MATH 5340 Fall '23

Lecture 22

Part III. Graphs, Laplacians, and Markov chains

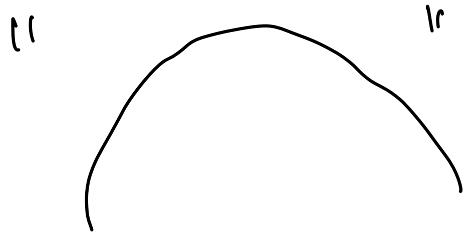
The geometric significance of the Laplacian (continued)

Last time one of the formulas we introduced for the Laplacian of a function was

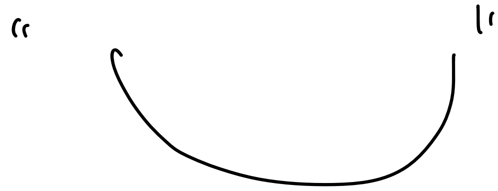
$$\Delta u = \mathrm{tr}(D^2 u)$$
$$= \sum_{i=1}^{d} (D^2 u)_{ii}$$
$$= \sum_{i=1}^{d} \lambda_i (D^2 u)$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $D^2 u$.

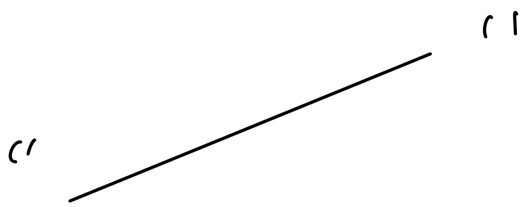
At a rough (if in general false) level, $\Delta u \geq 0$ means u looks "convex in average" and $\Delta u \leq 0$ means u looks "concave in average"



$\Delta u \leq 0$



$\Delta u \geq 0$



$\Delta u = 0$

The Dirichlet problem for domains in $\mathbb{R}^d$

(for us, $d = 2, 3$)

Data:

- $\Omega \subset \mathbb{R}^d$ (bounded, "nice" boundary)
  - " = piecewise graph of a C^1 funct
- $f : \Omega \to \mathbb{R}$
- $g : \partial\Omega \to \mathbb{R}$

both continuous.

(Dirichlet) **Problem**: Find a function $u: \overline{\Omega} \to \mathbb{R}$, continuous, and with $u \in C^2(\Omega)$, such that

$$\Delta u(x) = f(x) \quad , \quad x \in \Omega$$
$$u(x) = g(x) \quad , \quad x \in \partial\Omega$$

Theorem ($d=2$, similar results hold for $d>2$, but the statements are more complicated)

If $\partial\Omega$ is a finite union of C^2 curves, then there is a unique function u solving the Dirichlet problem for the given data.

Now we are going to discuss how this problem can be addressed in the context of scientific computing. We are going to create a discrete version of the problem living in a graph.

Let's make the following assumption: the function $g : \partial\Omega \to \mathbb{R}$ is the restriction to $\partial\Omega$ of a function $g : \mathbb{R}^2 \to \mathbb{R}$ which is also continuous.

Idea: Consider a graph G_h that closely resembles Ω, and place a corresponding Dirichlet problem on the graph

One concrete way of doing this is given by the finite difference schemes.

Let $G_h := \Omega \cap h\mathbb{Z}^d$. This will be a finite set with

$$\sim \frac{|\Omega|_d}{h^d} \text{ elements as } h \to 0^+.$$

(see: Richard Bellman's "curse of dimensionality")

Then in this graph, whose elements we index,

$$G_h = \{ x_1, \dots, x_N \}$$

we declare x_i and x_j to share an edge between them if

$$x_i - x_j \in \left\{ \begin{pmatrix} \pm h \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm h \\ 0 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm h \end{pmatrix} \right\}$$

Then, set $W_{ij} = \begin{cases} 0 & \text{if } x_i, x_j \text{ don't form an edge} \\ \frac{1}{2dh^2} & \text{if they do.} \end{cases}$