MATH 5340 Fall '23

Lecture 7

(Part I. Functions and their representation)

Fourier series

Given a function $f:(a,b)\longrightarrow \mathbb{C}$ for each $k\in\mathbb{Z}$ we define its corresponding Fourier coefficient

$$c_k(f)=\frac{1}{L}\int_a^b f(y)\,e^{-\frac{2\pi}{L}iky}\,dy$$

Here, $L=b-a$. Given these numbers, the series

$$\sum_{k=-\infty}^{+\infty} c_k(f)\,e^{\frac{2\pi}{L}ikx}$$

is called the Fourier series of the function f, this is denoted as

$$f \sim \sum_{k=-\infty}^{+\infty} c_k(f) e^{\frac{2\pi}{L} ikx}$$

(we will discuss in what sense does the series represent f)

Henceforth, take $a=0$, $b=1$, so $L=1$.

Consider the space

$$L^2(0,1) = \left\{ f : (0,1) \to \mathbb{C} \;\middle|\; \int_0^1 |f(x)|^2 dx < \infty \right\}$$

with the norm

$$\|f\|_{L^2(0,1)} \left(= \|f\|_2 \right) := \sqrt{\int_0^1 |f(x)|^2 dx}$$

Lemma: If $f \in L^2(0,1)$, then given $\varepsilon > 0$, there is a continuous function $g : [0,1] \to \mathbb{C}$, such that

$$\| f - g \|_2 < \varepsilon$$

(For the proof, see any intro to measure theory, or graduate analysis textbook)

Corollary: If $f \in L^2(0,1)$, and $\varepsilon > 0$, there is a $N > 0$ and complex numbers $c_{-N}, \dots, c_{-1}, c_0, c_1, \dots, c_N$ such that the trigonometric polynomial

$$P(x) = \sum_{k=-N}^{N} c_k e^{2\pi k i x}$$

is such that

$$\| f - P \|_2 < \varepsilon .$$

Proof. Fix $\varepsilon > 0$ and $f \in L^2(0,1)$, first, from the lemma, there is a function $g : [0,1] \to \mathbb{C}$, continuous, such that

$$\|f - g\|_2 < \varepsilon/2$$

The algebra generated by the functions $\{e^{2\pi i k x}\}_{k \in \mathbb{Z}}$ is separating any interval of the form $[1/m, 1]$ for $m \in \mathbb{N}$.

Choose m large enough so that

$$\int_0^{1/m} |g(x)|^2 \, dx < \varepsilon/4$$

Then apply the Weierstrass-Stone theorem on $[1/m, 1]$ to obtain a trig. polynomial

$$p = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$$

such that

$$\max_{\frac{1}{m} \le x \le 1} |g(x) - p(x)| < \varepsilon/8$$

$$\Rightarrow \left(\int_{1/m}^{1} |g(x) - p(x)|^2 dx \right)^{1/2}$$

$$\le \max_{\frac{1}{m} \le x \le 1} |g(x) - p(x)| < \varepsilon/8$$

exercise

Gathering everything together, we have

$$\|f - p\|_2 \le \|f - g\|_2 + \|p - g\|_2$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The space $L^2(0,1)$ is a complex Hilbert space because its norm arises from an inner product:

$$(f,g) := \int_0^1 f(x)\overline{g(x)}\,dx$$

This inner product, defined for any two $f, g \in L^2(0,1)$, has the following properties:

* $$(f,f) = \int_0^1 f(x)\overline{f(x)}\,dx = \int_0^1 |f(x)|^2\,dx \geq 0$$

* $$(f,g) = \overline{(g,f)}$$

$$\left(\overline{(g,f)} = \overline{\int_0^1 g(x)\overline{f(x)}\,dx} = \int_0^1 \overline{g(x)}\,\overline{\overline{f(x)}}\,dx = \int_0^1 f(x)\overline{g(x)}\,dx = (f,g) \right)$$

Reminder : If $f(x)$ is written as

$$f(x) = \alpha(x) + i\,\beta(x),$$

where $\alpha(x), \beta(x)$ are real valued, then

$$\overline{f(x)} = \alpha(x) - i\,\beta(x)$$

---

* $(\lambda_1 f_1 + \lambda_2 f, g) = \lambda_1 (f_1, g) + \lambda_2 (f_2, g)$

* $(f, \lambda_1 g_1 + \lambda_2 g_2)$
$$= \overline{\lambda}_1 (f, g_1) + \overline{\lambda}_2 (f, g_2)$$

* $\|f\|_2 = \sqrt{(f,f)}$

* $|(f,g)| \leq \|f\|_2 \|g\|_2$
(the Cauchy-Schwartz inequality)

As in linear algebra, if $\langle f, g\rangle = 0$ we say f and g are orthogonal, denoted $f \perp g$

Remark: (Fourier series with real-valued functions)

Suppose that $f \in L^2(0,1)$ only takes real values, that is $\overline{f(x)} = f(x)$, in that case its Fourier coefficients have a special property

$$\overline{c_k(f)} = \overline{\int_0^1 f(y)\, e^{-2\pi i k y}\, dy}$$

$$= \int_0^1 \overline{f(y)}\, e^{2\pi i k y}\, dy$$

$$= c_{-k}(\overline{f})$$

Since $\overline{c_k(f)} = c_{-k}(\bar{f})$ in general,

for f s.t. $f = \bar{f}$, we have

$$\overline{c_k(f)} = c_{-k}(f)$$

Take this and let's take a look

at the Fourier series,

$$\sum_{k=-\infty}^{k=+\infty} c_k(f)\, e^{2\pi i k x}$$

(assuming convergence)

$$= c_0(f) + \sum_{k=1}^{\infty} \left(c_k(f)\, e^{2\pi i k x} + c_{-k}(f)\, e^{-2\pi i k x} \right)$$

$$c_{-k}(f)\, e^{-2\pi i k x} = \overline{c_k(f)\, e^{2\pi i k x}}, \text{ so}$$

$$c_k(f)\, e^{2\pi i k x} + c_{-k}(f)\, e^{-2\pi i k x}$$
$$= 2\,\mathrm{Re}\left(c_k(f)\, e^{2\pi i k x} \right)$$

So, we expect (under convergence)

$$\sum_{k=-\infty}^{+\infty} c_k(f)\, e^{2\pi i k x}$$

$$= c_0(f) + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left(c_k(f)\, e^{2\pi i k x}\right)$$

Observe:

$$c_k(f) = \int_0^1 f(y)\, e^{-2\pi i k y}\, dy$$

$$= \int_0^1 f(y)\left(\cos(-2\pi k y) + i \sin(-2\pi k y)\right) dy$$

$$= \int_0^1 f(y) \cos(2\pi k y)\, dy - i \int_0^1 f(y) \sin(2\pi k y)\, dy$$

Then,

$$\mathrm{Re}\left(c_k(f) e^{2\pi i k x}\right)$$

$$= \left(\int_0^1 f(y)\cos(2\pi k y)\,dy\right)\cos(2\pi k x)$$

$$+ \left(\int_0^1 f(y)\sin(2\pi k y)\,dy\right)\sin(2\pi k x)$$

so, we see then that the series

$$\sum_{k=-\infty}^{k=+\infty} c_k(f) e^{2\pi i k x}$$

reduces to a real trigonometric when f is real.

Remark: (One way $\mathbb{C}$-valued function are practical)

Compare computing the integrals

$$\int_0^1 \cos(2\pi k x)\cos(2\pi l x)\,dx \quad ,$$

$$\int_0^1 \sin(2\pi k x)\sin(2\pi \ell x)\,dx,$$

$$\int_0^1 \cos(2\pi k x)\sin(2\pi \ell x)\,dx$$

versus computing the integrals

$$\int_0^1 e^{2\pi i k x}\cdot\overline{e^{2\pi i \ell x}}\,dx$$

$$= \int_0^1 e^{2\pi i (k-\ell)x}\,dx$$

$(k\neq \ell)$

$$= \frac{1}{2\pi i(k-\ell)} e^{2\pi i(k-\ell)x}\Big|_{x=0}^{x=1}$$

$$= 0.$$

This shows that if we write

$$e_k(x) := e^{2\pi i k x}, \quad k\in\mathbb{Z},$$

Then

$$(e_k, e_\ell) = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

so, the family of functions $\{e_k\}_{k \in \mathbb{Z}}$ is what we call an orthonormal family of functions.

Remark. Observe that the Fourier coefficients $c_k(f)$ of a function f are nothing but the inner products of f with the family $e_k(x)$.

$$c_k(f) = (f, e_k)$$

The partial sums $S_N(f)$

For each $N \in \mathbb{N}$ and $f \in L^2(0,1)$, we define

$$S_N(f)(x) = \sum_{k=-N}^{N} c_k(f)\, e^{2\pi i k x}$$

Theorem: Given $f \in L^2(0,1)$, we have

$$\lim_{N \to \infty} \| f - S_N(f) \|_2 = 0$$

Proof. First, a claim:

Given any trigonometric polynomial

$$P(x) = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$$

for some numbers $C_{-N}, \dots, C_N$

(not necessarily the Fourier coefficients of f)

then

$$\| f - S_N(f) \|_2 \le \| f - P \|_2$$

Proof of the claim:

By construction, for $k = -N, \dots, 1, \dots, N$

we have (check this!)

$$(f - S_N(f), e_k) = 0$$

This means that if g is a linear combination of $e_{-N}, \dots, e_N$, then

$$(f - S_N(f), g) = 0$$

In this case,

$\| f - P \|_2^2$

$= \| f - S_N(f) + (S_N(f) - P) \|_2^2$

The differen $S_N(f)-P$ is a linear combination of $e_{-N},\dots,e_N$, so

$$f - S_N(f) \perp S_N(f)-P$$

so, by the Pythagorean theorem,

$$\|f-P\|_2^2 = \|f-S_N(f)+(S_N(f)-P)\|_2^2$$

$$= \|f-S_N(f)\|_2^2 + \|S_N(f)-P\|_2^2$$

Therefore,

$$\|f-P\|_2 \geq \|f-S_N(f)\|_2 .$$

To prove the theorem, let $\varepsilon > 0$, Then we know there is a trigonometric polynomial P, s.t. for some N, $c_{-N},\dots$

$$P = \sum_{k=-N}^{N} c_k e^{2\pi i k x}$$

and $$\|f - P\|_2 < \varepsilon$$

But, by the claim,

$$\|f - S_N(f)\|_2 < \varepsilon$$

what's more, if $N' \geq N$, then

$$\|f - S_{N'}(f)\|_2 < \varepsilon .$$

$\Longrightarrow$

$$\lim_{N \to \infty} \|f - S_N(f)\|_2 = 0 .$$

Remark on Problem #10

All the functions $e^{2\pi i k x}$ have the following property:

$$\frac{d^2}{dx^2} e^{2\pi i k x} = -(2\pi k)^2 e^{2\pi i k x}$$

i.e. $$\frac{d^2}{dx^2} e_k(x) = -(2\pi k)^2 e_k(x)$$

We can make a finite / discrete analog of this using a uniform grid on the unit circle, and this is the purpose of problem 10.