MATH 5340 Fall '23

Lecture 18

Part III. Graphs, Laplacians, and Markov chains

Last time we introduced Dirichlet's principle, which states that if u_0 minimizes the functional

$$J(u) = \frac{1}{2}\mathcal{E}(u) + \sum_{i=1}^{N} u(x_i) f(x_i)$$

over the set V_g

$$V_g = \{ u \in C(G) \mid u = g \text{ in } G \setminus D \}$$

Then u_0 solves

$$\begin{cases} L u_0 = f & \text{in } D \\ u_0 = g & \text{in } G \setminus D \end{cases}$$

Today, we should prove, among other things, that there is a unique minimizer, provided D is connected.

Theorem : If D is connected, then there is a minimiser.

The philosophy of the proof we will present follows what is known as the direct method of the calculus of variations

In it, we first show there is a $C>0$ s.t.

$$J(u) \geq -C \quad \forall \; u \in V_g .$$

this means that

$$\inf_{u \in V_g} J(u)$$

is finite.

In such a case, it follows that $\forall n$ $\exists$ $u_n \in V_g$ s.t.

$$J(u_n) < \inf_{u \in V_g} J(u)$$

(about limits)

Then, it is an exercise to show that

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in V_g} J(u)$$

Notice that if we knew that

$$u_n \to u_* \quad \text{for some } u_* \in C(G)$$

then by the continuity of J, it would follow that

$$J(u_*) = \lim_{n\to\infty} J(u_n)$$

so u_* would be the minimum.

What we will show is that if $\{u_n\}$ is a minimizing sequence then $\exists$ $R>0$ $(R = R(G, D, f, g))$ s.t.

$$\sum_{i=1}^{N} |u_n(x_i)|^2 \leq R^2 \quad \forall n$$

so as vectors in $\mathbb{R}^N$, the $\{u_n\}$ lie inside a ball of radius R, and thus (by Heine-Borel theorem) a subsequence must converge to a u_*. One can show u_* belongs to V_g and thus that it is the minimizer.

That's the big picture. To put it to use in our problem we will need two important tools, and the first is called the strong maximum principle for harmonic functions

Definition: A function $u: G \to \mathbb{R}$ is said to be harmonic in $D \subset G$, if $Lu(x_i) = 0$ for all $x_i \in D$.

Theorem (strong maximum principle)

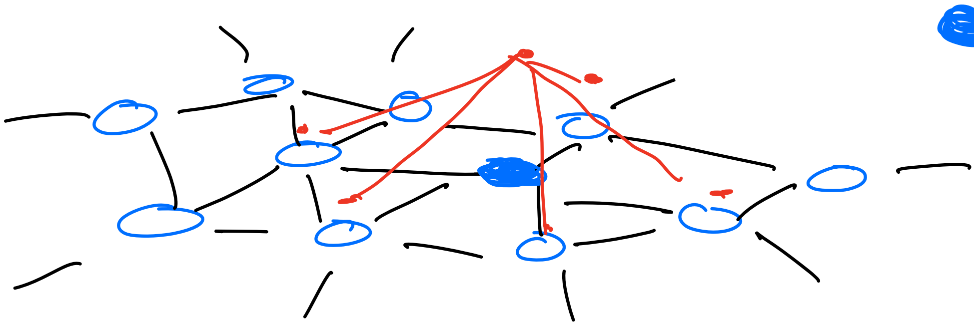
If $D \subset G$ is connected and u is harmonic in D, then the only way u can achieve its maximum value at a point in D is if u is constant in D.

Proof. Suppose $u : G \longrightarrow \mathbb{R}$, and that $x_{i_0} \in D$ is such that

$$u(x_{i_0}) = M = \max_{i=1,\dots,N} u(x_i)$$

We shall show that if x_i is a neighbor of x_{i_0}, then $u(x_i) = M$ as well.

$\bullet = x_i$



Since $Lu(x_{i_0})=0$, we have

$$\sum_{j=1}^{N} w_{i_0 j}\left(u(x_j)-u(x_{i_0})\right)=0$$

$$\parallel$$

$$\sum_{j=1}^{N} w_{i_0 j}\left(u(x_j)-M\right)=0$$

Recall that x_j is a neighbor of x_{i_0} $\equiv$ $w_{i_0 j}>0$, so we have

$$\sum_{\substack{x_j \text{ neighbor} \\ \text{of } x_{i_0}}} w_{i_0 j}\left(u(x_j)-M\right)=0$$

This is a sum of non-positive numbers that adds up to zero, therefore each summand must be zero,

$$w_{i_0 j}\left(u(x_j)-M\right)=0 \quad \forall j$$

and since $w_{i_0 j}>0$ for each neighbor x_j, it must be that

$$u(x_j) = M$$

for every neighbor of x_{i_0}.

To conclude the theorem, let $x_{i_*} \in D$. Then, since D is connected there is a sequence of nodes in D

$$x_{i_0}, x_{i_1}, \dots, x_{i_k}, \dots, x_{i_*}$$

such that x_{i_k} and $x_{i_{k+1}}$ are neighbors. From the first proof one sees that if the maximum of u is achieved in one node of the sequence then it must be achieved on the next one, then starting from $u(x_{i_0}) = M$, we conclude in succession that $u(x_{i_1}) = M$, $u(x_{i_2}) = M$, ... until $u(x_{i_*}) = M$. This shows that u is a constant in D.

This is a very important fact about harmonic functions and the Laplacian, let us see some immediate consequences.

Corollary (the comparison principle)
Suppose D is connected and that $u, v : G \to \mathbb{R}$ are such that

$$Lu = f \text{ in } D$$
$$Lv = f \text{ in } D$$
$$u \leq v \text{ in } G \setminus D$$

Then $u \leq v$ in D also, and if $u = v$ somewhere in D, then $u = v$ everywhere in D.

Proof By considering the function

$$w = u - v ,$$

we have

$$\begin{aligned} Lw &= Lu - Lv \\ &= f - f \\ &= 0 \quad \text{in } D \end{aligned}$$

$$w = u - v \le 0 \quad \text{in } G \setminus D .$$

So w is harmonic in D and ≤ 0 in $G \setminus D$. By the strong maximum principle either w is a constant $^{*}\le 0$ everywhere or else $w < 0$ in D, in either case $w \le 0$ everywhere

$$\Longrightarrow u \le v \quad \text{in } D .$$

* the constant is ≤ 0 because $w \le 0$ in $G \setminus D$, and G is connected.

∎

The next tool in our study of minimizers is something called a spectral gap or Poincaré inequality. It basically estimates the size of a function $u: G \to \mathbb{R}$ in the L^2-norm in terms of the Dirichlet energy of u.

<u>Lemma</u>: If D is connected (recall that always G is connected as well) there is a number $\lambda_0 > 0$ s.t.

$$\lambda_0 \sum_{i=1}^{N} u(x_i)^2 \leq \frac{1}{2} \sum_{i,j=1}^{N} \left(u(x_i) - u(x_j)\right)^2 \omega_{ij}$$

for every $u \in V_0 = \{u \mid u(x_i) = 0 \text{ if } x_i \notin D\}$

<u>Proof</u> It suffices to prove the lemma for those functions u with

$$\sum_{i=1}^{N} u(x_i)^2 = 1$$

because otherwise $u = 0$ everywhere (and then there is nothing to prove) we can divide u by a > 0 constant C so that

$$\sum_{i=1}^{N} \left(\frac{u(x_i)}{C} \right)^2 = 1$$

and apply the inequality for this case and obtain

$$\lambda_0 \sum_{i=1}^{N} \left(\frac{u(x_i)}{C} \right)^2 \leq \sum_{i,j=1}^{N} \left(\frac{u(x_i)}{C} - \frac{u(x_j)}{C} \right)^2 w_{ij}$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

but

$$\lambda_0 \frac{1}{\cancel{C^2}} \sum_{i=1}^{N} u(x_i)^2 \leq \frac{1}{\cancel{C^2}} \sum_{i,j=1}^{N} (u(x_i) - u(x_j))^2 w_{ij}$$

This reduces the proof to showing that

$$\min \left\{ \tfrac{1}{2}\,\mathcal{E}(u) \;\middle|\; u \in V_0,\ \sum u^2 = 1 \right\} =: \lambda_0 > 0$$

Clearly the minimum λ_0 must be ≥ 0.

Let's see that $\lambda_0 = 0$ is impossible.

For, if $\lambda_0 = 0$, there would be a function $u_* \in V_0$, with $\sum u_*^2 = 1$ and such that

$$\tfrac{1}{2}\,\mathcal{E}(u_*) = 0$$

i.e.

$$\sum_{i,j=1}^{N} \left(u_*(x_i) - u_*(x_j) \right)^2 w_{ij} = 0$$

it follows that $u_*(x_i) - u_*(x_j) = 0$ if i and j are any two neighbors. Because G is connected this means u_* must be a constant, but this contradicts that

$$u_* = 0 \quad \text{in } G \setminus D$$

and

$$\sum_{i=1}^{N} u_*^2(x_i) = 1$$

This contradiction shows $\lambda_0 > 0$.