MATH 5340 Fall '23

Lecture 3

(Part 1. Functions and their representation)

To start, let's add a few more definitions

Subspace : Given a Banach space X, a subspace of X is a family $Y \subset X$ such that

$$x_1, x_2 \in Y \Rightarrow x_1 + x_2 \in Y$$

$$x \in Y, \alpha \in \mathbb{R} \Rightarrow \alpha x \in Y$$

In addition, if Y has the property that for any sequence $\{x_n\}_n$ of vectors all in Y their limit x also belongs to Y then we say Y is a closed subspace.

Examples:

* For the space $X = C([0,1])$ each of the sets

$$P_n = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is a polynomial of degree} \leq n \}$$

are closed subspaces

(Exercise: i.e. show that if $\{f_K\}_K$ is an infinite sequence of functions each of which is a polynomial of degree $\leq n$, for some n independent of K, and if $\{f_K\}_K$ converges uniformly to some f, then f must be a polynomial of degree at most n)

* On the other hand, the set

$$P = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is a polynomial} \}$$

is a subspace of $C([0,1])$ but it is not a closed subspace (why! well, we know from calculus we know that the polynomials

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$$

converge uniformly to e^x in every interval $[-M, M]$, and e^x is not a polynomial.

Bases

* A finite family of vectors $x_1, \ldots, x_n$ is called a basis of the space X if

  1. the $x_1, \ldots, x_n$ are linearly independent
  2. every x in X can be expressed as

$$\alpha_1 x_1 + \cdots + \alpha_n x_n$$

for some scalars $\alpha_1, \ldots, \alpha_n$

In such a case the number n is called the dimension of X.

* If Y is a subspace of X then we

may consider Y as a vector space in its own right and with a dimension.

* A space X will be said to be infinite dimensional if for every $n \in \mathbb{N}$ one can find n linearly independent vectors.

Examples

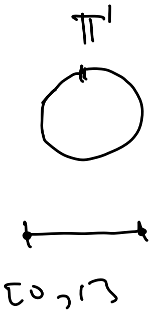
(1) $X = C([0,1])$

$P_n = \{$ Polynomials of degree at most $n \}$

Clearly, P_n is a $n+1$ dimensional subspace of $C([0,1])$, so it is infinite dimensional.

② $C(\mathbb{T}^1)$

$$= \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous, } f(x+1) = f(x) \ \forall x \}$$



$$\cong \{ f \in C([0,1]) \mid f(0) = f(1) \}$$

$$\mathcal{F}_n = \{ f \mid f = a_0 + \sum_{k=1}^{n} a_k \cos(2\pi k x) + b_k \sin(2\pi k x), \quad a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R} \}$$

Clearly $\mathcal{F}_n \subset C(\mathbb{T}^1)$ for every n, and one can show that

$$\dim(\mathcal{F}_n) = 2n+1$$

The class $\mathcal{F}_n$ is called the space of trigonometric polynomials of degree n.

③ Piecewise linear functions in $[0,1]$.

Consider for each N the set of all function in $C([0,1])$ s.t.

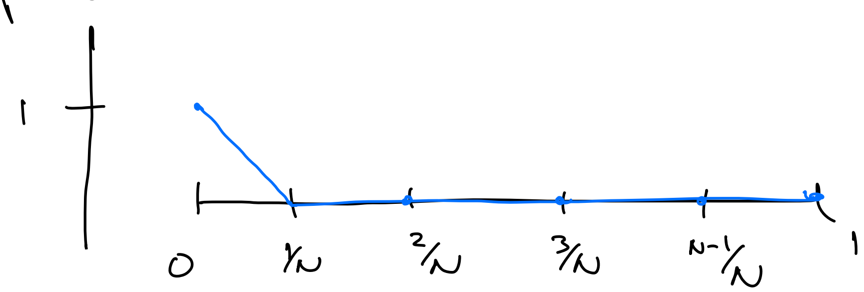
"f restricted to $[k/N, (k+1)/N]$"

$f\big|_{[k/N, \frac{k+1}{N}]}$ is an affine function $(k = 0, \dots, N-1)$

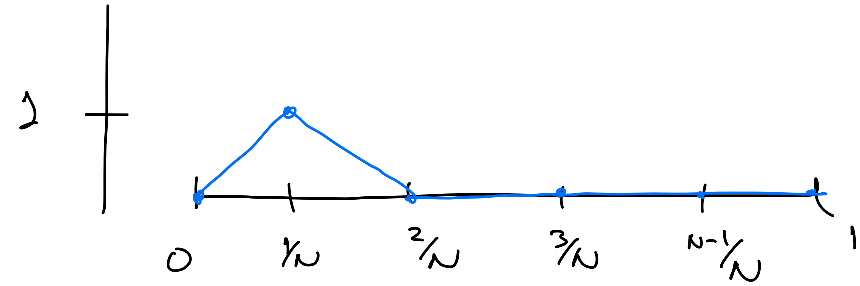
(affine means $f(x) = a \cdot x + b$, $a, b \in \mathbb{R}$)

In geometric terms, the graph of f is a polygonal line whose vertices have x-coordinates at $0, \frac{1}{N}, \frac{2}{N}, \dots \frac{N-1}{N}, 1$)

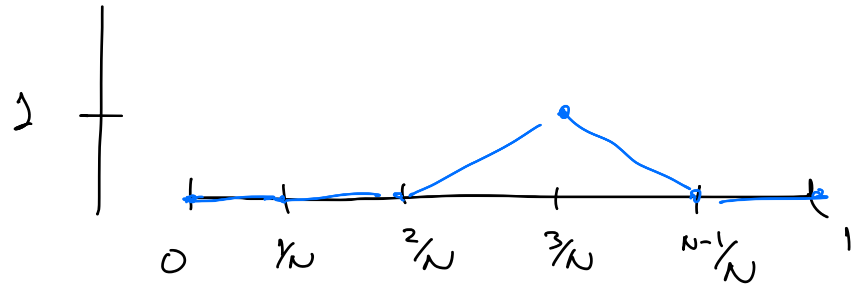
This space has a particularly interesting family of vectors:



$f_0(x)$



$f_1(x)$



⋮

$f_3(x)$

⋮

etc

These $N+1$ functions are called "tent functions", and they form a basis for this subspace.