Scientific Computing
MATH 5340

Lecture 25

$\Big(C^2(\Omega) = \{ u : \Omega \to \mathbb{R},$

$\partial_{x_i} u$ exist for each i

$\partial^2_{x_i x_j} u$ exist and are continuous for each i,j $\} \Big)$

Last time: For $u \in C^2(\Omega)$, the equation

$$\Delta u = f \text{ in } \Omega$$

holds if and only if

$$-\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \, \phi \in C_c^2(\Omega)$$

"ϕ is compactly supported, i.e. $\overline{\{\phi \neq 0\}}$ is a compact set inside Ω."

This can be used to pose the equation at the level of say, piecewise linear functions.

Divergence formulation of elliptic PDE

The same can be done with more general equations, such as

$$\mathrm{div}(a(x)\nabla u) + (b(x), \nabla u) + c(x)u = f \qquad (*)$$

(this is the general linear elliptic operator in divergence form, $a(x)$ is a $d\times d$ symmetric matrix, positive semidefinite at each x, $b(x)$ is a vector field, $c(x)$ is a scalar field).

If u is $C^2(\Omega)$, $a(x), b(x), c(x), f(x)$ are continuous, $a(x)$ is C^1, then $(*)$ holds if and only if

$$-\int_\Omega a(x)\nabla u\cdot\nabla\phi(x)\,dx + \int_\Omega (b(x)\cdot\nabla u(x))\phi(x) + c(x)u(x)\phi(x)\,dx$$

$$= \int_\Omega f(x)\phi(x)\,dx \quad \forall\ \phi\in C^2_c(\Omega)$$

Divergence formulation of elliptic PDE

Motivated by this, we introduce the bilinear functional,

$$B_L(u,v) = -\int_\Omega \nabla u \cdot \nabla v dx \qquad (\text{for } Lu = \Delta u)$$

$$B_L(u,v) = -\int_\Omega a\nabla u \cdot \nabla v \, dx + \int_\Omega (b\cdot\nabla u + cu)\phi \, dx$$

for $Lu = \operatorname{div}(a\nabla u) + (b\cdot\nabla u) + cu$

The Dirichlet problem:

$$Lu = f \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

is equivalent to

$$B_L(u,\phi) = \int_\Omega f\phi \, dx \quad \forall \; \phi \in C_c^2(\Omega)$$
$$u = g \text{ on } \partial\Omega.$$

Divergence formulation of elliptic PDE

Analytical problem

$$B(u,\phi) = \int_\Omega f\phi\, dx \quad \boxed{\forall \phi \in C^2_c(\Omega)}\,, \quad \boxed{u \in C^2(\Omega)}$$

and $u - g = 0$ on $\partial\Omega$

$\boxed{S_0}$ = some finite dimensional subspace of a function space (e.g. piecewise linear functions)

Finite element problem

$$B(u,\phi) = \int_D f\phi\, dx \quad \boxed{\forall\, \phi \in \mathcal{S}_0}\,, \quad \boxed{u \in S_0}$$

and $u - g$ ~~$\in \mathcal{S}_0$~~ $= 0$.

These stay the same!!!

(here, $B(u,\phi)$ is the problem's bilinear form just introduced)

Finite element methods: overview

Galerkin's idea

Assume function in S_0 "vanish" on $\partial\Omega$, and let $\phi_1, \ldots, \phi_N \in S_0$ be a basis, then every $u \in S_0$ can be represented thus

$$u = g + \sum_{i=1}^{N} z_i \phi_i$$

Plug in this u in $B(u, \phi) = \int f \phi \, dx$

$$B(g, \phi_j) + \sum_{i=1}^{N} z_i B(\phi_i, \phi_j) = \int_D f \phi_j \, dx \quad \forall\, j$$

$$\boxed{\begin{aligned} \mathrm{A}_{ij} &= B(\phi_i, \phi_j) \\ b_j &= \int_D f \phi_j \, dx - B(g, \phi_j) \end{aligned}} \quad (*)$$

The whole (approximate) Dirichlet problem is now

$$\boxed{\mathrm{A}\boldsymbol{z} = \boldsymbol{b}} \quad (*)$$

Finite element methods: overview

Elements and finite elements

• Decompose Ω into smaller and simpler shapes ("elements").
• Use this decomposition to make a respective decomposition of functions in a certain class ("finite elements").

Finite element methods: overview

Elements and finite elements

From this overview, there are 3 ideas we need to learn how to implement

1. How to choose and represent on a computer the function space S_0

Most delicate, challenging part.

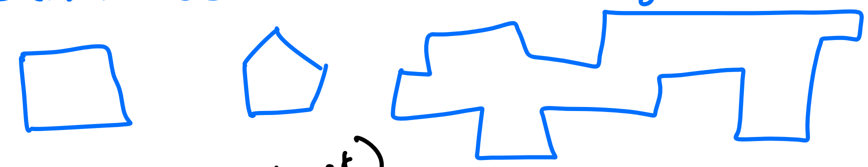
2. How to choose a basis $\phi_1, \ldots, \phi_N$ for S_0

3. How to compute the matrix coefficients $B(\phi_i, \phi_j)$.

Triangulations

To address ①, let us learn about triangulations.

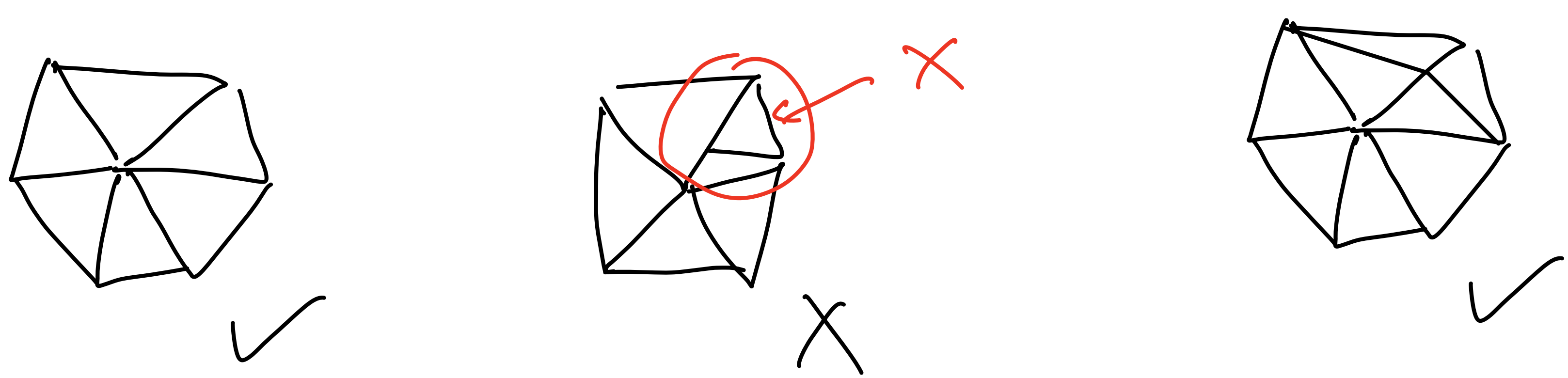
Let Ω be a polygonal domain.



A **triangular mesh** in Ω, denoted $\mathcal{T}$, is (almost) a partition of Ω made out of a finite number of triangles

$$\Omega = \tau_1 \cup \tau_2 \cup \ldots \cup \tau_M$$

where the triangles meet only at a single vertex or along a common side. Often, these triangles are referred to as **elements**.
Another term used for triangular mesh is triangulation.



Triangulations

Why we care about triangular meshes

A **finite element space** $\mathcal{S}$ is a finite dimensional space of functions associated to a triangular mesh $\mathcal{T}$, concretely, it is the set of functions ϕ satisfying the two properties

1. ϕ is continuous in $\overline{\Omega}$.
2. ϕ is an affine function in each triangle of $\mathcal{T}$.

Triangulations

Q: Why do we care about triangular meshes?

A: Convenient representation of piecewise linear functions!

Triangulations

For a triangular mesh $\mathcal{T}$ we denote the set of vertices of its triangles as $V(\mathcal{T})$.

Proposition

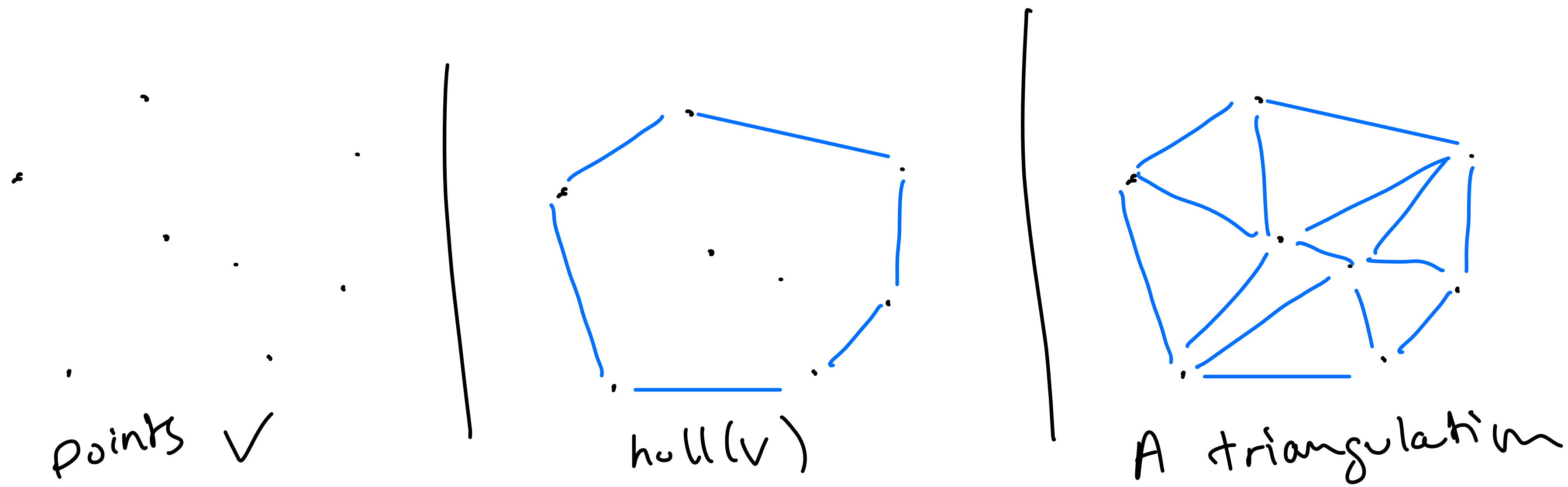
Every finite element is determined by its values in $V(\mathcal{T})$.

Conversely, any function in $V(\mathcal{T})$ defines, in a unique manner, a finite element.

(this follows from what we said last week about affine functions in triangles)

Triangulations

To any finite set $V \subset \mathbb{R}^2$ one can associate triangulations:



A triangulation for the finite set V is a triangulation T of $\Omega = \text{hull}(V)$ where the set of vertices of the triangulation, $\mathcal{V}(T)$, is equal to V.

In particular, this means that a point in V only meets a triangle of T when this point is one of the triangle's vertices.

Triangulations

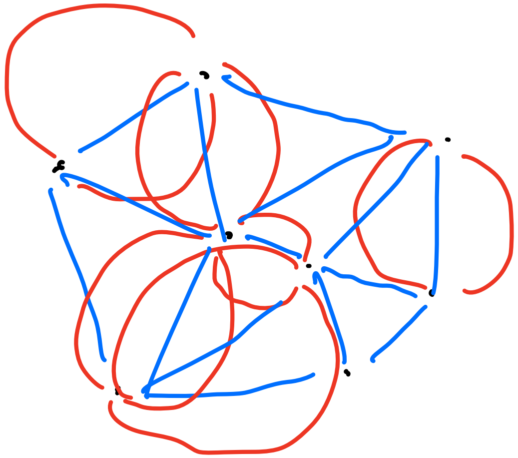
Delaunay triangulations

These are a special class of triangulations commonly used in graphics and finite elements.

If $x, y \in V$, draw an edge between them if the following property holds:

There is a circle passing through both x and y containing no other points of V in its interior

Surprisingly –at least at first thought– this simple rule always produces a triangulation!



Delaunay triangulations: extremal property
Among all triangulations of a vertex set V, the Delauney triangulation will maximize the value of the smallest angles among all its triangles.