one setting that has been used. In this case we look at prospects with relatively small amounts of money. **Problem:** stakes in these experiments are typically not larger than one month's income and thus do not provide evidence about risk attitudes regarding prospects that are significant in relation to lifetime wealth.
- **In the field:** empirical settings where we can measure risk aversion include asset and insurance markets. In this case we can observe choices over prospects with both relatively small and large amounts of money. **Problem:** researchers cannot directly observe risk preferences for most real-life problems, because the true probability distribution is not known to the subjects and the subjects' beliefs are not known to the researcher.

In «Deal or No Deal» Tv show game analysed by Post et al. (2008) the amounts at stake are larger than in experiments and the decision problems are often simpler and better defined than in real life.

Critique of expected utility

Expected utility cannot explain many things. For example:

- Allais Paradox.
- Risk aversion over small gambles.
- The reflection effect.

Which would you choose?

Prospect A

€100M with probability 100%

Prospect B

€500M with probability 10%
€100M with probability 89%
€0 with probability 1%

Prospect C

€100M with probability 11%
€0 with probability 89%

Prospect D

€500M with probability 10%
€0 with probability 90%

Allais paradox

It describes the empirically demonstrated fact that individuals' decisions can be inconsistent with expected utility theory.

Changing a sure thing to a risk matters more than changing a risk to a risk. Generalizing from this, the evidence suggests that people favor outcomes that are perceived certain rather than probable or possible. This is called the **certainty effect** .

Preferences do not satisfy independence

Which would you choose?

Prospect A

$2,500 with probability 0.33,
$2,400 with probability 0.66,
$0 with probability 0.01

Prospect B

$2,400 for sure

Prospect C

$2,500 with probability 0.33,
$0 with probability 0.67.

Prospect D

$2,400 with probability 0.34,
$0 with probability 0.66.

Allais paradox

Prospect C

\$2,500 with probability 0.33,
\$0 with probability 0.67.

Prospect D

\$2,400 with probability 0.34,
\$0 with probability 0.66.

Prospect A

\$2,500 with probability 0.33,
\$2,400 with probability 0.66,
\$0 with probability 0.01

Prospect B

\$2,400 for sure

To see why this is a paradox, first observe that

- prospect C is obtained from prospect A,
- and prospect D is obtained from prospect B,

by removing a 0.66 chance of winning \$2,400

If a consumer chooses B over A

$$EU(B) = U(2400) > \ EU(A) = 0.33u(2500) + 0.66u(2400)$$

Which removing a 0.66 chance of winning 2400 means that

$$0.34u(2400) > 0.33u(2500)$$

If, however, the same consumer chooses C over D:

$$EU(C) = 0.33u(2500) > EU(D) = 0.34u(2400)$$

these two statements cannot be both true.

Removing the 0.66 chance of winning $2,400 seems to matter more for prospect B than A because **changing a sure thing to a risk matters more than changing a risk to a risk.**

Generalizing from this, the evidence suggests that people favor outcomes that are perceived certain rather than probable or possible. This is called the **certainty effect** .

Reflection effect

Prospect O

$4,000 with probability 0.8, $0 otherwise.

Prospect P

$3,000 for sure.

Prospect Q

Lose $4,000 probability 0.8, $0 otherwise.

Prospect R

Lose $3,000 for sure.

This suggests that people are **risk-averse for gains**, preferring the $3,000 for sure, but **risk-loving for losses**, preferring to gamble on losing $0. This is called the **reflection effect** .

Can Expected utility explain Reflection Effect?

According to Expected Utility Theory, utility should be measured according to the absolute level of wealth. So, it is necessary that when asked to choose between these options, a person's current wealth just happened to be at a point where the utility function is concave for higher levels of wealth, and convex for lower levels of wealth. This is possible but seems a bit too much of a coincidence.

More plausible, particularly given what we saw about reference dependence, is that choices are made, in part, by measuring relative deviations from current wealth, and subjects are risk-averse in the gains, while they are risk-loving in the losses.

- The first thing we need for expected utility to work well is transitivity. **Preferences are transitive** if for any three prospects X, Y and Z,

  if $X \geq Y$ and $Y \geq Z$ then $X \geq Z$

- The second thing we need is independence. **Preferences satisfy independence** if for any three prospects X, Y and Z,

  if $X \geq Y$ then $(p, X; 1 - p, Z) \geq (p, Y; 1 - p, Z)$

Figure 3.3: A probability triangle diagram of the Allais Paradox

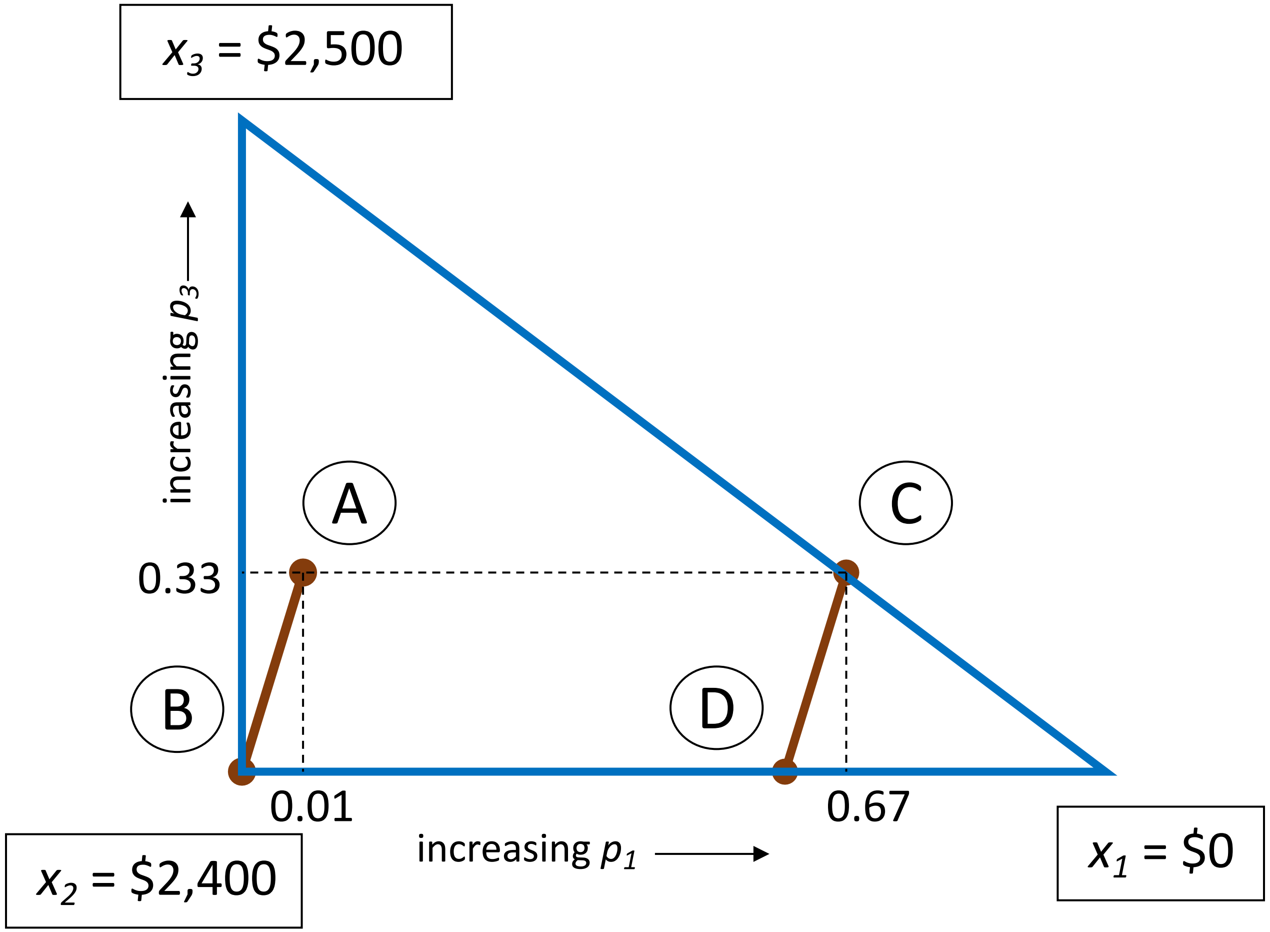


Figure 3.4: Indifference curves in a probability triangle diagram

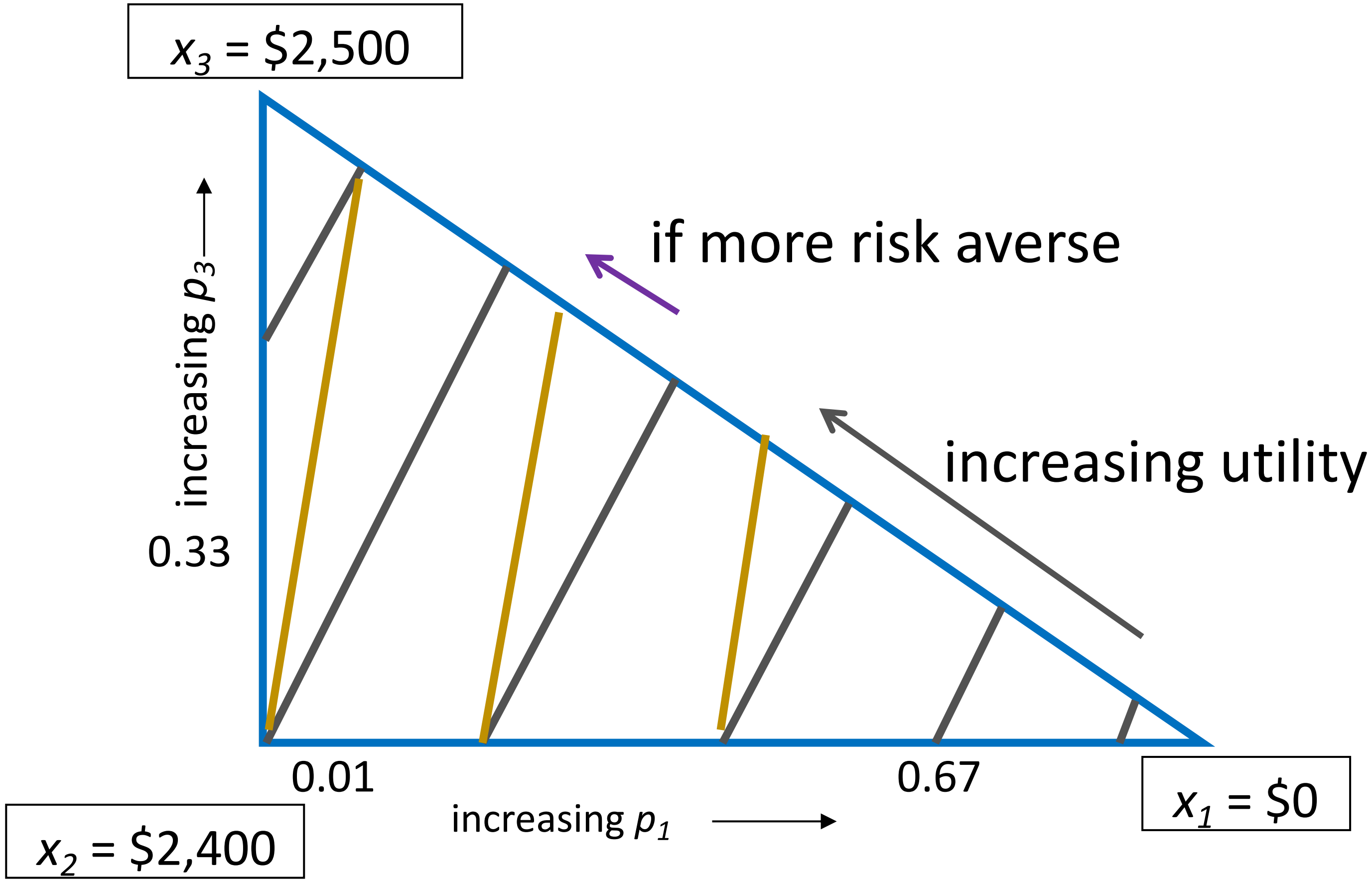


Figure 3.5: The Allais Paradox

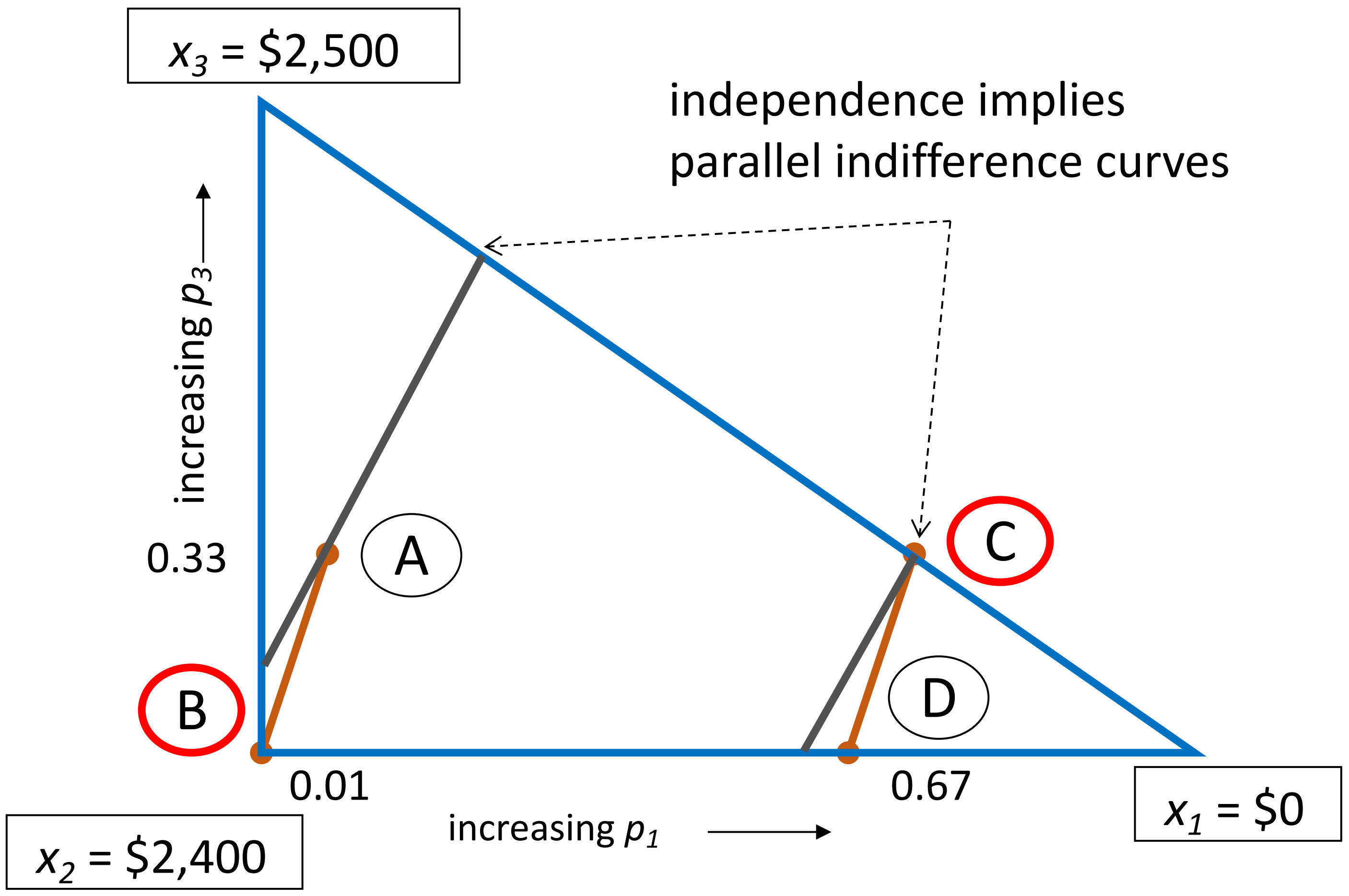
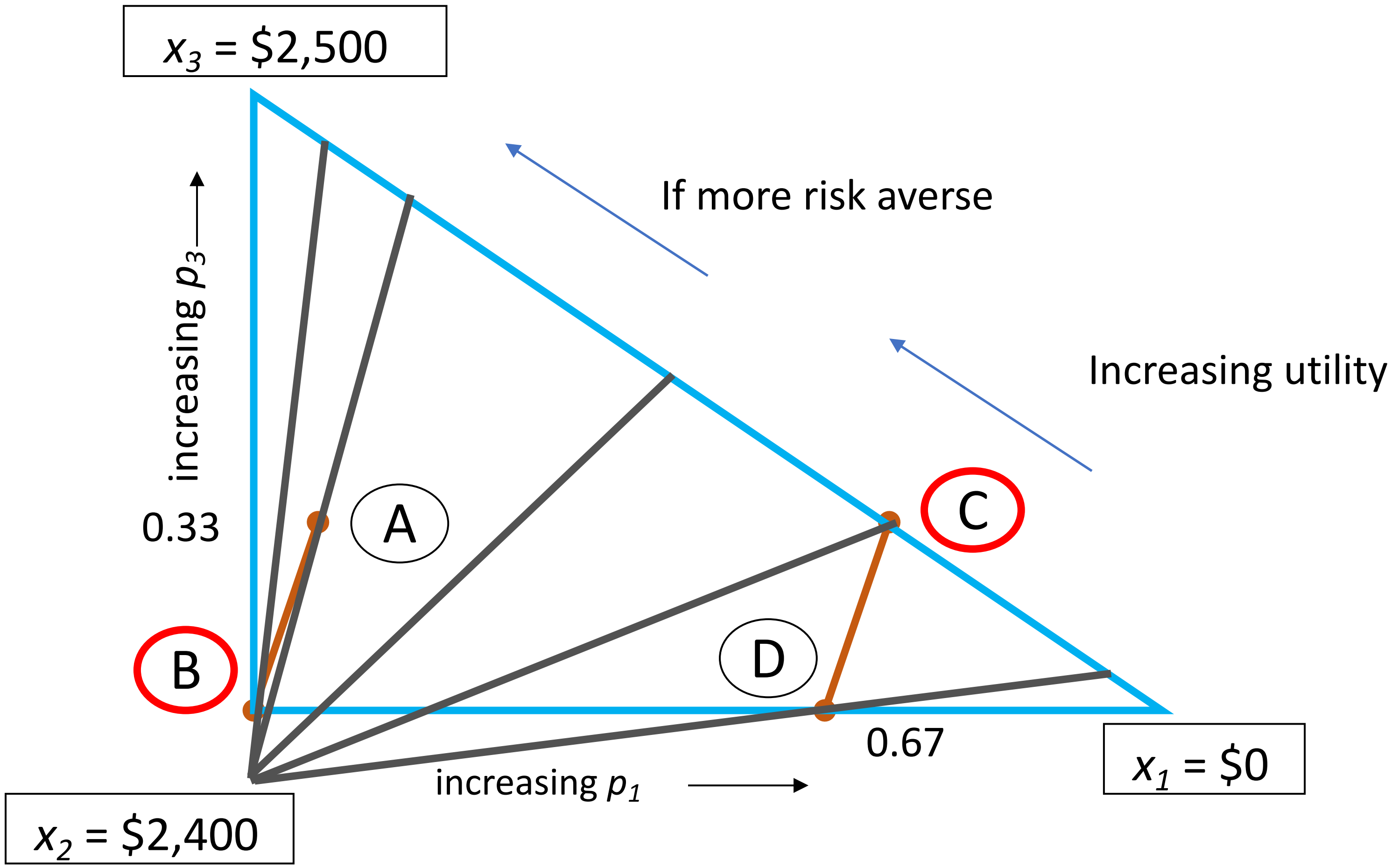


Figure 3.6: Fanning out of indifference curves



Model of disappointment

- A person forms a prior expectation of what her utility will be.
- She then experiences disappointment or elation.
- Maximize expected utility taking into account disappointment and elation

$$U(X) = \sum_{i=1}^{n} p_i \left[u(x_i) + D(u(x_i) - prior) \right].$$

Rank dependent expected utility

- Probabilities are weighted

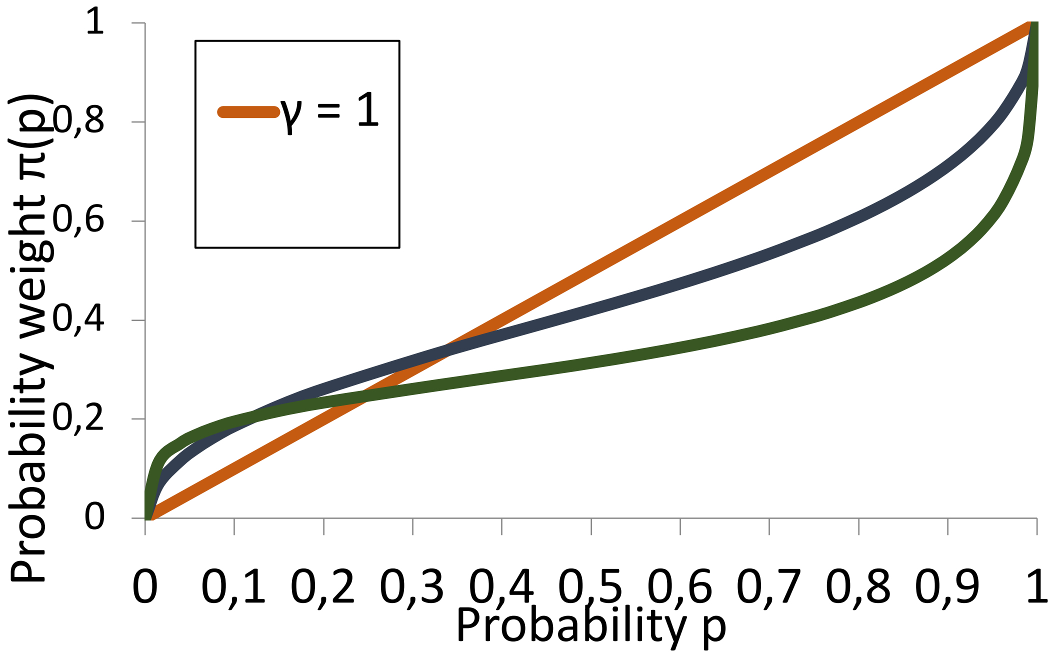
$$\pi(p) = \frac{p^\gamma}{\left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}}.$$

- If $\gamma < 1$ get overweighting of small probabilities.
- The rank dependent expected utility of X is

$$U(X) = \sum_{i=1}^{n} w_i u(x_i)$$

- where

$$w_i = \pi(p_i + \ldots + p_n) - \pi(p_{i+1} + \ldots + p_n).$$



Prospect theory

- The value function that introduces reference dependence

$$v(x)=\begin{cases}(x-r)^{\alpha} & \text{if } x\geq r\\ -\lambda(r-x)^{\beta} & \text{if } x<r\end{cases}.$$

The evalutation of **outcomes in relative terms** rather than in absolute terms is introduced: the value function is defined by the relative outcome $\boldsymbol{x-r}$, where $\boldsymbol{r}$ **is the reference points**.

- Potential asymmetry in the evaluation of losses and gains (**loss aversion**) is introduced via the parameter $\boldsymbol{\lambda}$**>1**
- Reflection effect: $\boldsymbol{\alpha<1}$ **risk aversion on gains;** $\boldsymbol{\beta>1}$ **risk seeking on losses**.

PROSPECT THEORY	EXPECTED UTILITY THEORY
$v(x)=\begin{cases}(x-r)^{\alpha} & \text{if } x\geq r\\ -\lambda(r-x)^{\beta} & \text{if } x<r\end{cases}.$	$u(x)=x^{\alpha}$

Prospect theory

- weighting of probabilities, distinguishing between gains and losses.
  - Gains are then given decision weight:

$$w_i = \pi^g(p_i + \ldots + p_n) - \pi^g(p_{i+1} + \ldots + p_n)$$

  - losses are given decision weight:

$$w_i = \pi^l(p_1 + \ldots + p_i) - \pi^l(p_1 + \ldots + p_{i-1})$$

  - Where the probability weights for gains and losses is:

$$\pi^g(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}, \quad \pi^l(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}$$

Rank dependent, reference dependent expected utility

- We can use a general formulation that encompasses the approaches seen so far

$$U(X) = \sum_{i=1}^{n} w_i \left[\eta u(x_i) + h(x_i, r) \right].$$

- This can allow for disappointment, weighting of probabilities, reference dependence, or a combination of all three.

Theory summary

- Expected utility is a convenient way to model risky choice.
- But, it cannot explain a lot of things we observe like the reflection effect.
- Prospect theory and rank dependent expected utility offer us a range of alternative models.