Scientific Computing
MATH 5340

Lecture 24

(contained in Part 3)

Part 4. Finite differences and finite elements

Finite elements have a very different philosophy than what we saw in Part 3. The emphasis is not on discretizing the pointwise application of a derivative operator, but to narrow it down to finite dimensional spaces of functions. In this class, such functions will be the so called linear elements

Finite element methods

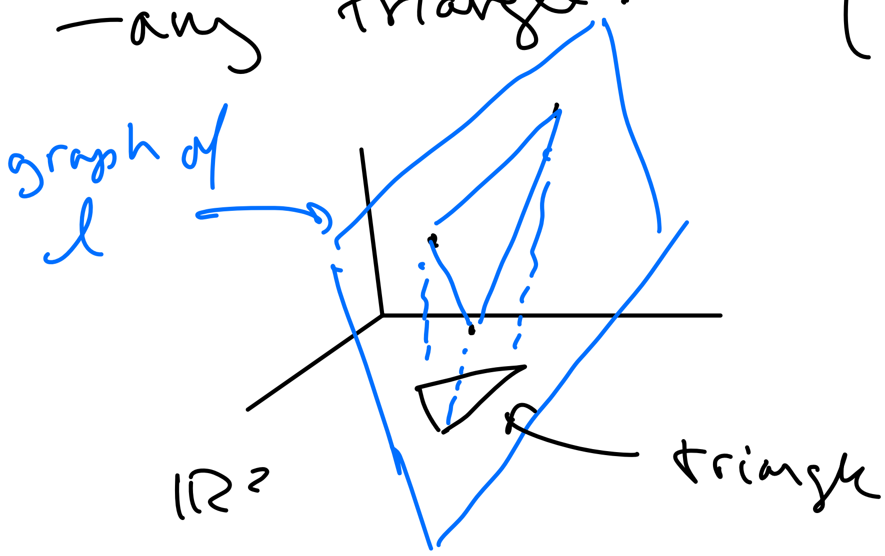
Linear elements — the basic idea

If a function in $\mathbb{R}^2$ is affine,

$$\ell(x_1, x_2) = a_1 x_1 + a_2 x_2 + a_0$$

then the function ℓ is completely determined by its values on the vertices of a triangle — any triangle.

(In $\mathbb{R}^n$, the corresponding fact is that an affine function is determined by its values in an n-simplex)



Finite element methods

Linear elements — the basic idea (continued)

Let's explain this in a second way. If a triangle T has vertices given by vectors $\bar{x}_1, \bar{x}_2, \bar{x}_3$ then any point inside T can be represented in a unique way as

$$\bar{x} = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \alpha_3 \bar{x}_3$$

where $0 \le \alpha_i \le 1$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. These α_i are called the barycentric coordinates of $\bar{x}$ w.r.t. to $\bar{x}_1, \bar{x}_2, \bar{x}_3$. With this we can express $\ell(\bar{x})$:

$$\boxed{\ell(\bar{x}) = \alpha_1 \ell(\bar{x}_1) + \alpha_2 \ell(\bar{x}_2) + \alpha_3 \ell(\bar{x}_3)}$$

Finite element methods

Linear elements — the basic idea (continued)

Now, we can talk about piecewise linear functions in $\mathbb{R}^2$. The idea (which we will formalize in a moment) is to consider sets which are made out of the union of triangles and then on such a set consider the space:

$$L = \{ f \mid f \text{ is affine on each triangle from the union, is continuous in the set formed by all the triangles} \}.$$

If there are N vertices total, then L is a N-dimensional vector space.

Finite element methods

That's the first idea for finite elements. The second helps us pose a respective PDE among piecewise linear functions.

Lemma: Let $\Omega \subset \mathbb{R}^2$ be a piecewise differentiable domain (e.g. a polygonal domain). A function $u : \Omega \to \mathbb{R}$ which is twice differentiable will solve

$$\Delta u = f$$

if and only if the following holds for every $\phi : \Omega \to \mathbb{R}$ which is C^2 and of compact support in Ω:

$$-\int_\Omega \nabla u \cdot \nabla \phi \, d\bar{x} = \int_\Omega f \phi \, d\bar{x}$$

Finite element methods

From this lemma we can define when is a piecewise linear function ϕ a solution to $\Delta u = f$.

"$\Delta u = f$" is said to hold if

$$-\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx$$

for all piecewise linear functions ϕ.

Divergence formulation of elliptic PDE

The same can be done with more general equations, such as

$$\operatorname{div}(a(x)\nabla u) + (b(x), \nabla u) + c(x)u = f$$

Divergence formulation of elliptic PDE

Analytical problem

$$B(u,\phi) = \int_D f\phi\, dx \;\; \forall\, \phi \in H_0^1(D)$$

and $u - g \in H_0^1(D)$

Finite element problem

$$B(u,\phi) = \int_D f\phi\, dx \;\; \forall\, \phi \in \mathcal{S}_0$$

and $u - g \in \mathcal{S}_0$

(here, $B(u,\phi)$ is the problem's bilinear form)

Finite element methods: overview

Galerkin's idea

$$u = g + \sum_{i=1}^{N} z_i \phi_i$$

$$B(g, \phi_j) + \sum_{i=1}^{N} z_i B(\phi_i, \phi_j) = \int_D f \phi_j \, dx \quad \forall\, j$$

$$\begin{aligned} \mathrm{A}_{ij} &= B(\phi_i, \phi_j) \\ b_j &= \int_D f \phi_j \, dx - B(g, \phi_j) \end{aligned}$$

$$\mathrm{A}\boldsymbol{z} = \boldsymbol{b}$$

Finite element methods: overview

Elements and finite elements

- Decompose D into smaller and simpler shapes (“elements”).
- Use this decomposition to make a respective decomposition of functions in a certain class (“finite elements”).

Triangulations

Let D be a polygonal domain.

A **triangular mesh** in D, denoted $\mathcal{T}$, is a partition of D made out of a finite number of triangles

$$D = \tau_1 \cup \tau_2 \cup \ldots \cup \tau_M$$

where the triangles meet only at a single vertex or along a common side. Often, these triangles are referred to as **elements**.
Another term used for triangular mesh is triangulation.

Triangulations

Why we care about triangular meshes

A **finite element space** $\mathcal{S}$ is a finite dimensional space of functions associated to a triangular mesh $\mathcal{T}$, concretely, it is the set of functions ϕ satisfying the two properties

1. ϕ is continuous in $\overline{D}$.
2. ϕ is an affine function in each triangle of $\mathcal{T}$.

Triangulations

Q: Why do we care about triangular meshes?

A: Convenient representation of piecewise linear functions!

Triangulations

For a triangular mesh $\mathcal{T}$ we denote the set of vertices of its triangles as $V(\mathcal{T})$.

Proposition

Every finite element is determined by its values in $V(\mathcal{T})$.

Conversely, any function in $V(\mathcal{T})$ defines, in a unique manner, a finite element.

Triangulations

To any finite set $V \subset \mathbb{R}^2$ one can associate triangulations:

A triangulation for the finite set V is a triangulation T of $D = \mathrm{hull}(V)$ where the set of vertices of the triangulation, $\mathcal{V}(T)$, is equal to V.

In particular, this means that a point in V only meets a triangle of T when this point is one of the triangle's vertices.

Triangulations

Delaunay triangulations

These are a special class of triangulations commonly used in graphics and finite elements.

If $x, y \in V$, draw an edge between them if the following property holds:

There is a circle passing through both x and y containing no other points of V in its interior

Surprisingly –at least at first thought– this simple rule always produces a triangulation!

Delaunay triangulations: extremal property
Among all triangulations of a vertex set V, the Delauney triangulation will maximize the value of the smallest angles among all its triangles.