

Combinatorial & Computational Aspects of Games

1. Reaching 100
2. Dr. Nim (aka Subtraction)
3. The Game of Nim

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Reaching 100

Two player & turn-based

Initially : Counter = 0

On a single turn:

current player increments

the counter by k $\leftarrow 1 \leq k \leq 10$

WIN : The first player to get the counter to 100 wins.

Reaching 100

players
Two player & turn-based
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Reaching 100

Two player & turn-based mechanics .

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Reaching 100

Two player & turn-based

Initially:

Counter = 0

Elements & starting state.

On a single turn:

current player increments

the counter by $k \leftarrow 1 \leq k \leq 10$

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Reaching 100

Two player & turn-based

Initially : Counter = 0

On a single turn:

ruleset &
dynamics

→ [current player increments
the counter by k] $\leftarrow 1 \leq k \leq 10$

WIN : The first player to get the counter to 100 wins.

Reaching 100

Two player & turn-based

Initially : Counter = 0

On a single turn:

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goal



The first player to get the counter to 100 wins.

Reaching 100 → # players
Two player & turn-based mechanics .

Initially : Counter = 0 ↗ elements & starting state .

On a single turn :

ruleset & dynamics → [current player increments the counter by $k \leftarrow 1 \leq k \leq 10$]

goal

WIN : The first player to get the counter to 100 wins .

Some other aspects:

- * Role of chance
- * Who knows what
- * Co-operative v/s. Competitive

Some other aspects:

In most scenarios that we will encounter:

- * Role of chance (None: deterministic moves)
- * Who knows what
- * Co-operative v/s. Competitive

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In most scenarios that we will encounter:

- * Role of chance (None: deterministic moves)
- * Who knows what (Perfect Information)
- * Co-operative v/s. Competitive

Reaching 100

Let's
play

Two player & turn-based

Initially : Counter = 0

On a single turn:

current player increments

the counter by k $\leftarrow 1 \leq k \leq 10$

WIN : The first player to get the counter to 100 wins.

Reaching 100

Observation.

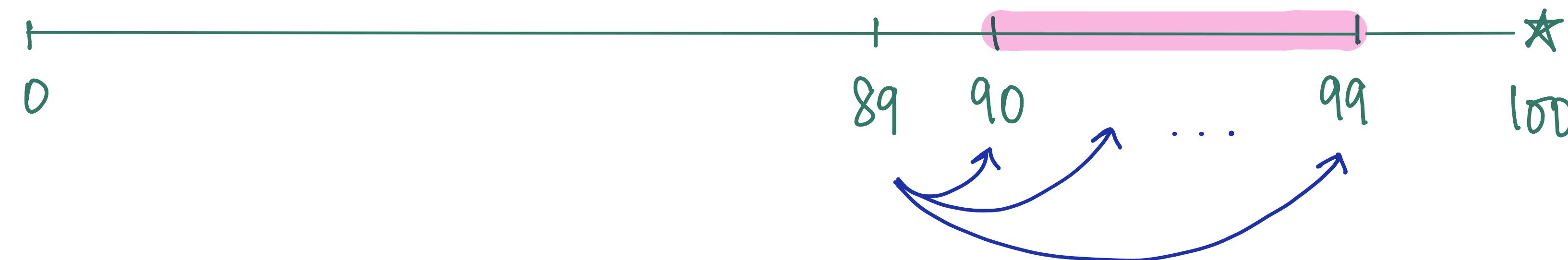
To reach 100, it suffices to reach 89.



Reaching 100

Observation.

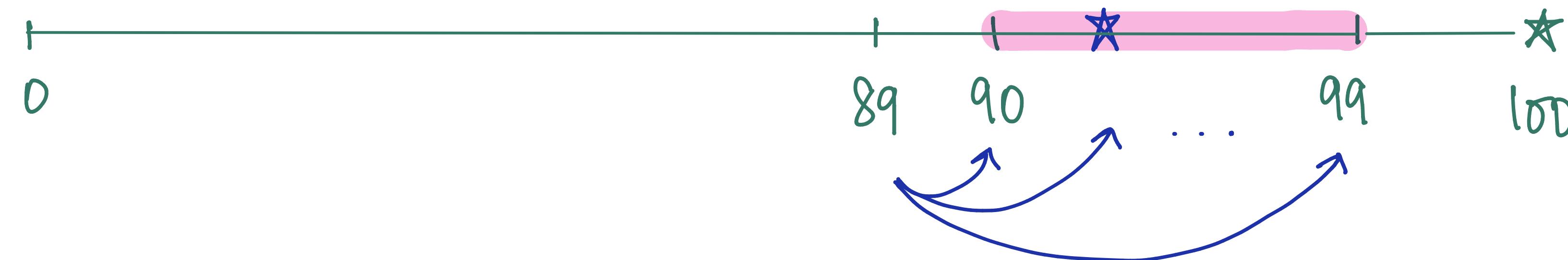
To reach 100, it suffices to reach 89.



Reaching 100

Observation.

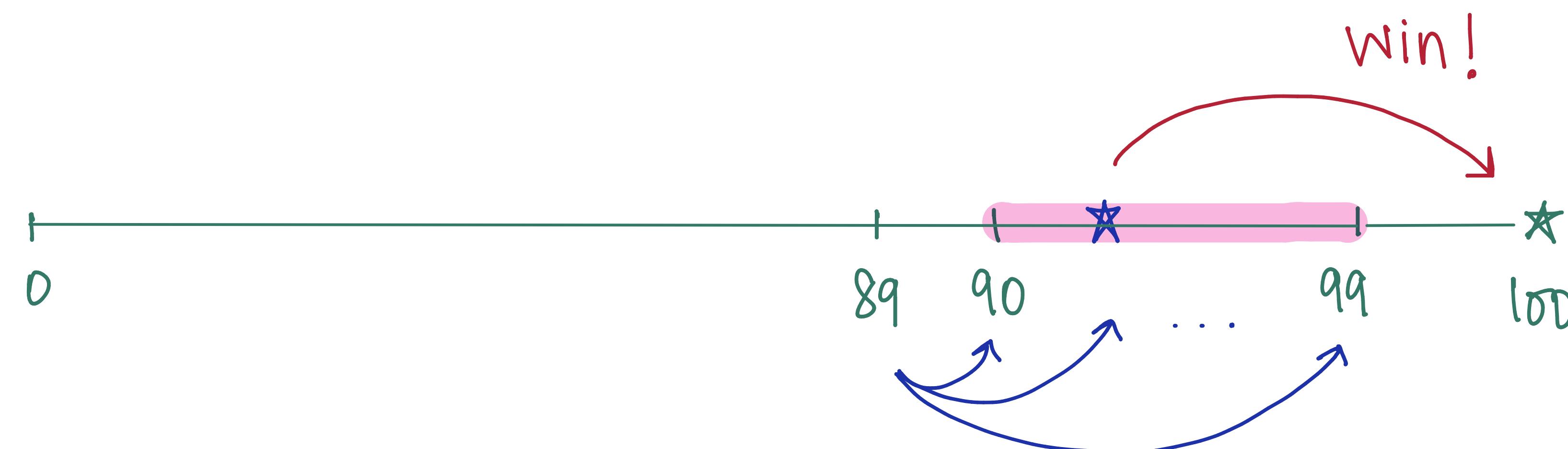
To reach 100, it suffices to reach 89.



Reaching 100

Observation.

To reach 100, it suffices to reach 89.



Reaching 100

Observation.

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

Reaching 100

Observation.

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

To reach 78, it suffices to reach 67.

Reaching 100

Observation.

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

To reach 78, it suffices to reach 67.

To reach 12, it suffices to reach 1.

Reaching 100

Observation.

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

To reach 78, it suffices to reach 67.

Start Here ↘

To reach 12, it suffices to reach 1.

Strategies



functions that map game states
to valid moves.

can always answer the question :

"what do I do now?"

Winning
🏆

(Strategies →)

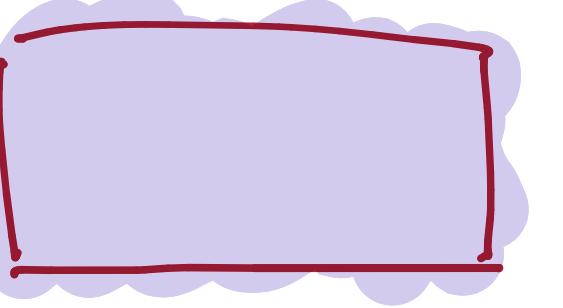
Guarantee a win.

functions that map game states

to valid moves.

can always answer the question :

"what do I do now?"

In the Reach 100 game, the  player has a winning strategy.

In the Reach 100 game, the [] player has a winning strategy.

if (State = 0)	inc by <u>1</u> .
if (2 ≤ state ≤ 11)	inc by <u>12 - state</u>
if (13 ≤ state ≤ 22)	inc by <u>23 - state</u>
.	
else	inc by <u>1</u>

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Dr. Nim



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Acknowledgement.

The narrative & proof sketch that follows is due to J. Vinay.

Nim

Two player, turn-based

Init : (n_1, n_2, \dots, n_k)

k heaps, i -th heap has n_i tokens

One move : pick a heap &
 $(1 \leq i \leq k)$

remove any # of tokens from it
 $1, 2, \dots, n_i$

Win :

The last player
who can make a
valid move wins

Nim (Examples)



- 1 Two heaps, one token each.

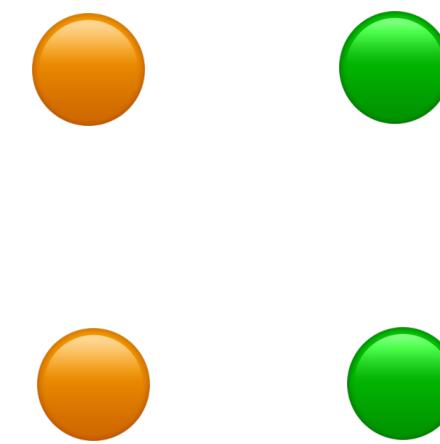
Nim
→
(Examples)



- 1 Two heaps, one token each.
→ First player loses.

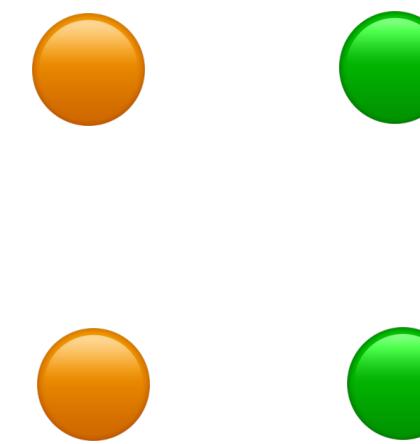
Nim

(Examples)



2. Two heaps, two tokens each.

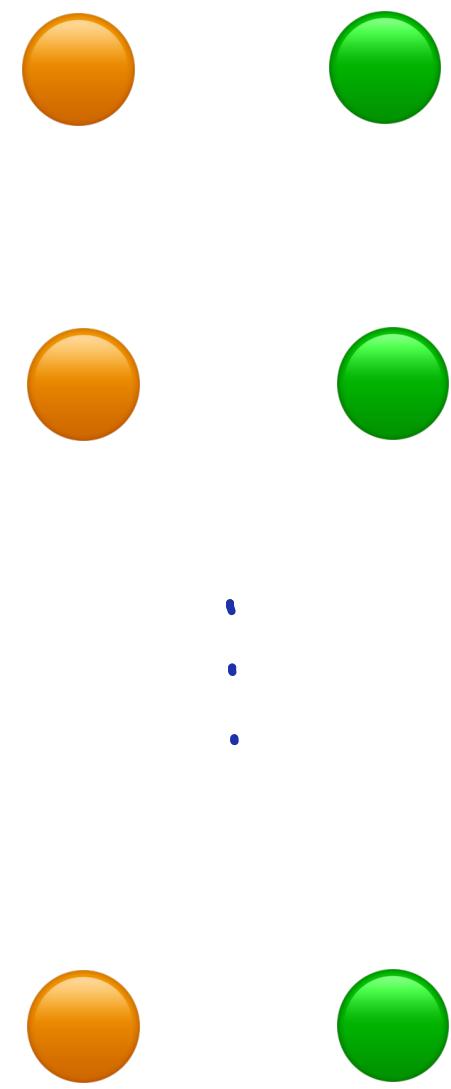
Nim
→
(Examples)



2. Two heaps, two tokens each.

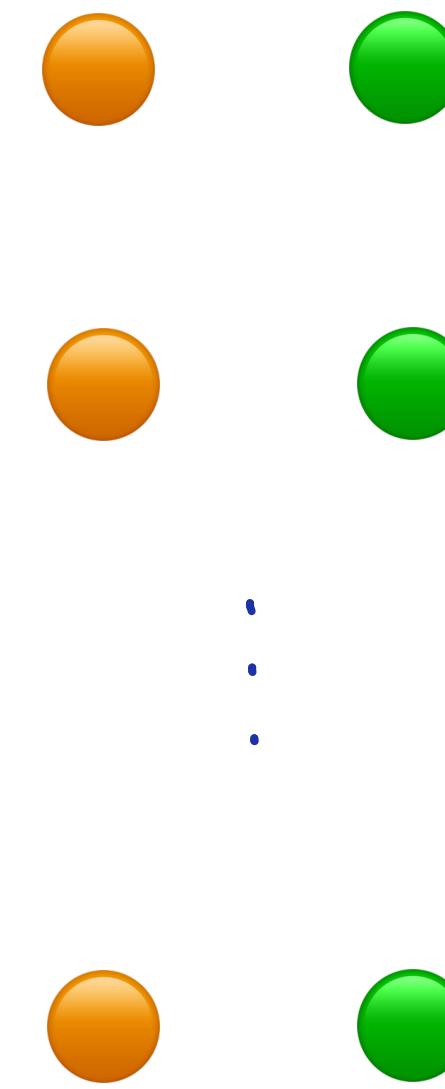
→ First player loses.

Nim
(Examples)



3. Two heaps, k tokens each.

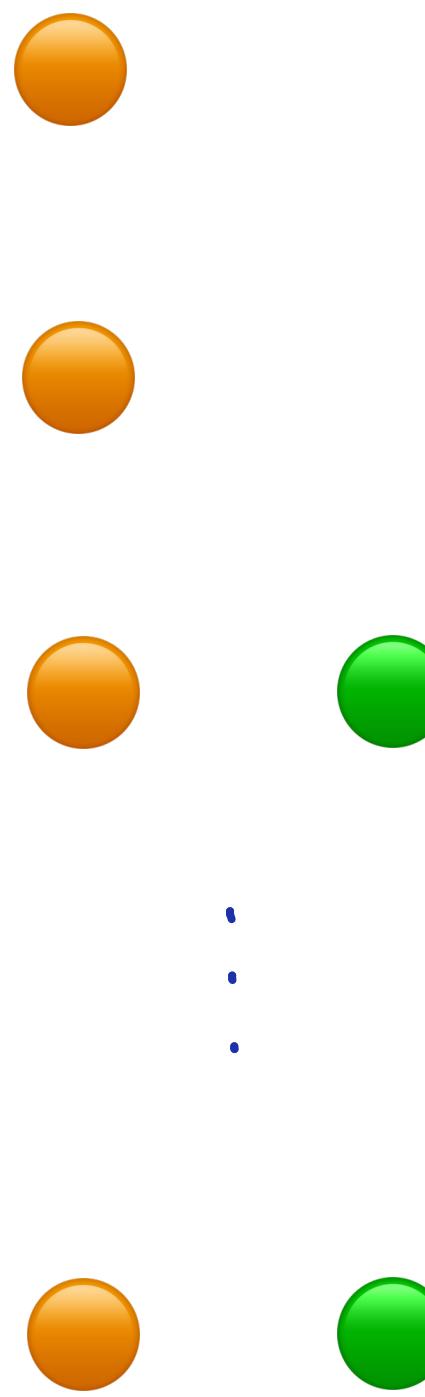
Nim
(Examples)



3. Two heaps, k tokens each.

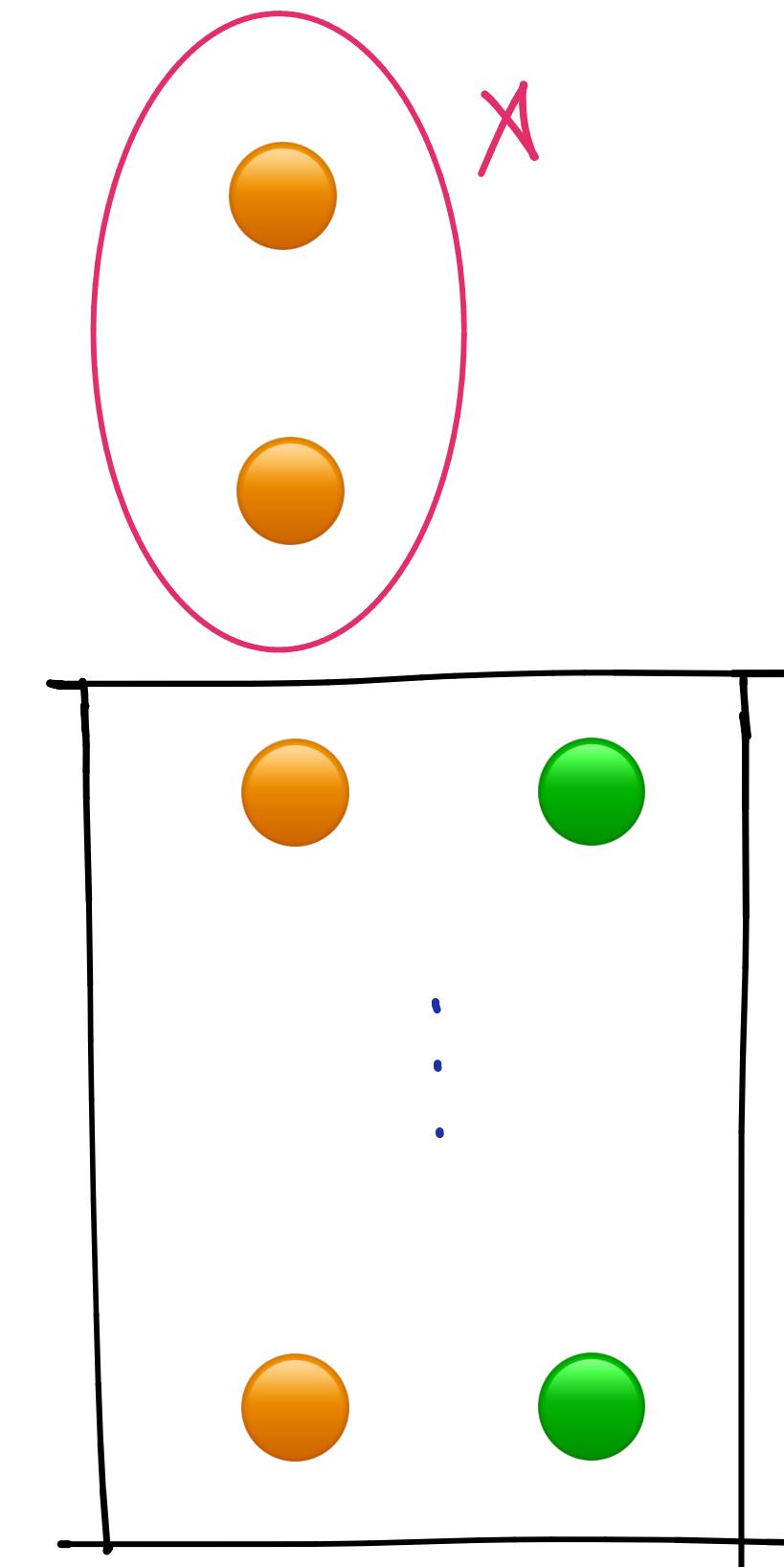
→ First player loses. (why?)

Nim
(Examples)



3. Two heaps, $p \& q$ tokens ; $p \neq q$.

Nim
(Examples)



3. Two heaps, $p \& q$ tokens ; $p \neq q$.

→ First player (finally) wins!

A Nim state is **HAPPY** if the player who starts there has a winning strategy.

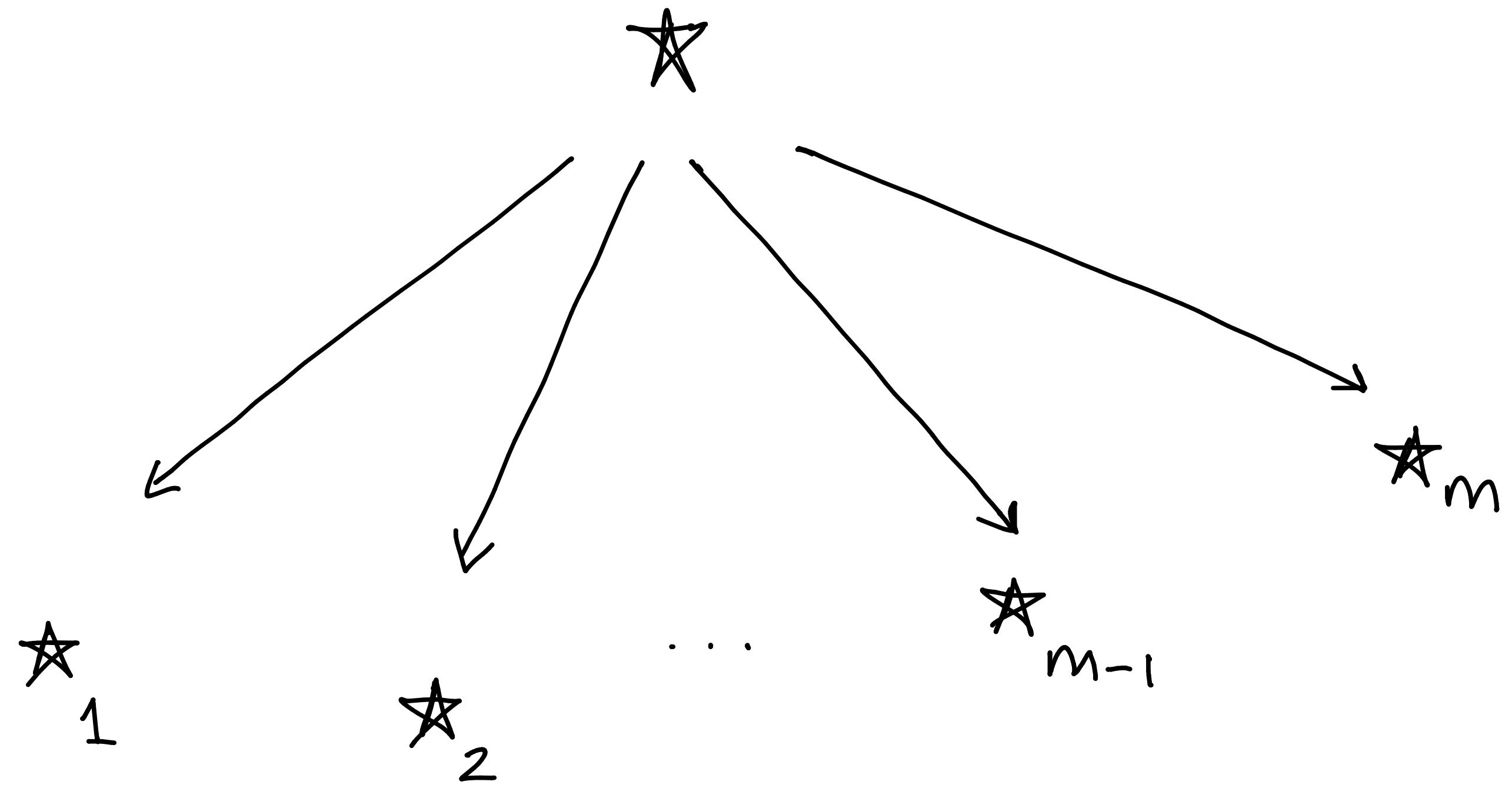
A Nim state is **SAD** if it is not **HAPPY**.

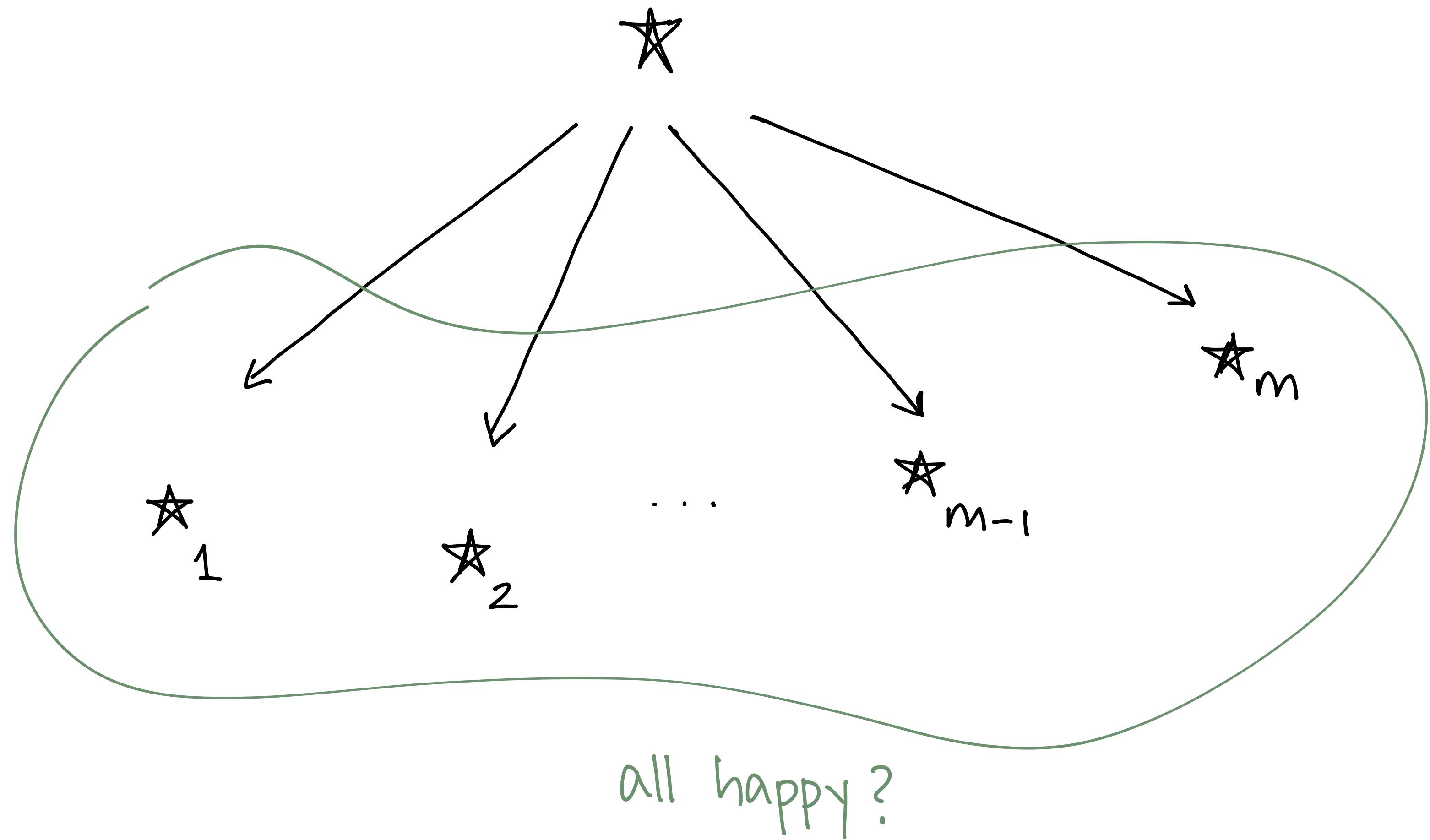
SAD

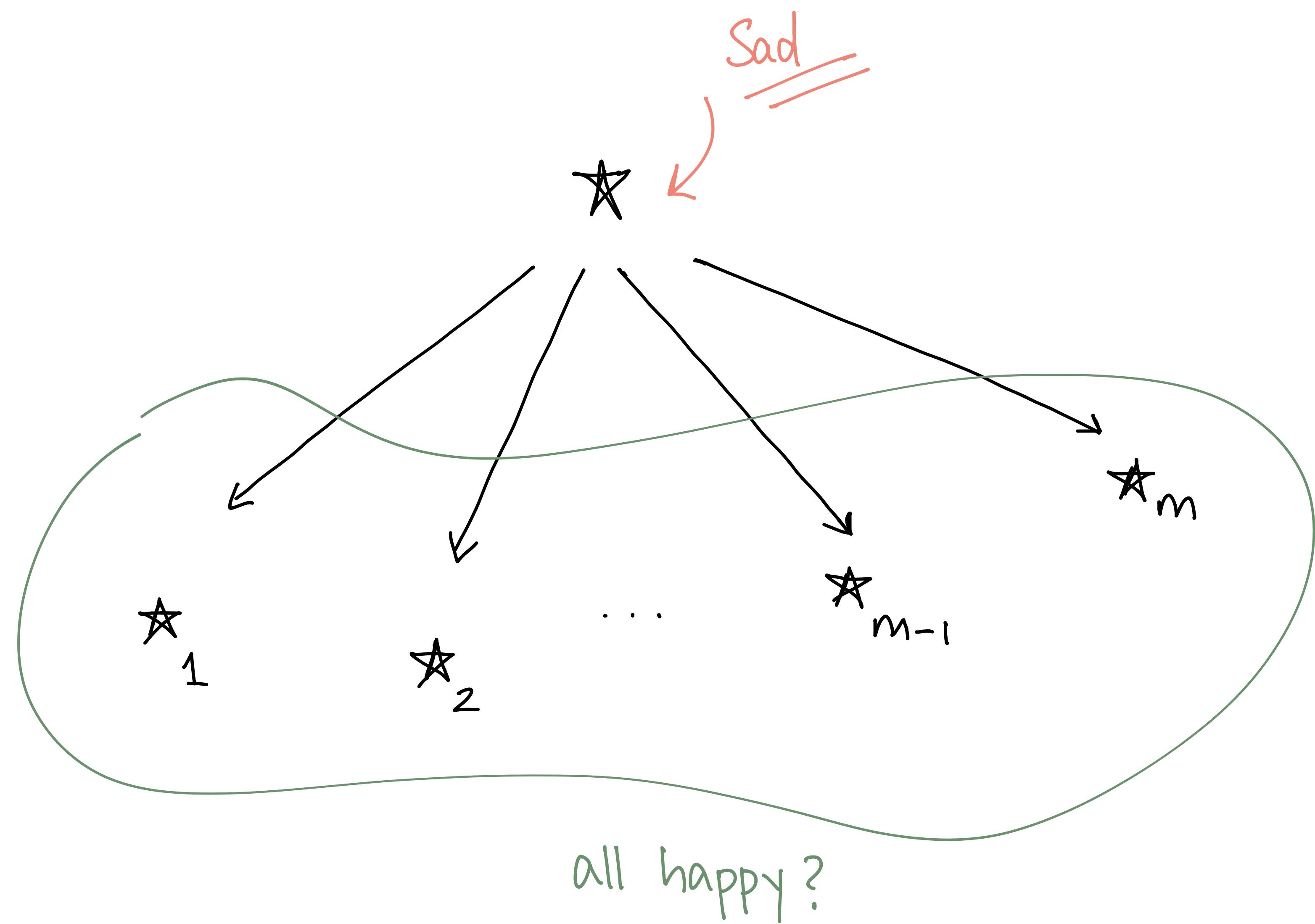


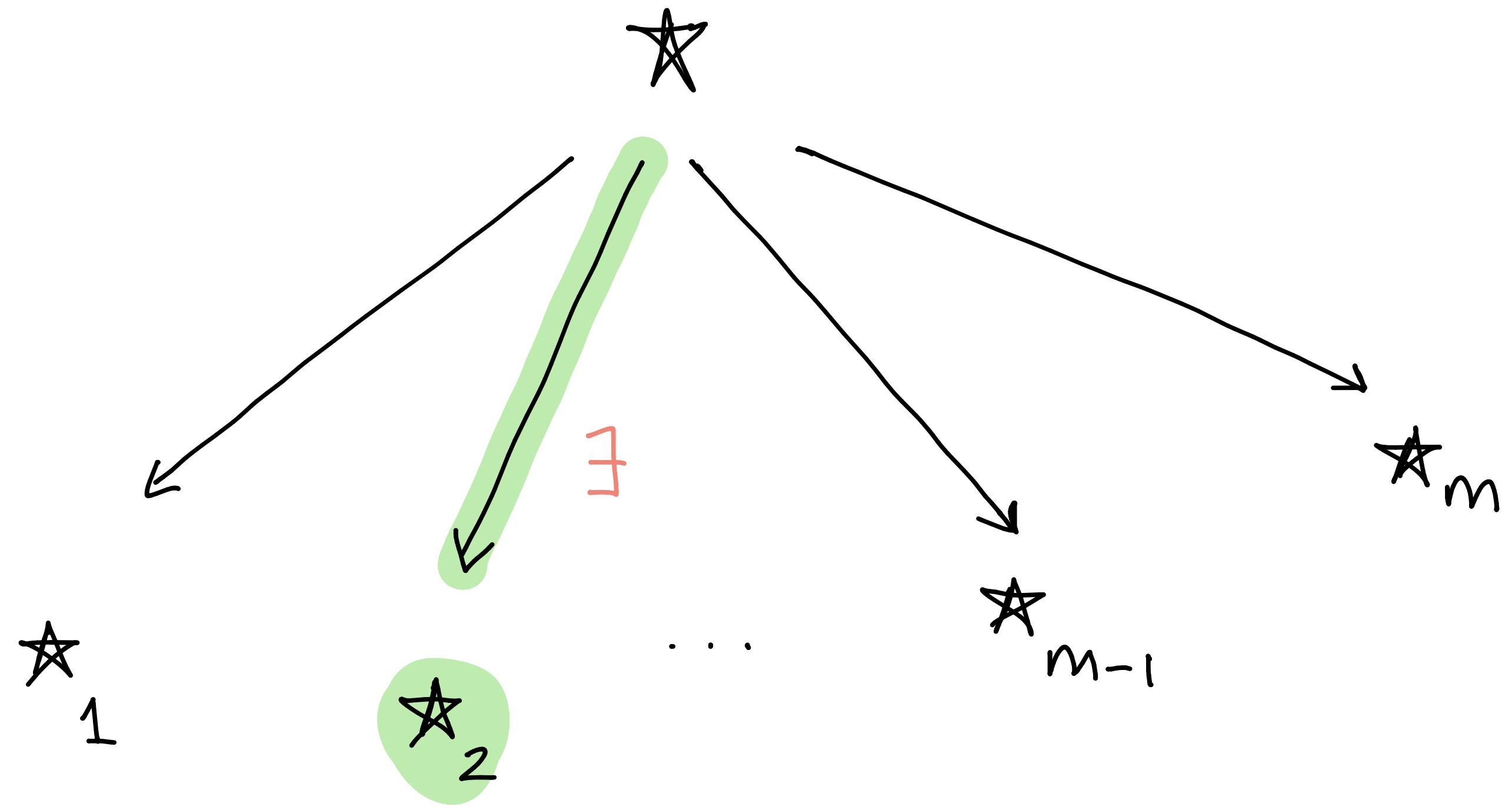
SAD

Can you have such a transition?

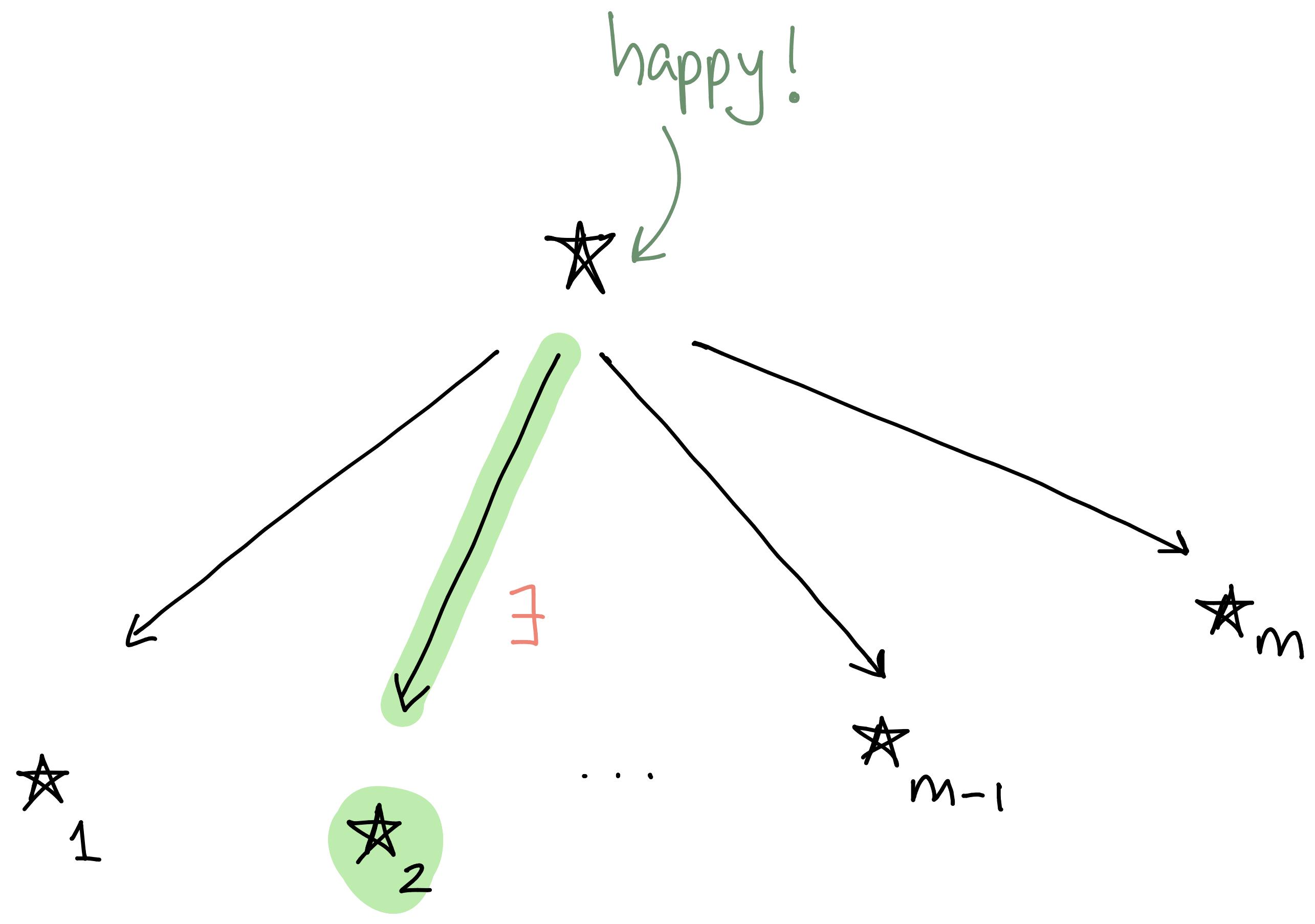








Sad?



Sad?

Suppose we can identify a property P such that :

if a Nim state satisfies P ,

any move leads to a Nim state that violates P ,

& if a Nim state violates P ,

\exists a move that leads to a Nim state that Satisfies P

Suppose we can identify a property P such that :

if a Nim state satisfies P ,

(& the "0"
State satisfies P)

& if a Nim state violates P ,

\exists a move that leads to a Nim state that Satisfies P

Then: the initial state violates P
if we have a first player win .

Then:

the initial state violates P

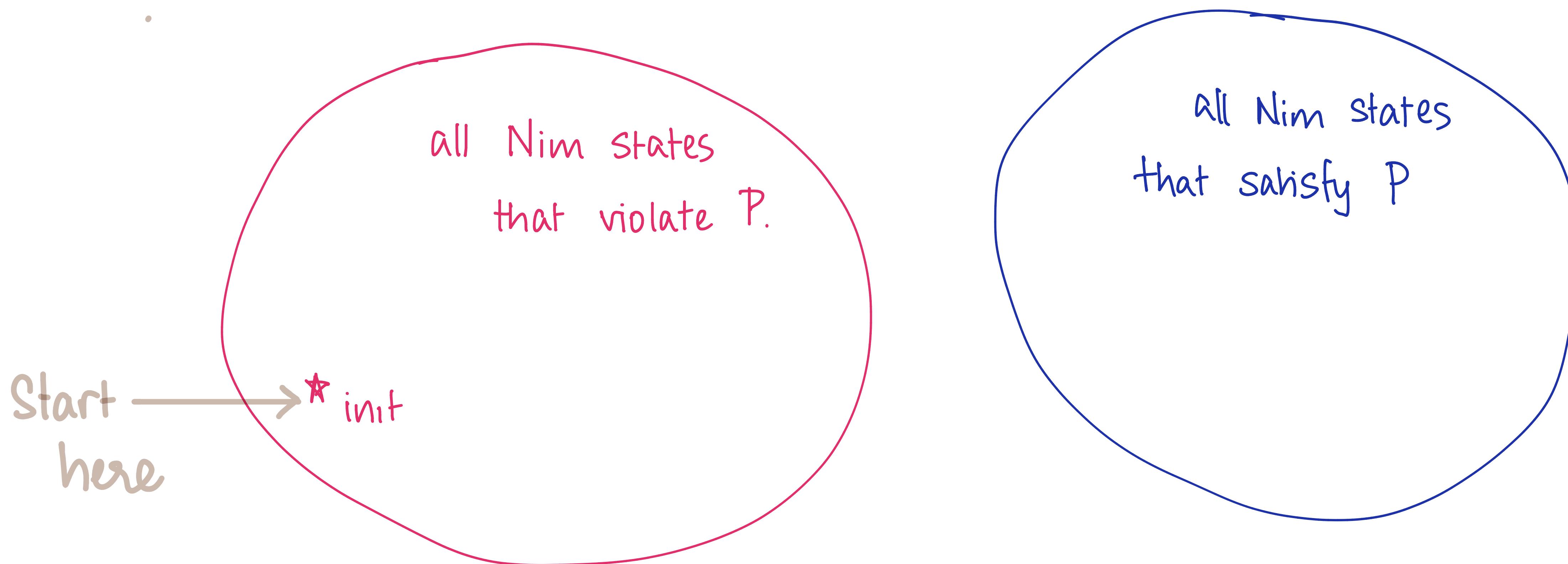
if we have a first player win .

all Nim states
that violate P.

All Nim states
that satisfy P

Then: the initial state violates P

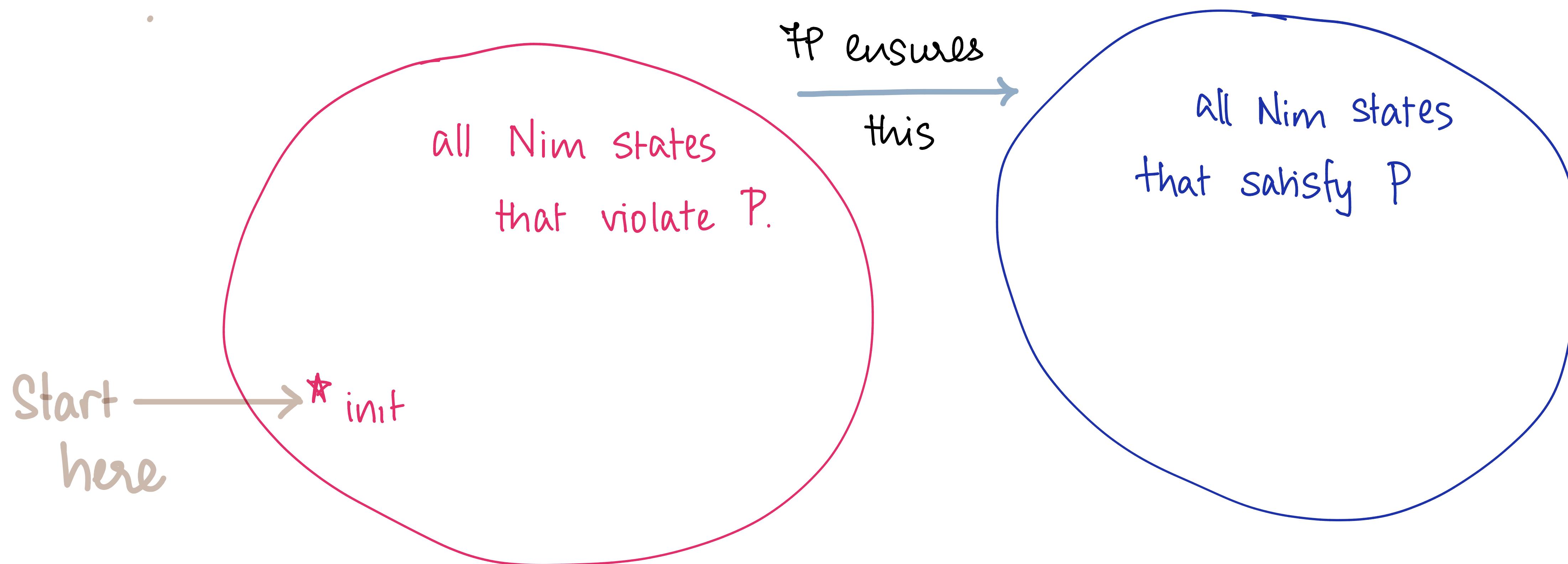
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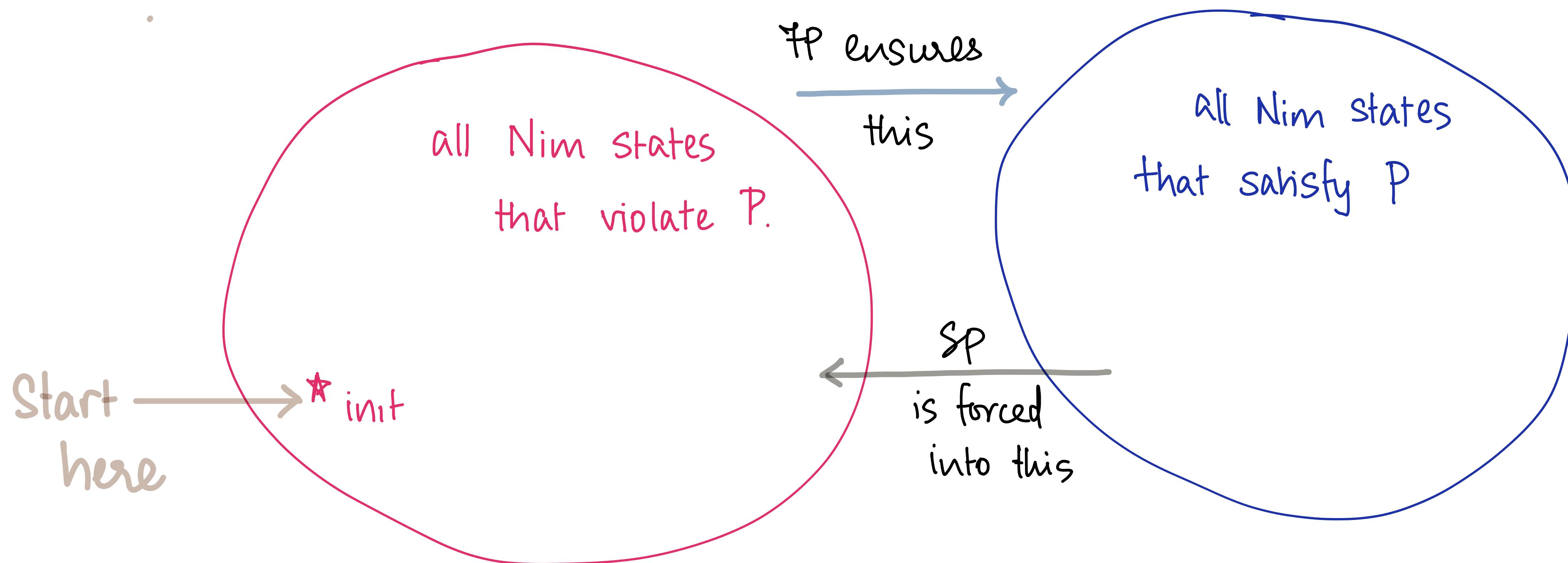
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Then:

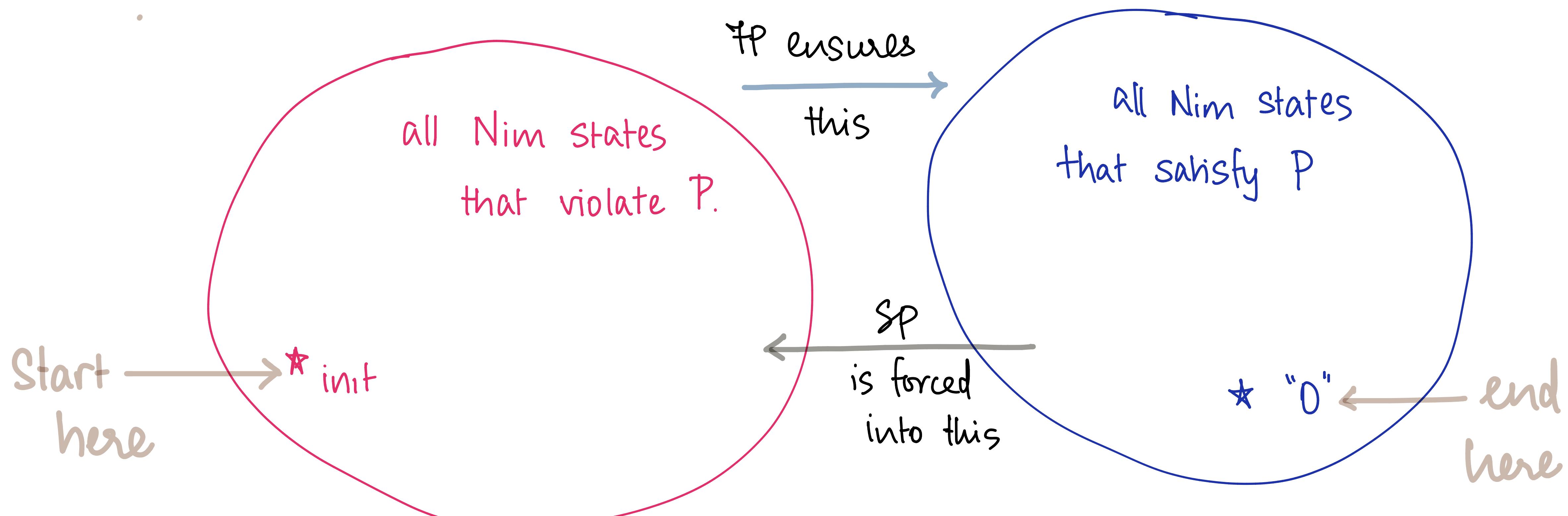
the initial state violates P

if we have a first player win .



Then: the initial state violates P

if we have a first player win .



The case of two heaps (p & q tokens WLOG p > q)

P : The two heaps have the same # of tokens.

balanced

if P holds : any move destroys P.

if P does not hold : \exists a move that restores P.

($P \rightarrow q$ by removing $p-q$ tokens)

The case of small heaps

(n_1, n_2, \dots, n_k) Each $n_i \in \{1, 2\}$.

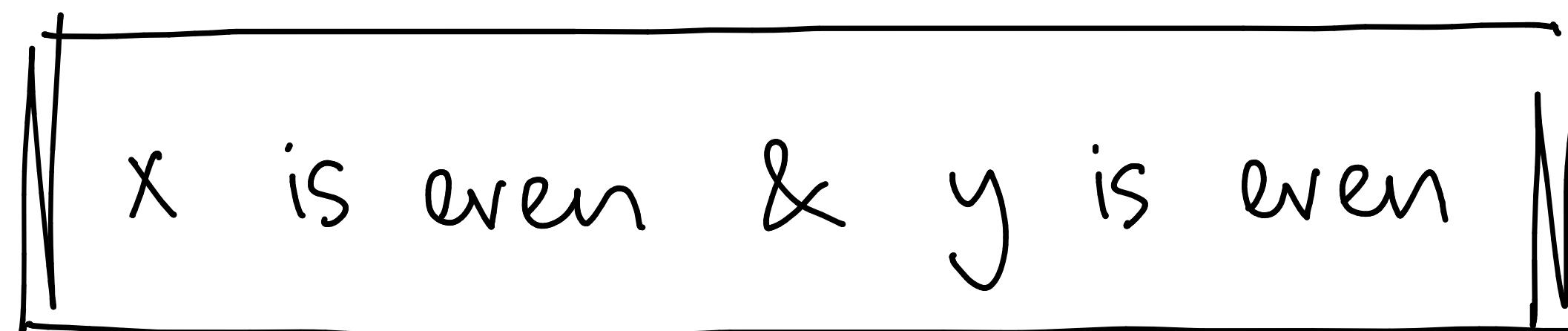
X heaps w/ 1 token Y heaps w/ 2 tokens

What's a property P in this setting?

The case of small heaps

(n_1, n_2, \dots, n_k) Each $n_i \in \{1, 2\}$.

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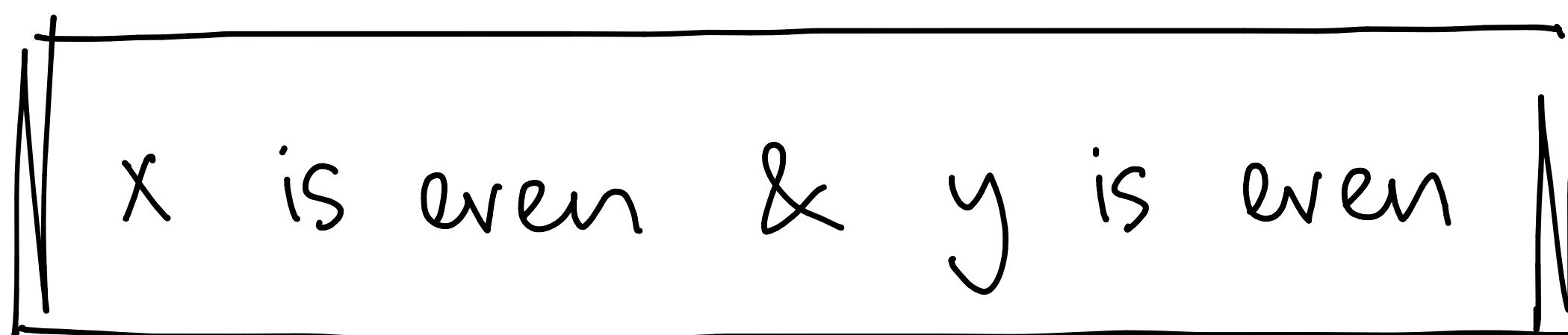


The case of small heaps

Remove $\underline{1}$ token from a 1-heap : $(x,y) \mapsto (x-1,y)$

Remove $\underline{1}$ token from a 2-heap : $(x,y) \mapsto (x+1,y-1)$

Remove $\underline{2}$ tokens from a 2-heap : $(x,y) \mapsto (x,y-1)$



The case of small heaps

Remove $\underline{1}$ token from a 1-heap : $(\cancel{x}, y) \mapsto (\underline{x-1}, \cancel{y})$

Remove $\underline{1}$ token from a 2-heap : $(\cancel{x}, y) \mapsto (\underline{x+1}, \cancel{y-1})$

Remove $\underline{2}$ tokens from a 2-heap : $(\cancel{x}, \cancel{y}) \mapsto (\underline{x}, \cancel{y-1})$

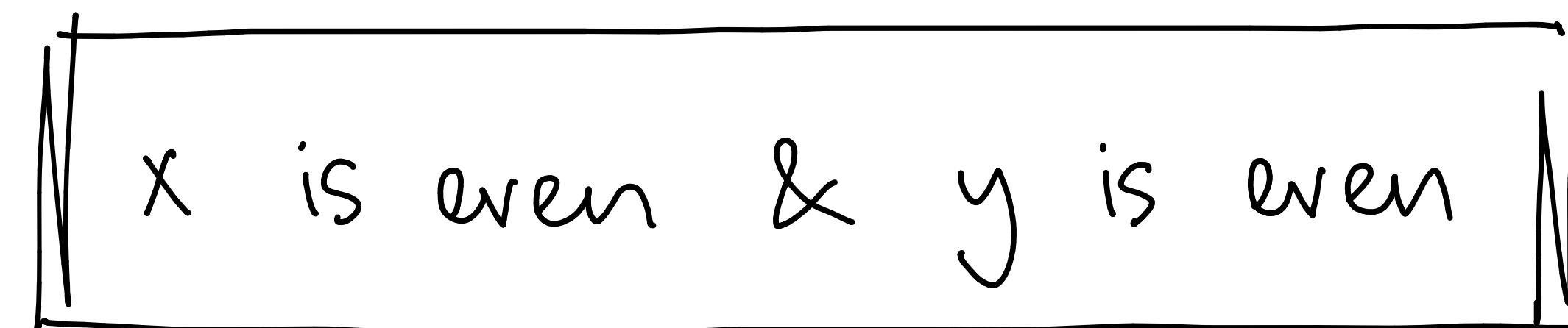
x is even & y is even

The case of small heaps

Remove $\underline{1}$ token from a 1-heap : $(x,y) \mapsto (\underline{x-1},y)$

Remove $\underline{1}$ token from a 2-heap : $(x,y) \mapsto (\underline{x+1},\underline{y-1})$

Remove $\underline{\geq 2}$ tokens from a 2-heap : $(x,y) \mapsto (\underline{x},\underline{y-1})$



The General Case.

Goal. Design the condition P that will allow us to determine if (n_1, \dots, n_k) is a first player win.

Detour: Combining Nim Games

$$(n_1, n_2, \dots, n_p)$$

*

$$(m_1, m_2, \dots, m_q)$$

=

$$(n_1, n_2, \dots, n_p, m_1, m_2, \dots, m_q)$$

{ Associate } w/ each Nim state

a # that captures

information about that state.

$$\{(n)\} \rightsquigarrow n \quad (\text{in particular, } \emptyset \rightsquigarrow 0)$$

$$\{(n,m)\} \rightsquigarrow \{(n) \star (m)\}$$

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$$\rightsquigarrow \{(n)\} \star \{(m)\} \rightsquigarrow n \star m$$

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?

WAT?

{ Associate } w/ each Nim state

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{(n)} \mapsto n (in particular, $\emptyset \mapsto 0$)

{(n,m)} \mapsto $n * m$

{ Associate } w/ each Nim state

a # that captures

information about that state.

$$\{(n)\} \rightsquigarrow n \quad (\text{in particular, } \emptyset \rightsquigarrow 0)$$

$$\{(n, m)\} \rightsquigarrow n * m$$

$$\text{WANT (possibly)} : n * n = 0.$$

{ Associate } w/ each Nim state

a # that captures

information about that state.

the very definition

of SAD

$$\{(n)\} \rightsquigarrow n$$

(in particular,

$$\emptyset \rightsquigarrow 0$$

$$\{(n, m)\} \rightsquigarrow n * m$$

SAD
when
 $n = m$.

WANT (possibly) : $n * n = 0$.

{ Associate } w/ each Nim state

a # that captures

$n \star n = 0$

information about that state.

{(n)} \mapsto n (in particular, $\emptyset \mapsto 0$)

{(n,m)} \mapsto n \star m

{ Associate } w/ each Nim state

a # that captures

$$n \star n = 0$$

information about that state.

$$\{(n)\} \mapsto n \quad (\text{in particular, } \emptyset \mapsto 0)$$

$$\{(n, m)\} \mapsto n \star m$$

Does \star commute?

{ Associate } w/ each Nim state

a # that captures

$$n \star n = 0$$

information about that state.

$$\{(n)\} \mapsto n \quad (\text{in particular, } \emptyset \mapsto 0)$$

$$\{(n,m)\} \mapsto n \star m$$

Does \star commute? $(n,m) \vee/s (m,n)$

{ Associate } w/ each Nim state

inverse



$$n \star n = 0$$

↑
ID

$$n \star m = m \star n$$

commutative.

feels like

a group?

a # that captures

information about that state.

$$\{(n)\} \mapsto n \quad (\text{in particular, } \emptyset \mapsto 0)$$

$$\{(n, m)\} \mapsto n \star m$$

When can we say : $a \star b = c$?

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$$c \star c = 0$$

When can we say :

$$\boxed{a \star b = c} ?$$

$$c \star c = 0$$



$$\equiv (a \star b) \star c = 0$$

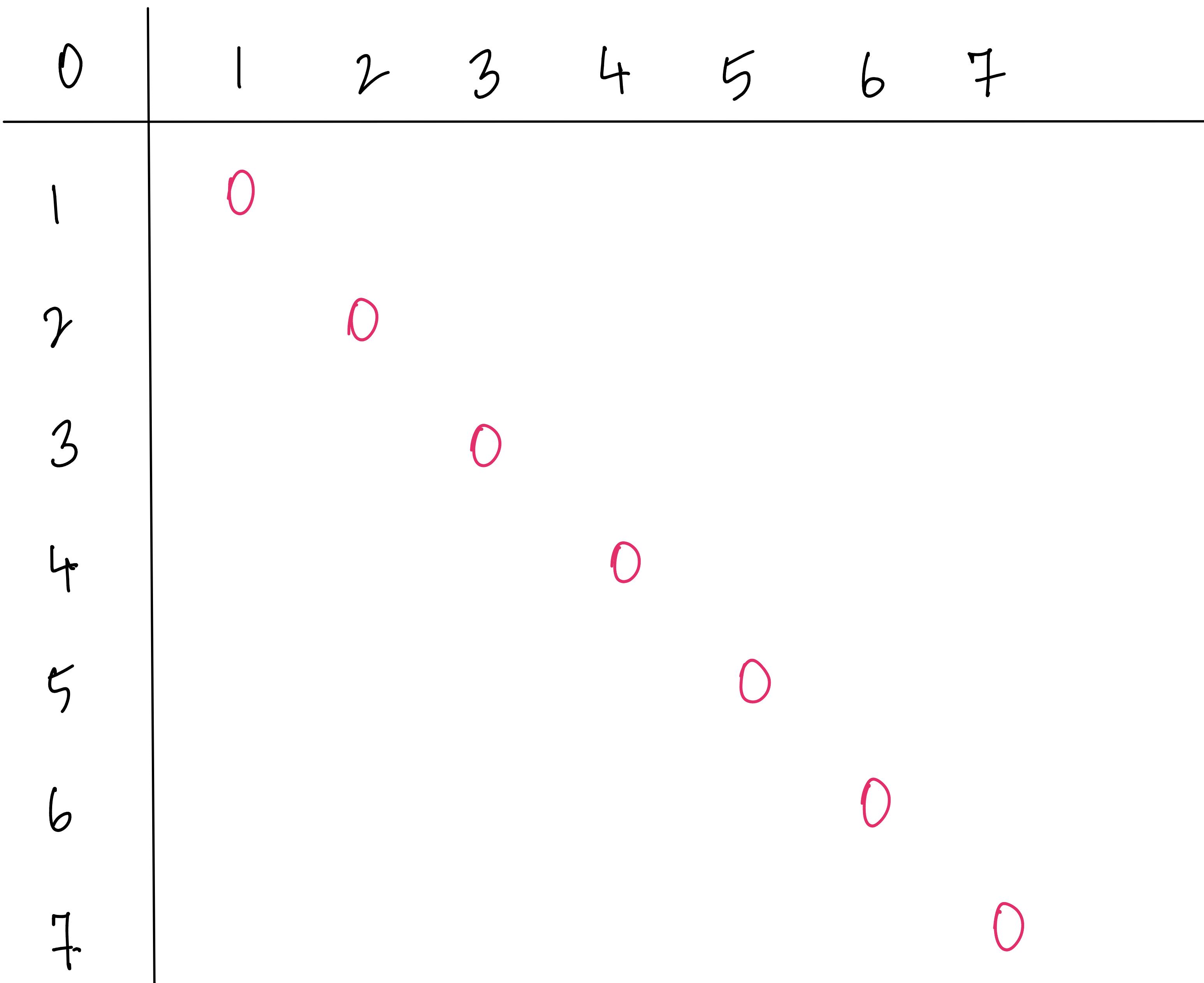
When can we say : $a \star b = c$?

$$c \star c = 0$$



$$\equiv (a \star b) \star c = 0$$

So: $a \star b = c$ iff $(a \star b) \star c = 0$



(what
we
know
so
far)

\star is associative

$$(a \star b) \star c = a \star (b \star c)$$

$\underbrace{ \quad \quad}_{p}$ $\underbrace{}_{q}$

$$p = a \star b$$

$$p = a \star b \star c \star c$$

$$p \star c = a \star b \star c$$

$$p \star c = a \star q$$

What is $1 \star 2$?

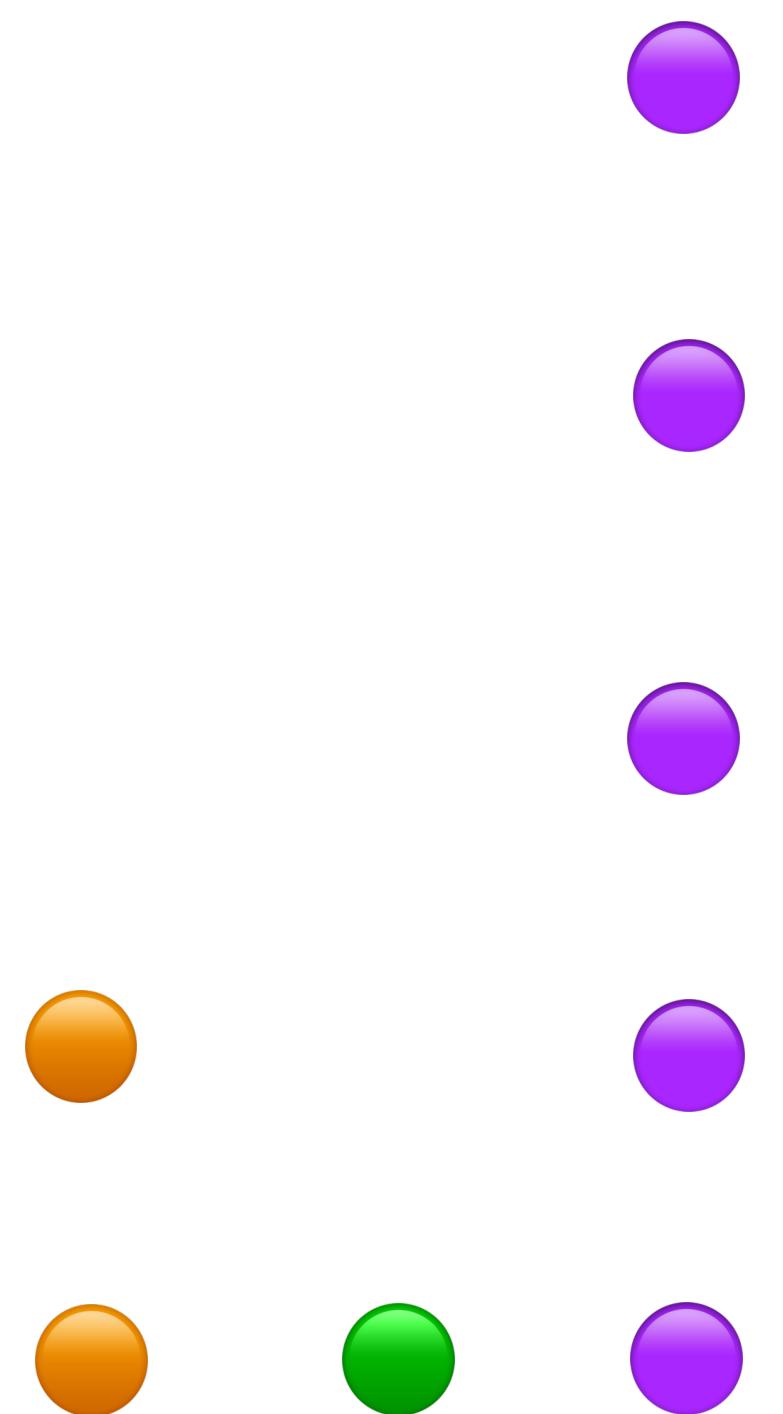
If we claim that

$$1 \star 2 = n,$$

then it must be that

$$1 \star 2 \star n = 0.$$

SAD



(i.e. every move \Rightarrow happy)

What is $1 \star 2$?

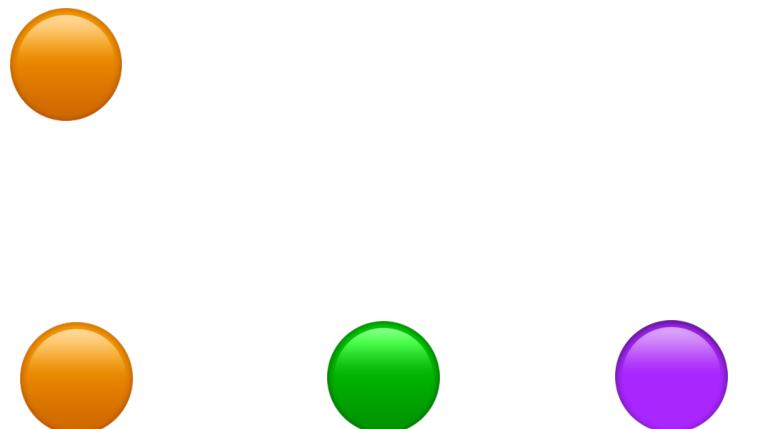
If we claim that

$$1 \star 2 = 1,$$

then it must be that

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SAD



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What is $1 \star 2$?

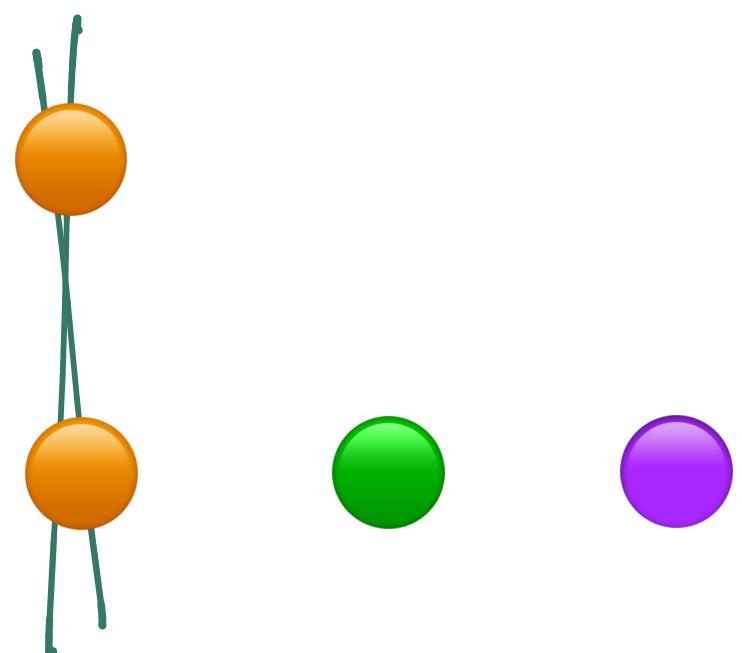
If we claim that

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~~SAD~~ happy



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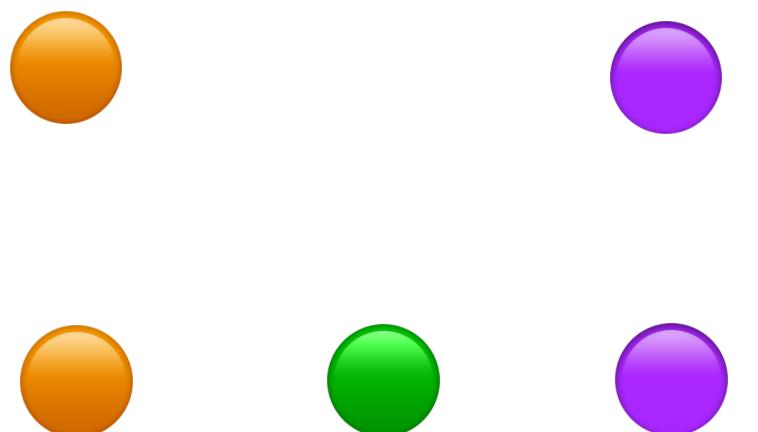
If we claim that

$$1 \star 2 = 2,$$

then it must be that

$$1 \star 2 \star 2 = 0.$$

SAD



(i.e. every move \Rightarrow happy)

What is $1 \star 2$?

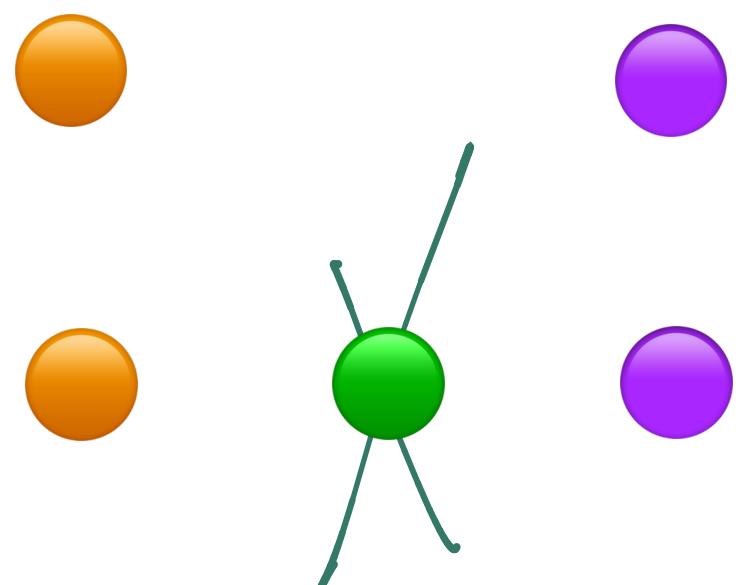
If we claim that

$$1 \star 2 = 2,$$

then it must be that

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What is $1 \star 2$?

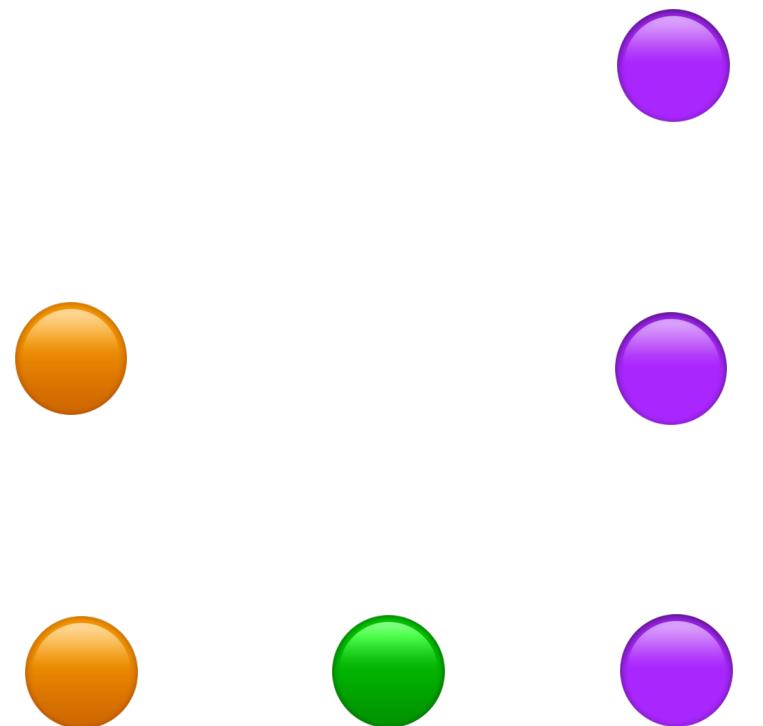
If we claim that

$$1 \star 2 = 3,$$

then it must be that

$$1 \star 2 \star 3 = 0.$$

SAD



(i.e. every move \Rightarrow happy)

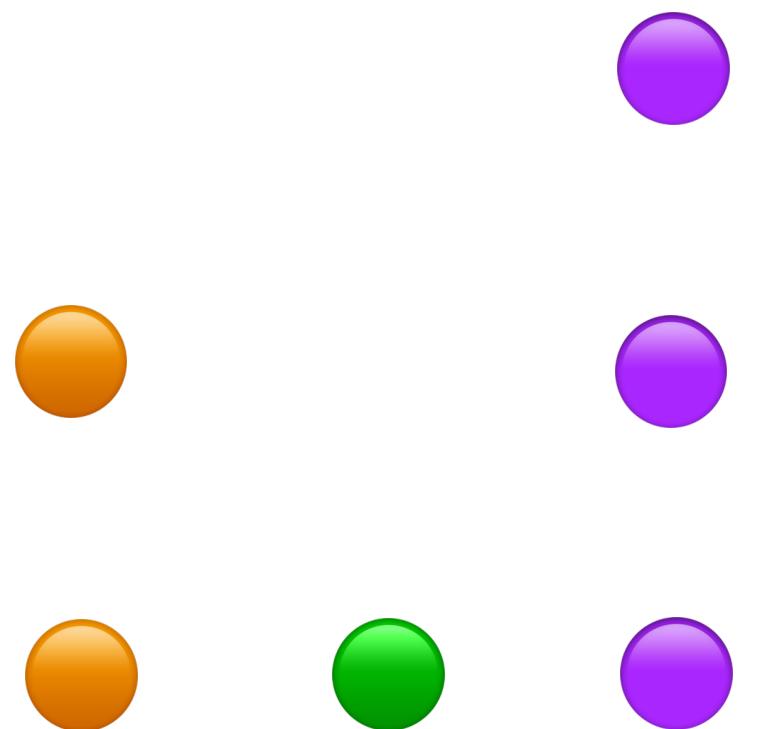
What is $1 \star 2$?

If we claim that

$$1 \star 2 = 3,$$

* by case

analysis



$$1 \star 2 \star 3 = 0.$$

SAD

(i.e. every move \Rightarrow happy)

this actually works! *

	0	1	2	3	4	5	6	7
0	0	3	2					
1	0	3	2					
2	3	0	1					
3	2	1	0					
4				0				
5					0			
6						0		
7							0	

$$1 \star 2 = 3$$

$$1 \star 2 \star 3 = 0$$

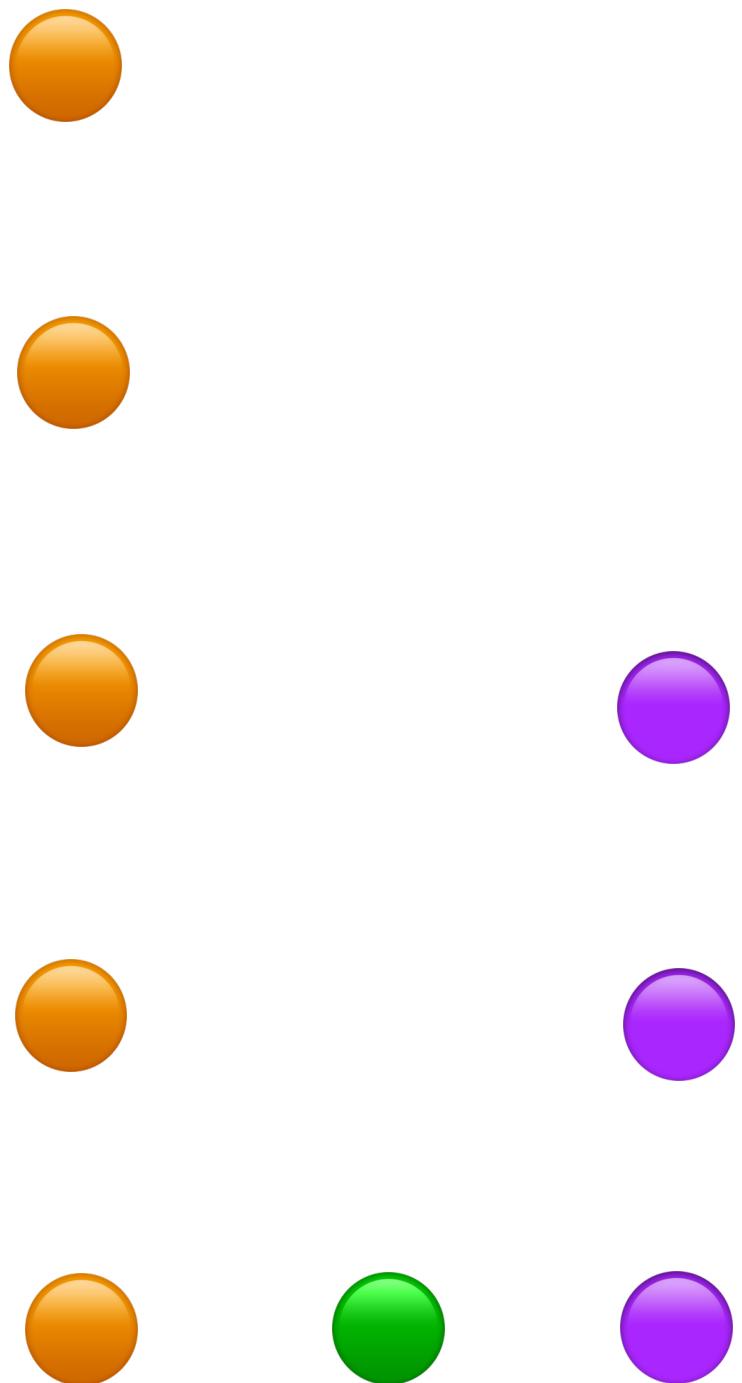
$$1 \star 3 = 2$$

$$2 \star 3 = 1$$

What is $1 \star 5$?

If we claim that

$$1 \star 5 = n,$$



then it must be that

$$1 \star 5 \star n = 0.$$

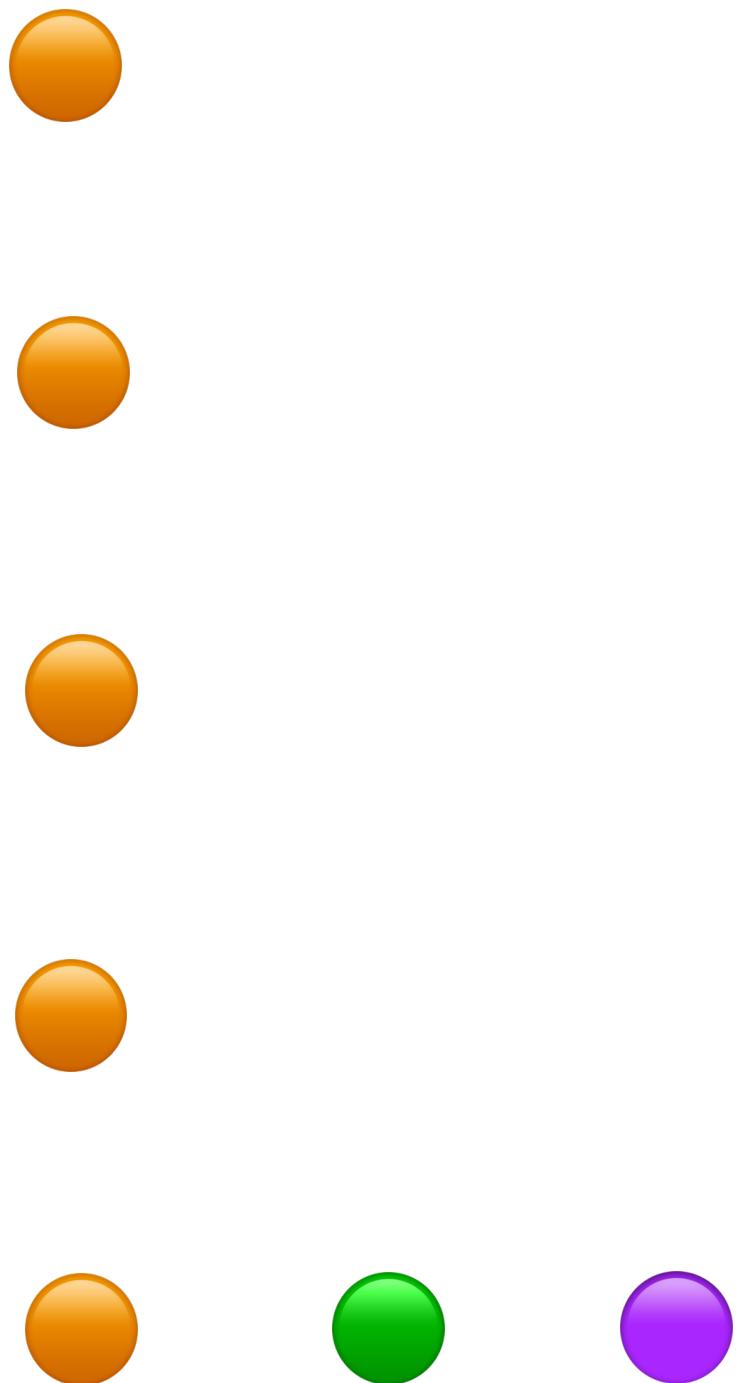
SAD

(i.e. every move \Rightarrow happy)

What is $1 \star 5$?

If we claim that

$$1 \star 5 = 1,$$



then it must be that

$$1 \star 5 \star 1 = 0.$$

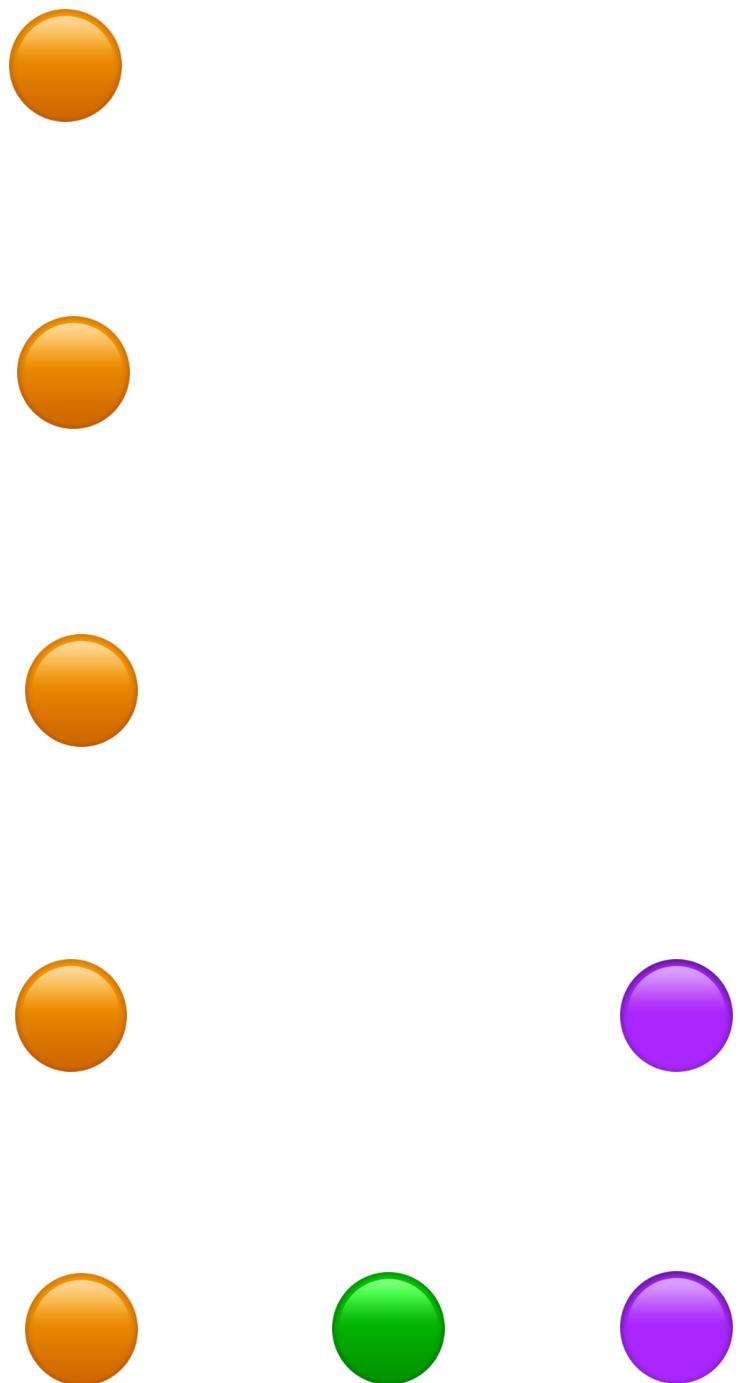
SAD

(i.e. every move \Rightarrow happy)

What is $1 \star 5$?

If we claim that

$$1 \star 5 = 2,$$



then it must be that

$$1 \star 5 \star 2 = 0.$$

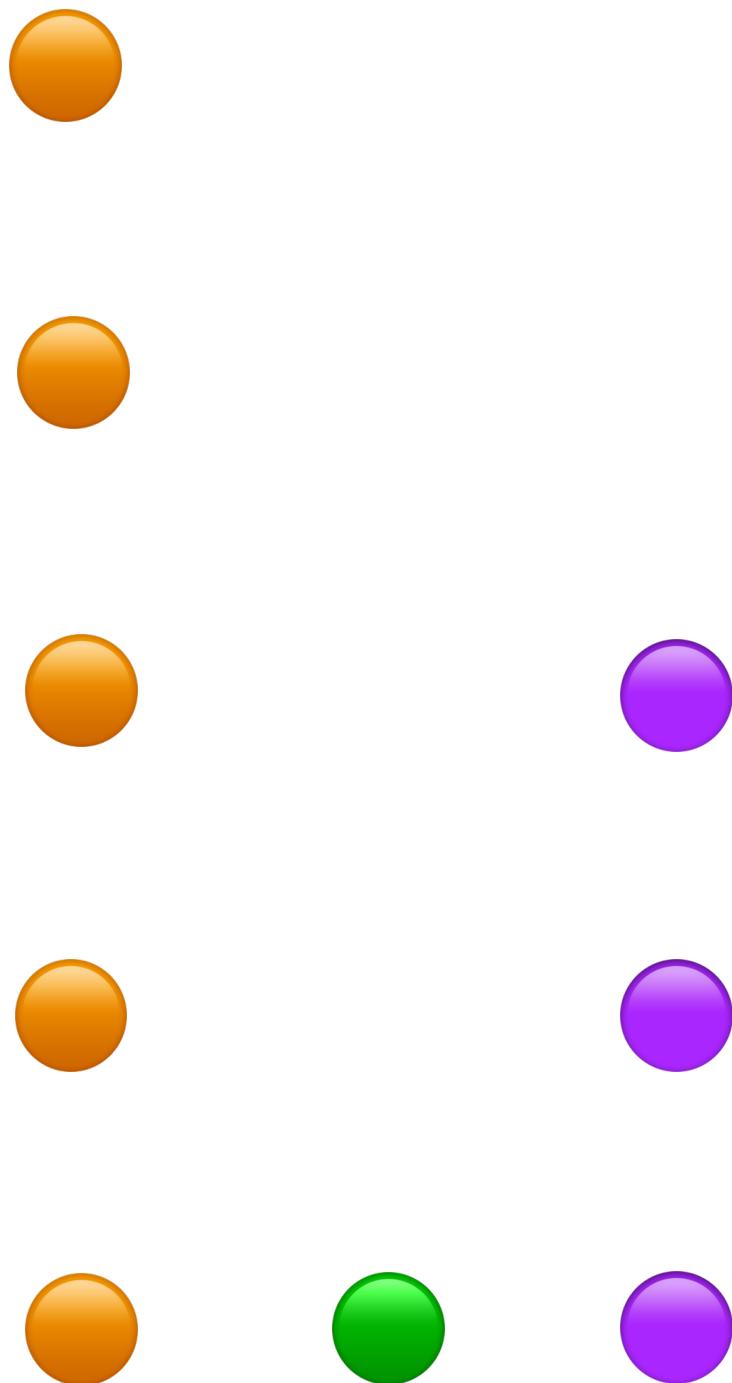
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What is $1 \star 5$?

If we claim that

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then it must be that

$$1 \star 5 \star 3 = 0.$$

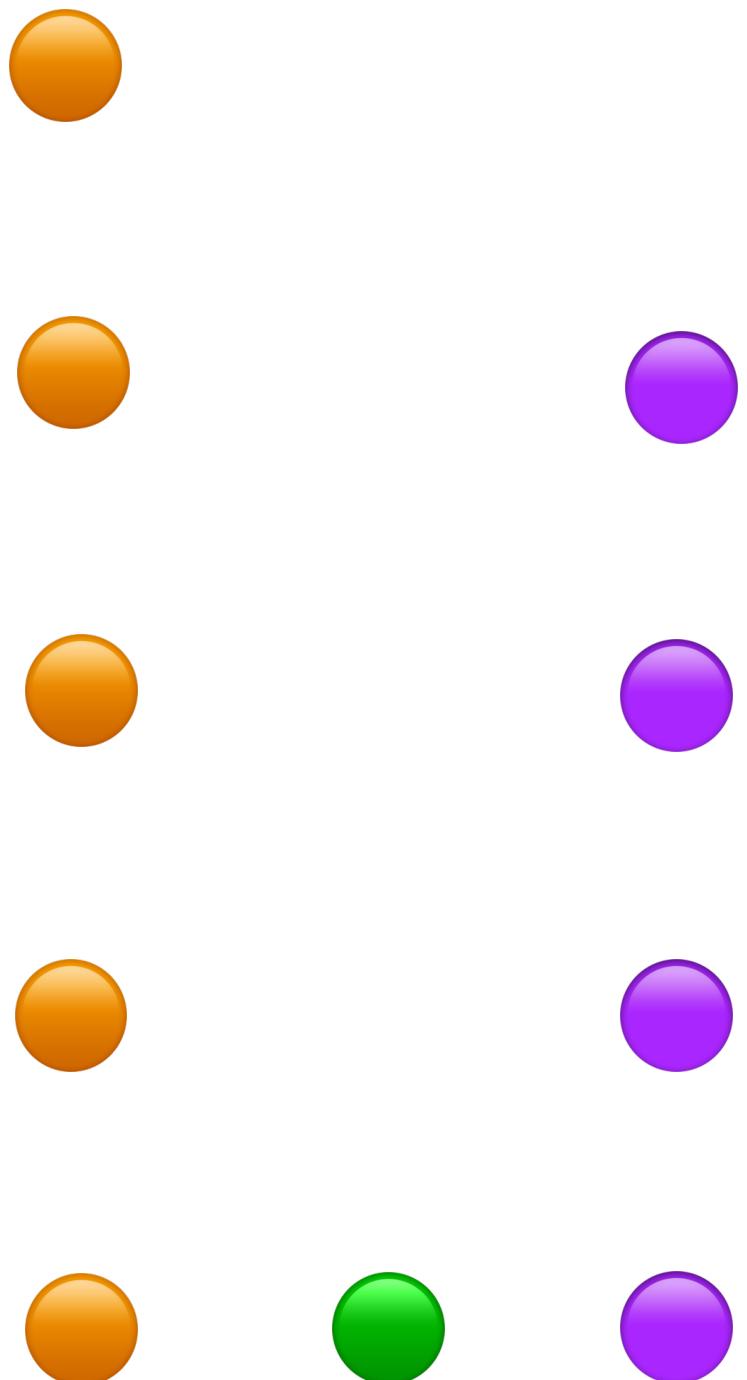
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What is $1 \star 5$?

If we claim that

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then it must be that

$$1 \star 5 \star 3 = 0.$$

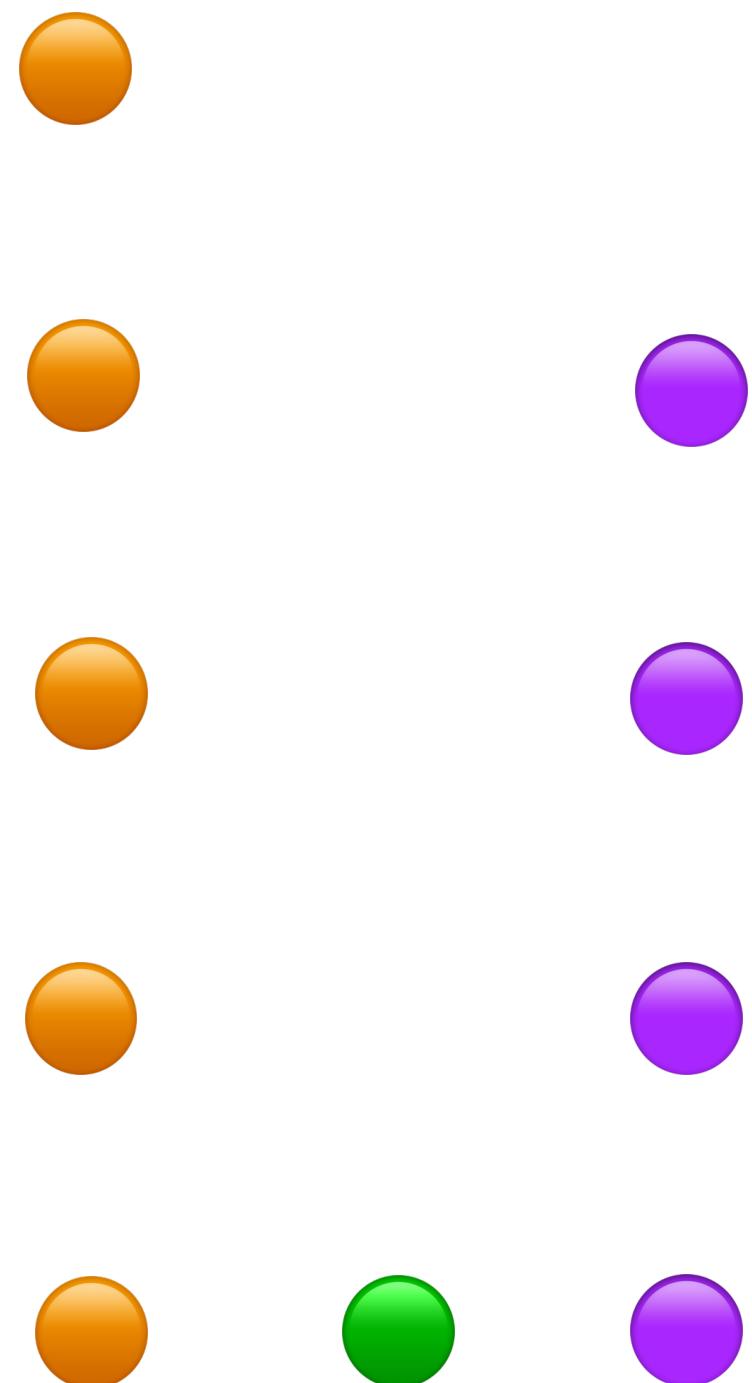
SAD

(i.e. every move \Rightarrow happy)

What is $1 \star 5$?

If we claim that

$$1 \star 5 = 4,$$



then it must be that

$$1 \star 5 \star 4 = 0.$$

SAD

(i.e. every move \Rightarrow happy)

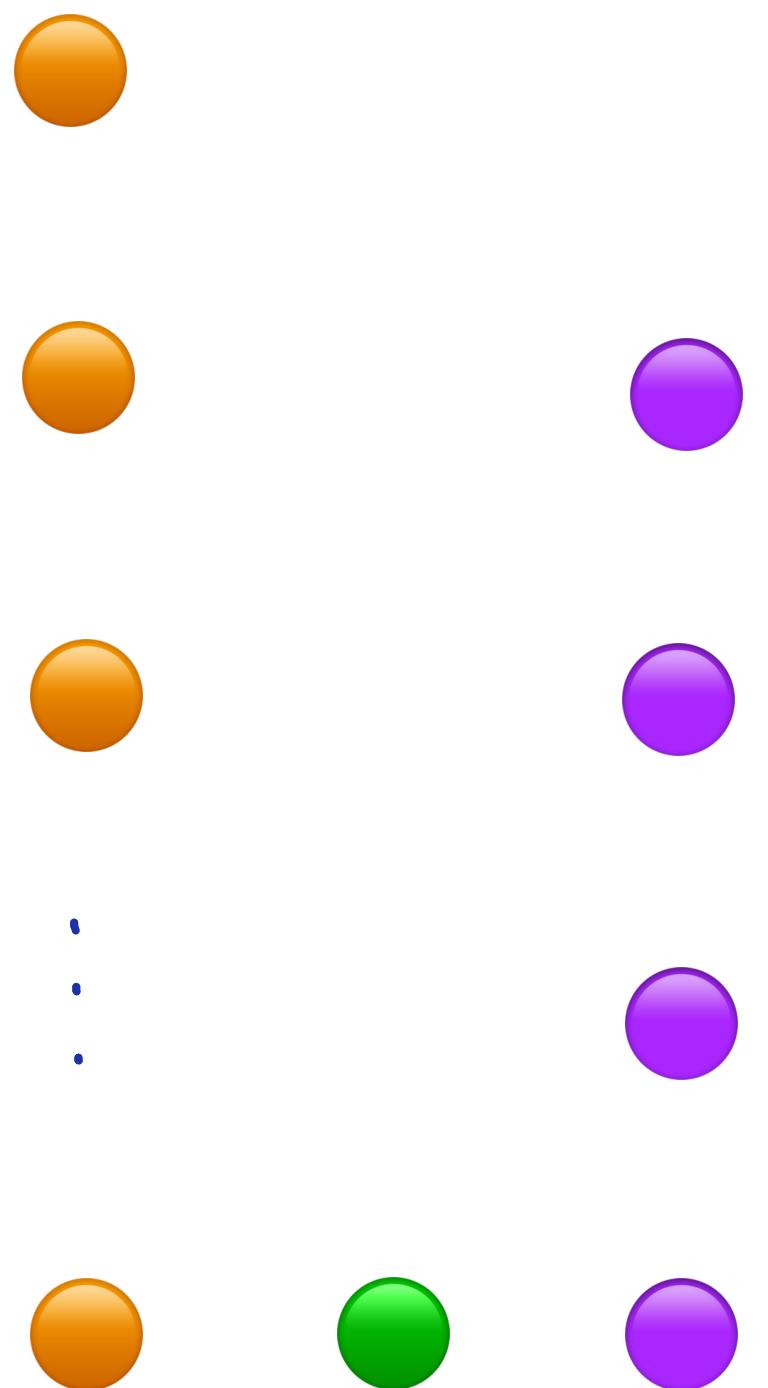
What is $1 \star k$?



⋮



What is $1 \star k$?



$$1 \star k = n$$

$$\equiv 1 \star k \star n = 0$$

SAD

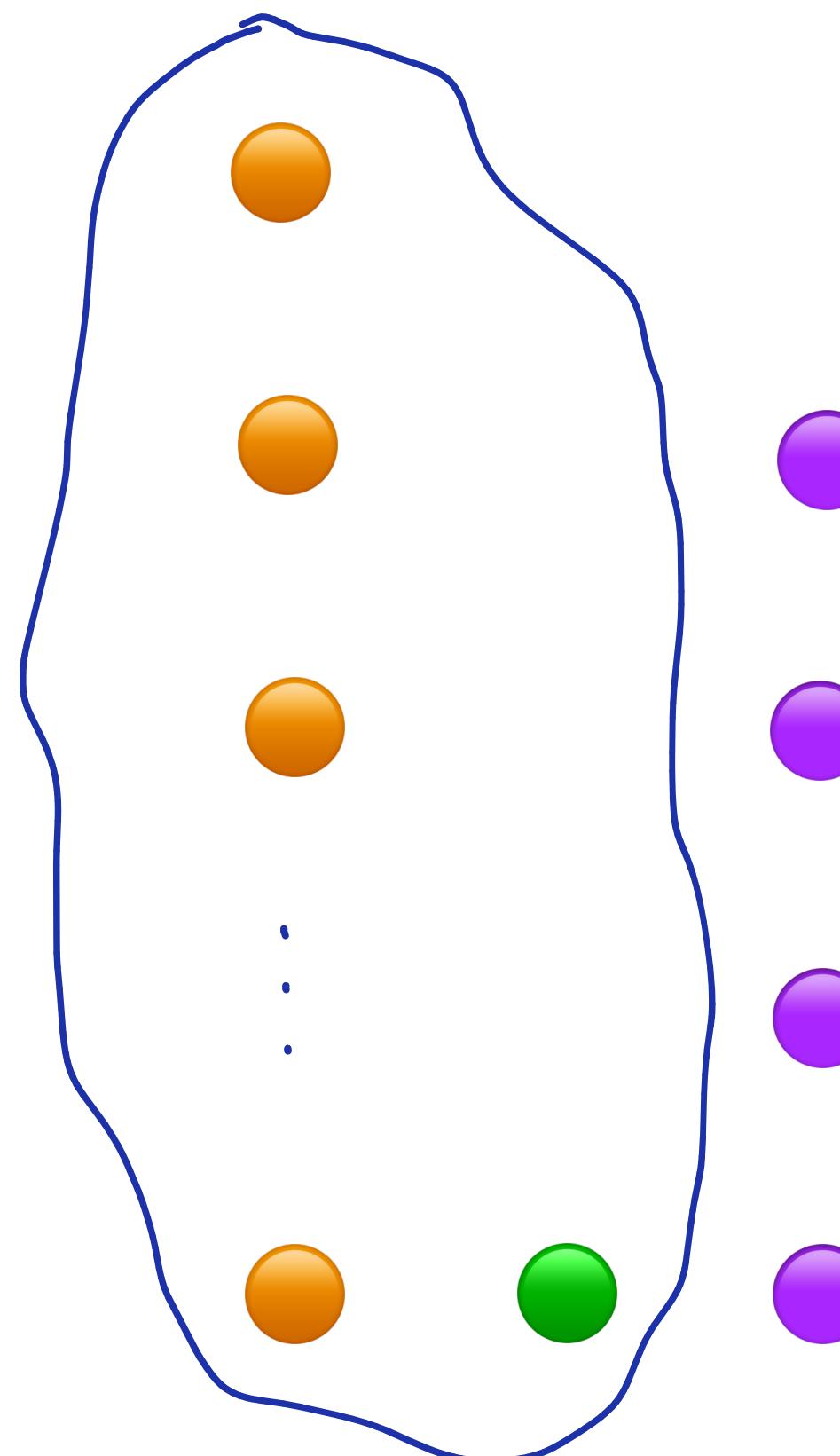
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SAD

Suppose
 \exists a move
in this game
leading to a
game of value s



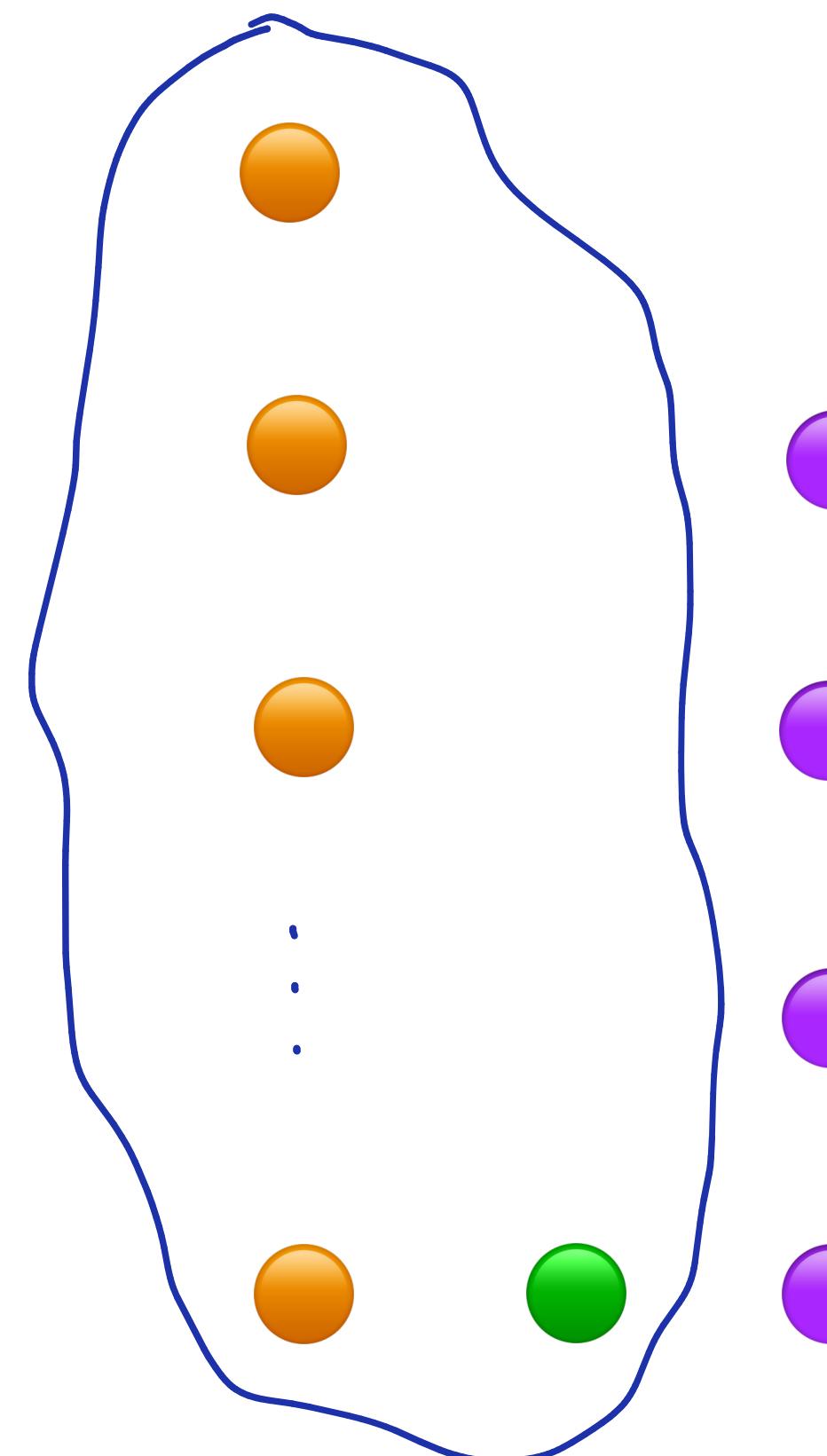
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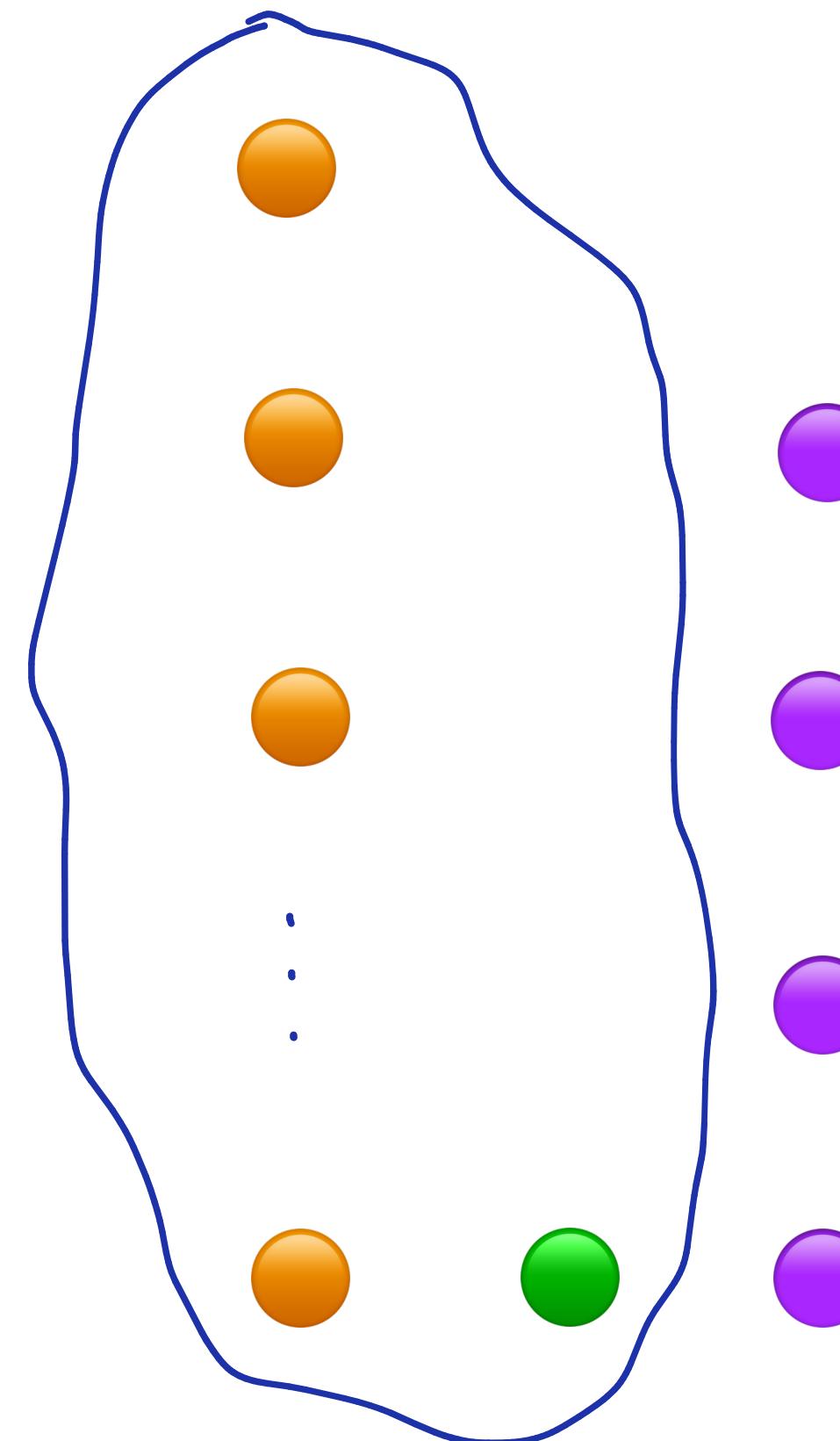
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Then $n \neq s$,
because \exists a move
to $\underbrace{s \star s}_{\text{SAD}}$

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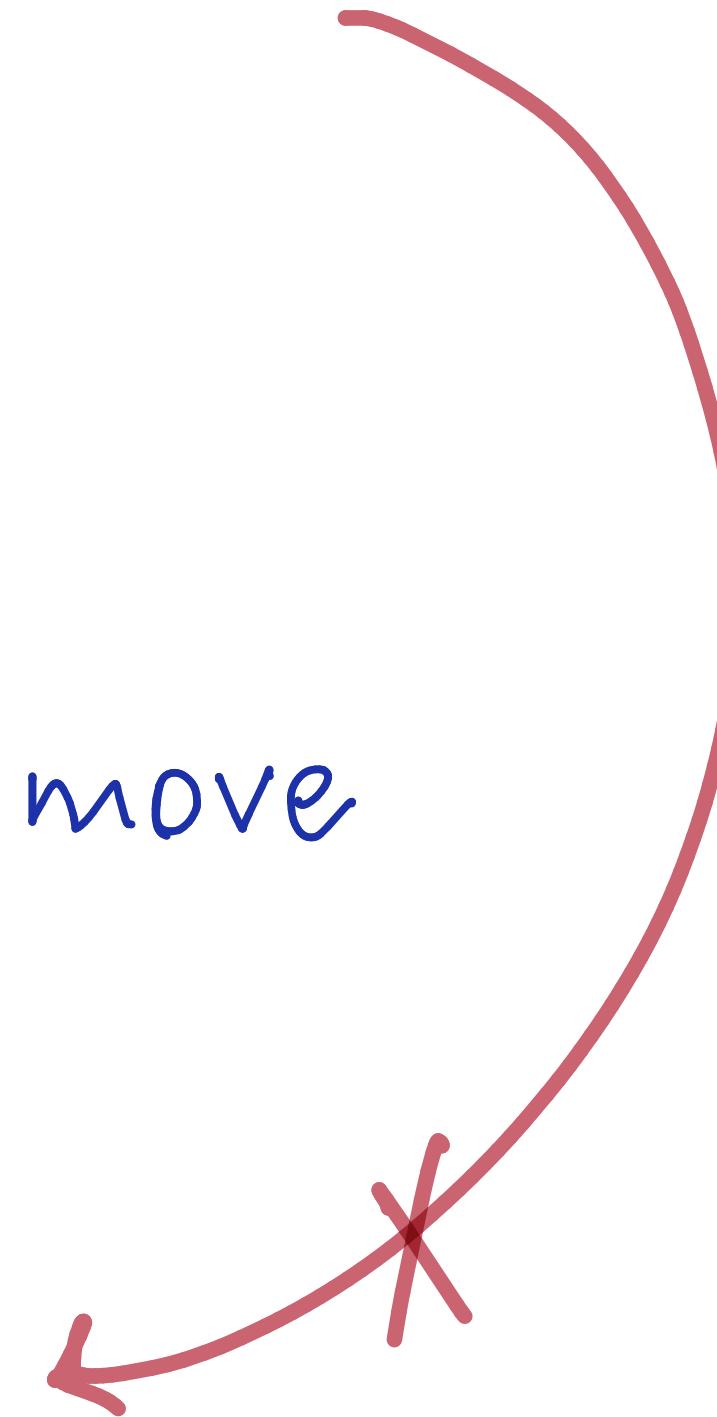


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$$\equiv 1 \star k \star n = 0$$

SAD

Then $n \neq s$,
because \exists a move
to $\underbrace{s \star s}_{\text{SAD}}$



$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

$$1 \star 4 = ?$$

$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

$$1 \star 4 = ?$$

cannot be 0, 1, 2, 3, or 4

$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

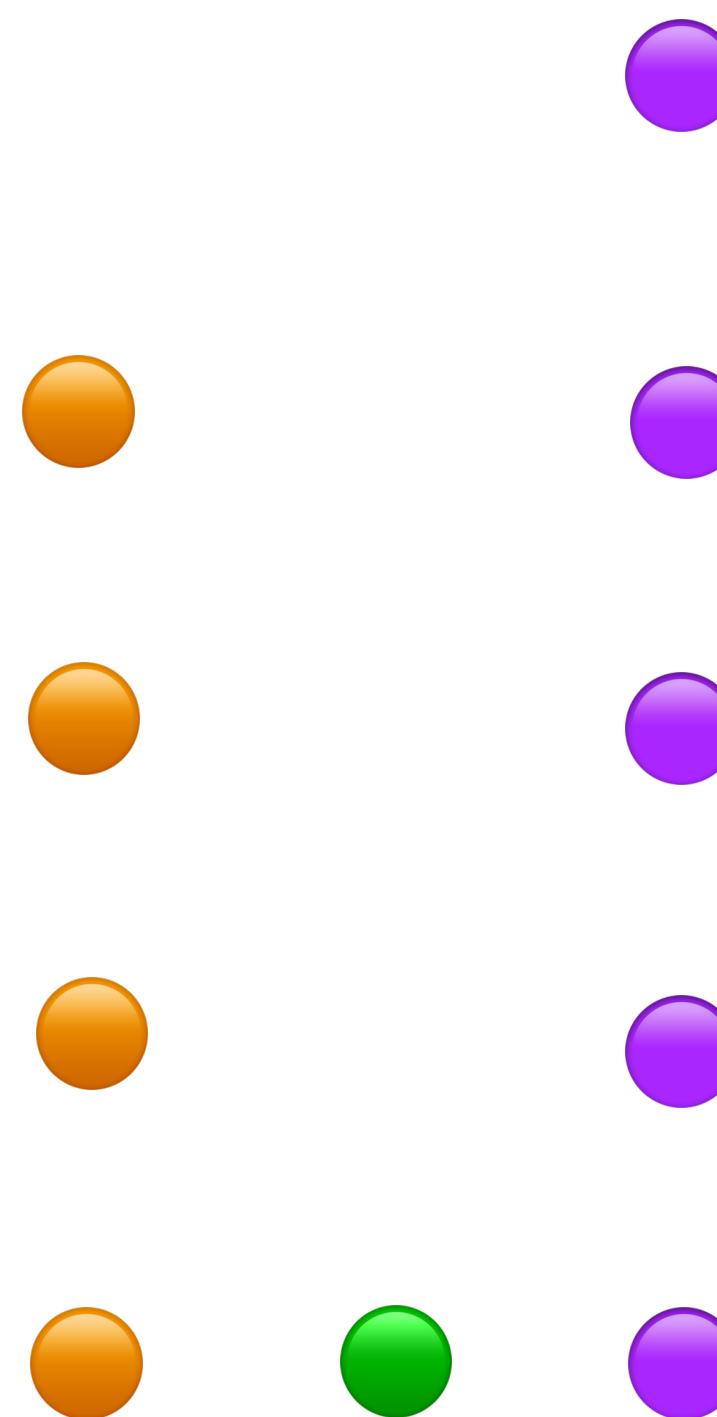
$$1 \star 4 = ?$$

cannot be 0, 1, 2, 3, or 4

earliest possibility : 5

$$1 \star 4 \star 5 = 0$$

WTS: every move is happy



by a case analysis into "smaller scenarios" that we understand

$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

$$1 \star 4 = 5$$

$$1 \star 5 = ?$$

$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

$$1 \star 4 = 5$$

$$1 \star 5 = ? \quad \text{cannot be } 0, 1, 2, 3, 5 .$$

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let's try ... 4?

$$1 \star 0 = 1$$

$$1 \star 1 = 0$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

$$1 \star 4 = 5$$

$$1 \star 5 = ?$$

Turns out

$$1 \star 5 = 4$$

(and we have, in fact,

implicitly argued this already!)

cannot be $0, 1, 2, 3, 5$.

let's try ... 4?

	0	1	2	3	4	5	6	7
0								
1	0	3	2	5	4			
2	3	0	1					
3	2	1	0					
4	5			0				
5	4				0			
6					0			
7						0		

	0	1	2	3	4	5	6	7	
0									
1	0	3	2	5	4	7	6	..	
2	3	0	1						
3	2	1	0						
4	5			0					
5	4				0				
6	7					0			
7	6						0		

0	1	2	3	4	5	6	7	
1	0	3	2	5	4	7	6	..
2	3	0	1					
3	2	1	0					(minimum excluded #)
4	5			0	←	★		MEX of all entries
5	4				0			above & to the left
6	7					0		
7	6						0	



- ① more on the operation \star
(in direct terms)
- ② the property P for general Nim games
- ③ Sprague - Grundy Theorem
- ④ Hackenbush