MATH 5340 Fall '23

Lecture 5

Part I. Functions and their representation.

(picking up where we left last time)

Last time we showed the following:

① If $f: \mathbb{R} \to \mathbb{R}$ is continuous, vanishes outside the interval $[-M, M]$ for some $M > 0$, and p is a polynomial of degree n, then $f * p$ is a polynomial of degree $\leq n$.

② We introduced the Gauss distribution,

$$K_\delta(y) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{y^2}{2\delta}}$$

and claimed (refer to Notebook 1 problem 5–7) that if f is as before then $\forall \, \varepsilon > 0 \; \exists$

$\delta > 0$ s.t.

$$|f(x) - (f * K_\delta)(x)| < \varepsilon \quad \forall\, x \in [-M, M]$$

③ We showed that the residual for the power series representation of e^x has a nice bound,

$$|R_N(x)| \leq |x|^{N+1} \frac{e^{|x|}}{(N+1)!}$$

where
$$R_N(x) = e^x - \sum_{k=0}^{N} \frac{1}{k!} x^k$$

Lastly, the last formula leads to a respective formula for the Gaussian kernel

$$K_\delta(y) = \frac{1}{\sqrt{2\pi\delta^2}} e^{\left(-\frac{y^2}{2\delta}\right)}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\delta^2}} \sum_{k=1}^{N} \frac{1}{k!} \left(-\frac{y^2}{2\delta}\right)^k}_{=:\, P_{N,\delta}(y)} + \frac{1}{\sqrt{2\pi\delta^2}} R_N\left(-\frac{y^2}{2\delta}\right)$$

The ~~bound~~ bound in (3) means that

$$\left| K_\delta(y) - P_{N,\delta}(y) \right| \leq \frac{1}{\sqrt{2\pi\delta^2}} \left(\frac{y^2}{2\delta} \right)^{N+1} \frac{e^{\frac{y^2}{2\delta}}}{(N+1)!}$$

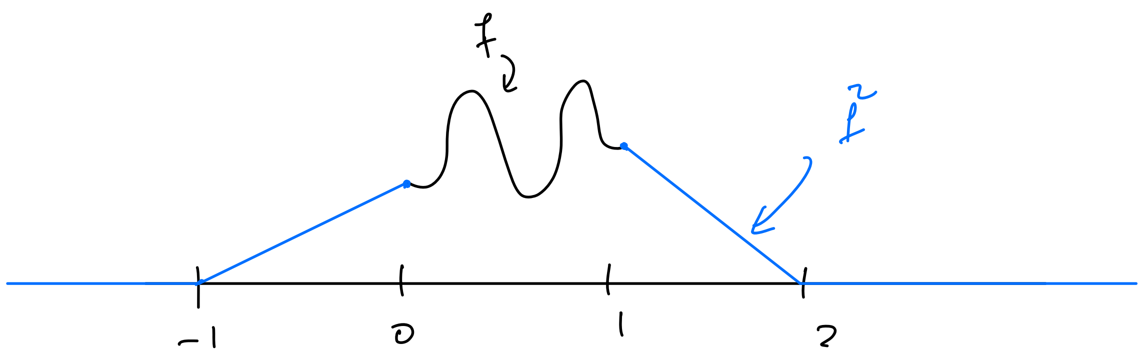
Now we are ready to prove Weierstrass' theorem.

Theorem: If $f : [0,1] \longrightarrow \mathbb{R}$ is continuous and $\varepsilon > 0$ there is a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon \qquad \forall \quad x \in [0,1].$$

Proof. Fix $\varepsilon > 0$. 1. Define a function $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\tilde{f}(x) = \begin{cases} 0 & x < -1 \\ f(0)\,(x+1) & -1 \leq x \leq 0 \\ f(x) \quad \text{in} & 0 \leq x \leq 1 \\ -f(1)(x-1) + f(1) & 1 \leq x \leq 2 \\ 0 & x > 2 \end{cases}$$



We are going to approximate $\tilde{f}$ with a polynomial, since $\tilde{f} = f$, that will achieve our goal.

2. With the ε that we chose, we pick $\delta > 0$ small enough so that

$$\left| \tilde{f} * K_\delta(x) - \tilde{f}(x) \right| < \varepsilon/2 \quad \text{in } [-2,2]$$

3. We know that for $\delta > 0$ and every N we have the polynomial $P_{N,\delta}$ and that in this case the function

$$\tilde{f} * P_{N,\delta}$$

is a polynomial (of degree at most $2N$)

4. We are going to combine everything, noting that

$$\tilde{f} - \tilde{f} * P_{N,\delta} = \tilde{f} - \tilde{f} * (K_\delta - R_{N,\delta})$$
$$= \tilde{f} - \tilde{f} * K_\delta + \tilde{f} * R_{N,\delta}$$

The triangle inequality then says that

$$|\tilde{f}(x) - (\tilde{f} * P_{N,\delta})(x)| \le |\tilde{f}(x) - (\tilde{f} * K_\delta)(x)| + |(\tilde{f} * R_{N,\delta})(x)|$$
$$\le \frac{\varepsilon}{2} + |(\tilde{f} * R_{N,\delta})(x)|$$

for all $x \in [-2,2]$

Let's look at the last term $\tilde{f} * R_{N,\delta}$,

$$\tilde{f} * R_{N,\delta}(x) = \int_{-2}^{2} \tilde{f}(y) R_{N,\delta}(x-y)\, dy$$

The bound for $R_{N,\delta}$ states that if $z \in [-L, L]$ (for some L) then

$$|R_{N,\delta}(z)| \le \frac{1}{\sqrt{2\pi\delta^2}} \left(\frac{z^2}{2\delta}\right)^{N+1} \frac{e^{\frac{z^2}{2\delta}}}{(N+1)!}$$

$$\le \frac{1}{\sqrt{2\pi\delta^2}} \frac{L^{2(N+1)}}{(2\delta)^{N+1}} \frac{e^{\frac{L^2}{2\delta}}}{(N+1)!} \quad \forall z \in [-L, L]$$

We apply this inequality with $z = x-y$ $(x, y \in [-2,2])$ and $L = 4$, we have

$$\left| \int_{-2}^{2} \tilde{f}(y) R_{N,\delta}(x-y)\, dy \right|$$

$$\le \int_{-2}^{2} |\hat{f}(y)|\, |R_{N,\delta}(x-y)|\, dy$$

$$\le \int_{-2}^{2} |\hat{f}(y)| \frac{1}{\sqrt{2\pi\delta^2}} \frac{L^{2(N+1)}}{(2\delta)^{N+1}} \frac{e^{\frac{L^2}{2\sigma^2}}}{(N+1)!}\, dy$$

Exercise: Show that $\lim_{N \to \infty} \frac{(L^2)^{N+1}}{(2\delta)^{N+1}} \frac{1}{(N+1)!} = 0$

note that you can write this as

$$\lim_{N\to\infty} \frac{\alpha^{N+1}}{(N+1)!} = 0 \quad , \text{ when } \alpha = \frac{L^2}{2\delta} \, .$$

From the exercise, choose N large enough so that (with $L=4$)

$$\frac{\left(\frac{L^2}{2\delta}\right)^{N+1}}{(N+1)!} \le \frac{\varepsilon}{2} \, \frac{1}{\int_{-2}^{2} |\hat{f}(y)|\,dy \; \frac{e^{\frac{L^2}{2\delta}}}{\sqrt{2\pi\delta^2}}}$$

in this case, we have

$$|\tilde{f} * R_{N,\delta}(x)| \le \varepsilon/2 \qquad \forall\, x \in [-2,2]$$

$$\Rightarrow \quad \left| f(x) - (\tilde{f} * P_{N,\delta})(x) \right| < \varepsilon \qquad \forall \quad x \in [0,1]$$