MATH 5340 Fall '23

Lecture 2

Part 1 : Functions and their representation

Banach spaces

A vector space (typically infinite-dimens.) with a norm $\|\cdot\|$ is called a normed space. In that space we have a topology given by a distance function:

$$d(x,y) = \|x-y\|$$

If a normed space is complete* in this metric then it is called a <u>Banach space</u>.

(* meaning every Cauchy sequence has a limit)

A Banach space X is called separable if it contains a countable subset D which is dense in X, that is given $x \in X$ $\exists$ an infinite sequ $\{x_n\}$ in D s.t.

$$\lim_{n \to \infty} \| x - x_n \| = 0 .$$

Example

① $\mathbb{R}^n$ (n-dimensional Euclidean space)

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

② $C([a,b])$

$$= \{ f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is continuous} \}$$

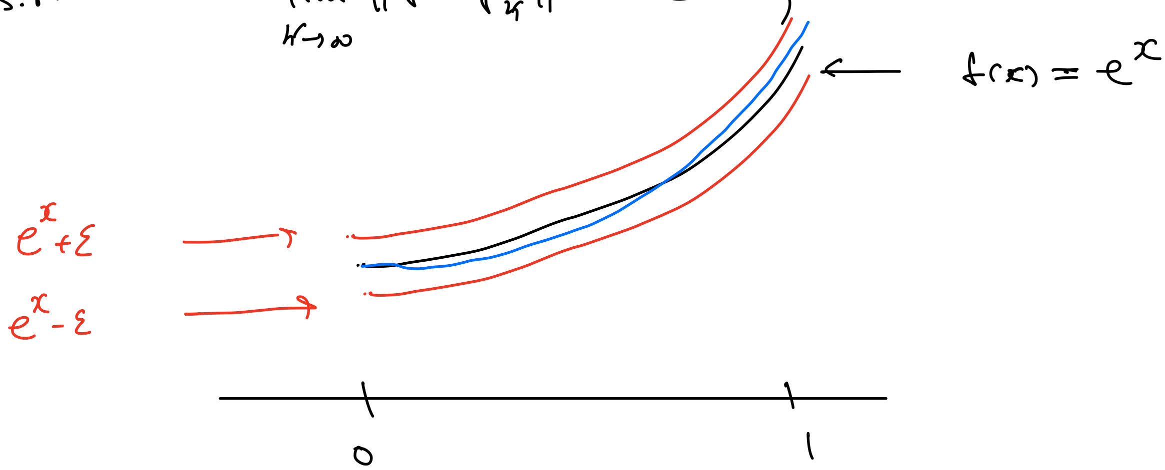
$$\| f \| := \max_{a \le x \le b} |f(x)|$$

(If $\{f_k\}$ is a sequence of elements of $C([a,b])$ s.t. $\forall\ \varepsilon > 0$ $\exists$ k_0 s.t.

$$\| f_K - f_j \| < \varepsilon \quad \text{if} \quad K, j > K_0$$

then there is a continuous function $f : [a,b] \to \mathbb{R}$

s.t. $\displaystyle \lim_{K \to \infty} \| f - f_K \| = 0$



$$f_K(x) = \sum_{n=0}^{K} \frac{1}{n!} x^n$$

③ $C(K) = \{ f : K \longrightarrow \mathbb{R} \mid f$ is continuous $\}$

Here, $K \subset \mathbb{R}^d$ is a compact set (closed and bounded) e.g.

$$K = [-L, L]^d, \quad K = \{ x \in \mathbb{R}^d \mid \|x\| \le R \}$$

$$\| f \| = \max_{x \in K} | f(x) |$$

④ $L^2(\mathbb{R})$

$$= \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f \text{ is (Lebesgue) integrable}, \; \int_{-\infty}^{+\infty} |f|^2 dx < \infty \right\}$$

$$\|f\|_{L^2} = \sqrt{\int_{-\infty}^{+\infty} |f|^2 dx}$$

This is not just a Banach space, it is a Hilbert space.

⑤ $$L^2(0,1) = \left\{ f : (0,1) \to \mathbb{C} \;\middle|\; \int_0^1 |f|^2 dx < \infty \right\}$$

$$\|f\|_{L^2} = \sqrt{\int_0^1 |f|^2 dx}$$

⑥ $\ell^2(\mathbb{Z})$ ("little ℓ-2")

$$\ell^2(\mathbb{Z}) = \left\{ (a_n)_{n\in\mathbb{Z}} \;\middle|\; a_n \in \mathbb{C} \; \forall n \text{ and } \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\}$$

$$\|(a_n)\|_{\ell^2} = \sqrt{\sum_{n \in \mathbb{Z}} |a_n|^2}$$

These are some examples of function spaces. For each, one can produce finite dimensional subspaces that can be used to approximate them.

Example of finite dimensional spaces

(1) $C([0,1])$ Consider

$$P_n = \{ f = \text{polynomials of degree at most } n \}$$

So, $f \in P_n$ if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

for some numbers $a_0, \ldots, a_n$

$\mathcal{P} := \bigcup_{n \in \mathbb{N}} \mathcal{P}_n \leftarrow$ This is the set of all polynomials

Theorem (Weierstrass)

Given any $f : [a,b] \to \mathbb{R}$, continuous, and $\varepsilon > 0$, there exists a polynomial P of sufficiently high degree such that

$$\max_{a \le x \le b} |f(x) - P(x)| < \varepsilon .$$