MATH 5340 Fall '23

Lecture 19

Part III. Graphs, Laplacians, and Markov chains

Last time we studied when is the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } D \subset G \\ u = g & \text{in } G \setminus D \end{cases}$$

uniquely solvable.

In particular, we have the following.

Remark. If $u_1, u_2 : G \longrightarrow \mathbb{R}$ are such that

$$\begin{cases} Lu_1 = f \\ u_1 = g \end{cases} \qquad \begin{matrix} Lu_2 = f & \text{in } D \subset G \\ u_2 = g & \text{in } G \setminus D \end{matrix}$$

and D, G satisfy the assumptions from the previous lecture, then $u_1 = u_2$.

The Dirichlet Laplacian

Given G and $D \subset G$ we define a new linear transformation L_D, which acts on scalar functions in G, via the following formula

$$L_D(u)(x_i) = \begin{cases} Lu(x_i) & \text{if } x_i \in D \\ u(x_i) & \text{if } x_i \in G \setminus D. \end{cases}$$

Exercise: The matrix for L_D (in the standard basis for $C(G)$) is obtained by taking the matrix for L and for each row with index i s.t. $x_i \in G \setminus D$ we replace it with a row made out of zeroes everywhere except for a 1 on the diagonal.

The previous remark can be restated by saying that under the assumptions stated in the remark the matrix L_D has a trivial kernell, i.e.,

$$L_D u = 0 \iff u = 0.$$

Since L_D is a square matrix, we conclude that $\forall$ function $\hat{f}$ in G there is a u such that

$$L_D u = \hat{f}.$$

This mean the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } D \\ u = g & \text{in } G \setminus D \end{cases}$$

always has a unique solution.

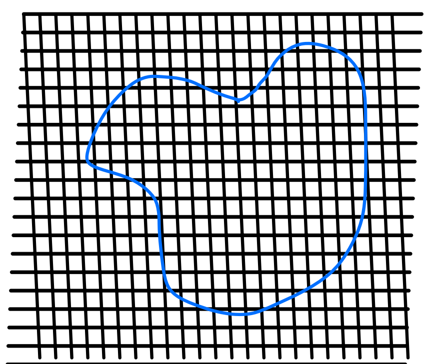
Problem

We are given a domain $\Omega \subset \mathbb{R}^2$, which we represent via a defining function $\Phi(x,y)$:

$$\Omega = \{ (x,y) \mid \Phi(x,y) < 0 \}$$

We are also given $h > 0$. Then let me define the following graph

$$G = \{ x_1, \dots, x_N \} = \Omega \cap h\mathbb{Z}^2$$



Define $D \subset G$ as follows:

$x_i \in D \equiv \{x_i \pm (h,0), x_i \pm (0,h)\} \subset \Omega$

Lastly, $w_{ij} = \begin{cases} \frac{1}{h^2} & \text{if } x_i - x_j \in \{(h,0),(-h,0),(0,h),(0,-h)\} \\ 0 & \text{otherwise.} \end{cases}$

This defines a graph.

Write a function that takes as input a function Φ and a float $h > 0$ and returns 3 things:

1. A list $x_1, \dots, x_N$ of 2D vectors (or 1-dim arrays of len 2)

2. An int N_0 s.t.

$$D = \{x_1, \dots, x_{N_0}\}$$

$$G \setminus D = \{x_{N_0+1}, \dots, x_N\}$$

3. The Dirichlet Laplacian L_D for D.

ALTERNATIVELY, return

1. The list $\{x_1, \ldots, x_N\}$
2. A 1-d array of length N, char_D, where $\text{char\_D}[i] = \begin{cases} 1 & \text{if } x_i \in D \\ 0 & \text{otherwise} \end{cases}$
3. L_D.