MATH 5340 Fall '23

Lecture 6

(Part I. Functions and their representation)

Announcements:

* Problem Set 1 now due Friday week 5 (midnight)
* Quiz 1 now will be Thursday week 6.
* Come to my DEAM talk this Friday (noon @ DH 334)

The Weierstrass–Stone Theorem

(for the most general version of this theorem, look up "Banach algebras")

$C(K)$: Let K be a compact metric space (e.g. $K \subset \mathbb{R}^d$ is closed and bounded), the set

$$C(K) = \{ f : K \to \mathbb{R} \}$$

is a Banach space with norm

$$\|f\|_\infty := \sup_{x \in K} |f(x)|.$$

A subset $A \subset C(K)$ is called a subalgebra if it is a vector space and whenever $f_1, f_2 \in A$ it follows that $f_1 f_2 \in A$.

A subset $B \subset C(K)$ is said to separate K if given any two distinct points $x, y \in K$ there exists $f \in B$ s.t.

$$f(x) \neq f(y)$$

Theorem (Weierstrass - Stone)

A subalgebra A of $C(K)$ is dense if it contains the constant function $\mathbb{1}$ and separates K.

In other words, if A is such a set, then for any $f \in C(K)$ and any $\varepsilon > 0$ there is some $p \in A$ such that

$$|f(x) - p(x)| < \varepsilon \quad \text{for all } x \in K.$$

$$\left(\text{i.e. } \|f - p\|_{L^\infty(K)} < \varepsilon\right)$$

Examples

* For $C([0,1]^2)$, the class

$$A = \left\{ p \;\middle|\; p(x,y) = \sum c_{k\ell}\, x^k y^\ell \text{ with finitely many terms} \right\}$$

is a dense subalgebra by the

Weierstrass - Stone Theorem

* Likewise for any $K \subset \mathbb{R}^d$ compact and

$$A = \{ p(x_1, \dots, x_d) \mid p \text{ is a polynomial in } x_1, \dots, x_d \}$$

* Consider the set

$$C([a,b])$$

where $b-a<1$. Then take

$$A = \{ p \mid p \text{ is a trigonometric polynomial} \}$$

$$\left(\text{i.e. } p(x) = a_0 + \sum_{k=1}^{n} a_k \cos(2\pi k x) + \sum_{k=1}^{n} b_k \sin(2\pi k x) \right)$$

for some n, a_k, b_k

Show:

* Thanks to $b-a<1$, the set A separates $[a,b]$
* A is closed under multiplication and it is therefore an algebra.

Hint: Recall the identities

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \sin\beta \cos\alpha$$

so

$$\cos(\alpha+\beta) - \cos(\alpha-\beta)$$

$$= 2\cos\alpha\cos\beta$$

etc...

From where it follows that $\forall\, k_1, k_2$, the function

$$\cos(2\pi k_1 x)\cos(2\pi k_2 x)$$

is a linear combination of the functions

$\cos(2\pi(\nu_1 + \nu_2)x)$ and $\cos(2\pi(\nu_1 - \nu_2)x)$