# MATH 5340 Fall '23

## Lecture 14

Last time

① **Local Truncation Error for Backward and Forward Euler:**

(Fix $h>0$, $n \in \mathbb{N}$ and let $x(t)$ solve the IVP)

(Forward Euler)

$$\varepsilon_n^{(n)} := \frac{x(t_{n+1}) - x(t_n) - h\, f(x(t_n), t_n)}{h}$$

(Backward Euler)

$$\varepsilon_n^{(n)} := \frac{x(t_{n+1}) - x(t_n) - h\, f(x(t_{n+1}), t_{n+1})}{h}$$

(2) We introduced the maps

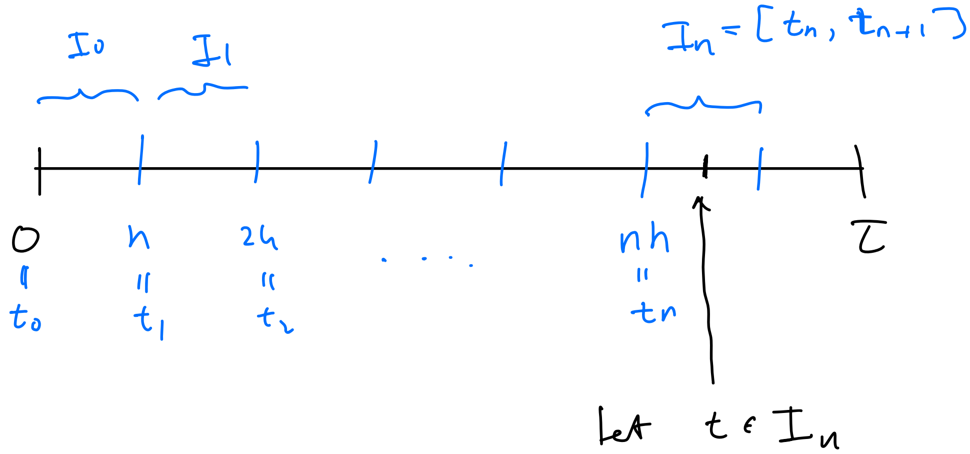
$$T^{(n)} : C([0,\infty)) \to C([0,\infty))$$

(or, if you prefer,

$$T^{(n)} : C([0,\tau]) \to C([0,\tau])$$

for some $\tau > 0$ )

We define these maps in a not so explicit recurrent fashion, let's write them down in an equivalent form,




Now let $x(t)$ be any function in $C([0,\tau])$, then define $T^{(n)}(x)(t)$ as follows:

If $t \in [0, T]$, let $n$ be such that $t_n \leq t < t_{n+1}$,

$$T_{FE}^{(h)}(x)(t) := x_0 + \sum_{k=0}^{n-1} h f(x(t_k), t_k) + (t - t_n) f(x(t_n), t_n)$$

this is a continuous and piecewise linear function.

and likewise we define

$$T_{BE}^{(h)}(x)(t) := x_0 + \sum_{k=0}^{n-1} h f(x(t_{k+1}), t_{k+1}) + (t - t_n) f(x(t_{n+1}), t_{n+1})$$

for $t \in [t_n, t_{n+1}]$,

Also last time, we proved:

<u>Lemma</u>: If $TL < 1$, then $T_{FE}^{(h)}$ and $T_{BE}^{(h)}$ are contraction mappings.

As a consequence, for every $h > 0$ there is a unique function $x^{(h)}$ such that (technically, $x_{FE}^{(h)}$, $x_{BE}^{(h)}$)

$$T^{(h)}(x^{(h)}) = x^{(h)}$$

Today, we prove

**Theorem** If $\tau L < 1$, then

$$\|x - x^{(h)}\|_{L^\infty([0,\tau])} \leq \frac{\tau}{1-\tau L}\left(\max |\varepsilon_n^{(h)}| + O(h^2)\right)$$

Here, $x$ is the analytical solution, which is assumed to be twice differentiable in $[0,\tau]$.

**Proof**. We shall use that

$$T(x) = x, \quad T^{(h)}(x^{(h)}) = x^{(h)}.$$

Indeed,

$$\begin{aligned} x - x^{(h)} &= T(x) - T^{(h)}(x^{(h)}) \\ &= T(x) - T^{(h)}(x) + T^{(h)}(x) - T^{(h)}(x^{(h)}) \end{aligned}$$

by the triangle inequality

(all the following norms are $\| \cdot \|_{L^\infty([0,\tau])}$ )

$$\| x - x^{(n)} \| \leq \| T(x) - T^{(n)}(x) \| + \| T^{(n)}(x) - T^{(n)}(x^{(n)}) \|$$

Last time we saw $T^{(n)}$ is a contraction mapping with contraction constant $\tau L < 1$, so

$$\| x - x^{(n)} \| \leq \| T(x) - T^{(n)}(x) \| + \tau L \, \| x - x^{(n)} \|$$

$$\Rightarrow \quad (1 - \tau L) \| x - x^{(n)} \| \leq \| T(x) - T^{(n)}(x) \|$$

and because $1 - \tau L > 0$,

$$\| x - x^{(n)} \| \leq \frac{1}{1 - \tau L} \| T(x) - T^{(n)}(x) \|$$

Since $T(x) = x$, the quantity $\| T(x) - T^{(n)}(x) \|$ estimates how far is $x$ from being a fixed point of $T^{(n)}$.

Let me now estimate $\| x - T^{(n)}(x) \|$ for $T^{(n)} = T^{(n)}_{FE}$. Well, we have

(for $t \in [t_n, t_{n+1}]$)

$$T^{(n)}(x)(t) = x_0 + \sum_{k=0}^{n-1} h f(x(t_k), t_k) + (t - t_n) f(x(t_n), t_n)$$

and

$$\begin{aligned} T(x)(t) &= x_0 + \int_0^t f(x(s), s)\, ds \\ &= x_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(x(s), s)\, ds + \int_{t_n}^{t} f(x(s), s)\, ds \end{aligned}$$

since $\dot{x} = f(x(t), t)$,

$$T(x)(t) = x_0 + \sum_{k=0}^{n-1} x(t_{k+1}) - x(t_k) + \left(x(t) - x(t_n)\right)$$

Now we can compare, and arrive at

$$\begin{aligned} & x(t) - T^{(n)}(x)(t) \\ &= \cancel{x_0} + \sum_{k=0}^{n-1} \underline{x(t_{k+1}) - x(t_k)} + \left(x(t) - x(t_n)\right) \\ &\quad - \left[ \cancel{x_0} + \sum_{k=0}^{n-1} \underline{h f(x(t_k), t_k)} + (t - t_n) f(x(t_n), t_n) \right] \end{aligned}$$

$$= \sum_{k=0}^{n-1} \overbrace{x(t_{k+1}) - x(t_k) - h f(x(t_k), t_k)}^{h\,\varepsilon_k^{(h)}}$$

$$+ \quad x(t) - x(t_n) - (t - t_n) f(x(t_n), t_n)$$

Then, by the triangle inequality

$$|x(t) - T^{(n)}(x)(t)|$$
$$\leq h \sum_{k=0}^{n-1} |\varepsilon_k^{(h)}| + |x(t) - x(t_n) - (t - t_n) f(x(t_n))|$$

Let's bound each term, on one hand

$$h \sum_{k=0}^{n-1} |\varepsilon_k^{(h)}| \leq h n \max_{k : t_k \in [0, \tau]} |\varepsilon_k^{(h)}|$$

$$(\text{since } hn = t_n \leq \tau) \quad \leq \tau \max_{k : t_k \in [0, \tau]} |\varepsilon_k^{(h)}|$$

for the second order term, by Taylor's residual Theorem

$$x(t) - x(t_n) - \dot{x}(t_n)(t - t_n) = \frac{1}{2}\ddot{x}(\xi)(t - t_n)^2$$

for some $\xi = \xi(t)$ in $[t_n, t_{n+1}]$.

$$\Rightarrow \left| x(t) - x(t_n) - \dot{x}(t_n)(t-t_n) \right|$$
$$\leq \underbrace{\frac{1}{2}\left( \max_{[0,\tau]} |\ddot{x}| \right)}_{=:C} h^2$$

so

$$\|x - x^{(h)}\| \leq \frac{1}{1-\tau L} \| x - T^{(h)} x \|$$
$$\leq \frac{\tau}{1-\tau L}\left( \max_{\substack{k \\ t_k \in [0,\tau]}} |\varepsilon_k^{(h)}| + C h^2 \right)$$

$\blacksquare$

<u>Remark</u>: If $x(t)$, the solution to the IVP, is twice differentiable, then

$$\varepsilon_n^{(h)} = \frac{x(t_{n+1}) - x(t_n) - h f(x(t_n), t_n)}{h}$$

$$= \frac{x(t_{n+1}) - x(t_n) - h\,\dot{x}(t_n)}{h}$$

Then, again, using the formula for the 2nd order residual in Taylor's formula, we can show that

$$\left| x(t_{n+1}) - x(t_n) - h\,\dot{x}(t_n) \right| \le \frac{1}{2} \max_{[0,T]} |\ddot{x}(t)|\, h^2$$

$$\Longrightarrow \qquad \varepsilon_n^{(h)} \le \left( \frac{1}{2} \max_{[0,T]} |\ddot{x}(t)| \right) h$$

Putting this observation and the previous theorem together,

$$\| x - x^{(h)} \| \le \frac{1}{2} \frac{T}{1 - TL} \left( \max_{[0,T]} |\ddot{x}(t)| \, (h + h^2) \right)$$

For example, if $h \le 1$, then

$$\frac{h + h^2}{2} \le \frac{h + h}{2} = h$$

so, for $h \le 1$ we have

$$\| x - x^{(h)} \| \le \frac{\tau}{1 - \tau L} \| \ddot{x} \| \, h$$

Both norm are $L^{\infty}([0,\tau])$.