MATH 5340 Fall '23

Lecture 17

Part III : Graphs, Laplacians, and Markov chains.

The Dirichlet Problem (in a graph)

Data: We are given

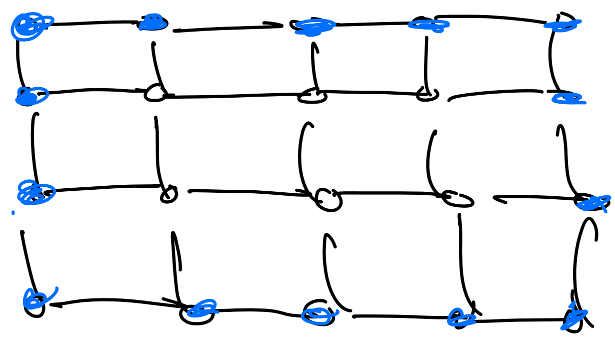
* A graph (G, w)
* A subset $D \subset G$
* A function $f : D \to \mathbb{R}$

and a function $g : G \setminus D \to \mathbb{R}$

Example

$\bullet = G \setminus D$

$\circ = D$



The Dirichlet problem asks, given the above data, to find a function

$$u : G \longrightarrow \mathbb{R}$$

solving the following:

$$\begin{cases} Lu = f & \text{in } D \\ u = g & \text{in } G \setminus D \end{cases}$$

meaning

$$Lu(x_i) = f(x_i) \quad \text{if} \quad x_i \in D$$

$$u(x_i) = g(x_i) \quad \text{if} \quad x_i \notin D$$

Here, L is the Laplacian of the graph, which we recall is defined by

$$Lu(x_i) = \sum_{j=1}^{N} \left(u(x_j) - u(x_i)\right) W_{ij}$$

Regarding the Dirichlet problem we want to know

1) Are there solutions?
2) If there is a solution, is it unique?
3) What are the special properties of such solutions?
4) How do the solutions change if we change the data?
5) How reliably and quickly can we compute the solution from the given data?

The variational approach to Dirichlet's problem

Given a graph (G, w) we shall define the Dirichlet energy as a function of any $u \in C(G)$, via the formula

$$\mathcal{E}(u) := \frac{1}{2} \sum_{i,j=1}^{N} (u(x_i) - u(x_j))^2 W_{ij}$$

This quantity gives us a measure of how much is u changing from neighbor to neighbor. (weighted by W_{ij})

Remark. The original Dirichlet energy was defined for differentiable function $u : D \to \mathbb{R}$ for some domain $D \subset \mathbb{R}^d$, by the integral

$$\int_D |\nabla u(x)|^2 \, dx$$

Dirichlet's principle refers to the fact that if D is a bounded domain with a C^1 boundary and $g : \partial D \longrightarrow \mathbb{R}$ is a continuous function equal to the restriction of a differentiable function, THEN the unique minimizer of $\int_D |\nabla u|^2 dx$ among all function which equal g on ∂D solves

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

$\blacksquare$

To explain/prove Dirichlet's principle on a graph, let's introduce some notation.

<u>Definition</u>: Given $D \subset G$ and $g : G \setminus D \to \mathbb{R}$ we will denote by V_g the class of function in $C(G)$ which equal g in $G \setminus D$, that is

$$V_g := \{ u : G \to \mathbb{R} \mid u(x_i) = g(x_i) \text{ if } x_i \notin D \}$$

Moreover, given $f : D \to \mathbb{R}$, define a function

$$J : C(G) \to \mathbb{R}$$

by the formula

$$J(u) = \underbrace{\frac{1}{4}\sum_{i,j=1}^{N}(u(x_i)-u(x_j))^2 w_{ij}}_{\left(\text{Note: } = \frac{1}{2}\mathcal{E}(u)\right)} + \sum_{x_i\in D} u(x_i)\, f(x_i).$$

Lemma: Given the data above, suppose that $u_0 \in V_g$ is such that

$$J(u_0) \le J(u) \quad \forall\, u \in V_g$$

Then, u_0 solves the Dirichlet problem, i.e.

$$\begin{cases} Lu_0(x_i) = f(x_i) & \text{if } x_i \in D \\ u_0(x_i) = g(x_i) & \text{if } x_i \notin D \end{cases}.$$

Remark. This lemma is closely connected to an important basic observation in numerical linear algebra and optimization: that given a symmetric $N\times N$ matrix A, the equation

$$Az = b$$

is solved by minimizers of the quadratic function

$$J(z) = \frac{1}{2}(Az, z) - (b, z)$$

at least when A is positive semidefinite

Proof of the lemma

Idea: Take $\phi : G \to \mathbb{R}$ s.t. $\phi(x_i) = 0$ if $x_i \notin D$,

and then let

$$u_t := u_0 + t\phi$$

Since $u_0 \in V_g$, $u_t \in V_g$ for all t, and

thus the function

$$t \longrightarrow J(u_t)$$

has its minimum at $t=0$, therefore,

$$\frac{d}{dt}\Big|_{t=0} J(u_t) = 0$$

Let's write $\frac{d}{dt}\Big|_{t=0} J(u_t)$ and see what

it means for it to be 0.

$$\frac{d}{dt} J(u_t) = \frac{1}{4}\frac{d}{dt}\sum_{i,j=1}^{N} (u_t(x_i) - u_t(x_j))^2 w_{ij}$$
$$+ \frac{d}{dt}\sum_{x_i \in G} u_t(x_i) f(x_i)$$

Observe,

$$\frac{d}{dt}\left((u_t(x_i) - u_t(x_j))^2 \right)$$
$$= 2(u_t(x_i) - u_t(x_j)) \frac{d}{dt}(u_t(x_i) - u_t(x_j))$$
$$= 2(u_t(x_i) - u_t(x_j))(\phi(x_i) - \phi(x_j))$$

and in particular

$$\frac{d}{dt}\Big|_{t=0}\left((u_t(x_i) - u_t(x_j))^2 \right)$$
$$= 2(u_0(x_i) - u_0(x_j))(\phi(x_i) - \phi(x_j))$$

Going back to $\frac{d}{dt}\big|_{t=0} J(u_t)$, we have that it equals

$$\frac{1}{2}\sum_{i,j=1}^{N} (u_0(x_i) - u_0(x_j))(\phi(x_i) - \phi(x_j)) w_{ij}$$
$$+ \sum_{i=1}^{N} \phi(x_i) f(x_i)$$

Note: The change to the last index used that $\phi(x_i) = 0$ if $x_i \notin D$.

Let's take a step back, what we have so far is that if $\phi \in C(G)$ is zero in $G \setminus D$, then

$$\frac{1}{2}\sum_{i,j=1}^{N}\left(u_0(x_i)-u_0(x_j)\right)\left(\phi(x_i)-\phi(x_j)\right)W_{ij}$$
$$+\sum_{i=1}^{N}\phi(x_i)f(x_i) = 0$$

To further make use of this, we are going to use a trick we already used in the last class and break the first sum in two parts and use that $W_{ij} = W_{ji}$ $\forall\, i,j$:

$$\frac{1}{2}\sum_{i,j}^{N}\left(u(x_i)-u(x_j)\right)\left(\phi(x_i)-\phi(x_j)\right)W_{ij}$$

$$= \frac{1}{2}\sum_{i,j}^{N}\left(u(x_i)-u(x_j)\right)\phi(x_i)W_{ij}$$
$$- \frac{1}{2}\sum_{i,j=1}^{N}\left(u(x_i)-u(x_j)\right)\phi(x_j)W_{ij}$$

flipping the labels in the 2nd sum,

$$= \frac{1}{2}\sum_{i,j=1}^{N}\left(u(x_i)-u(x_j)\right)\phi(x_i)W_{ij}$$

$$- \frac{1}{2} \sum_{i,j=1}^{N} (u(x_j) - u(x_i)) \phi(x_i) w_{ji}$$

and since $w_{ij} = w_{ji}$,

$$= \frac{1}{2} \sum_{i,j=1}^{N} (u(x_i) - u(x_j)) \phi(x_i) w_{ij}$$

$$- \frac{1}{2} \sum_{i,j=1}^{N} (u(x_j) - u(x_i)) \phi(x_i) w_{ij}$$

$$= \sum_{i,j=1}^{N} (u(x_i) - u(x_j)) \phi(x_i) w_{ij}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} (u(x_i) - u(x_j)) \phi(x_i) w_{ij}$$

$$= - \sum_{i=1}^{N} \phi(x_i) Lu(x_i)$$

Putting everything together,

$$- \sum_{i=1}^{N} \phi(x_i) Lu(x_i) + \sum_{i=1}^{N} \phi(x_i) f(x_i)$$

$$= \sum_{i=1}^{N} \phi(x_i) \left(f(x_i) - Lu(x_i) \right) = 0$$

This happens $\forall$ $\phi \in C(G)$ vanishing in $G \setminus D$,

so, given $x_{i_0} \in D$, let

$$\phi_0(x_i) = \begin{cases} 1 & \text{if } i = i_0 \\ 0 & \text{otherwise,} \end{cases}$$

Then putting these ϕ_0 above, we have

$$f(x_{i_0}) - Lu(x_{i_0}) = 0$$

This shows that $Lu = f$ in D.