Scientific Computing
MATH 5340

Lecture 26

Part 4. Finite differences and finite elements

This week

- Quick review: bilinear form for a BVP and triangulations
- Implementing the FEM (computing A_{ij} and b_i)
- Convergence and visualization of solutions

Review: bilinear form for a BVP

We have seen how $\Delta u = f$ in Ω

is equivalent to

$$(*) \quad B(u, \phi) = \int_\Omega f \phi \, dx \qquad \forall \phi \in C^2_c(\Omega)$$

where

$$B(u, \phi) := - \int_\Omega \nabla u \cdot \nabla \phi \, dx$$

From here we formulated our finite element approximation to

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

by modifying $(*)$ in relation to a given triangular mesh.

Review: bilinear form for a BVP

The formulation is thus:

① We take Ω_h, an approximation to Ω made out of a triangular mesh T

② We look for $u : \Omega_h \longrightarrow \mathbb{R}$ which is piecewise linear wrt. to the mesh T, and such that

$$B(u, \phi) = \int_{\Omega_h} f \phi \, dx \quad \forall \phi \in S_0$$

$$\left(S = \{ \phi \mid \text{piecewise linear wrt to mesh } T \} \right.$$

$$\left. S_0 = \{ \phi \mid \phi \in S \text{ and } \phi|_{S_0} = 0 \} \right)$$

Review: bilinear form for a BVP

Because S and S_0 are finite dimensional subspaces, the above problem (as we will see) reduces to a finite linear system of equations: for suppose $\phi_1, \dots, \phi_N$ is a basis of S s.t. $\phi_1, \dots, \phi_n$ is a basis of S_0 (so, $n < N$) then, we look for u:

$$u = \sum_{j=1}^{N} z_j \phi_j$$

unknowns!

s.t.

$$\sum_{j=1}^{N} \boxed{B(\phi_j, \phi_i)} z_j = B(u, \phi_i) = \boxed{\int_{\Omega_h} \phi_i f \, dx} \quad \boxed{\forall i = 1, \dots, n}$$

and $u = g$ on the nodes on $\partial\Omega_h$

Review: triangulations

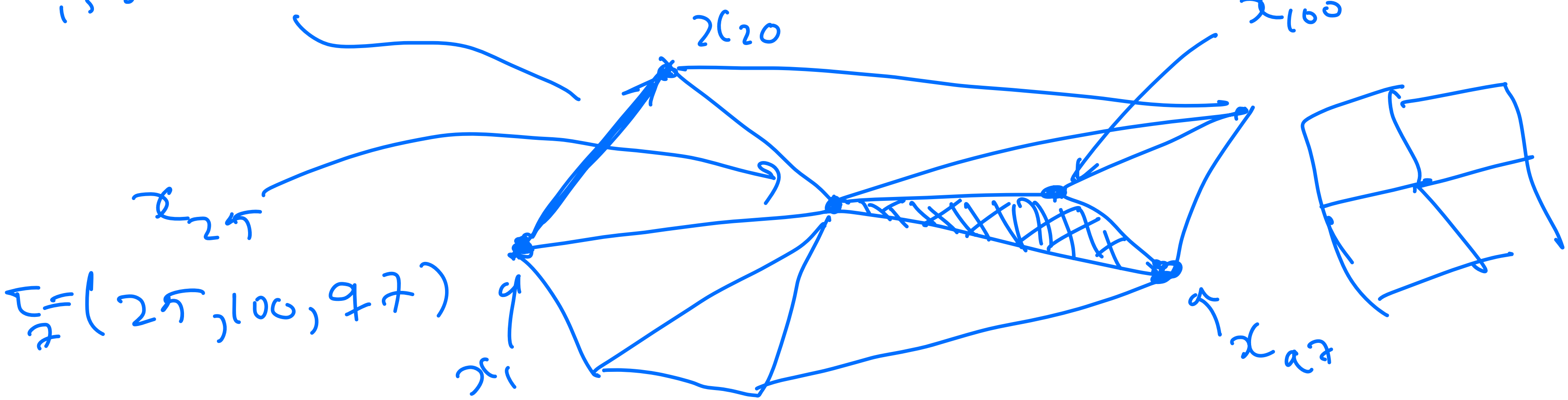
To represent a triangular mesh for a domain $\Omega \subset \mathbb{R}^2$ we need

1. A list of vertices: $x_1, x_2, \ldots, x_N$

2. A list of edges: $e_1, e_2, \ldots$
($e = (i, j)$ means an edge connecting x_i to x_j)

3. A list of triangles: $\tau_1, \tau_2, \ldots$
($\tau = (i, j, k)$ means x_i, x_j, x_k form a triangle)

$e = (1, 20)$ is an edge



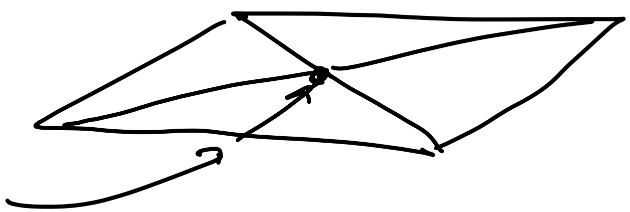
Review: triangulations

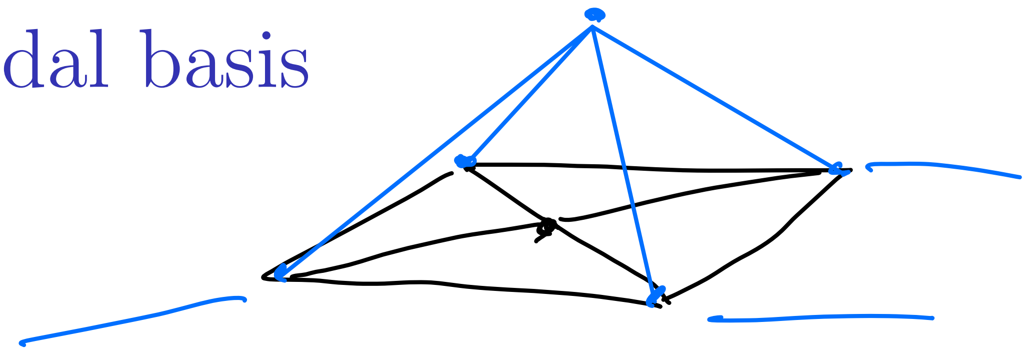
In the list of vertices $x_1, \ldots, x_N$ we will take the convention that all the boundary points of the mesh are listed at the end.

As a result, we may talk about n interior vertices, $x_1, \ldots, x_n$.

The boundary vertices will be the remaining ones, $x_{n+1}, \ldots, x_N$.

The nodal basis





x_i

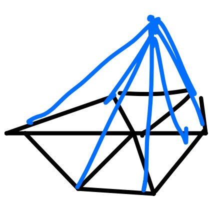
The finite elements is the family of functions in $\mathcal{S}$ defined by

$$\phi_i(x_j) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases}$$

The N functions $\phi_1, \ldots, \phi_N$ form the **nodal basis** of $\mathcal{S}$.

The n functions $\phi_1, \ldots, \phi_n$ form a basis of $\mathcal{S}_0$, where

$$\mathcal{S}_0 = \{\phi \in \mathcal{S} \mid \phi \text{ vanishes on } \partial D\}$$



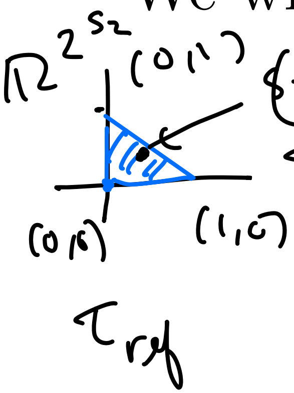
The nodal basis

The reference triangle

The **reference triangle** τ_{ref} is the one given by the vertices

$$(1, 0), (0, 1), (0, 0)$$

We will use (s_1, s_2) to parametrize elements of τ_{ref}.

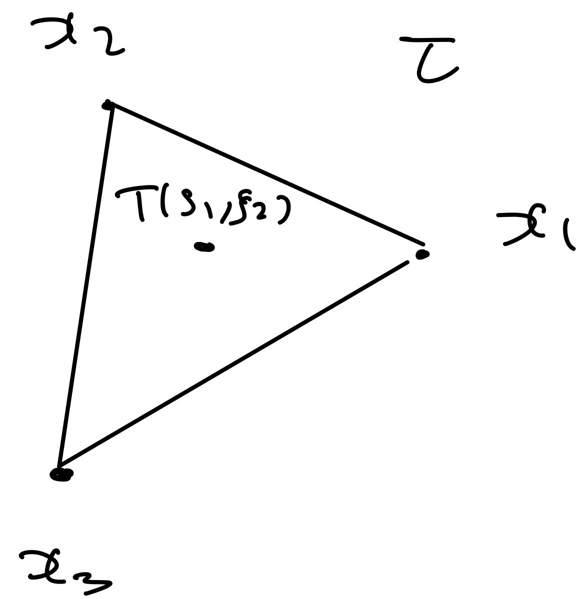


$$s_1 + s_2 \le 1, \quad s_1, s_2 \ge 0, \qquad T = T_\tau$$

$$T : \tau_{ref} \Longrightarrow \tau$$

(note, this is a vector equation)

$$T(s_1, s_2) = s_1(x_1 - x_3) + s_2(x_2 - x_3) + x_3$$



The nodal basis

The reference triangle

Take a triangle τ given by vertices x_1, x_2, x_3.

Consider the affine map $T_\tau : \tau_{\text{ref}} \to \tau$ defined by

$$T_\tau((1,0)) = x_1,\ T_\tau((0,1)) = x_2,\ T_\tau((0,0)) = x_3$$

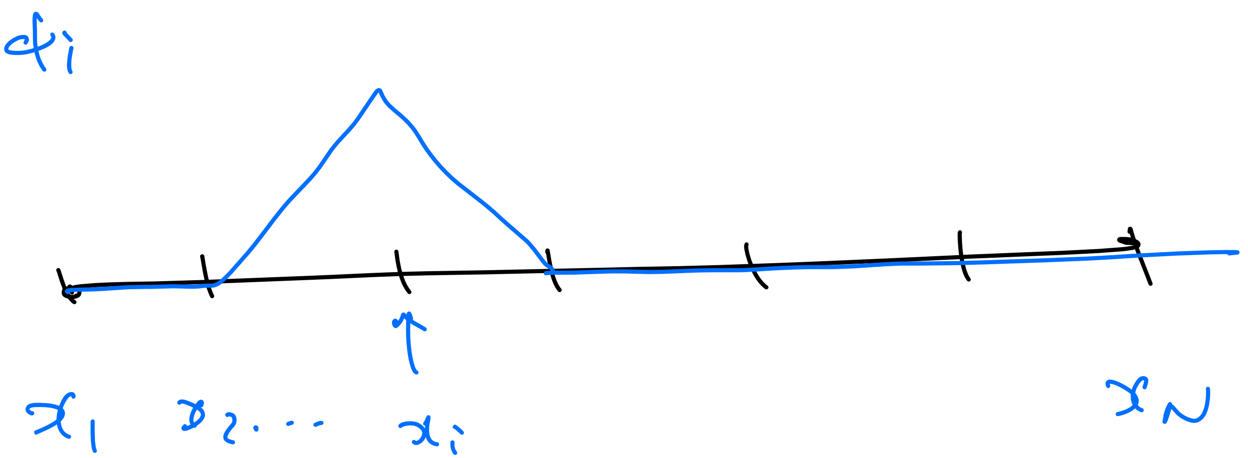
This amounts to

$$T_\tau((s_1, s_2)) = s_1(x_1 - x_3) + s_2(x_2 - x_3) + x_3$$

The nodal basis

The nodal basis ϕ_i resemble "tents" made out of triangles.

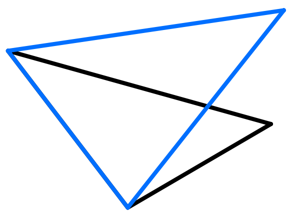
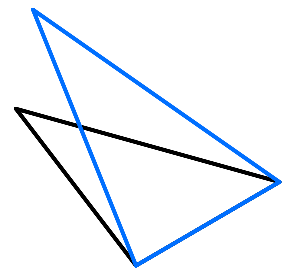
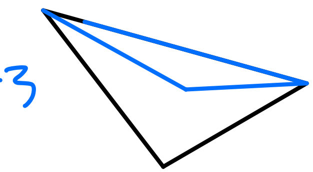
Compare with 1D:



The nodal basis

Through T_τ, $\phi_{i,\tau}$ corresponds to an affine function in τ_{ref},

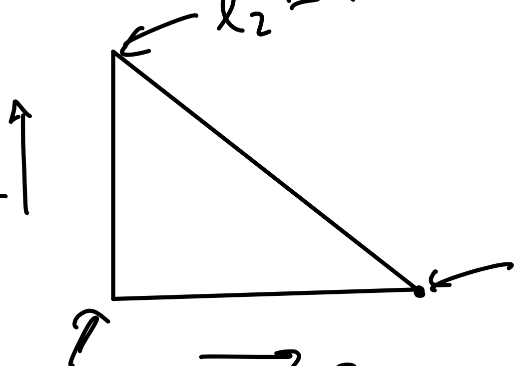
$$\ell(s_1, s_2) = \phi_{i,\tau}(T_\tau(s_1, s_2))$$

ℓ_1  ℓ_2  ℓ_3 

It will be exactly one of the following three functions:

(s_1, s_2)

$\ell_2 = 1$ here



s_2

$\ell_1 = 1$ here

$\ell_3 = 1$ here

s_1

$$\ell_1(s_1, s_2) = s_1$$
$$\ell_2(s_1, s_2) = s_2$$
$$\ell_3(s_1, s_2) = 1 - s_1 - s_2$$

Review: finite elements general setup

We have recast our (approximate) linear Dirichlet problem as

$$\mathrm{A}\boldsymbol{z} = \boldsymbol{b}$$

where $\boldsymbol{z} = (z_1, \ldots, z_N)$ corresponds to

$$u = \sum_{i=1}^{N} z_i \phi_i$$

The functions ϕ_i are the basis of our finite element space, and

$$\mathrm{A}_{ij} = B(\phi_i, \phi_j) = -\int_\Omega (\nabla\phi_i, \nabla\phi_j)\, dx$$

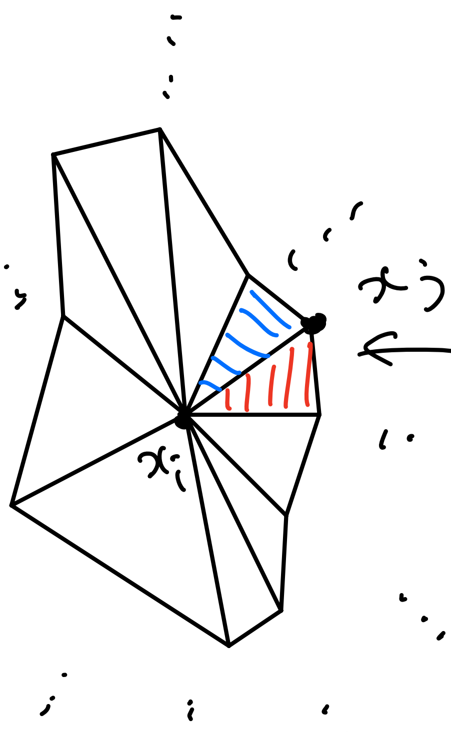
$$b_i = \int_\Omega f\phi_i\, dx$$

$$\int_\Omega = \sum_{\tau \text{ in the triangulation}} \int_\tau$$

Computing the coefficient matrix. . .

For $i, j = 1, \dots, N$ we defined $A_{ij} = B(\phi_i, \phi_j)$, and this becomes

$$
\begin{aligned}
A_{ij} &= -\int_\Omega (\nabla\phi_i(x), \nabla\phi_j(x))\, dx \\
&= -\sum_{\tau : x_i, x_j \in \tau} \int_\tau (\nabla\phi_i(x), \nabla\phi_j(x))\, dx
\end{aligned}
$$



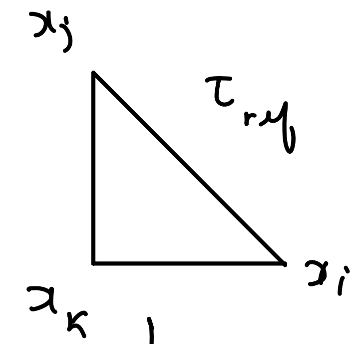
only two triangles

. . . and computing the right hand side

Likewise, for $i = 1, \dots, n$ we have

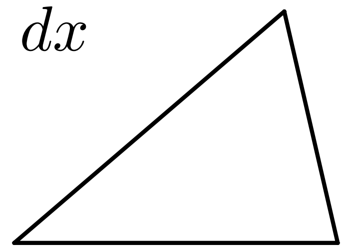
$$
\begin{aligned}
b_i &= \int_D f\phi_i \, dx \\
&= \sum_{\tau : x_i \in \tau} \left\{ \int_\tau f\phi_i \, dx \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right\} \\
&= \sum_{\tau : x_i \in \tau} \left\{ \int_\tau f\phi_i \, dx \qquad\qquad\qquad\qquad \right\}
\end{aligned}
$$

Computing the coefficient matrix



First, let us understand for each triangle, the integral

$$\int_\tau (\nabla\phi_i, \nabla\phi_j)\, dx$$



Since

$$\phi_i(T_\tau(\boldsymbol{s})) = (\boldsymbol{e}_{i,\tau}, \boldsymbol{s}) + \ell_{i,\tau}(\boldsymbol{0})$$
$$\phi_j(T_\tau(\boldsymbol{s})) = (\boldsymbol{e}_{j,\tau}, \boldsymbol{s}) + \ell_{j,\tau}(\boldsymbol{0})$$

we have, by the chain rule

constant in s

constant in s

$$(DT_\tau)^t \boxed{\nabla\phi_i\;(T_\tau(\boldsymbol{s}))} = \boldsymbol{e}_{i,\tau}, \quad \forall \boldsymbol{s} \in \tau_{\text{ref}}.$$

constant in s

$e_{i,\tau}, e_{j,\tau}, e_{k,\tau} \in \{(1,0), (0,1), (-1,-1)\}$

Computing the coefficient matrix

$\nabla\phi_i = M_\tau^{-1} e_{i,\tau}$

$\nabla\phi_j = M_\tau^{-1} e_{j,\tau}$

For brevity, let us write $M_\tau = DT_\tau$.
The change of variable formula says that

$$\int_\tau (\nabla\phi_i, \nabla\phi_j)\, dx = \int_{\tau_{\mathrm{ref}}} ((M_\tau^{-1})^t M_\tau^{-1} e_{i,\tau}, e_{j,\tau}) \det(M_\tau)\, dx$$

All the terms in the integral are constant, and $\mathrm{Area}(\tau_{\mathrm{ref}}) = \frac{1}{2}$, so

$$\int_\tau (\nabla\phi_i, \nabla\phi_j)\, dx = \tfrac{1}{2} \det(M_\tau)((M_\tau^{-1})^t M_\tau^{-1} e_{i,\tau}, e_{j,\tau})$$

Computing the coefficient matrix

In terms of the triangle τ, the matrix M_τ has the form

$$M(x_1, x_2, x_3) = \begin{pmatrix} (x_1 - x_3, e_1) & (x_2 - x_3, e_1) \\ (x_1 - x_3, e_2) & (x_2 - x_3, e_2) \end{pmatrix}$$

Its determinant is given by

$$\begin{aligned} &\det(M(x_1, x_2, x_3)) \\ &= (x_1 - x_3, e_1)(x_2 - x_3, e_2) - (x_1 - x_3, e_2)(x_2 - x_3, e_1) \end{aligned}$$

and its inverse also has a straightforward formula

$$M(x_1, x_2, x_3)^{-1} = \frac{1}{\det(M(x_1, x_2, x_3))} \begin{pmatrix} (x_2 - x_3, e_2) & (x_3 - x_2, e_1) \\ (x_3 - x_1, e_2) & (x_1 - x_3, e_1) \end{pmatrix}$$

Computing the coefficient matrix

Algorithm to compute A_{ij}

Input data: x, τ
(τ is a list of triplets of indices, so $\tau_k = (\tau_{k,1}, \tau_{k,2}, \tau_{k,3}) \ \ \forall k$)

$N =$`len`(x)
$A = (0)_{N \times N}$
`For` $k = 1, \ldots,$`len`(τ)
  $M_\tau = M(x_{\tau_{k,1}}, x_{\tau_{k,2}}, x_{\tau_{k,3}})$
  `For` $i = 1, 2, 3$
    `For` $j = 1, 2, 3$
      $A_{\tau_{k,i}\tau_{k,j}} + = \frac{1}{2}\det(M_\tau)((M_\tau^{-1})^t M_\tau^{-1} e_{i,\tau}, e_{j,\tau})$
`Return` A

Computing the right hand side

It remains to compute the right hand side, first, observe that

$$\int_\tau f\phi_i \, dx = \int_\tau f(x_{center,\tau})\phi_i \, dx + \int_\tau (f(x) - f(x_{\text{center},\tau}))\phi_i \, dx$$

where

$$x_{center,\tau} := \frac{1}{3}(x_{\tau_1} + x_{\tau_2} + x_{\tau_3}).$$

Let us analyze the two terms above.

Computing the right hand side

For the first term, let $\rho(\cdot)$ be a modulus of continuity for f, then

$$\left|\int_\tau (f(x) - f(x_{\text{center},\tau}))\phi_i \; dx\right| \le \frac{1}{2}\det(M_\tau)\rho(\text{diam}(\tau))$$

Then, if every node belongs to at most K triangles,

$$\sum_\tau \left|\int_\tau (f(x) - f(x_{\text{center},\tau}))\phi_i \; dx\right| \le K \max_\tau \det(M_\tau) \max_\tau \rho(\text{diam}(\tau))$$

This means that if f is smooth and τ is a fine triangulation, then this term will be very small.

Finite elements setup

Integrals over triangles –lots of integrals over triangles

As for the term,

$$\int_{\tau} f(x_{\mathrm{center},\tau})\phi_i \, dx$$

we see this is equal to

$$= f(x_{\mathrm{center},\tau}) \int_{\tau} \phi_i \, dx = f(x_{\mathrm{center},\tau}) \det(M_\tau) \int_{\tau_{\mathrm{ref}}} \ell_{\tau,i} \, dx$$

The last integral is elementary, it equals $1/6$ (for all τ and i). This yields the simple expression

$$\int_{\tau} f(x_{\mathrm{center},\tau})\phi_i \, dx = \frac{1}{6} f(x_{\mathrm{center},\tau}) \det(M_\tau)$$

Finite elements setup

Integrals over triangles –lots of integrals over triangles

Input data: x, τ

$b = (0, \ldots, 0)$ (same length as x)

`For` $k = 1, \ldots,$`len(`τ`)`

`Compute` M_τ

`Compute` $\det(M_\tau)$

`For` $i = 1, 2, 3$

$$x_c = \tfrac{1}{3}\left(x_{\tau_{k,1}} + x_{\tau_{k,2}} + x_{\tau_{k,3}}\right)$$

$$b_{\tau_{k,i}} = b_{\tau_{k,i}} + \frac{1}{6} f(x_c) \det(M_\tau)$$

`Return b`