



ON THE RONEN METHOD IN SIMPLE 1-D GEOMETRIES

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ON THE RONEN METHOD IN SIMPLE 1-D GEOMETRIES

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Abstract

In this work we apply the Ronen method to resolve the neutron transport in simple homogeneous problems. Slab, cylindrical and spherical geometries are studied. This method demands successive resolutions of the diffusion equation, where the local diffusion constants are modified in order to reproduce new estimates of the currents by a transport operator. The diffusion solver employs here finite differences and the transport-corrected currents are forced in the numerical scheme by means of drift terms, like in the CMFD scheme. Boundary conditions are discussed introducing proper approximations to save the particle balance in case of reflection in the slab. The solution from the Ronen iterations is compared against reference results provided by the collision probability method (CPM). More accurate estimates of the currents are provided by integral equations using first flight escape probabilities. Slow convergence on the scalar flux is analyzed, although the results match the reference solutions in the limit of fine meshes and far from the bare boundary.

Keywords — Ronen method, Collision probability method, Neutron transport, CMFD

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1 INTRODUCTION

In 2004 Ronen suggested to calculate more accurate currents by means of an integral transport operator still using a neutron flux computed in diffusion theory [1]. This allowed having new estimates of the diffusion coefficient still using Fick’s law. Since these estimates need a known flux distribution, it was also suggested to execute new diffusion calculations, thus updating iteratively the diffusion coefficient in the global calculation. The main reason motivating this method was to overcome the underlying limitation of Fick’s law requiring low flux gradients, and so low neutron absorption and high scattering in general. Nevertheless, isotropic scattering remained as a basic postulate.

This idea was later used by Tomatis and Dall’Osso, who provided a numerical demonstration in a simple slab problem [2]. Instead of updating the diffusion coefficient by the ratio of the current exchange and the flux gradient, as in Fick’s law, they recurred to the scheme of the coarse mesh finite differences method (CMFD) for taking into account the new currents estimated by the integral transport operator in the diffusion solver. This technique, largely adopted in the literature of nodal methods [3, 4], can avoid indeterminate divisions in case of vanishing current and flux gradient. They tested this implementation in a bare slab with two-group cross sections homogenized in a realistic PWR assembly. It was observed that the Ronen method (RM) could drive the flux from diffusion towards the reference of the integral Boltzmann transport equation, regardless of the initial formulation used for the diffusion coefficient. Remarkably, the unphysical zero-flux boundary condition was not preventing to correct the flux near the boundary, that is within a few mean free paths. However, higher errors were noticed at the boundary with vacuum, slowly decreasing even after many iterations.

In this work we investigate further the convergence on the first flux moments, aiming to extend the study to the cylindrical and spherical geometries with homogeneous media. The details of the implementation of the Ronen method in the diffusion solver are presented in section 2, while the procedure to estimate the currents by the integral transport operator follows in section 3. Boundary conditions are derived in section 4. The results on a few characteristic configurations are discussed in section 5. Finally, the article ends with the conclusion in section 6.

2 IMPLEMENTATION

The cross sections, as well as the diffusion coefficient, are usually available as volume-averaged data per cell in the mesh. A lattice transport computer code can prepare these data by homogenization. Once the scalar flux is known from the finite differences solver using the original diffusion coefficients, the integral expressions derived in section 3 can be used to get new estimates of the current J at the cell interfaces. Instead of computing new diffusion coefficients on the same interfaces straight by Fick’s law, $J = -D\partial_x\phi$, we obtain new corrective currents $\delta J = J - \Upsilon$ that are supplied next to the neutron balance in diffusion. Here Υ is the current obtained with the original values of the diffusion coefficient and with the derivative

approximated by finite differences. In 1-D geometry and using the notation in Fig. 1, it is:

$$\Upsilon_{i+1/2} = -2D_{i+1/2} \frac{\phi_{i+1} - \phi_i}{\Delta_{i+1} + \Delta_i}, \text{ for } i = 0, \dots, I-1, \quad (1)$$

where $\Delta_i = (x_{i+1/2} - x_{i-1/2})$. Integer and rational subscripts indicate cell-averaged and interface quantities, respectively. Since the input diffusion coefficients are provided as volume-averaged, and that they are always needed at interfaces, we approximate them by local volume averages:

$$D_{i+1/2} = \frac{D_{i+1}V_{i+1} + D_iV_i}{V_{i+1} + V_i}, \text{ for } i = 0, \dots, I-1. \quad (2)$$

The values of the diffusion coefficient at $-1/2$ and $I-1/2$ are simply the coefficients of the boundary cells. V_i is the reduced volume in the i -th cell, that is determined along the only dimension of interest. It is then per unit of transverse surface in the slab, the unit azimuthal angle in the cylinder and per unit cone in the sphere. Specifically, $V_i = \Delta_i c_i$ with c_i depending on the system of coordinates. The c coefficients are unitary in the Cartesian frame. In the cylinder, $c_i = x_i$ for all i with $x_i = (x_{i-1/2} + x_{i+1/2})/2$ (arithmetic mean), whereas $c_i = (4x_i^2 - \hat{x}_i^2)$ in the sphere with $\hat{x}_i = \sqrt{x_{i-1/2}x_{i+1/2}}$ (geometric mean). The discretized form of the current δJ must involve the flux in order to be taken into account in the finite differences equations. Its representation is changed into a drift-advection term determined by the neighboring cell fluxes, thus avoiding possible undefined division by zeros in case of flat flux [3]:

$$\delta J_{i+1/2} = -2\delta D_{i+1/2} \frac{\phi_{i+1} + \phi_i}{\Delta_{i+1} + \Delta_i}, \text{ for } i = 0, \dots, I-1. \quad (3)$$

This allows determining the new numerical corrections δD to use in the finite differences solver, together with the diffusive currents from Eq. (1). If necessary, the spatial differences at the denominator of Eq. 3 can be removed by reason of the arbitrary definition used for δD . Finally, the neutron balance resolved by the CMFD takes into account both types of currents Υ and δJ . Non-linear iterations with new corrections given by δJ are needed because of their dependence on the unknown flux.

The divergence operator used in the multi-group balance equation can be described with the general form $d_x(x^b J_g)/x^b$, with $b = 0, 1$ and 2 respectively for the slab, for the cylinder and for the sphere. We solve then for the volume-integrated flux,

$$\hat{\phi}_{g,i} = \int_{x_{i-1/2}}^{x_{i+1/2}} x^b \phi_g(x) dx = \bar{\phi}_{g,i} V_i,$$

making the approximation that the average group flux $\bar{\phi}_{g,i} \approx \phi_{g,i}$ in the equations above.

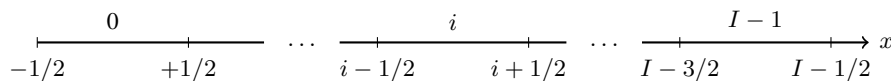


Figure 1: Notation of the 1-D mesh.

3 CURRENT ESTIMATION

3.1 Slab geometry

The angular flux from the integral transport equation (with standard notation) is:

$$\varphi(x, \mu) = \varphi(x_b, \mu) e^{-\tau(x_b, x)/\mu} + \int_{x_0}^x dx' \frac{q(x', \mu)}{\mu} e^{-\tau(x', x)/\mu}, \quad (4)$$

with the optical length $\tau(x_1, x_2) = \int_{x_1}^{x_2} dx' \sigma_t(x')$. Contrary to other formulations, we do not include the direction cosine of the neutron velocity μ in the definition of the optical length to ease the numerical integrations in the following. Eq. 4 holds for both positive directions ($\mu > 0$) where $x_0 = x_{-1/2} = a$, and negative ones ($\mu < 0$) where $x_0 = x_{I-1/2} = b$. The inclusion of the energy variable in Eq. 4 is straightforward; we use hereafter the multi-group theory, writing the source q_g as*:

$$q_g = \sum_{g'=1}^G \left[\sum_{l=0}^{\infty} \frac{2l+1}{2} \sigma_{s,l,g' \rightarrow g}(x) P_l(\mu) \varphi_{l,g'}(x) + \frac{\chi_g}{2\{k\}} \nu \sigma_{f,g'}(x) \phi_{g'}(x) \right] + S_g, \quad (5)$$

where the subscript l refers to the moments of the expansions of the flux and of the scattering cross section on the Legendre polynomials P_l . Fission emission is isotropic, so that only the scalar flux ϕ being equal to the first moment φ_0 is retained for the fission term. Without the external source S , the multiplication factor k is used as eigenvalue to avoid the only trivial vanishing solution. Since q is computed by the results of the diffusion equation in the Ronen method, l can only assume the values 0 and 1, with the second moment φ_1 equal to the net current J .

Multiplication of Eq. 4 by μ and the integration over $[-1, 1]$ provides the expression for the current, see [2]. The integration on μ yields integral exponential functions by a simple change of variable ($\mu = \pm 1/u$, according to the sign of μ) [5]:

$$E_n(\tau) = \int_0^1 d\mu e^{-\tau/\mu} \mu^{n-2} = \int_1^\infty du e^{-\tau u} u^{-n}, \quad n \geq 0.$$

These special functions are then used to compute the current at the cell interfaces for all i :

$$\begin{aligned} J_{g,i+1/2} = & \int_0^1 d\mu \mu \varphi_g(a, \mu) e^{-\tau_g(a, x_{i+1/2})/\mu} - \int_0^{-1} d\mu \mu \varphi_g(b, \mu) e^{-\tau_g(b, x_{i+1/2})/\mu} \\ & + \sum_{j=0}^{I-1} \left[\operatorname{sgn}(x_{i+1/2} - x_{j+1/2}) \frac{q_{0,g,j}}{2} \int_{x_{j-1/2}}^{x_{j+1/2}} dx' E_2(|\tau_g(x', x_{i+1/2})|) \right. \\ & \left. + \frac{3}{2} q_{1,g,j} \int_{x_{j-1/2}}^{x_{j+1/2}} dx' E_3(|\tau_g(x', x_{i+1/2})|) \right], \end{aligned} \quad (6)$$

*Curly brackets remind the possible use of the k -eigenvalue in case of vanishing external sources.

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where $q_{(\cdot),g,j}$ are the moments of the source q_g from Eq. 5 averaged in the volume of the j -th cell:

$$q_{0,g,j} = \sum_{g'} (\sigma_{s,0,g' \rightarrow g,j} + \chi_{g'} \nu \sigma_{f,g',j}) \phi_{g',j}, \quad (7a)$$

$$q_{1,g,j} = \sum_{g'} \sigma_{s,1,g' \rightarrow g,j} J_{g',j}. \quad (7b)$$

The current appearing in $q_{1,g,j}$, that is in case of linearly anisotropic scattering, is also to be considered as volume-averaged in the cell j . The optical lengths show the subscript g because they are evaluated with the corresponding total cross section $\sigma_{t,g}$. For the property $E'_{n+1}(\ell) = -E_n(\ell)$, the spatial integrals of the integral exponential functions can be solved analytically by integrating on τ :

$$\int_{x_{j-1/2}}^{x_{j+1/2}} dx' E_n(|\tau_g(x', x_{i+1/2})|) = \frac{\text{sgn}(x_{i+1/2} - x_{j+1/2})}{\sigma_{t,g,j}} \cdot [E_{n+1}(|\tau_g(x_{j+1/2}, x_{i+1/2})|) - E_{n+1}(|\tau_g(x_{j-1/2}, x_{i+1/2})|)].$$

These quantities can be computed for $i > j$ only, because the others are simply opposite in sign

(anti-symmetric). Eq. 6 can be reformulated for the partial currents, thus considering only the contributions coming from the different sides of $x_{i+1/2}$. Omitting the index on groups and keeping only isotropic sources, this is:

$$J_{i+1/2}^{\pm} = \sum_{j=1}^I q_{0,j} \Delta_i \tilde{e}_{i+1/2,j}^{\pm} + J^{\pm}(x_b) \tilde{t}_{i+1/2,x_b}. \quad (8)$$

$\tilde{e}_{i+1/2,j}$ represents the probability of a neutron emitted isotropically in Δ_i to escape uncollided at $x_{i+1/2}$, along its positive or negative direction of flight. $\tilde{t}_{i+1/2,x_b}$ is the transmission probability of a neutron to enter isotropically at x_b and to cross the interface at $x_{i+1/2}$ without colliding hitherto.

3.2 Cylindrical geometry

The expression of the current in curvilinear geometries is derived in this section from the 3-D Cartesian geometry, where the angular flux at the point \mathbf{r} along the direction of flight Ω is given by the general expression [6]:

$$\varphi(\mathbf{r}, \Omega) = \int_0^{S'} ds' q(\mathbf{r} - s'\Omega, \Omega) \exp(-\tau(\mathbf{r}, \mathbf{r} - s'\Omega)) + \varphi(\mathbf{r} - S'\Omega, \Omega) \exp(-\tau(\mathbf{r}, \mathbf{r} - S'\Omega)), \quad (9)$$

that is considering the contribution of the source q within the distance $S'(\mathbf{r}, \Omega)$ from the boundary and the possible entering amount of particles therein. All contributions are collected at \mathbf{r} by exponential attenuation along the travelled optical path, that is according to the probability of still continuing the first flight after emission. We examine first the case of the cylinder as a particular case of the 2-D frame. In planar geometry the angular flux and the source do not depend on the axial coordinate z . Hence, the position on the characteristic line identified by Ω is projected on the x - y plane (see Fig. 2): $\varphi(\mathbf{r} - s'\Omega, \Omega) = \varphi(\mathbf{r} - s\Omega_p, \Omega)$ and likewise for q , with the unit vector Ω_p lying on the x - y plane. θ is the polar angle

measured between \hat{e}_z and Ω . In absence of incoming particles, this allows to rewrite Eq. 9 as:

$$\varphi(\mathbf{r}, \Omega) = \int_0^S ds \frac{q(\mathbf{r} - s\Omega_p, \Omega)}{\sin \theta} \exp \left[-\frac{\tau(\mathbf{r}, \mathbf{r} - s\Omega_p)}{\sin \theta} \right]. \quad (10)$$

As well, integration of Eq. 10 over the solid angle $d\Omega = \sin \theta d\omega d\theta$ yields the scalar flux at the point \mathbf{r} . The theory of collision probability methods originates from the use of flat isotropic sources in the cells of the spatial mesh. The use of anisotropic sources with spatial variation in the same cells, for instance polynomial-like, is possible but leads to much more difficult expressions to solve. Two integrations in space arise for each direction Ω , connecting the source in region j to the flux (or its total reaction rate) in region i by means of its collision probability. The incoming flux is usually considered as isotropic too, or linearly anisotropic in angle as in [7] to relate the partial entering and outgoing currents to the only first flux moments. These partial currents on the boundary must satisfy a general condition of albedo.

Using standard nomenclature of collision probability methods, transfer probabilities refer to particles entering the problem domain and leaving it without incurring into any collision. They can provide the probability of reflection in case the entering and the leaving surfaces are the same. The probability of escape deals with neutrons produced in a given region and leaving a surface, still uncollided. These probabilities are employed to write the integral transport equations for the volume-averaged scalar flux (or its total reaction rate) in each cell, and the surface-averaged currents at the cell interfaces.

The projections of partial currents on the outward normal \hat{n} to a given surface are obtained from integration of Eq. 10 in $d\Omega$ with the weight $|\hat{n} \cdot \Omega| = \sin \theta |\hat{n} \cdot \Omega_p|$ on $\hat{n} \cdot \Omega \leq 0$. Higher moments can be obtained by the weights associated to the corresponding spherical harmonics. Integration along the polar angle is resolved analytically by the Bickley-Naylor functions [8]:

$$\text{Ki}_n(\tau) = \int_0^{\pi/2} d\theta \sin^{n-1} \theta \exp \left(-\frac{\tau}{\sin \theta} \right), \quad n \geq 0. \quad (11)$$

The properties used for the computation of these functions are available elsewhere [6, 7]. We use the Fortran library by Amos from 1983 in this work [9].

After introducing these arguments for a bare cylinder, the current leaving the cylindrical surface at \mathbf{r} (with direction \hat{n}) given by neutrons flying along Ω_p is:

$$J^+(\mathbf{r}) = \frac{1}{2\pi} \int_{W^+} d\omega |\hat{n} \cdot \Omega_p| \int_0^S ds \text{Ki}_2[\tau(\mathbf{r}, \mathbf{r} - s\Omega_p)] q(\mathbf{r} - s\Omega_p), \quad (12a)$$

with $W^+ = \{\omega \mid \hat{n} \cdot \Omega_p > 0\}$. Using the surface element $dA' = s d\omega ds$ and $\mathbf{r}'_p = \mathbf{r} - s\Omega_p$, it is also:

$$J^+(\mathbf{r}) = \int_{W^+ \times [0, S]} dA' \hat{n} \cdot (\mathbf{r} - \mathbf{r}'_p) \frac{\text{Ki}_2[\tau(\mathbf{r}, \mathbf{r}'_p)]}{2\pi |\mathbf{r} - \mathbf{r}'_p|^2} q(\mathbf{r}'_p). \quad (12b)$$

J^- is obtained with integration over $w \in W^- = \{\omega \mid \hat{n} \cdot \Omega_p < 0\}$ instead.

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Another integration on the total surface $S(r) = \int r d\theta' = 2\pi r$, with $\theta' = \hat{n} \cdot \Omega_p$, is necessary to obtain the current density in the 1-D frame. For a given angle ω , we have then volume integrals to solve numerically along many parallel lines called tracks, see Fig. 4, for $rd\theta' = dh/\cos\theta'$. Furthermore, the tracks are identical for any angle ω in the 1-D cylindrical geometry factoring out 2π from the integrals in Eqs. 12, which can also be limited to half portion of the cylinder thanks to its symmetry. Finally, the outgoing and incoming currents at r become:

$$J^\pm(r) = \pm \frac{1}{\pi r} \int_0^r dh \int_{-Y}^{\pm y} d\ell \text{Ki}_2[\tau(y, y - \ell)] q(h, y - \ell) \quad (13)$$

with $y(r, h) = \sqrt{r^2 - h^2}$, $Y = y(R, h)$ and the outer radius R . After multiplication by $S(r)$ and still with uniform sources, Eq. 13 can be written in terms of the escape probability $\tilde{e}_j(r)$ for a neutron to be emitted isotropically in ring j and to leave uncollided along the normal direction the semi-cylinder surface at r , being crossed with angle $\omega \in W^\pm$:

$$J^\pm(r)S(r) = \sum_j q_j V_j [\pm \tilde{e}_j^\pm(r)] \quad \text{with} \quad \tilde{e}_j^\pm(r) = \frac{2}{V_j} \int_0^r dh \int_{L_j^\pm} d\ell \text{Ki}_2[\tau(y, y - \ell)], \quad (14)$$

$L_j^\pm(h) = [-Y, \pm y] \cap V_j$ and where V_j is the ring area. $\text{Ki}_2[\tau(y, y - \ell)]$ is the transport kernel expressing the probability of a neutron to travel uncollided along ω for a length τ , given in mean free path units, from $(y - \ell)$ to y [10]. The integral on the track ℓ is written differently for the convex and for the concave parts of the rings with respect to the direction of flight, see for instance track 1 and 2 in Fig. 4 where the same notation of Fig. 1 is used for the radial mesh. The current leaving the i -th ring at $r_{i+1/2}$ gets all source contributions weighted with the quantities $\varepsilon_{i+1/2,j}^+ = V_j \tilde{e}_{i+1/2,j}^+$:

$$\varepsilon_{i+1/2,j}^+ = 2 \cdot \left\{ \begin{array}{l} \int_0^{r_{i-1/2}} dh \int_0^{\ell_j} d\ell \left[\begin{array}{l} \text{Ki}_2(2\sigma_i \ell_i + \tau_{ii} + \tau_{ij} + \sigma_j \ell) \text{ if } i < j \\ + \text{Ki}_2(\sigma_i \ell_i + (1 - \delta_{ij})(\tau_{ij} + \sigma_j \ell_j) + \tau_{jj} + \sigma_j \ell) \text{ if } i \geq j \\ + \text{Ki}_2((1 - \delta_{ij})(\sigma_i \ell_i + \tau_{ij}) + \sigma_j \ell) \text{ if } i \geq j \end{array} \right. \\ \left. \right] + \int_{r_{i-1/2}}^{r_{i+1/2}} dh y \int_0^{\ell_j} d\ell \text{Ki}_2((1 - \delta_{ij})(\sigma_i \ell_i + \tau_{ij}) + \sigma_j \ell) \text{ if } i \leq j, \end{array} \right. \quad (15)$$

using the Kronecker function $\delta_{ij} = 1$ only if $i = j$ and null otherwise. τ_{ij} and τ_{ii} are respectively the optical lengths between the rings i and j , and across the outer diameter of ring i . The radial integral includes all rings within $r_{i+1/2}$. The integrals involving the Ki function are solved thanks to the property $d_\tau \text{Ki}_n(\tau) = -\text{Ki}_{n-1}(\tau)$.

The corresponding form for the reduced escape probability for the incoming current is:

$$\varepsilon_{i+1/2,j}^- = 2 \int_0^{r_{i+1/2}} dh \int_0^{\ell_j} d\ell \text{Ki}_2(\tau_{ji} + \sigma_j \ell), \quad (16)$$

for $i < j$, whereas $\varepsilon_{i+1/2,j}^- \equiv 0$ for $i \geq j$ since in this case a neutron born in volume j (or in volume $j = i$) cannot enter volume i through its outer surface at radius $r_{i+1/2}$ without undergoing a collision. The useful reciprocity and conservation properties commonly used in CPM solvers to save the total computational effort can be applied only after calculating the first flight collision probabilities.

Singularity of the integrand for the h -integration is known in the convex part while computing the collision probabilities [7, 10]; this notably happens at the outer radius of the j -th ring where \hat{n} is perpendicular to Ω_p . However, that integral can be regularized by a suitable change of variable, see appendix A. The calculation of the escape probabilities as presented in this section is also affected by this endpoint singularity. Because of this singularity, the integral in h can then be resolved numerically with weights and points by the Gauss-Jacobi quadrature, which is one order more accurate than the common Gauss-Legendre quadrature.

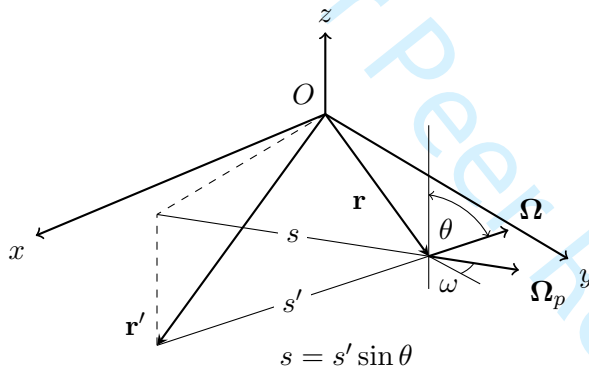


Figure 2: Projection of vectors on the x - y plane.

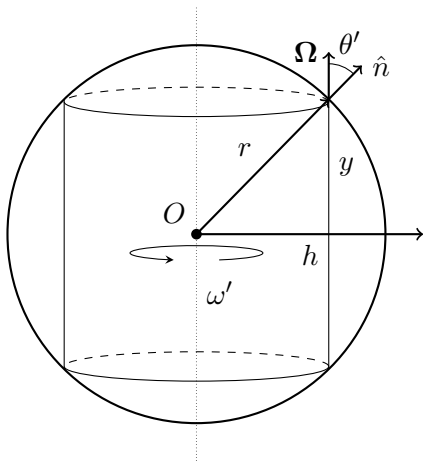


Figure 3: Use of symmetry in the sphere for integration over the surface.

3.3 Spherical geometry

The derivation of the scalar flux and of the current in the sphere follows the same rational adopted in section 3.2 by angular integration of Eq. 9 and exploiting the symmetries available in the given coordinate

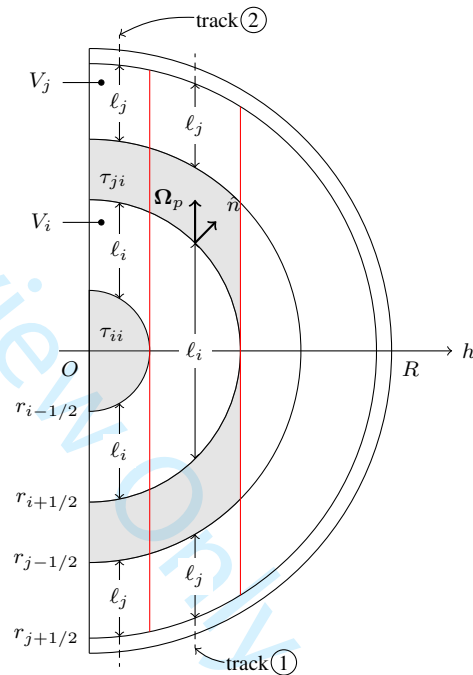


Figure 4: Integration along the tracks in the 1-D cylindrical geometry (suggested by Fig. 3.17 of reference [7]).

frame. Because of the rotational symmetry any direction of flight lays on a spherical cut passing through the origin ($\Omega_p = \Omega$), see Fig. 3. The integration to carry for each direction Ω over the spherical surface $S(r) = \int r^2 \sin \theta' d\theta' d\omega' = 4\pi r^2$ shows a case similar to the cylinder of Fig. 4, with integration on the semi-circle along the coordinate h , thanks to the axial symmetry around ω' that factors out 2π , but with the new weight $h = r \sin \theta'$. The tracking integrals up to the spherical surface are the same for all directions, so that the integration over the unit solid angle yields simply 4π , cancelling out the probability of isotropic emission used for the source. Fig. 3 is specific to the calculation of the positive radial partial currents. Negative partial currents need to consider the sources outside the depicted sphere of integration.

Using the same quantities defined in Eq. 13, Eq. 14 holds also for the sphere but using the new reduced escape probability:

$$\varepsilon_j(r) = 2\pi \int_0^r dh h \int_{L_j} d\ell \exp[-\tau(y, y - \ell)]. \quad (17)$$

Endpoint singularities at integration stand also in this case. The derivation of the equation for the current leaving the i -th spherical surface can be obtained straightforwardly from Eq. 15.

3.4 Derivation of collision probabilities

The probabilities used to solve the integral transport equation can treat a volume or surface source, and similarly for the target quantity after emission. Indeed, they consider only the first flight of particles until collision in the target or leakage from its boundary (being still uncollided). The first flight collision probability is then the probability that a neutron emitted isotropically in a cell volume suffers its first collision in another or within itself. This is equivalent to count the number of first collisions in the same target volume produced by a unitary source in the starting volume. This number cannot be an integer of course.

After all escape and inscape probabilities are known, the first flight collision probabilities can be determined by conservation properties in order to reduce the amount of total operations. The escape probabilities obtained from Eqs. 15 and 16 yield the number of neutrons leaving a surface uncollided. For instance, the probability \tilde{p} for a neutron emitted isotropically in the region i to collide in the same region must fulfill the following relation:

$$\tilde{p}_{ii} + \tilde{e}_{i+1/2,i}^+ + \tilde{e}_{i-1/2,i}^- = 1, \quad \forall i. \quad (18a)$$

\tilde{e} denotes a total escape probability introduced above. This can be generalized with neutrons coming from region j and colliding in region i as:

$$\tilde{p}_{ij} = \tilde{e}_{i-1/2,j}^+ - \tilde{e}_{i+1/2,j}^+ + \tilde{e}_{i+1/2,j}^- - \tilde{e}_{i-1/2,j}^- + \delta_{ij}, \quad (18b)$$

with $\tilde{e}_{\mp 1/2,j}^\pm = 0, \forall j$ in the sphere and in the cylinder. A formulation resolving the integral Boltzmann equation by first-flight escape probabilities was already introduced by Bitelli and Turrin[11, 12].

Particle conservation on the full domain requires $\sum_i \tilde{p}_{ij} + \tilde{e}_{I+1/2,j}^+ = 1, \forall j$. At last, reciprocity implies

$$\frac{\tilde{p}_{ij}}{V_i \sigma_i} = \frac{\tilde{p}_{ji}}{V_j \sigma_j} \quad \text{and} \quad (19a)$$

$$\frac{\tilde{p}_{j,i+1/2}}{V_j \sigma_j} = 4 \frac{\tilde{e}_{i+1/2,j}^+ + \tilde{e}_{i+1/2,j}^-}{S_{i+1/2}}, \quad (19b)$$

for $1 \leq i, j, \leq I$. Eq. 19a can also be used to verify the implementation of the escape probabilities.

Collision probabilities with surface source emission from Eq. 19b are needed with non-zero entering currents of neutrons at the boundary $x_{I+1/2}$. This probability refers again to an angular flux entering the surface $(i + 1/2)$ isotropically, yielding the cosine current typical of white reflection.

Transmission probabilities between surfaces are also needed in response matrix formulations and with boundary conditions different from vacuum. The response matrix formulation partitions the calculation domain in separate blocks, which exchange out-going and entering partial currents to update their internal distributions of reaction rates. The method to solve the integral transport equation by the collision probability method providing reference solutions for the Ronen method is reported in appendix B.

It might not be possible to compute the escape probabilities as derived in this work using existing computer codes that employ the collision probability method without code modifications. In fact, existing codes usually calculate directly the collision probabilities, by which all other probabilities are determined next using reciprocity and conservation properties. Besides, these codes only need to calculate the escape probability from the outer boundary surface.

4 BOUNDARY CONDITIONS

A generalized form for the boundary condition of the diffusion equation applying at the left side of the slab follows as $\Upsilon = -D_0 \phi_0 / (\Delta_0/2 + \zeta)$, where ζ is the extrapolation length in case of vacuum. $\zeta = 2D$ yields the Marshak boundary conditions for instance, but the value $\zeta = 2.13D$ is usually recommended in slab geometry [10]. Reflection can be reproduced by $\zeta \rightarrow \infty$, whereas the condition of zero-flux comes with $\zeta = 0$. The boundary δJ takes the simpler form $\delta J = -\delta D_{-1/2} \phi_0$ without dividing by the spatial width, since no particular extrapolation length is meant for the correction. Indeed, it will be possible to determine the actual extrapolation length only at convergence. The expression for the right boundary is straightforward, implying a non-negative current. Reflection is always reproduced at the center in curvilinear geometries.

About the currents from Eq. 6, the boundary conditions must be implemented through the first two terms at the right side due to the incoming flux. They are zero by definition only in case of vacuum. The flux expansion on the Legendre polynomials is necessary to reproduce other types of boundary condition. Again, we can only approximate the flux up to the first order, that is $\varphi \approx \phi/2 + 3/2 J \mu$. This approximation may not reproduce a vanishing current with reflection reproducing a symmetric distribution.

Only situations with perfectly isotropic angular flux at the boundary will be described correctly. At the left boundary for instance, the current can be expressed as $J(a) = J^+ + J^- = 0$ with the entering current $J^+ = \int_0^1 d\mu \mu \phi(a, \mu) = -J^-$. J^- takes into account all contributions coming from the right of a up to b , see Eq. 6. In the absence of higher moments, the angular flux at the boundary becomes simply $\phi(a, \mu) \approx \phi_0(a)/2$, yielding $J^+(x) = E_3(\tau(a, x)) \phi_0(a)/2$, or $J^+(x) = \tilde{t}_a(x) \phi(a)/4$ using the CPM formalism (white reflection at a). Here, \tilde{t} is the transmission probability of neutrons crossing uniformly and isotropically the surface a to get to the position x uncollided. The value of this probability can be obtained through reciprocity and conservation properties, as shown in section 3.4. Hence, this approximation does not guarantee to obtain the expected vanishing current at the boundary when computing the same current at the boundary with symmetric flux distributions. However, it is possible to determine the difference between the partial currents (remind that J^- is negative). The higher (even) flux moments are responsible for this residual quantity, but they are not available unfortunately. A possible solution is considering that this quantity comes artificially from the only second moment, like[†]

$$J^+(x) += \frac{5}{4} \tilde{\phi}_2(a) [3E_5(\tau(a, x)) - E_3(\tau(a, x))],$$

$$\text{with } \tilde{\phi}_2(a) = -\frac{16}{5} \left(\frac{1}{4} \phi_0(a) + J^- \right).$$
(20)

The flux expansion at the left boundary is then

$$\phi(a, \mu) = \frac{1}{2} \phi_0 + \frac{3}{2} \mu \phi_1 + \frac{5}{4} (3\mu^2 - 1) \tilde{\phi}_2,$$

with $\phi_1 = J = 0$ with reflection. The derivation of the correction at the other boundary of the slab can easily be derived.

Transmission probabilities of neutrons entering with anisotropic (even) distributions are now needed, but this time they can not be derived from the escape probabilities based on isotropic emission. We only give the expression to calculate these quantities in the slab, since white reflection is generally preferred in the 1-D curvilinear geometries for reproducing more accurate physical results.

Although this correction definitely improves the results, it cannot yield exactly the same results of the unfolded geometry, that is without reproducing the half symmetry by reflection. This occurs because the small current difference is allotted only to the second moment, neglecting the others which may become relevant on a case-dependent basis.

5 RESULTS

New computer programs have been developed in Python v3.8.2 with the numerical libraries NumPy v1.18.2 and SciPy v1.3.2 in order to solve the diffusion and the transport problems after calculating the first

[†]The operator += assigns to J the sum of itself and quantity to the right.

flight escape probabilities in the three geometries. The programs can treat heterogeneous media and use the multigroup formalism in energy, being intended mainly for educational purposes.

We present in this section one-group homogeneous critical problems with isotropic scattering to study the behavior of the Ronen method with angular redistribution in the curvilinear geometries. These cases belong to a test set of analytical benchmarks for code verification [13]. The setup of the complete test suite with problems of increasing level of complexity is planned as future research action. Although analytical solutions exist in literature, reference solutions are here provided by the collision probability method, whose implementation takes advantage of the escape probabilities derived in section 3, and whose numerical solution is explained in appendix B. Besides, reference values of the fundamental flux are provided by the benchmark problems at a few positions.

The problems are resolved in units of optical length according to the change of variable $x \leftarrow \sigma x$. The isotropically multiplying source becomes $q = c\phi_0/4\pi$ with the number of secondaries from scattering and fission events $c = (\sigma_s + \nu\sigma_f)/\sigma$. The transport equation to solve is then:

$$\left(\mu \frac{\partial}{\partial x} + 1\right) \varphi(x, \mu) = \frac{c}{4\pi} \int_{-1}^1 \varphi(x, \mu) d\mu$$

The multiplicity ν recovers the role of eigenvalue in absence of external sources. The size of the problem is simply $b = r_c$ with $a = 0$. The material specifications of the benchmark cases are reported in Table I. The naming convention of the test cases follows by Sood et al., with the critical lengths listed in Table II. All problems use the vacuum boundary condition.

Table I: One-group cross section values used in the benchmark problems.

Material	ν	σ_f	σ_c	σ_s	σ_t	c
Pu-239 (PUa)	3.24	0.081600	0.019584	0.225216	0.32640	1.50
Pu-239 (PUB)	2.84	0.081600	0.019584	0.225216	0.32640	1.40
H20 (refl)	0.0	0.0	0.032640	0.293760	0.32640	0.90

Table II: Reference critical lengths r_c of the test cases, with the multiplication factor k_D and the geometrical buckling B_G^2 of the analytical solution in diffusion.

Problem Id.	Geometry type	r_c		k_D	B_G^2
		(mfp)	(cm)		
PUa-1-0-SL	Slab	0.605055	1.853722	0.891973	0.191159
PUB-1-0-SL	Slab	0.736603	2.256751	0.913739	0.149267
PUB-1-0-CY	Cylinder	1.396979	4.279960	0.956090	0.138266
PUB-1-0-SP	Sphere	1.985343	6.082547	0.964129	0.136287

The analytical solutions from diffusion theory are available in Table III, where the geometrical buckling B_G^2 comes as root of the transcendental equations in the last column, see Table II. The multiplication factor k_D of the analytical diffusion problem is obtained by equating the geometrical buckling and the material

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buckling $B_M^2 = (\nu\sigma_f/k - \sigma_a)/D = (k_\infty/k - 1)/L^2$, where $k_\infty = \nu\sigma_f/\sigma_a$ is the multiplication factor in the infinite medium and $L^2 = D/\sigma_a$ is the diffusion area. The benchmark problems communicate the reference transport fundamental flux at given positions, see Table IV. The scalar fluxes are all normalized to be unitary at the center.

Table III: Analytical solutions from diffusion in centered, bare and homogeneous one-group problems of length X (vacuum boundary with extrapolation distance $d = \zeta D$).

Geometry type	Solution	B_G , s.t.
Slab	$\cos(B_G x)$	$\tan(B_G X) = 1/(dB_G)$, $B_G \in (0, \pi/(2X))$
Cylinder	$J_0(B_G x)$	$J_0(B_G X) - dB_G J_1(B_G X)$, $B_G \in (0, 2.404825/X)$
Sphere	$\sin(B_G x)/x$	$B_G X - (1 - X/d) \tan(B_G X)$, $B_G \in (\pi/(2X), \pi/X)$

Table IV: Fundamental flux from the reference transport solution of the benchmark.

Problem Id.	Geometry type	$r = x/r_c$			
		$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
PUB-1-0-SL	Slab	0.970173	0.881054	0.731813	0.490259
PUB-1-0-CY	Cylinder	-	0.8093	-	0.2926
PUB-1-0-SP	Sphere	0.935380	0.755753	0.498843	0.192226

After computing the numerical solution of the diffusion problem by finite differences without corrections at the cell interfaces, we evaluate new current estimates by using escape, transmission and reflection probabilities according to the given geometry type, as derived from the integral equation in section 3. Then, the solution by the Ronen method is sought iteratively as explained in section 2. Anderson acceleration through the DAAREM algorithm (Damped Anderson Acceleration with Epsilon Monotonicity) is also used during the iteration process [14], reducing considerably the number of iterations to achieve residuals on the relative flux differences below 10^{-6} . Its use is crucial to find the fixed point solution throughout the non-linear iterations of the RM.

The calculation of half slab by CPM after setting (white) reflection at the center yields poor results due to missing contributions from the higher flux moments, see Figure 5. The solution obtained by RM takes into account an artificial correction applied to the only second moment, reducing the flux error, but its eigenvalue solution is higher than 1 of 836 pcm on converged meshes. This issue, discussed in section 4, is more severe with very thin slab, like in the case PUA-1-0-SL, and disappears when the slab thickness increases beyond a few mean free paths (mfp). This forces the modelling of the full slab when verifying the flux distribution for the case PUB-1-0-SL. Such issue does not appear at the center of the solid cylinder or sphere. All methods computes a critical configuration for PUA-1-0-SL after unfolding the half-slab. The study to verify the fundamental flux is then only shown for the case PUB-1-0-SL.

About the cases PUB, the relative error on the critical flux computed by CPM against the values from Table IV are shown in Figure 6 with increasing number of cells (I) always using equivolume meshes. The same

The Ronen Method in simple 1-D geometries

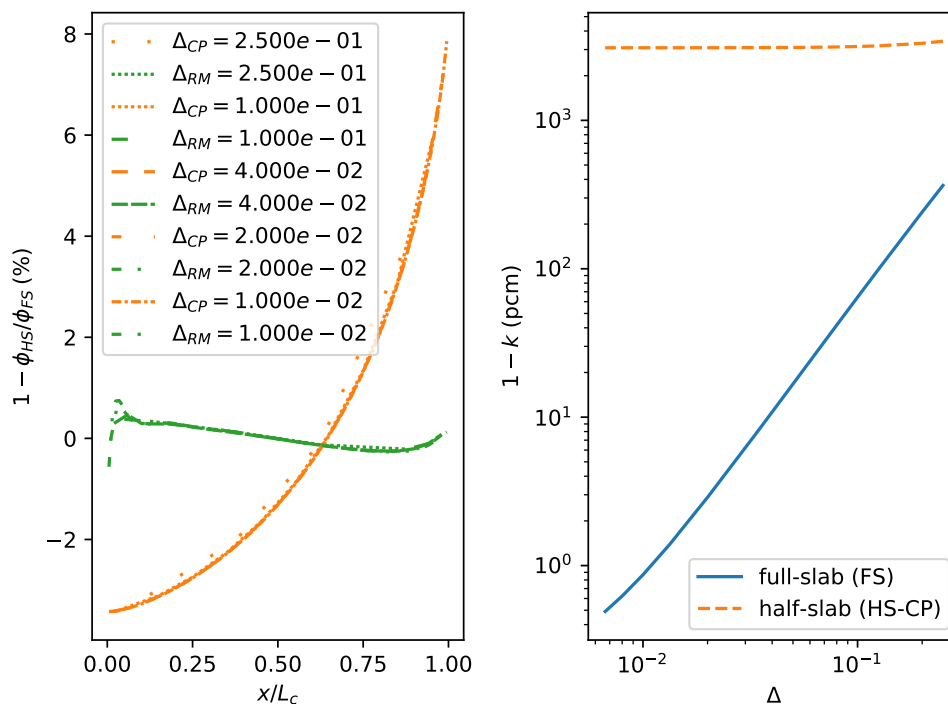


Figure 5: Error in the half slab (PUa-1-0-SL) with white reflection at the right and vacuum at the left.

figure shows also the results obtained by RM. The spherical geometry needs finer meshes to decrease the error, which is always higher near the vacuum boundary. Refined meshes near the boundary, drawn for instance by a geometric progression with ratio less than one, can lower this error, see the program released with this publication. The RM results practically overlaps those obtained by the CPM, but at the vacuum boundary where the monotonic behavior of the error trend is broken with fine meshes, leaving a residual error of about 1-2% in the slab and in the cylinder, and higher than 3% in the sphere. The flux at the boundary used to reproduce the current by diffusion is approximated by a quadratic polynomial fitting the midpoint fluxes at the nearest three cells, instead of the only flux of the last cell.

The number of cells in the slab is doubled indeed for taking a thickness of $2r_c$ in the calculations. The deviation of the multiplication factor from unity by refining the mesh is shown in Figure 7, instead. The same trend for the three geometries is noticed, with the spherical one needing finer meshes to achieve the same level of error. The multiplication factor computed by RM is about the same as the one given by CPM. Four integration points are used in the Gauss-Jacobi quadratures, yielding an error lower than 0.1%.

The solution by CPM needs a single inversion of a full matrix with the direct solving scheme from appendix B. This scheme may not be suitable to problems with a high number of energy groups and spatial cells. The RM needs to invert several times a tridiagonal matrix, which is efficiently performed in our code by Thomas TDMA algorithm, a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations. Both methods share the calculation part of the first flight escape/collision

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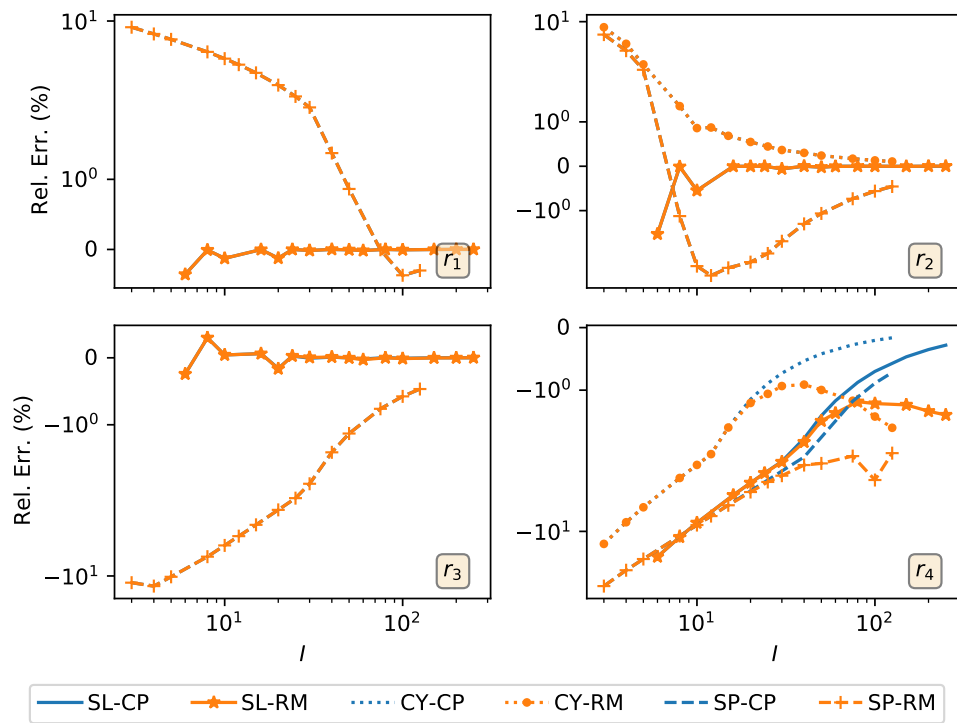
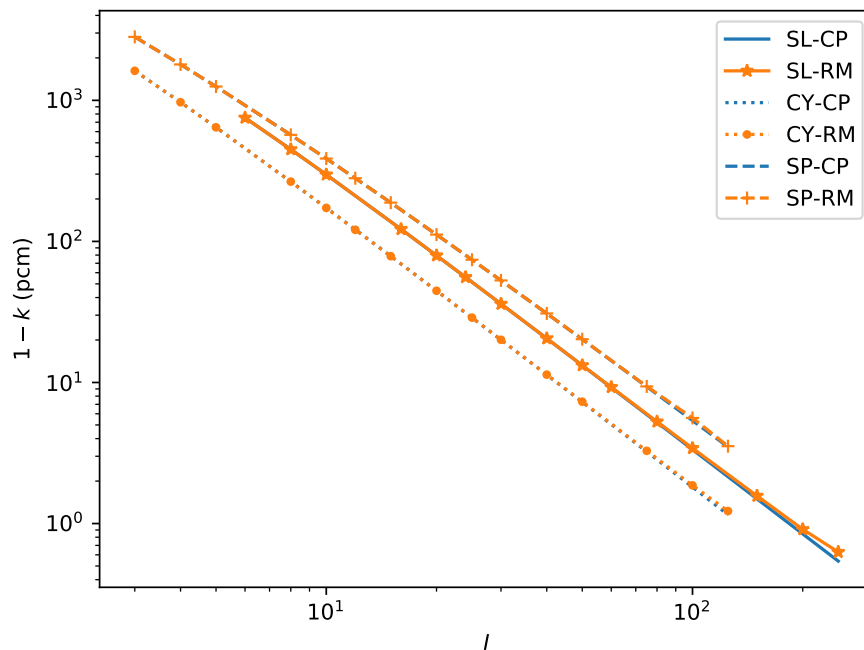


Figure 6: Relative error in percent of the fundamental flux for all PUB cases.

Figure 7: Error in pcm of the multiplication factor k in the PUB critical benchmark problems.

probabilities. Typical runtimes of the benchmark problems treated in this article are quite fast and they can be run on any laptop computers. The calculation of the first flight probabilities is the most expensive part, especially in curvilinear geometries, and it should be reprogrammed in a compiled language to save runtime.

6 CONCLUSION

The Ronen Method was firstly proposed for better estimates of the diffusion coefficient by calculating the current with a higher-order transport operator and a known best-estimate neutron flux. This originates an iterative scheme leading to new flux distributions solved by a diffusion solver yet with the aim to fulfill the integral transport equation. The direct resolution of the integral equation implies the inversion of full matrices, with poor control of their conditioning. The solution of the diffusion equation offers many numerical advantages instead, solution speed and robustness above all.

Nonetheless, the same current can be enforced in the discretized form of the diffusion equation as suggested by the CMFD that avoids the numerical issues arising with flat flux. This is the option adopted in this work. The implementation of the boundary conditions is improved to reproduce the expected vanishing current in case of reflection. However, this correction is not sufficient to fix the limitations due to the use of white reflection, which is particularly serious in problems smaller than a mean free path.

The RM is developed in this work by following the CPM formalism and theory. Only first flight escape probabilities are computed, by which all other kinds of probabilities are derived using conservation and reciprocity properties. The solutions are verified with simple one-group homogeneous problems from the benchmark test suite by Sood and al. [13]. More benchmark problems will be analyzed in a future work to study the RM with material heterogeneity and with many energy groups. The application of the RM to problems with anisotropic scattering suffers from the lack of the flux moments higher than the first, since diffusion can count only on the net current. This article shows the extension of the RM to linearly anisotropic scattering in the source terms in the case of the slab. The study of anisotropy still needs further developments.

The use of Anderson acceleration in the RM iterations was crucial to find converged results, fixing the issue of slow convergence noticed in the previous published works. The large discrepancy on the scalar flux observed in previous works at bare vacuum boundaries has now been reduced to a few percent with respect to the reference solutions. This suggests that same test problems could have been affected in the past implementations by false convergence for very slowly decreasing convergence rates. Additional tests are on-going to verify this assertion. The upgrade of the RM to coarser meshes and different implementations of the CMFD scheme will be the objects of future research and development. These are considered as central topics for the advancement of the methodology in practical applications.

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ADDITIONAL MATERIAL

The theory and results of this article were produced by the computer program available at <https://github.com/ndamage/RonenMethod1D.git>. This program is distributed for educational purposes and basic training in transport theory, without containing any confidential material on nuclear data.

A APPENDIX. Gauss-quadratures

The integrals used in the calculation of the probabilities in the cylindrical and spherical frames are in the form:

$$I = \int_{r_{i-1/2}}^{r_{i+1/2}} dh K(\ell(h)),$$

and they are often evaluated numerically by Gauss-quadratures. The integrand can show an endpoint singularity in the first derivative when integrating in the convex regions, since $d_h K = \partial_\ell K \partial_\ell h$ and

$$\lim_{h \rightarrow r_{i+1/2}} \frac{-2h}{\sqrt{r_{i+1/2}^2 - h^2}} = -\infty,$$

for $\ell = 2\sqrt{r_{i+1/2}^2 - h^2}$ [7]. The use of a Gauss-Legendre quadrature with N points in the integration interval is a good choice only if the integrand is continuous over the interval up to order $2N$.

Gauss-Legendre quadrature integrates exactly polynomials of degree less than $2N$. The definite integrals can be regularized by a proper change of variable following the Flurig scheme, which yields a Gauss-Jacobi quadrature scheme [15]. We propose hereafter a general change of variable to obtain both schemes at once:

$$\varrho = 1 - 2 \left(\frac{r_{i+1/2} - h}{\Delta r_i} \right)^{(1-\alpha/2)} \quad (21)$$

with $\Delta r_i = r_{i+1/2} - r_{i-1/2}$ that yields

$$I = \frac{\Delta r_i}{2} \int_{-1}^1 d\varrho (1 - \varrho)^\alpha K(\ell(\varrho)) = \sum_{n=1}^N w_n K(\ell(h_n)), \quad (22)$$

with the Gauss-Legendre scheme for $\alpha = 0$ and the Gauss-Jacobi scheme for $\alpha = 1$. After retrieving the weights v_n and the roots ϱ_n of the selected N -point quadrature in $[-1, +1]$, the weights and roots to use in Eq. 22 are respectively:

$$w_n = \frac{\Delta r_i}{2} v_n \quad \text{and} \quad h_n = r_{i+1/2} - \left(\frac{1 - \varrho_n}{2} \right)^{(1+\alpha)} \Delta r_i. \quad (23)$$

Weights and roots are obtained by the Scipy v1.4.1 library (special.roots_jacobi) and Numpy v1.18.2 (polynomial.legendre.leggauss). We note that the values from these libraries are different from the ones printed in the cited textbooks [7, 10].

B APPENDIX. Direct solution of integral transport by the collision probability method

The integral transport equation for the scalar flux with isotropic sources can be written using the first flight collision probabilities as [6, 7]:

$$V_i \sigma_i \phi_i = \sum_{j=1}^I q_j V_j \tilde{p}_{ij} + J_b^{(-)} S_b \tilde{p}_{ib}, \quad (24a)$$

where b stands for any boundary with a non-zero entering current $J_b^{(-)} = \phi_b/4$, given as well by an isotropic angular flux. \tilde{p}_{ij} and \tilde{p}_{ib} are the probabilities of a neutron to have its first collision in the volume V_i after being emitted isotropically in the volume V_j or from the surface S_b , respectively. $b = I + 1/2$ for the cylindrical and for the spherical geometries after placing the center at $x_a = 0$ ($a = -1/2$), whereas two incoming currents can be present in the slab. This equation can be simplified further by using the reciprocity property from Eqs. 19,

$$\phi_i = \sum_{j=1}^I q_j p_{ji} + 4J_b^{(-)} \tilde{e}_{bi} \quad (24b)$$

with the reduced escape probability $p_{ji} = \tilde{p}_{ji}/\sigma_j$ and the escape probability counting for both the signed terms. The incoming current at the boundary can be given as fixed external source, or being expressed as proportional to the out-going current by means of an albedo $0 \leq \beta \leq 1$, like

$$J_b^- = \beta_b J_b^+ = \beta_b \left(\sum_{j=1}^I \frac{q_j p_{jb}}{4} + \tilde{r}_b J_b^- + \tilde{t}_{ba} \frac{S_a}{S_b} J_a^+ \right), \text{ and} \quad (25a)$$

$$J_a^+ = \beta_a J_a^- = \beta_a \left(\sum_{j=1}^I \frac{q_j p_{ja}}{4} + \tilde{r}_a J_a^+ + \tilde{t}_{ab} \frac{S_b}{S_a} J_b^- \right), \quad (25b)$$

with the reduced collision probability from the surface at x_b thanks again to reciprocity. About 19b, reciprocity always considers entering neutrons distributed according to the cosine described by the direction of flight and the normal to the same surface, yielding a cosine current by $J^\pm = \phi/4$, suitable to reproduce white reflection with $\beta = 1$. The reflection \tilde{r} is the probability that a neutron enters the outer surface with the cosine distribution, travels inwards and leaves it back again without colliding. It must be zero in the slab, but not in the cylinder or in the sphere. Transmission is used instead when the leaving surface to be crossed is different from the entering one, and it is non-vanishing here only in the slab.

Conservation applied in the cylinder or in the sphere implies that $\tilde{r}_b = 1 - \sum_j \tilde{p}_{jb}$, where we have also assumed implicitly no possible transmission in the derivation of their escape probabilities. Similarly in the slab, it is $\tilde{t}_{ab} = 1 - \sum_j \tilde{p}_{jb}$ for instance. Eq. 25b can be used for Eq. 25a to obtain an expression for J_b^-

that is suitable for substitution in Eq. 24a. Eqs. 25 are in a general form for 1-D geometries, which simplify either in the slab or in the others.

The source term considers only isotropic emission, and it is composed by scattering and fission production as in Eq. 7a. In absence of external sources, the problem becomes homogeneous with the multiplication factor k assuming the role of eigenvalue to avoid the only vanishing solution. Eqs. 24 hold for any energy group, whose index is dropped for simplicity.

We setup hereafter an algebraic system equations for the unknown flux array $\vec{\phi}$ of size $(I \times G)$, flattened with the index of the spatial mesh running first. The extension of the solution method to problems with external sources in submultiplicative media is straightforward. Eq. 24b for bare bodies, that is in presence of vacuum boundary condition, is rewritten as:

$$(\mathbf{I} - \mathbf{PS})\vec{\phi}^{(n+1)} = \mathbf{P}(\chi \otimes \mathbf{F})\vec{\phi}^{(n)} / k^{(n)}, \quad (26)$$

where the index n addresses power method iterations to solve the eigenvalue problem. All matrices in Eq. 26 are constructed by blocks. The matrix \mathbf{P} is block diagonal with all (transposed) matrices of reduced collision probabilities per group, placed in increasing order from the fast groups, which are assigned to the lowest values of the index g as usual. The matrix \mathbf{S} is composed by $G \times G$ diagonal blocks with elements $[\sigma_{s,g \leftarrow g',j} \delta_{ij}, i, j = 1, \dots, I]$, contrary to the matrix \mathbf{F} which has only G blocks set by columns and with elements $[\nu \sigma_{f,g,j} \delta_{ij}, i, j = 1, \dots, I]$. The new estimate of the flux vector is computed by multiplying the left side of the equation by the inverse of the removal matrix, i.e. $(\mathbf{I} - \mathbf{SP})^{-1}$, assuming that it is invertible. The new estimate of the eigenvalue can be obtained by enforcing neutron conservation through the iterations, while achieving the convergence:

$$k^{(n+1)} = k^{(n)} \frac{\langle \vec{\phi}^{(n+1)} | \mathbf{F} \vec{\phi}^{(n+1)} \rangle}{\langle \vec{\phi}^{(n+1)} | \mathbf{F} \vec{\phi}^{(n)} \rangle}.$$

The full eigen-spectrum can be obtained by resolving the problem for the fission production reaction rates, showing smaller rank. First, note that $(\chi \otimes \mathbf{F}) = \mathbf{XF}$, with \mathbf{X} built by G diagonal blocks stacked per row of the form $[\chi_{g,j} \delta_{ij}, i, j = 1, \dots, I]$. The eigen-solver operates then on the matrix $\mathbf{F}(\mathbf{I} - \mathbf{SP})^{-1}\mathbf{PX}$.

In case the partial current is given by an albedo relation at the boundary of the cylinder (or of the sphere), the same solving scheme still holds after redefinition of the matrix \mathbf{P} :

$$\mathbf{P} \leftarrow \mathbf{P} + \frac{\beta_b}{1 - \beta_b \tilde{r}} \mathbf{E}_b,$$

with $\mathbf{E}_b = [\tilde{e}_{bi} p_{jb}, 1 \leq i, j \leq I]$. This redefinition holds also for the slab in case of periodic boundary condition, i.e. $J_b^- = J_a^+$. Lastly for the slab, a general relation for the right boundary is

$$\mathbf{P} \leftarrow \mathbf{P} + \frac{\beta_b}{1 - \beta_a \beta_b \tilde{t}^2} (\mathbf{E}_b + \beta_a \tilde{t} \mathbf{E}_a)$$

with $\mathbf{E}_a = [\tilde{e}_{bi} p_{ja}, 1 \leq i, j \leq I]$. The corresponding expression at the left boundary is easy to obtain.

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