# MATH 5340 Fall '23

## Lecture 11

## Part II. Ordinary differential equations

Last time we talked about ODE's in general, today we will talk about the existence and uniqueness of solutions to IVP's for system of the form

$$(\mathrm{IVP}) \quad \begin{cases} \dot{x}(t) = f(x(t), t), & t \in [\tau_0, \tau_1] \\ x(\tau_0) = x_0, \end{cases}$$

The way we will construct a solution (following an idea of Emile Picard) is to analyze the equivalent integral equation

**Proposition:**

A continuously differentiable $x : [0, \tau] \longrightarrow \mathbb{R}^d$ solves (IVP) if and only if for every $t \in [0, \tau]$ we have

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds$$

Proof: It follows from the fundamental theorem of calculus

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From the above proposition we make a definition

**Definition:** A continuous function $x : [0, \tau] \to \mathbb{R}^d$ is called a weak solution of the (IVP) if for every $t \in [0, \tau]$,

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds$$

To show a solution to the IVP exists we will focus on finding a funct $x(t)$, $t \in [0, \tilde{T}]$ which is continuous and solves the integral equation.

For this we need to talk about contraction mappings (see also Kolmogorov and Fomin Chapter 2).

## Contraction Mappings in $C([a,b])$

(This idea works in general complete metric spaces)

**Definition**: A mapping $T : C([a,b]) \to C([a,b])$ is called a contraction mapping if there is a number $\lambda \in (0,1)$ such that

$\forall \; f_1, f_2 \in C([a,b])$,

$$\| T(f_1) - T(f_2) \|_{L^\infty([a,b])} \leq \lambda \| f_1 - f_2 \|_{L^\infty([a,b])}$$

**Theorem (Banach's contraction mapping theorem)**

If $T : C([a,b]) \to C([a,b])$ is a contraction mapping then there is exactly one $f_* \in C([a,b])$ such that

$$T(f_*) = f_*.$$

Moreover, for any $f_0 \in C([a,b])$, the sequence $f_k$ defined by

$$f_{k+1} = T(f_k) \quad \forall k$$

is such that $\|f_* - f_k\|_{L^\infty([a,b])} \to 0$ as $k \to \infty$.

Proof: We will do it in a later class $\boxtimes$

Let's see how this theorem is useful for us.

Given the (IVP) as above we define for every $\tau > 0$ a mapping from $C([0,\tau])$ to itself:

Take $x : [0,\tau] \to \mathbb{R}^d$, then define

$$T(x)(t) = x_0 + \int_0^t f(x(s), s)\, ds$$

clearly a function $x(t)$ is a solution of the IVP in $[0,\tau]$ if and only if $x(t)$ is a fixed point of the mapping $T$.

Lemma: Let $T$ be as above. If the function $f(x,t)$ is Lipschitz continuous in $x$ with Lipschitz constant $L$ for every $t$, then the mapping $T$ is a contraction provided $\tau L < 1$.

**Remark:** Recall that $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is said to be Lipschitz with (Lipschitz) constant $L$ if

$$\| f(x) - f(y) \| \leq L \| x - y \| \quad \forall x, y \in \mathbb{R}^m$$

so, for the above lemma the assumption is there is a $L > 0$ s.t.

$$\| f(x,t) - f(y,t) \| \leq L \| x - y \| \quad \forall x, y \in \mathbb{R}^d \quad \forall t \in [0, \tau]$$

**Proof:** Take two functions $x_1(t), x_2(t) \in C([0,\tau])$

Then

$$(Tx_1)(t) - (Tx_2)(t)$$
$$= \cancel{x_0} + \int_0^t f(x_1(s), s)\, ds - \left( \cancel{x_0} + \int_0^t f(x_2(s), s)\, ds \right)$$
$$= \int_0^t f(x_1(s), s) - f(x_2(s), s)\, ds$$

by the triangle inequality

$$|(Tx_1)(t) - (Tx_2)(t)|$$
$$\leq \int_0^t |f(x_1(s), s) - f(x_2(s), s)|\, ds$$
$$\leq \int_0^t L\, |x_1(s) - x_2(s)|\, ds \quad \text{(by the assumption on } f\text{)}$$
$$\leq L \int_0^\tau |x_1(s) - x_2(s)|\, ds$$
$$\leq L\tau\, \|x_1 - x_2\|_{L^\infty([0,\tau])}$$

We have shown that $\forall\ t \in [0,\tau]$

$$|T(x_1)(t) - T(x_2)(t)| \leq L\tau\, \|x_1 - x_2\|_{L^\infty([0,\tau])}$$

$\Rightarrow$

$$\|T(x_1) - T(x_2)\|_{L^\infty([0,\tau])} \leq L\tau\, \|x_1 - x_2\|_{L^\infty([0,\tau])}$$

since $L\tau < 1$, $T$ is a contraction mapping.