MATH 5340 Fall '23

Lecture 8

In linear algebra we learn that if A is a symmetric $n\times n$ matrix then there are vectors $v_1, v_2, \dots, v_n$ in $\mathbb{R}^n$ such that

$$(v_i, v_j) = \delta_{ij}$$

and $\quad Av_i = \lambda_i v_i \quad$ for some $\lambda_i \in \mathbb{R}$

In particular, $\{v_1, \dots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ and this among other things means that if $x \in \mathbb{R}^n$, then

$$x = (x, v_1)v_1 + (x, v_2)v_2 + \cdots + (x, v_n)v_n$$

$$Ax = \lambda_1 (x, v_1)v_1 + \lambda_2 (x, v_2)v_2 + \cdots + \lambda_n (x, v_n)v_n$$

Remark   In the context of Fourier series, we have an analogous situation in a vector space of functions, for, we saw that if $f \in L^2(0,1)$, then

$$f(x) = \sum_{k \in \mathbb{Z}} \left(f, e^{2\pi i k \cdot y}\right) e^{2\pi i k x}$$

where $(f,g) := \int_0^1 f(y)\,\overline{g(y)}\,dy$ as discussed last week.

The role of the matrix A in this analogy is played by the differential operator $\frac{d^2}{dx^2}$, since

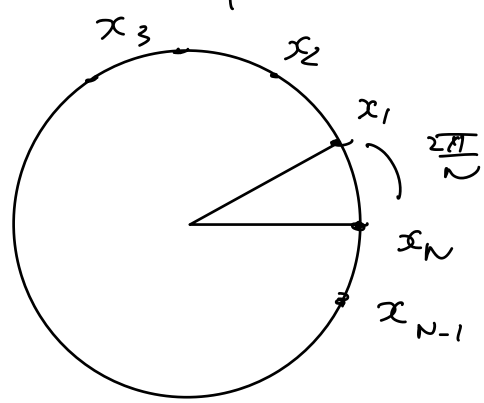
$$\frac{d^2}{dx^2} e^{2\pi i k x} = -(2\pi)^2 k^2 e^{2\pi i k x}$$

Finite dimensional Fourier series

From now on let

$$G_N = \{ \underbrace{(\cos(k h_N), \sin(k h_N))}_{\overset{x_k}{\parallel}} \mid k = 1, 2, .., N \}$$

where $h_N = \frac{2\pi}{N}$



Let us denote by $C(G_N)$ the space of all $\mathbb{C}$-valued functions in G_N, so once we fix a basis of $C(G_N)$ there is a clear isomorphism between $C(G_N)$ and $\mathbb{C}^N$

Now we introduce the operator

$$L : C(G_N) \longrightarrow C(G_N)$$

If $f \in C(G_N)$ then

$$(Lf)(x_k) = \frac{f(x_{k+1}) + f(x_{k-1}) - 2f(x_k)}{h_N^2}$$

Observation: For each $l = 0, 1, .., N-1$ define the function $e_l \in C(G_N)$ by

$$e_l(x_k) = e^{ilkh_N} = e^{ilk\frac{2\pi}{N}}$$

Observe, for each k and l,

$$e_l(x_{k\pm 1}) = e^{\pm il\frac{2\pi}{N}} e_l(x_k)$$

Gathering these, we see that

$$Le_l = \left(e^{il\frac{2\pi}{N}} + e^{-il\frac{2\pi}{N}} - 2\right)\frac{1}{h_N^2}\, e_l$$

writing $\lambda_\ell := \dfrac{\left(e^{i\ell\frac{2\pi}{N}} + e^{-i\ell\frac{2\pi}{N}} - 2\right)}{h_N^2}$

we have

$$L e_\ell = \lambda_\ell e_\ell \qquad \ell = 0, 1, \dots, N-1$$

Since $e^{i\ell\frac{2\pi}{N}} = \cos\left(\frac{2\pi\ell}{N}\right) + i\sin\left(\frac{2\pi\ell}{N}\right)$,

$$\lambda_\ell = \frac{2\left(\cos\left(\frac{2\pi\ell}{N}\right) - 1\right)}{\left(\frac{2\pi}{N}\right)^2}$$

Using L'Hopital's rule, one can see that

$$\lambda_\ell = -\ell^2 + \left(\underset{\text{as } N\to\infty}{\text{small error}}\right)$$

for $\ell = 0, 1, \dots, N$

As $N \to \infty$, the functions $e^{i\ell\left(\frac{2\pi}{N} n\right)}$ correspond to $e^{i\ell\theta}$, $\theta \in [0, 2\pi)$, and

$$\frac{d^2}{d\theta^2} e^{i\ell\theta} = -\ell^2 e^{i\ell\theta}$$