MATH 5340 Fall '23

Lecture 12

Part II. Ordinary Differential Equations

Backward and Forward Euler

(wrapping up from last time)

The following theorem gives one way of addressing the potential short time interval of existence.

Theorem (Picard plus a little extra, see notebook 2)

Suppose $f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ is such that

* $|f(x,t) - f(y,t)| \leq L|x-y| \quad \forall t \in [0,\infty)$, $\forall x,y \in \mathbb{R}^d$

* $|f(0,t)| \leq C e^{Lt}$

Then, for any initial vector $x_0 \in \mathbb{R}^d$

There is a unique continuous function $x(t)$, $x : [0,\infty) \to \mathbb{R}^d$

such that

$$\dot{x}(t) = f(x(t), t) \quad \text{for } t \in (0,\infty)$$

$$x(0) = x_0$$

$$e^{-Lt}|x(t)| \in L^\infty((0,\infty)) \quad , \text{ i.e.}$$

There is a $M > 0$ s.t.

$$e^{-Lt}|x(t)| \leq M \quad \forall t \in [0,\infty)$$

PE

A remark about the proof

For the proof one works with the following functional space

$$X_\lambda := \{ x : [0,\infty) \to \mathbb{R}^d \mid x(t) \text{ is continuous and } |x(t)| \leq Me^{\lambda t} \text{ for some } M > 0 \}$$

This is a Banach space with the norm

$$\|x\|_{\infty,\lambda} := \sup_{0\le t<\infty} \{ e^{-\lambda t} |x(t)| \}$$

The proof of the theorem will follow from the Banach contraction mapping theorem applied to

$$T : X_\lambda \longrightarrow X_\lambda$$

defined by

$$T(x)(t) = x_0 + \int_0^t f(x(s), s)\, ds \quad , \quad t \in [0,\infty)$$

Then you must show

* $T(x) \in X_\lambda$ if $x \in X_\lambda$
* If $\lambda > L$, then T is a contraction mapping in X_λ.

Forward and Backward Euler schemes

(AKA Explicit / implicit Euler)

At their most basic these schemes produce discrete sequences of vectors that are meant to approximate the values of the solution to an IVP at a discrete set of times.

Given the IVP

$$\begin{cases} \dot{x}(t) = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

and given a number $h > 0$, called the time step, we produce a sequence of vectors as follows

(Define $t_k := hk$, $k = 0, 1, 2, \ldots$)

Forward Euler

We generate a sequence x_k $(k = 0, 1, \cdots)$ recursively as follows

- x_0 is given
- $x_{k+1} = x_k + h f(x_k, t_k)$

Backward Euler

As before, we generate a sequence x_k recursively as follows,

* x_0 is given
* Given x_k, we define x_{k+1} as any vector such that

$$x_{k+1} = x_k + h f(x_{k+1}, t_{k+1})$$

(If we define $g(x) = x - h f(x, t_{k+1})$, and g happens to be invertible, then

$$x_{k+1} = g^{-1}(x_k)$$)

Remark: One could consider a transformation

$$T_h : \mathbb{R}^d \to \mathbb{R}^d$$

defined by $T_h(x) = x_k + h f(x, t_{k+1})$

(here, x_k, t_{k+1} are already given)

The way x_{k+1} is defined is as a fixed point of T_h. Conveniently, T_h will be a contraction mapping if $h < \frac{1}{L}$, because $\forall\, y_1, y_2 \in \mathbb{R}^d$

$$
\begin{aligned}
& T_h(y_1) - T_h(y_2) \\
&= \cancel{x_k} + h f(y_1, t_{k+1}) - \cancel{x_k} - h f(y_2, t_{k+1}) \\
&= h\left(f(y_1, t_{k+1}) - f(y_2, t_{k+1}) \right)
\end{aligned}
$$

$$
\Rightarrow \quad |T_h(y_1) - T_h(y_2)| = h\, |f(y_1, t_{k+1}) - f(y_2, t_{k+1})| \le hL\, |y_1 - y_2|
$$

so if $hL < 1$, T_h is a contraction mapping and there is a unique fixed point.

The hope with either of these schemes is that if h is very small and $x(t)$ solves the IVP (this is called sometimes the analytical solution) then

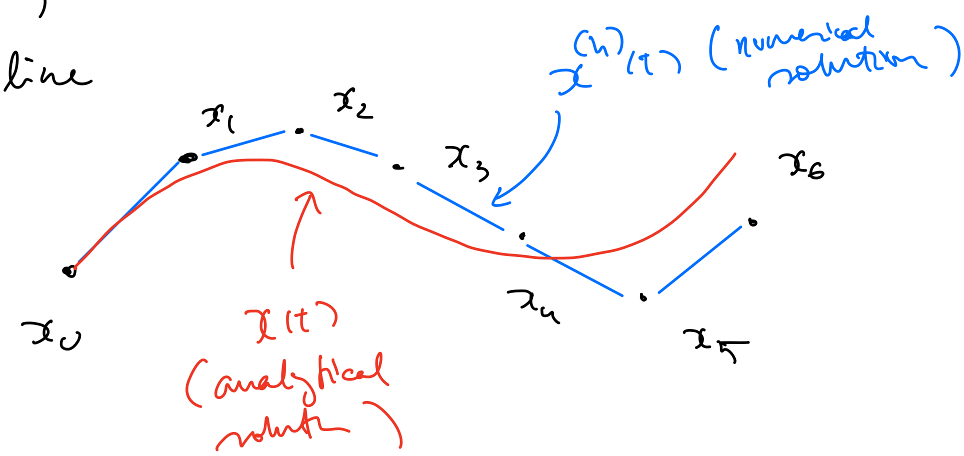
x_k is close to $x(t_k)$

Let's make this hope into a theorem. To do so it will be convenient to work with the piecewise linear function generated by the sequence $\{x_k\}$ generated above.

Definition: Let $\{x_k\}$ be a sequence generated by Forward or Backward Euler with time step $h>0$, then define the function

$$x^{(h)}(t) := x_k + \frac{t-t_k}{h}(x_{k+1}-x_k)$$
$$= \left(1-\frac{t-t_k}{h}\right)x_k + \frac{t-t_k}{h}\,x_{k+1}$$

For $d=2$, this can be visualized via a polygonal line



Next time we will show that for either backward or forward Euler

$$\max_{0 \le t \le T} | x^{(h)}(t) - x(t) | \xrightarrow[h \to 0]{} 0$$

for any time T.