MATH 5340 Fall '23

Lecture 9

Errata : Q5 :

Part a) B is a $M \times N$ matrix

(M rows , N columns)

Part b) Here, take only the

case $M = N$

Q13 : For this problem you must assume that $f(0) = f(1)$, otherwise the formula as stated is incorrect; and it should read

$$\int_0^1 f'(x) e^{-2\pi i n x} \, dx = 2\pi i n x \int_0^1 f(x) e^{-2\pi i n x} \, dx$$

Part 1. Functions and their representation

(final lecture)

Regularity / Vanishing of coefficients / Approximation rates

A well known important fact in Fourier analysis is that given

$$f = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k x}$$

Then the faster $|c_k| \to 0$ as $|k| \to \infty$, the smoother (i.e. the more derivatives) f has, and the faster the partial sums $S_N f$ approach f. The following observations are meant to illustrate this.

(1) Suppose $f : [0,1] \to \mathbb{C}$ with $f(0) = f(1)$ and suppose f is differentiable in $(0,1)$, Then the following holds:

$$\int_0^1 |f'(x)|^2 dx < \infty \Rightarrow \sum_{k\in\mathbb{Z}} |c_k|^2 |k|^2 < \infty$$

(see EX 13/14)

In this case we have $\forall N \in \mathbb{N}$

$$(*) \quad \int_0^1 |(S_N f)'|^2 dx \leq \sum_{k\in\mathbb{Z}} |c_k|^2 |k|^2$$

(prove this! Hint: Use that $\forall f$,

$$S_N f - f \perp S_N f$$
$$S_N f' - f' \perp S_N f'$$
)

The bound in $(*)$ is significant.

Theorem (see: Sobolev embeddings)

If $\phi : [0,1] \to \mathbb{C}$ is continuous and ϕ' exists in $(0,1)$, and for some $M > 0$ we have $\int_0^1 |\phi'(x)|^2 dx \leq M^2$ then

$$|\phi(x_1)-\phi(x_2)| \leq M|x_1-x_2|^{1/2}$$

In other words, ϕ is Hölder continuous with modulus of continuity $\omega(r) = M r^{1/2}$.

Proof We are going to use the following inequality for integrals:

$\forall\ \psi \in L^2(0,1)$ and $\forall\ a,b \in (0,1)$,

$$\left|\int_a^b \psi(x)\,dx\right|^2 \leq \left(\int_a^b |\psi(x)|^2\,dx\right)(b-a)$$

(see: Jensen's inequality, or Cauchy–Schwartz)

Take $x_1, x_2 \in (0,1)$ and wlog $x_1 < x_2$, then

$$|\phi(x_1)-\phi(x_2)| = \left|\int_{x_1}^{x_2} \phi'(x)\,dx\right|$$

$$\leq \left(\int_{x_1}^{x_2} |\phi'(x)|^2\,dx\right)^{1/2} (x_2-x_1)^{1/2}$$

By the inequality above, since $x_2 > x_1$, we have $|x_2 - x_1| = x_2 - x_1$, so

$$|f(x_1) - f(x_2)| \le \left(\int_{x_1}^{x_2} |f'(x)|^2 dx\right)^{1/2} |x_1 - x_2|^{1/2}$$
$$\le \left(\int_0^1 |f'(x)|^2 dx\right)^{1/2} |x_1 - x_2|^{1/2}$$
$$\le M |x_1 - x_2|^{1/2}$$

As a consequence of this theorem, we have that if f is such that

$$M^2 = \sum_{k \in \mathbb{Z}} |c_k(f)|^2 |k|^2 < \infty$$

Then the sequence $\{S_N f\}_N$ has a common modulus of continuity, namely $\omega(r) = M r^{1/2}$, from here we will show next class that in this case

$$\lim_{N \to \infty} \|S_N f - f\|_\infty = 0$$

i.e. the $S_N f$ converge uniformly to f.