MATH 5340 Fall '23

Lecture 4

(Part I. Functions and their representation)

Today: Convolutions, proof of the Weierstrass approximation theorem, statement of the Weierstrass-Stone theorem

Convolutions

The convolution is an operation that takes two functions over $\mathbb{R}$ and return a new function, called their convolution.

Definition: If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are two bounded, integrable functions (meaning that the integrals $\int_{-\infty}^{+\infty} |f| \, dx$ and $\int_{-\infty}^{+\infty} |g| \, dx$ are both finite) their convolution, denoted $f * g$, is the function defined by

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy .$$

Some properties:

* $(f_1+f_2)*g = f_1*g + f_2*g$

  for any functions f_1, f_2, g

* $(\lambda f)*g = \lambda(f*g)$

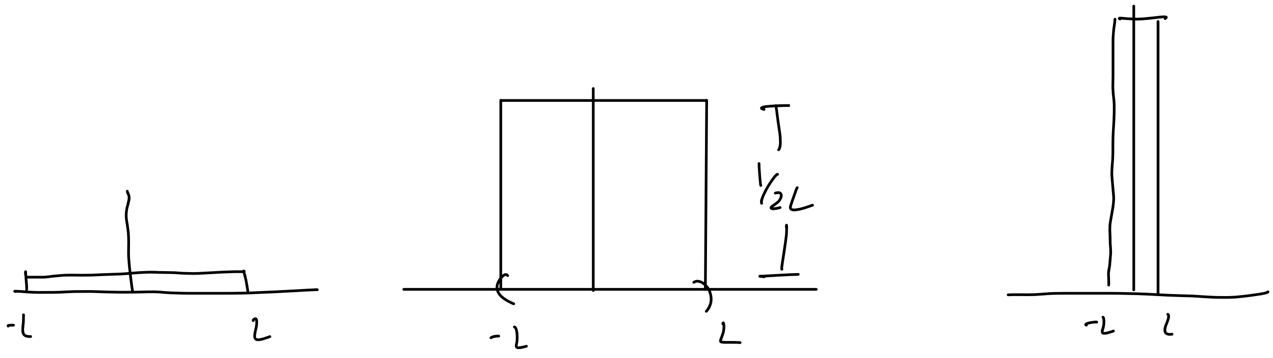
  for any function f, g and $\lambda \in \mathbb{R}$

* $f*g = g*f$

  for any function f, g

Convolutions are averages / or rather, weighted averages of functions. Consider for $L>0$, the function:

$$g(x) = \begin{cases} \frac{1}{2L} & \text{in } (-L, L) \\ 0 & \text{outside } (-L, L) \end{cases}$$

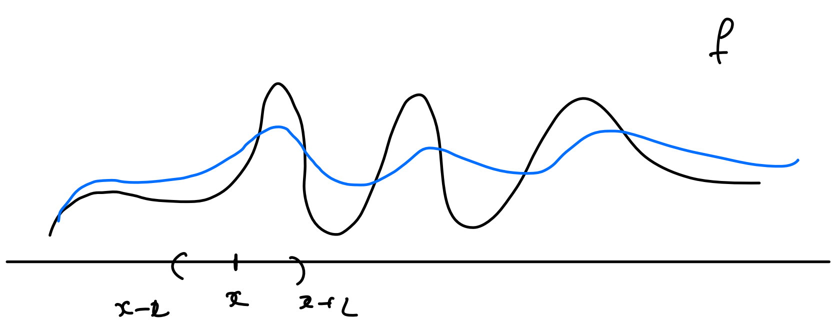


(L large)　　　　　　　　　　　　　　　　　　　　L small

Observe:
$$\int_{-\infty}^{+\infty} g(x)\,dx = \int_{-L}^{L} \frac{1}{2L}\,dx = 1$$

In this case, given f, we have

$$f*g(x) = \int_{-\infty}^{+\infty} f(y)\,g(x-y)\,dy$$
$$= \int_{x-L}^{x+L} f(y)\,\frac{1}{2L}\,dy$$
$$= \frac{1}{2L}\int_{x-L}^{x+L} f(y)\,dy$$



If L is small and f is continuous, we expect $f*g$ to close to f, and in fact, this is the content of one of the problem in Notebook 1.

In general, $f*g$ will have "more smoothness" (i.e. more derivatives) than f or g. For example, if in addition to the above we assume g has a continuous and bounded derivative, then $f*g$ will have a derivative.

<u>Sketch of a proof</u> : Let's assume $g'(x)$ exists and that there is a $M>0$ s.t. $|g'(x)| \leq M$ $\forall x \in \mathbb{R}$. Let's show $f*g$ is differentiable for all x.

$$\lim_{h\to 0} \frac{(f*g)(x+h) - (f*g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{-\infty}^{+\infty} f(y)\,g(x+h-y)\,dy - \int_{-\infty}^{+\infty} f(y)\,g(x-y)\,dy\right]$$

$$= \lim_{h\to 0} \int_{-\infty}^{+\infty} f(y)\,\frac{g(x+h-y) - g(x-y)}{h}\,dy$$

Under the assumption on f and g one can show the limit of this integral exists and equals the integral of the pointwise limit, that is

$$\lim_{h\to 0} \frac{1}{h}\big((f*g)(x+h) - (f*g)(x)\big)$$

$$= \int_{-\infty}^{+\infty} \lim_{h\to 0} f(y)\,\frac{g(x-y+h) - g(x-y)}{h}\,dy$$

$$= \int_{-\infty}^{+\infty} f(y)\,g'(x-y)\,dy$$

In other words

$$\frac{d}{dx}(f*g) = f * \frac{dg}{dx}$$

Clearly we can iterate this, resulting in the following theorem

<u>Theorem</u>: If $f, g : \mathbb{R} \to \mathbb{R}$ are bounded and integrable and g has derivatives of order up to k, all bounded, then $f * g$ has derivatives of order k, and

$$\frac{d^k}{dx^k}(f * g) = f * \frac{d^k g}{dx^k}$$

<u>Exercise</u>: Suppose f is some integrable function such that for some $M > 0$ f vanishes outside $[-M, M]$, and suppose P is a polynomial of degree n, show then that

$$f * P(x) = \int_{-\infty}^{+\infty} f(y) P(x-y)\, dy$$

is a polynomial of degree at most n.

Solution: A bit of preliminary work shows it is enough to answer this in the special case where $P(x)$ is a monomial, i.e. $P(x) = x^n$ for some $n = 0, 1, \ldots$

$$(f * y^n)(x) = \int_{-\infty}^{+\infty} f(y)\,(x-y)^n\,dy$$

(since f vanishes outside $[-M, M]$)
$$= \int_{-M}^{M} f(y)\,(x-y)^n\,dy$$

Since $(x-y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (-y)^{n-k}$,

$$(f * y^n)(x) = \sum_{k=0}^{n} \int_{-M}^{M} f(y) \binom{n}{k} x^k (-y)^{n-k}\,dy$$

$$= \sum_{k=0}^{n} a_k x^k$$

where
$$a_k = \binom{n}{k} \int_{-M}^{M} f(y)\,(-y)^{n-k}\,dy$$

Smoothing kernels

Consider functions $K : \mathbb{R} \to \mathbb{R}$ s.t.

$$K \geq 0, \quad \int_{-\infty}^{+\infty} K \, dy = 1, \quad K \in C^\infty$$

For such functions we introduce a 1-parameter family of rescalings:

$$K_s(y) := \frac{1}{s} K\left(\frac{y}{s}\right)$$

Check that $\int_{-\infty}^{+\infty} K_s(y) \, dy = 1 \quad \forall \, s > 0$,

and K_s is still ≥ 0, and C^∞.

Theorem (see Problem 5-7 in Notebook 1)

If f vanishes outside $[-M, M]$ and has a modulus of continuity, and K is as above,

$$\max_{-M \le x \le M} \left| f * K_s(x) - f(x) \right| \xrightarrow[\text{as } s \to 0]{} 0$$

The Gaussian Kernel

This is kernel given by a normal distribution

$$K_\sigma(y) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

We are going to use this kernel in our proof of the Weierstrass theorem, and for that we need to use its power series representation.

First, the exponential itself.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

We can write this as

$$e^x = \sum_{k=0}^{N} \frac{1}{k!} x^k + \sum_{k=N+1}^{\infty} \frac{1}{k!} x^k$$

The term, $\displaystyle\sum_{k=N+1}^{\infty} \frac{1}{k!} x^k$ is called the residual

$$\sum_{k=N+1}^{\infty} \frac{1}{k!} x^k = x^{N+1} \sum_{k=N+1}^{\infty} \frac{1}{k!} x^{k-(N+1)}$$

(write $v = k-(N+1)$)

$$= x^{N+1} \sum_{v=0}^{\infty} \frac{1}{(v+N+1)!} x^v$$

$(a+b)! \geq a!\, b!$ (check this is true!)

triangle inequality

Then $$\left| \sum_{v=0}^{\infty} \frac{1}{(v+N+1)!} x^v \right| \leq \sum_{v=0}^{\infty} \frac{1}{(v+N+1)!} |x|^v$$

$$\leq \left(\sum_{v=0}^{\infty} \frac{1}{v!} |x|^v \right) \frac{1}{(N+1)!}$$

$$= \frac{1}{(N+1)!} e^{|x|}$$

So,

$$\left| \sum_{k=N+1}^{\infty} \frac{1}{k!} x^k \right| \leq |x|^{N+1} \frac{e^{|x|}}{(N+1)!}$$

This shows that $\forall$ $M>0$,

$$\max_{-M \leq x \leq M} \left| e^x - \sum_{k=0}^{N} \frac{1}{k!} x^k \right| \xrightarrow[\text{as } N \to \infty]{} 0$$

In particular, given $M>0$ and $\sigma>0$,

$$\max_{-M \leq y \leq M} \left| \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} - \sum_{k=0}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{(-1)^k}{k!} \frac{y^{2k}}{(2\sigma^2)^k} \right| \xrightarrow[N \to \infty]{\text{as}} 0$$

With this inequality we have all the tools to prove the Weierstrass theorem (next class)