MATH 5340 Fall '23

Lecture 21

Part III. Graphs, Laplacians, and Markov chains

Continuous problem and finite differences

The Laplacian is an operation of twice-differentiable functions. If $u = u(x_1, \dots, x_d)$ is a real valued function in $\mathbb{R}^d$, then its Laplacian is defined as

$$\Delta u = \partial_{x_1}^2 u + \cdots + \partial_{x_d}^2 u$$

Equivalent definitions are

- $$\Delta u = \mathrm{tr}(D^2 u)$$

where for a $N \times N$ matrix $\mathrm{tr}(M) = \sum_{i=1}^{N} M_{ii}$, and $D^2 u$ is the Hessian matrix of u

$$D^2 u = \begin{pmatrix} \partial^2_{x_1 x_1} u & \cdots & \partial^2_{x_1 x_d} u \\ \vdots & & \\ \partial^2_{x_d x_1} u & \cdots & \partial^2_{x_d x_d} u \end{pmatrix}$$

(this definition is popular in applications involving stochastic processes like Brownian motion)

- $\Delta u = \operatorname{div}(\nabla u)$

(this definition emphasizes the relevance of the Laplacian in continuum mechanics)

The previous two ways of writing Δu are called the non-divergence form and the divergence form, respectively.

These two different ways come with corresponding and different ways of generaligiving the Laplacian operator. Thus, one can consider the so-called elliptic operators.

* Non-divergence case: if for every $x \in \Omega \subset \mathbb{R}^d$ we have a matrix $A(x)$, and $A(x)$ is symmetric and positive definite for every x, then we define the following operator:

$$Lu(x) = \operatorname{tr}\left(A(x) D^2 u(x)\right)$$

This is a generalization of the Laplacian (when $A(x) = \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ for every x we get $L = \Delta$)

* Divergence case: Now with $A(x)$ as before, we define

$$Lu(x) = \operatorname{div}\left(A(x) \nabla u(x)\right)$$

this is called a divergence-form elliptic operator.

The geometric significance of the Laplacian

Let u be a twice differentiable function defined near some point x_0.

By Taylor's formula,

$$u(x) = u(x_0) + (\nabla u(x_0), (x-x_0)) + \frac{1}{2}(D^2u(x_0)(x-x_0), x-x_0) + \text{Error}(x-x_0)$$

where

$$\lim_{x \to 0} \frac{|\text{Error}(x-x_0)|}{|x-x_0|^2} = 0$$

Let's use this formula to estimate the average of $u(x)$ on a sphere of radius ε centered at x_0

Say, take $d=2$ for concreteness (the following will hold for $d>2$ with the necessary modifications)

Then one can see that $(d=2)$

$$\frac{1}{2\pi\varepsilon}\int_{|x-x_0|=\varepsilon} u(x(\theta))\,d\theta = u(x_0) + \frac{1}{2}\Delta u(x_0)\,\varepsilon^2 + \frac{1}{2\pi\varepsilon}\int_{|x-x_0|=\varepsilon} \text{Error}\,(x(\theta)-x_0)\,d\theta$$

But then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\frac{1}{2\pi\varepsilon}\int_{|x-x_0|=\varepsilon} u(x(\theta)) - u(x_0)\,d\theta\right) = \frac{1}{2}\Delta u(x_0)$$

So the Laplacian measures at an infinitesimal scale the average oscillation of u.