Combinatorial & Computational Aspects of Games

1. Reaching 100
2. Dr. Nim (aka Subtraction)
3. The Game of Nim

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Reaching 100

Two player & turn-based

Initially : counter = 0

On a single turn :

current player increments the counter by k ← $1 \leq k \leq 10$

WIN : The first player to get the counter to 100 wins.

Reaching 100

\# players

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Reaching 100

Two player & turn-based

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elements & starting state.

On a single turn :

current player increments

the counter by k ← $1 \leq k \leq 10$

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Reaching 100

Two player & turn-based

Initially : $\boxed{\text{counter} = 0}$

On a single turn :

ruleset & dynamics →

current player increments
the counter by k ← $1 \leq k \leq 10$

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Reaching 100

Two player & turn-based

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On a single turn :

current player increments

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Reaching 100

\# players

Two player & turn-based mechanics.

Initially : counter = 0

elements & starting state.

On a single turn:

ruleset & dynamics

current player increments the counter by k ← $1 \leq k \leq 10$

goal

WIN : The first player to get the counter to 100 wins.

Some other aspects:

* Role of chance
* Who knows what
* Co-operative v/s. Competitive

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In most scenarios that we will encounter:

* Role of chance (None: deterministic moves)
* Who knows what
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Reaching 100

Two player & turn-based

Initially : counter = 0

Let's play

On a single turn :

current player increments

the counter by k ← $1 \leq k \leq 10$

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Reaching 100

Observation To reach 100, it suffices to reach 89.

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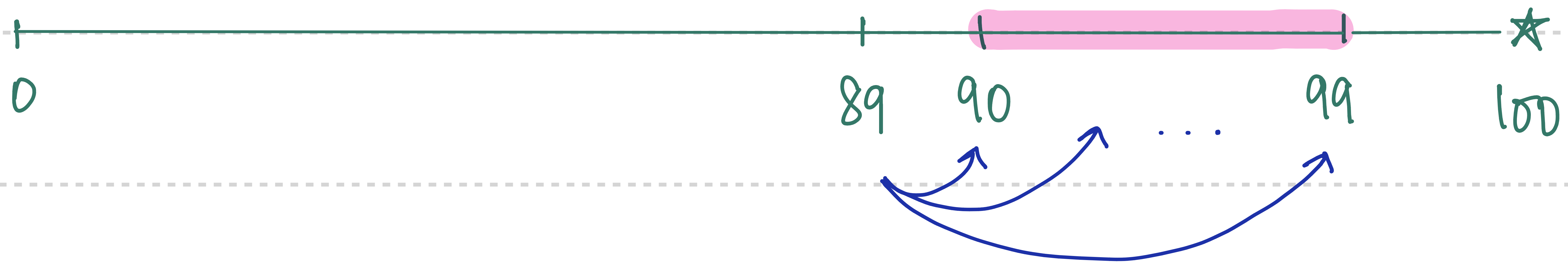

Reaching 100

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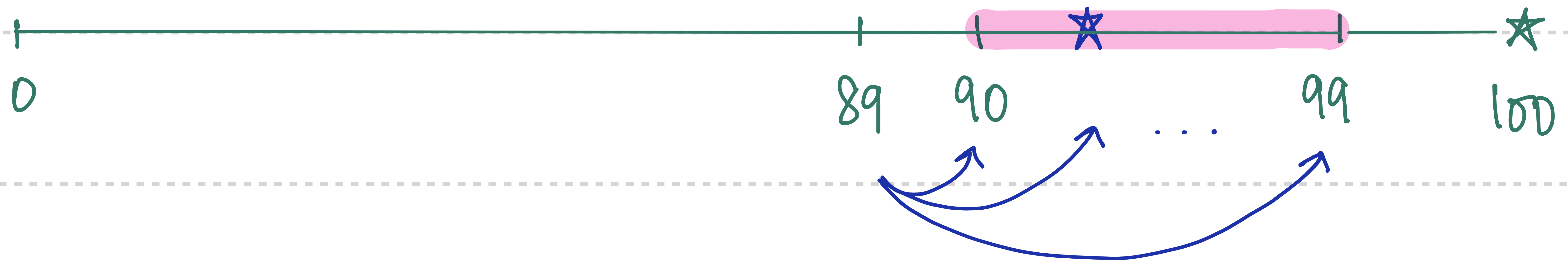

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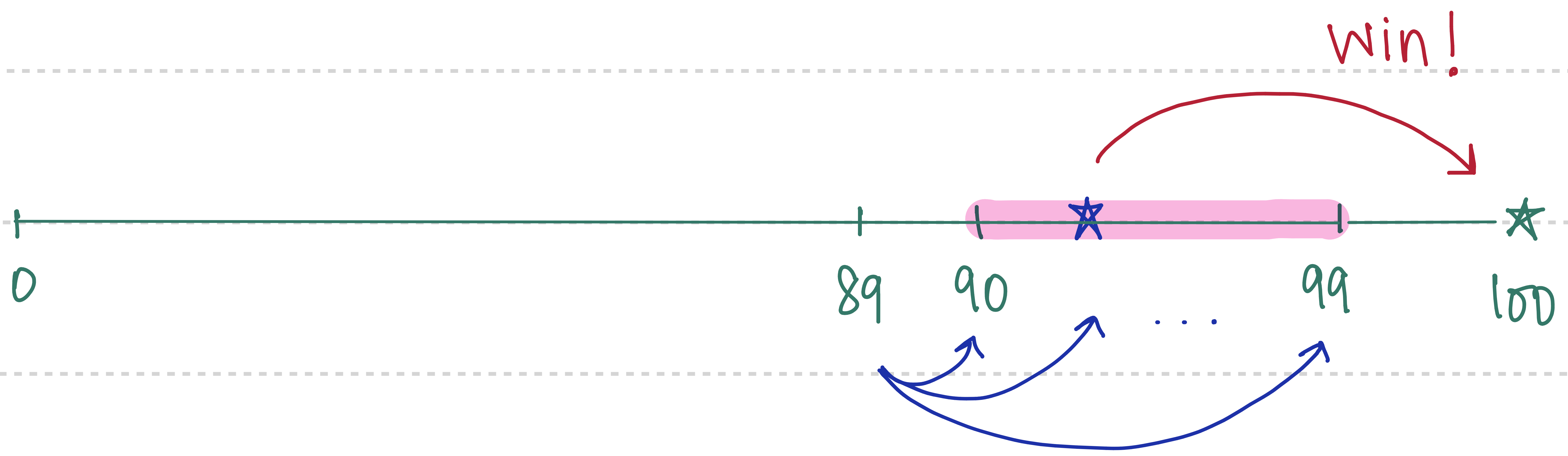

Reaching 100

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To reach 89, it suffices to reach 78.

Reaching 100

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To reach 78, it suffices to reach 67.

Reaching 100

Observation

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

To reach 78, it suffices to reach 67.

To reach 12, it suffices to reach 1.

Reaching 100

Observation

To reach 100, it suffices to reach 89.

To reach 89, it suffices to reach 78.

To reach 78, it suffices to reach 67.

Start Here

To reach 12, it suffices to reach 1.

Strategies

Functions that map game states to valid moves.

can always answer the question:

"what do I do now?"

Winning 🏆

Strategies

Guarantee a win.

Functions that map game states to valid moves.

can always answer the question:

" what do I do now? "

In the Reach 100 game, the $\boxed{\qquad\qquad}$ player has a winning strategy.

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if (state = 0) inc by $\underline{1}$.

if ($2 \leq$ state ≤ 11) inc by $\underline{12 - \text{state}}$

if ($13 \leq$ state ≤ 22) inc by $\underline{23 - \text{state}}$

$\vdots$

else inc by $\underline{1}$

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Dr. Nim

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Nim

Two player, turn-based

Init : $(\underbrace{n_1, n_2, \ldots, n_k}_{})$

k heaps, i-th heap has n_i tokens

One move : pick a heap &
$(1 \le i \le k)$

remove any # of tokens from it
$1, 2, \ldots, n_i$

Win :

The last player who can make a valid move wins

Nim

(Examples)

1 Two heaps, one token each.

Nim

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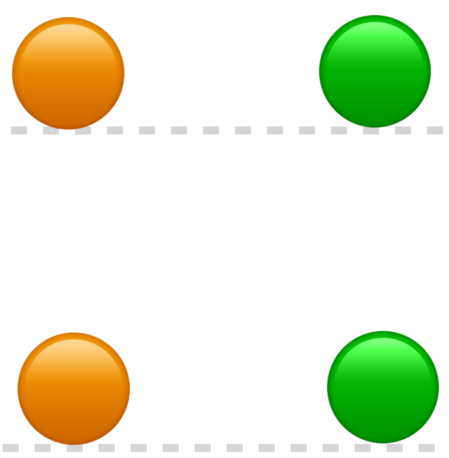
→ First player loses.

Nim

(Examples)



2. Two heaps, two tokens each.

Nim

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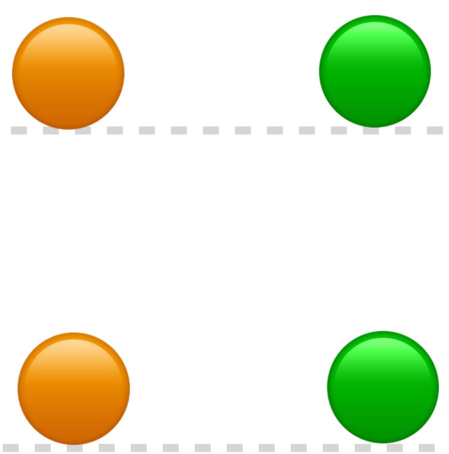

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Nim

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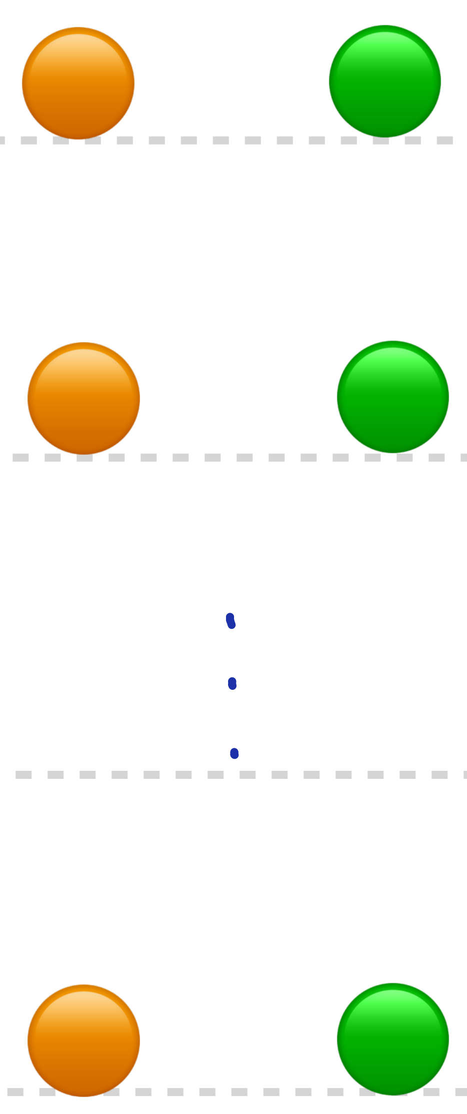
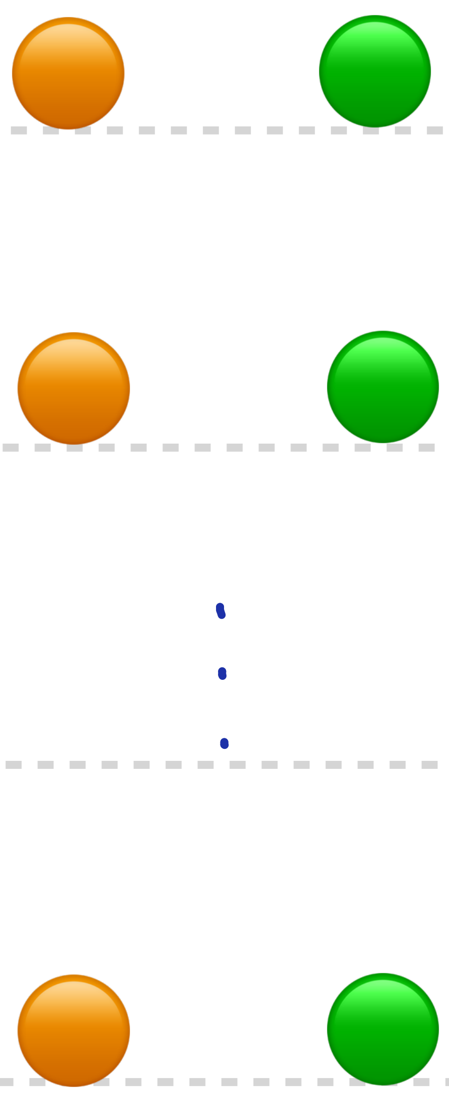


3. Two heaps, k tokens each.

Nim

(Examples)

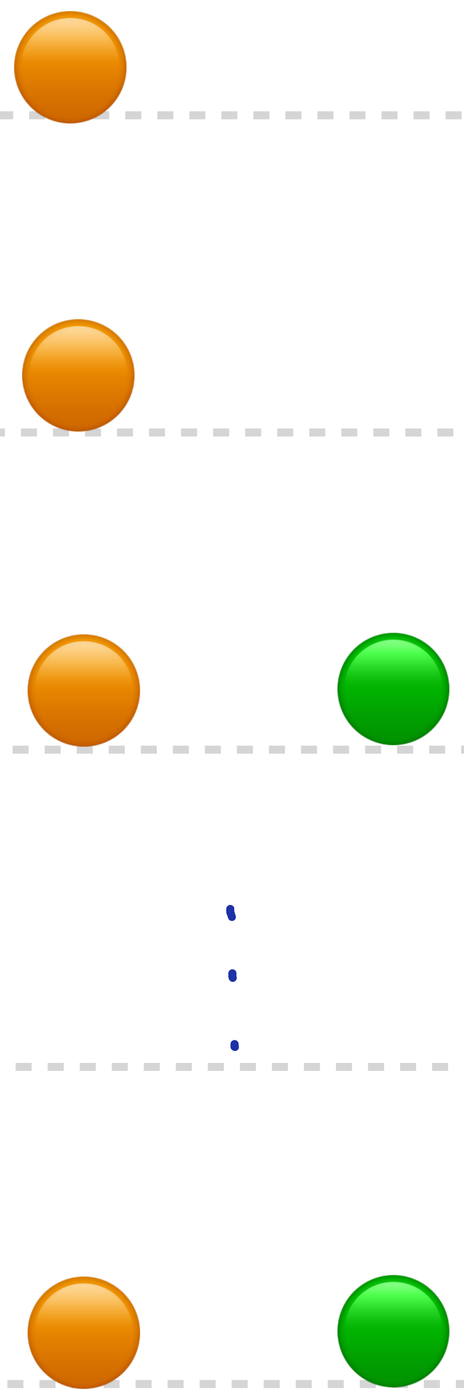


3. Two heaps, k tokens each.

→ First player loses. (why?)

Nim

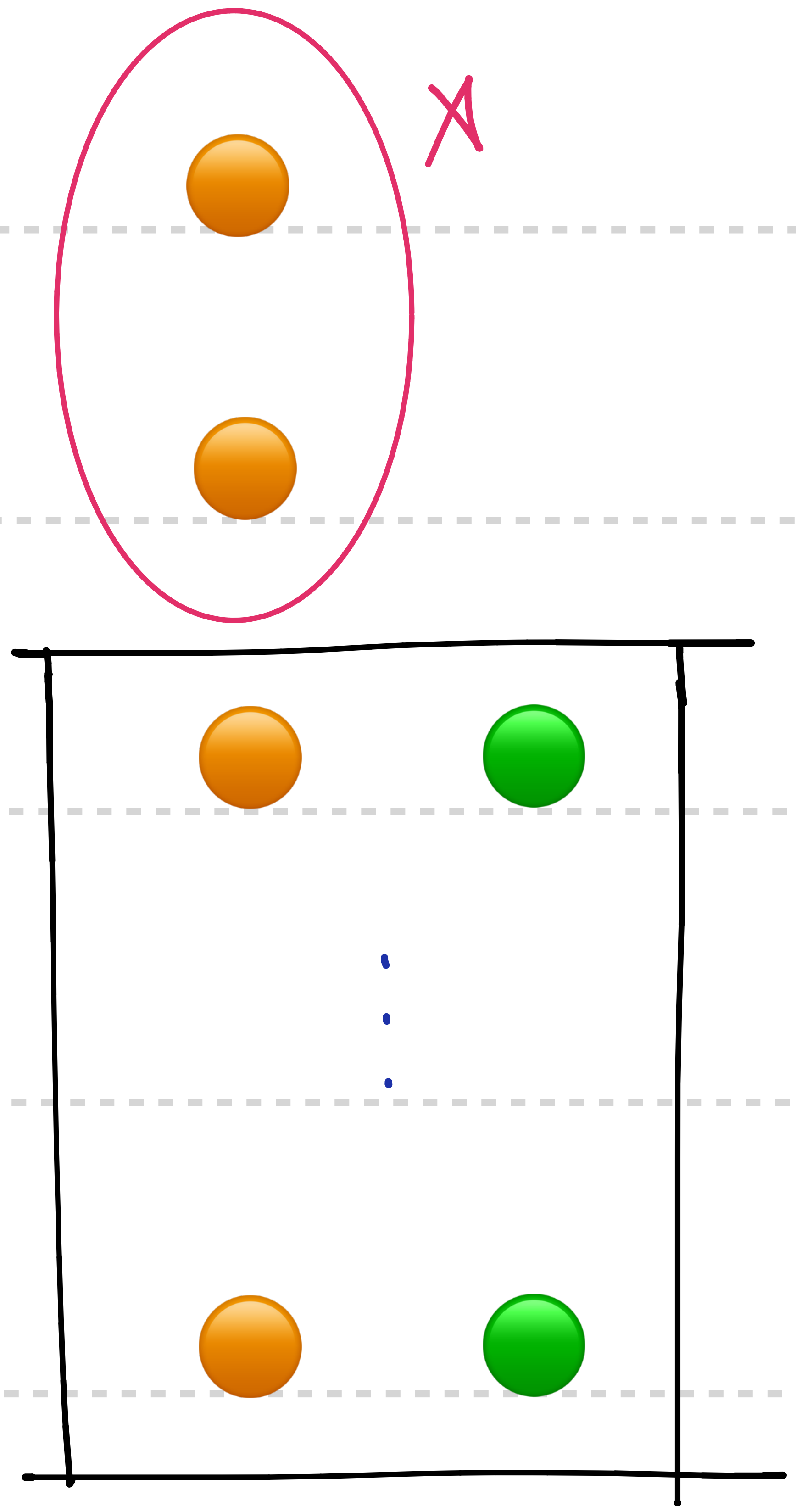
(Examples)



3. Two heaps, p & q tokens; $p \neq q$.

Nim

(Examples)



3. Two heaps, p & q tokens; $p \neq q$.

→ First player (finally) wins!

A Nim state is HAPPY if the player who starts there has a winning strategy.

A Nim state is SAD if it is not HAPPY.

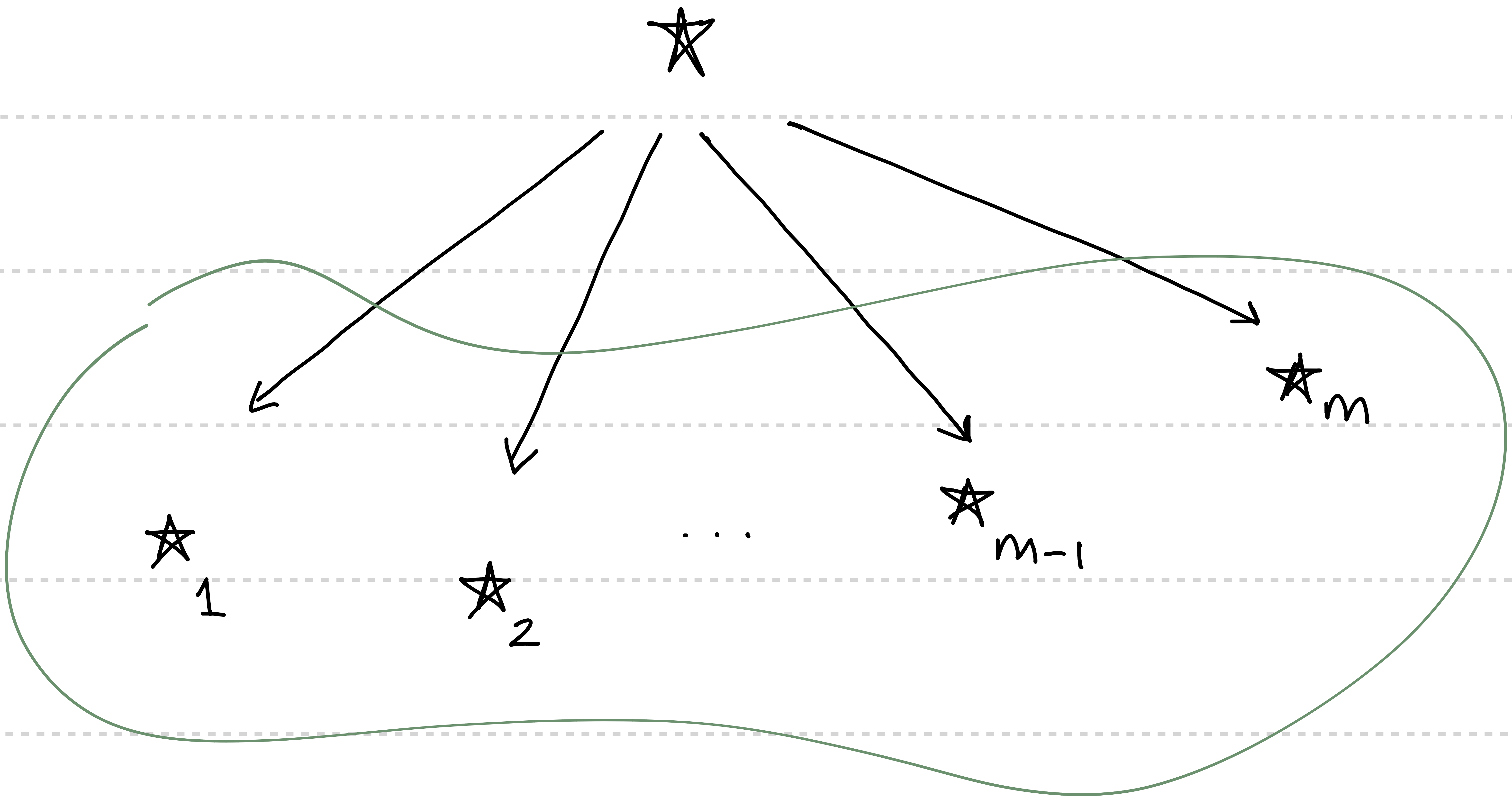
SAD

$\downarrow ?$

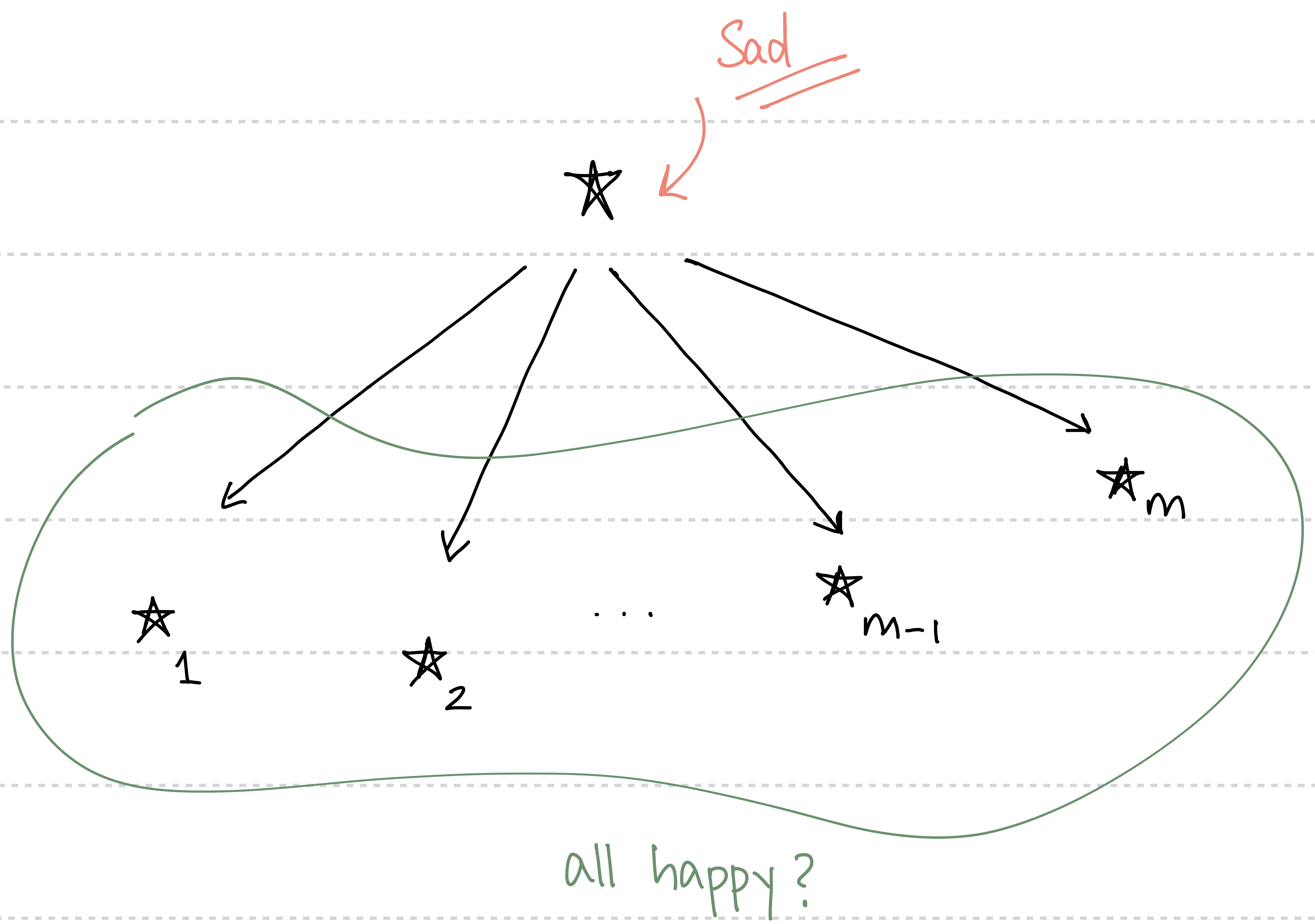
SAD

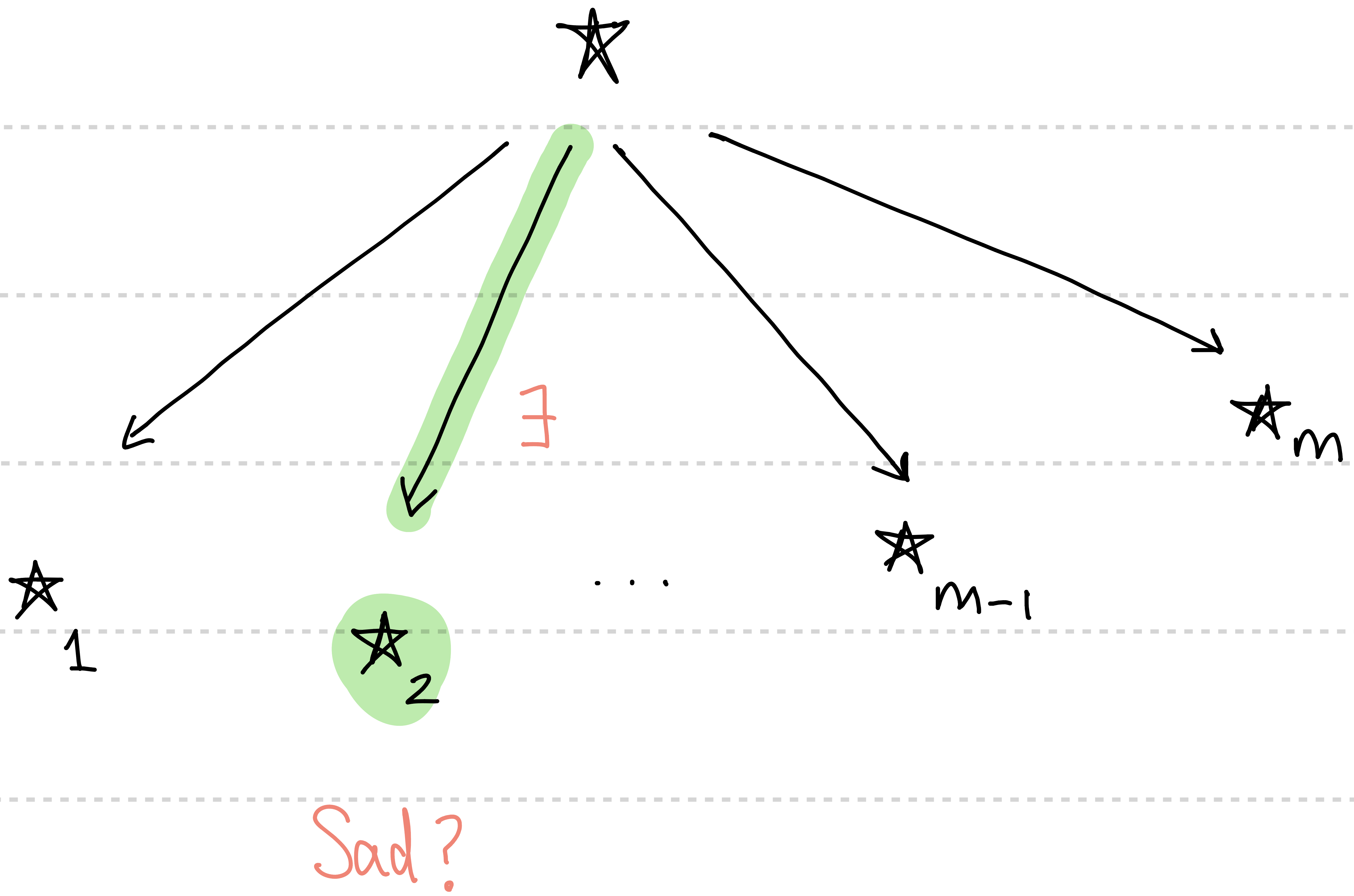
Can you have such a transition?

$$
\begin{tikzcd}
 & & \star \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\star_1 & \star_2 & \cdots & \star_{m-1} & \star_m
\end{tikzcd}
$$

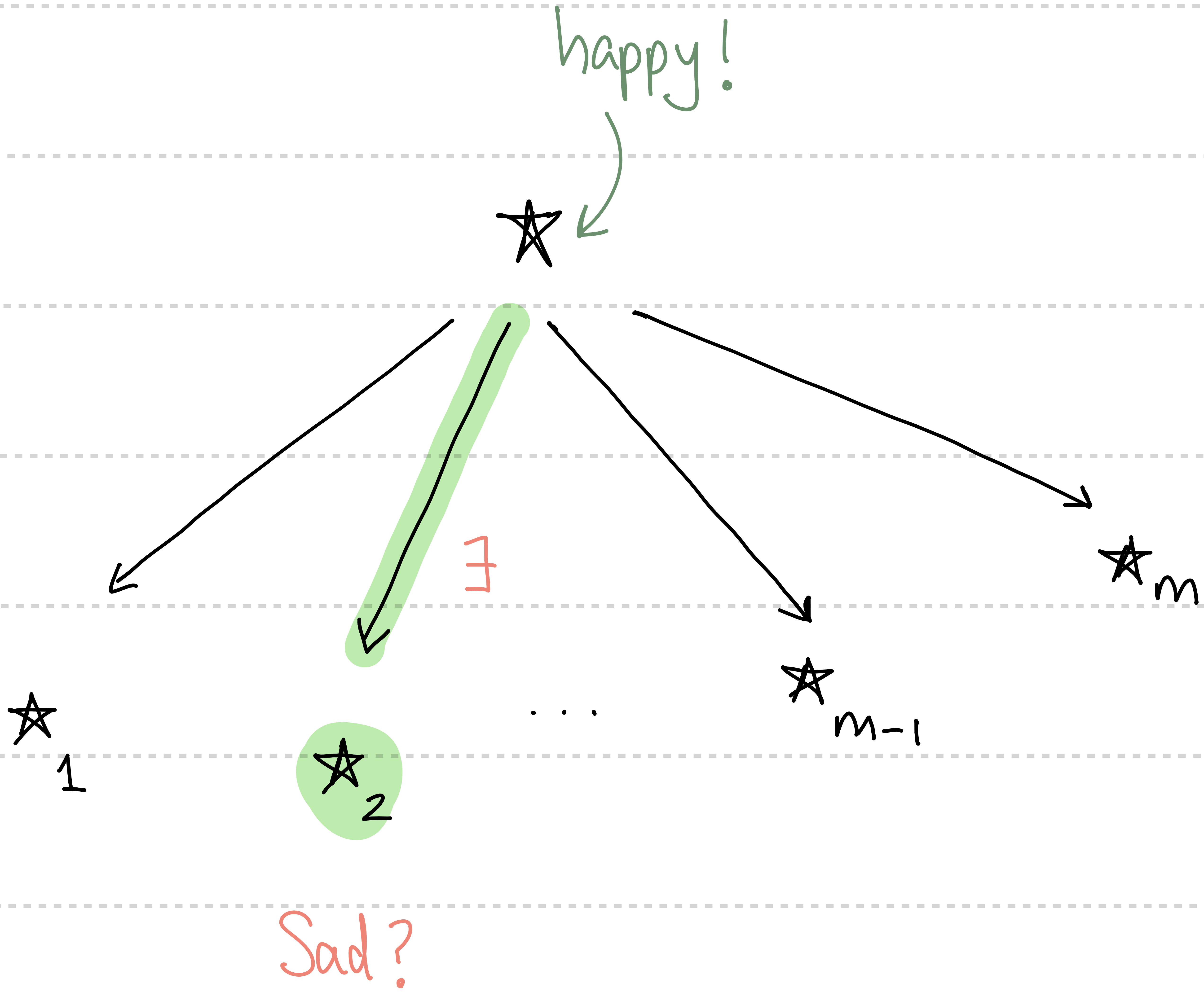


all happy?


Sad
1
2
. . .
m-1
m
all happy ?



∃
1
2
. . .
m−1
m
Sad?



happy!
∃
1
2
. . .
m-1
m
Sad?

S'pose we can identify a property P such that :

if a Nim state satisfies P,

any move leads to a Nim state that violates P,

& if a Nim state violates P,

$\exists$ a move that leads to a Nim state that satisfies P

S'pose we can identify a property P such that :

if a Nim state satisfies P,

(& the "0" state satisfies P)

any move leads to a Nim state that violates P,

& if a Nim state violates P,

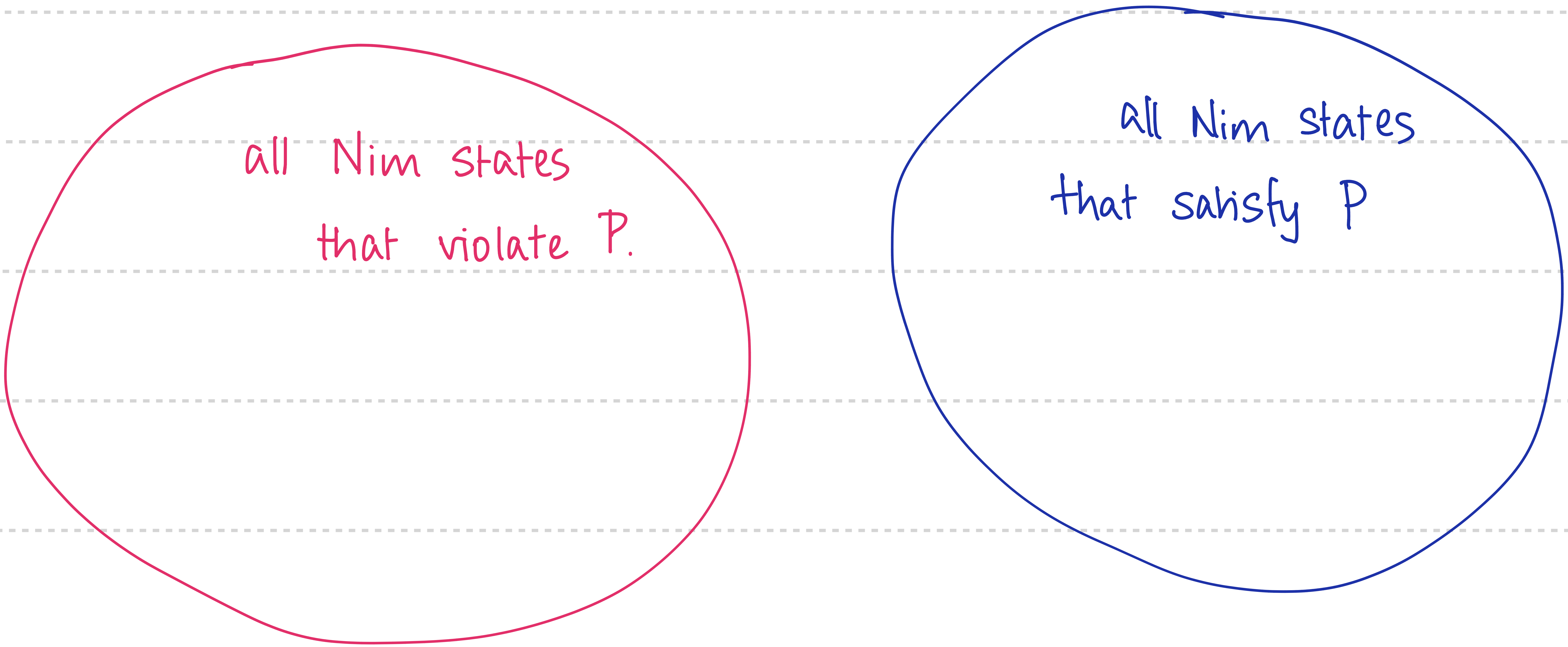
$\exists$ a move that leads to a Nim state that satisfies P

Then: the initial state violates P

if we have a first player win.

Then: the initial state violates P

iff we have a first player win.



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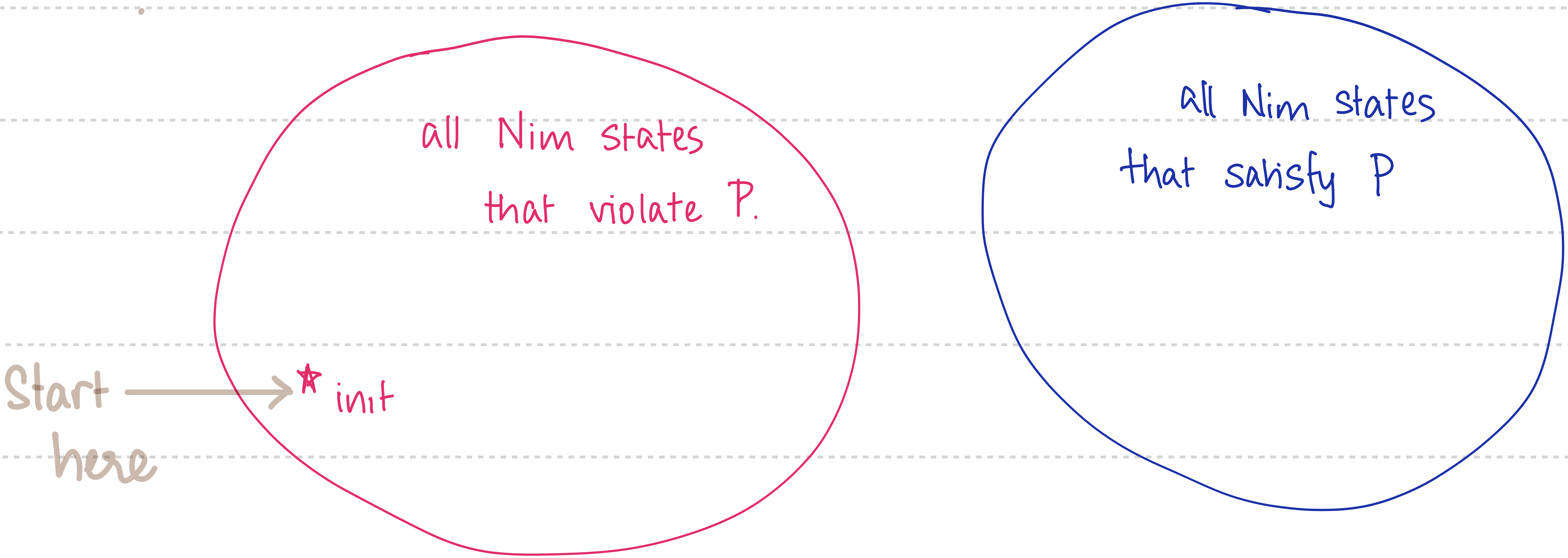

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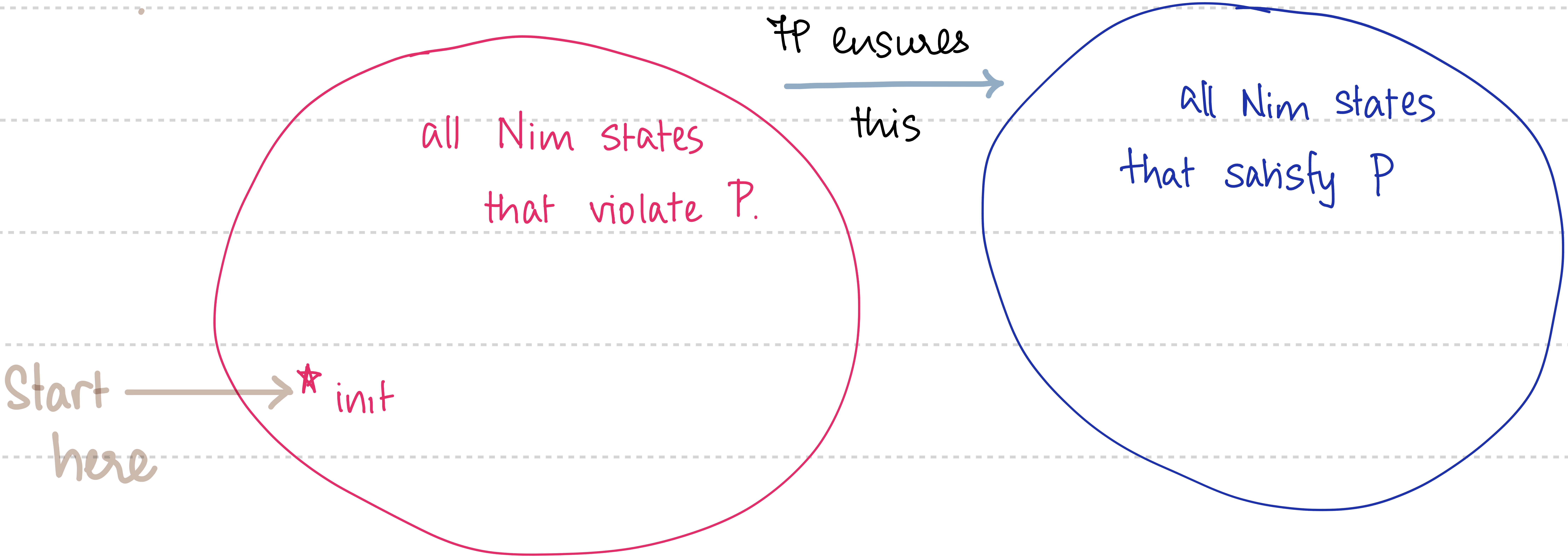

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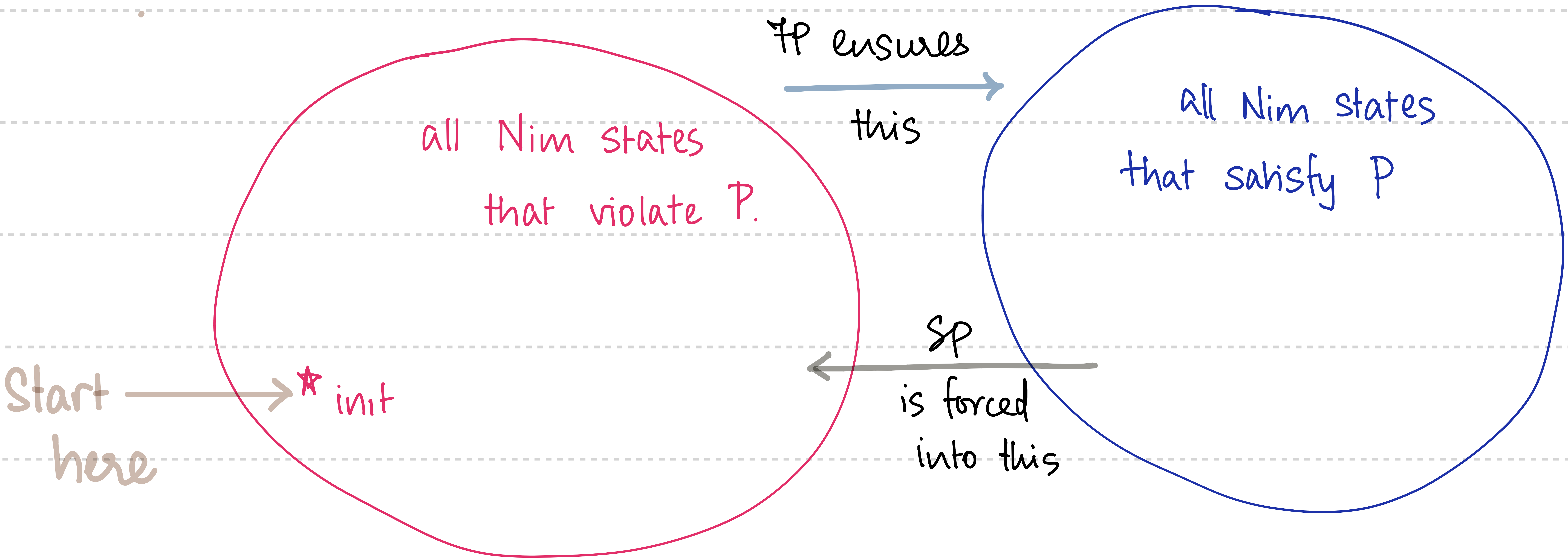

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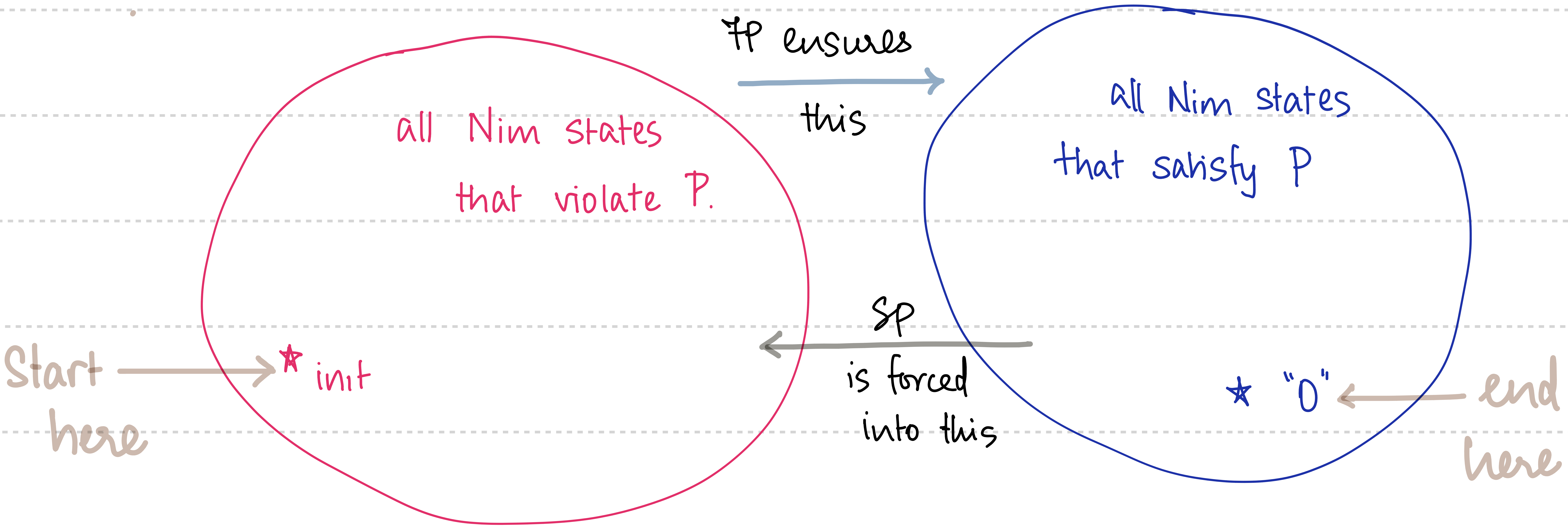

The case of two heaps (p & q tokens WLOG $p \geq q$)

P : The two heaps have the same # of tokens.

balanced

if P holds : any move destroys P.

if P does not hold : $\exists$ a move that restores P.

($p \rightsquigarrow q$ by removing $p-q$ tokens)

The case of small heaps

$(n_1, n_2, \ldots, n_k)$ Each $n_i \in \{1,2\}$.

x heaps w/ 1 token y heaps w/ 2 tokens

What's a property P in this setting?

The case of small heaps

$(n_1, n_2, \ldots, n_k)$ Each $n_i \in \{1, 2\}$.

x heaps w/ 1 token y heaps w/ 2 tokens

$$\boxed{x \text{ is even } \& \ y \text{ is even}}$$

The case of small heaps

Remove $\underline{1}$ token from a 1-heap : $(x, y) \mapsto (x-1, y)$

Remove $\underline{1}$ token from a 2-heap : $(x, y) \mapsto (x+1, y-1)$

Remove $\underline{2}$ tokens from a 2-heap : $(x, y) \mapsto (x, y-1)$

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The General Case.

Goal. Design the condition P that will allow us to determine if $(n_1, \dots, n_k)$ is a first player win.

Detour: Combining Nim Games

$$(n_1, n_2, \ldots, n_p)$$

$$\star$$

$$(m_1, m_2, \ldots, m_q)$$

$$=$$

$$(n_1, n_2, \ldots, n_p, m_1, m_2, \ldots, m_q)$$

{Associate} w/ each Nim state
a # that captures
information about that state.

$\{(n)\} \rightsquigarrow n$ (in particular, $\emptyset \rightsquigarrow 0$)

$\{(n,m)\} \rightsquigarrow \{(n) \star (m)\}$

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WAT?

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$\{(n, m)\} \rightsquigarrow n \star m$

WANT (possibly): $n \star n = 0$.

{Associate} w/ each Nim state
a # that captures
information about that state.

the very definition of SAD

$\{(n)\} \rightsquigarrow n$ (in particular: $\emptyset \rightsquigarrow 0$)

$\{(n,m)\} \rightsquigarrow n \star m$

SAD when $n=m$.

WANT (possibly): $n \star n = 0$.

$\{$ Associate $\}$ w/ each Nim state a # that captures information about that state.

$n \star n = 0$

$\{(n)\} \rightsquigarrow n$ (in particular, $\emptyset \rightsquigarrow 0$)

$\{(n, m)\} \rightsquigarrow n \star m$

$\{$Associate$\}$ w/ each Nim state

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$n \star n = 0$

information about that state.

$$\{(n)\} \rightsquigarrow n \quad \text{(in particular, } \emptyset \rightsquigarrow 0\text{)}$$

$$\{(n, m)\} \rightsquigarrow n \star m$$

Does $\star$ commute?

$n \star n = 0$

{Associate} w/ each Nim state a # that captures information about that state.

$\{(n)\} \rightsquigarrow n$ (in particular, $\emptyset \rightsquigarrow 0$)

$\{(n,m)\} \rightsquigarrow n \star m$

Does $\star$ commute? (n,m) v/s (m,n)

{Associate} w/ each Nim state a # that captures information about that state.

$\{(n)\} \rightsquigarrow n$ (in particular, $\emptyset \rightsquigarrow 0$)

$\{(n, m)\} \rightsquigarrow n \star m$

inverse

$n \star n = 0$

ID

$\underbrace{n \star m = m \star n}_{\text{commutative.}}$

feels like a group?

When can we say: $a \star b = c$?

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$$c \star c = 0$$

When can we say : $\boxed{a \star b = c}$?

$$c \star c = 0$$

$$\Updownarrow$$

$$\equiv \quad (a \star b) \star c = 0$$

When can we say: $\boxed{a \star b = c}$?

$$c \star c = 0$$

$$\Updownarrow$$

$$\equiv \quad (a \star b) \star c = 0$$

So: $a \star b = c$ iff $(a \star b) \star c = 0$

0	1	2	3	4	5	6	7
1	0						
2		0					
3			0				
4				0			
5					0		
6						0	
7							0

(what we know so far)

$\star$ is associative

$$\underbrace{(a \star b)}_{p} \star c = a \star \underbrace{(b \star c)}_{q}$$

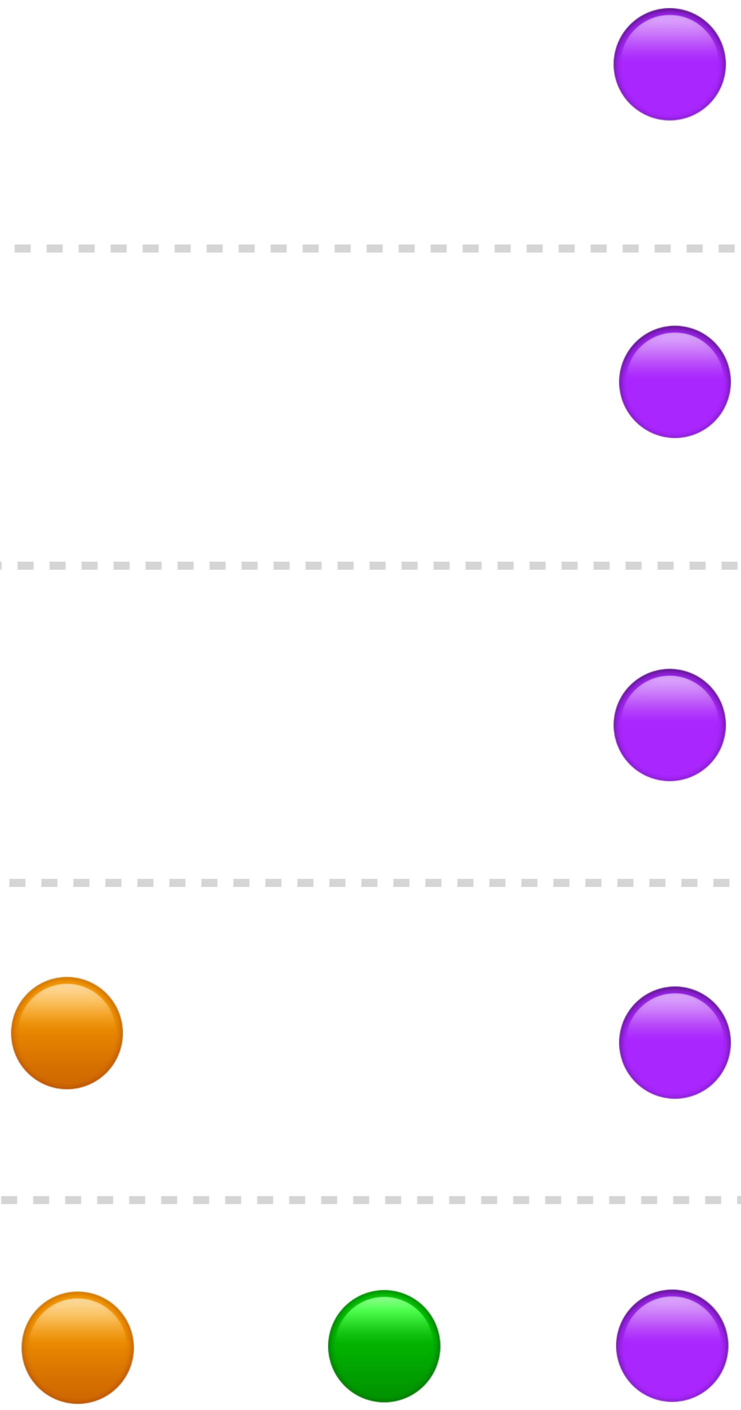
$$p = a \star b$$

$$p = a \star b \star c \star c$$

$$p \star c = a \star b \star c$$

$$p \star c = a \star q$$

What is $1 \star 2$?



If we claim that

$$1 \star 2 = n,$$

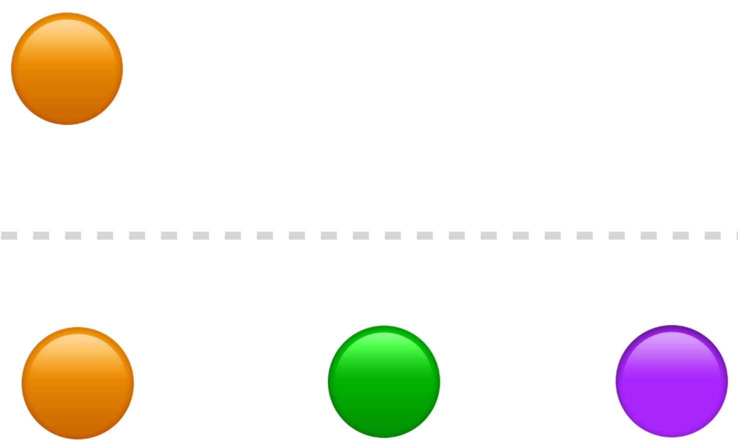
then it must be that

$$1 \star 2 \star n = 0.$$

SAD

(ie, every move $\Rightarrow$ happy)

What is $1 \star 2$?



If we claim that

$$1 \star 2 = 1 \,,$$

then it must be that

$$1 \star 2 \star 1 = 0 \,.$$

SAD

(i.e. every move $\Rightarrow$ happy)

What is $1 \star 2$?

If we claim that

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~~SAD~~ happy

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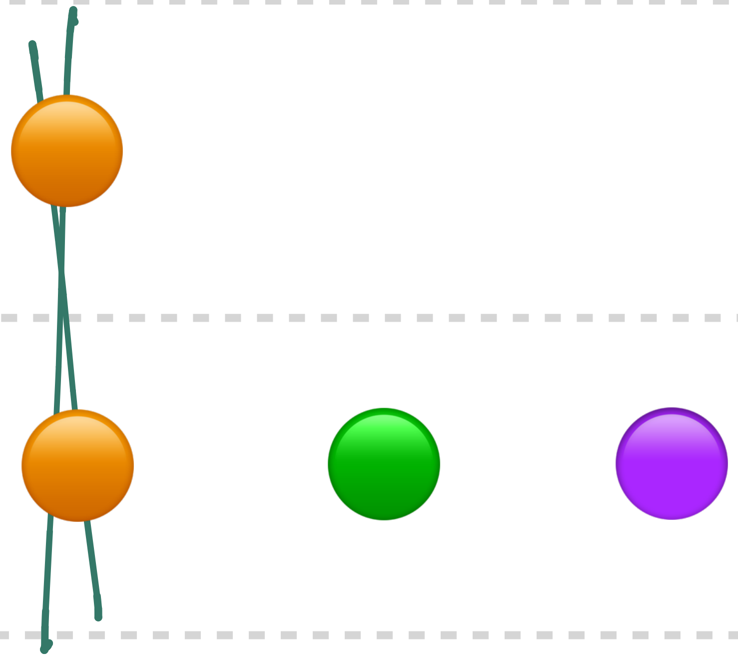

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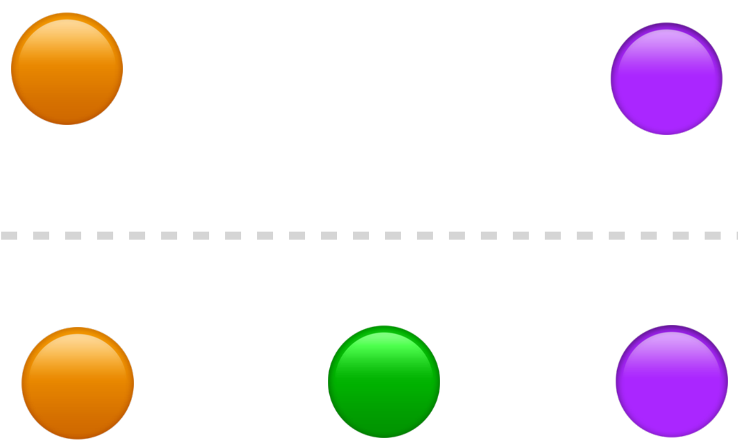


If we claim that

$1 \star 2 = 2$,

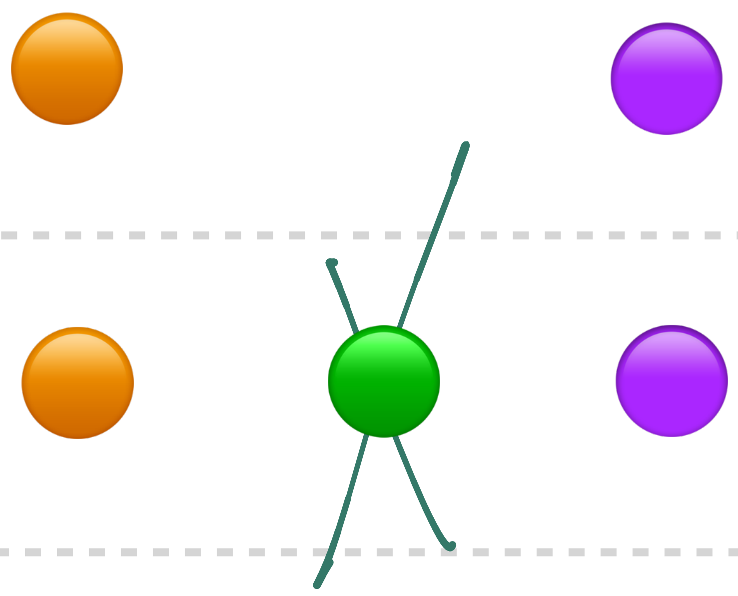
then it must be that

$1 \star 2 \star 2 = 0$.

SAD

(ie. every move $\Rightarrow$ happy)

What is $1 \star 2$?



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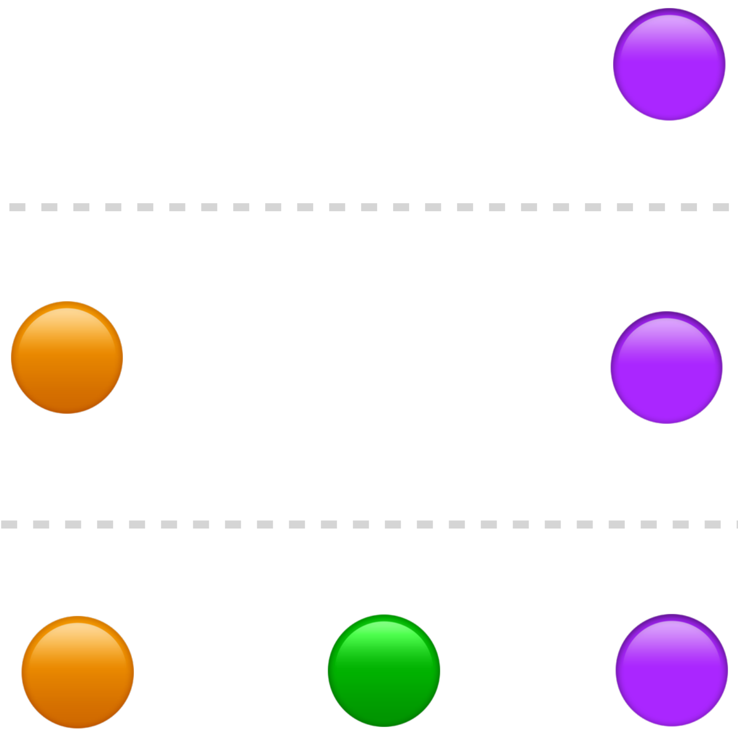
What is $1 \star 2$?



If we claim that

$$1 \star 2 = 3,$$

then it must be that

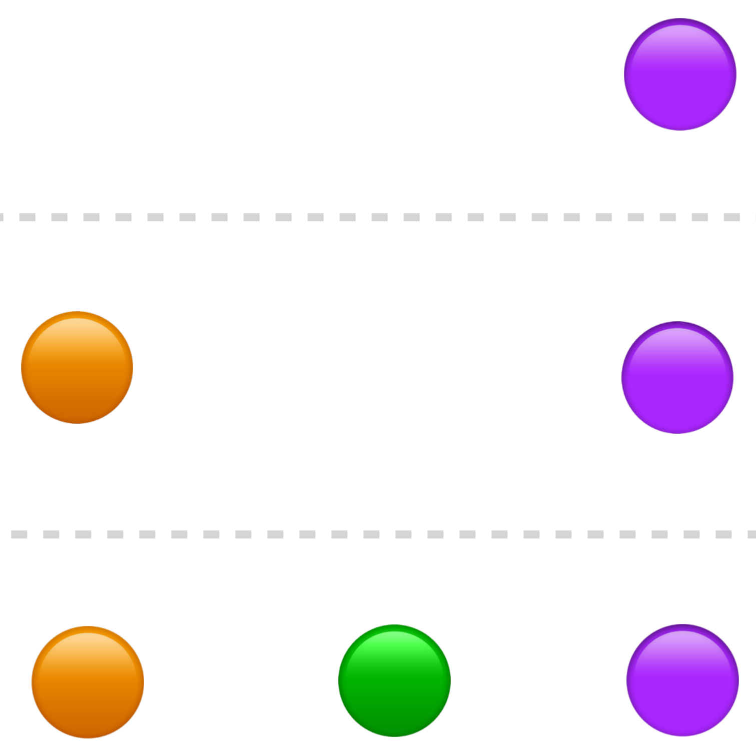
$$1 \star 2 \star 3 = 0.$$

SAD

(ie. every move $\Rightarrow$ happy)

What is $1 \star 2$?

$\star$ by case analysis



If we claim that

$$1 \star 2 = 3,$$

then it must be that

$$1 \star 2 \star 3 = 0.$$

SAD

(i.e. every move $\Rightarrow$ happy)

this actually works! $\star$

0	1	2	3	4	5	6	7
1	0	3	2				
2	3	0	1				
3	2	1	0				
4				0			
5					0		
6						0	
7							0

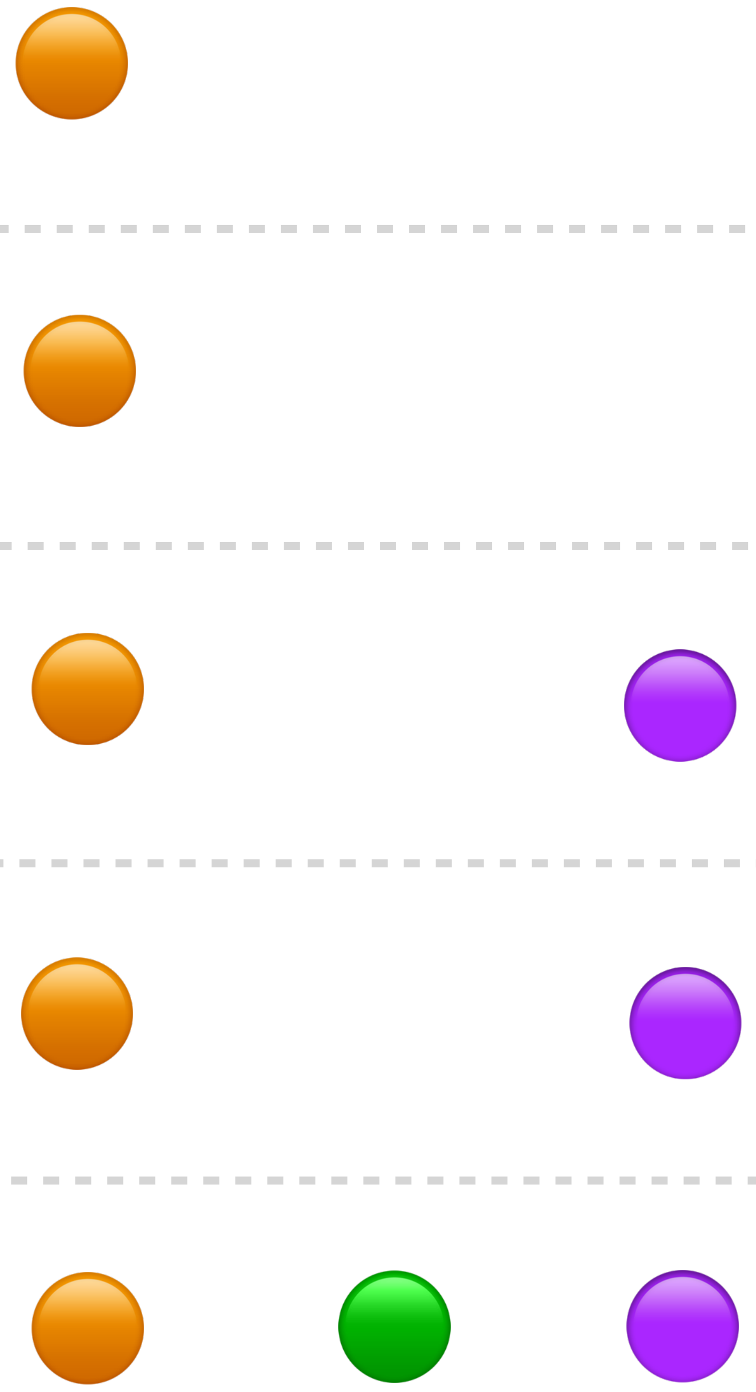
$1 \star 2 = 3$

$1 \star 2 \star 3 = 0$

$1 \star 3 = 2$

$2 \star 3 = 1$

What is $1 \star 5$?



If we claim that

$$1 \star 5 = n,$$

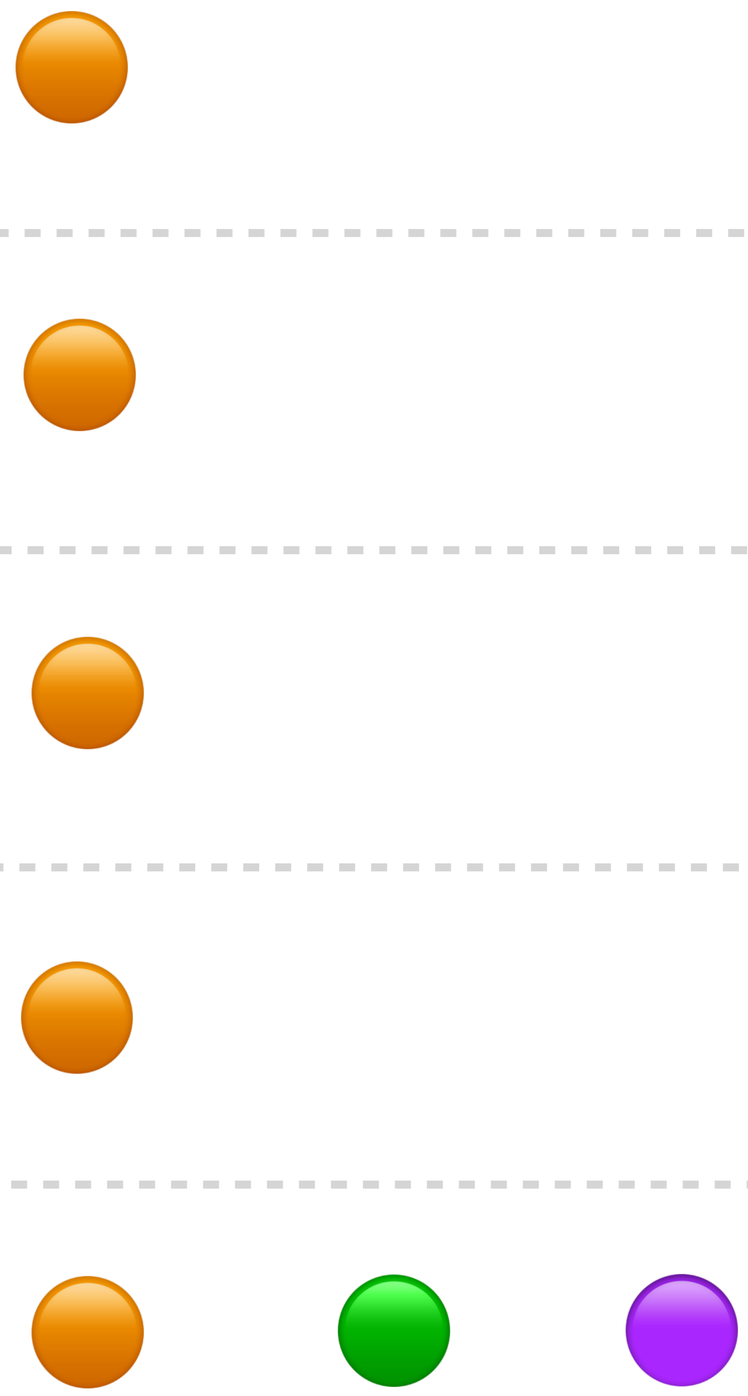
then it must be that

$$1 \star 5 \star n = 0.$$

SAD

(ie. every move $\Rightarrow$ happy)

What is $1 \star 5$?



If we claim that

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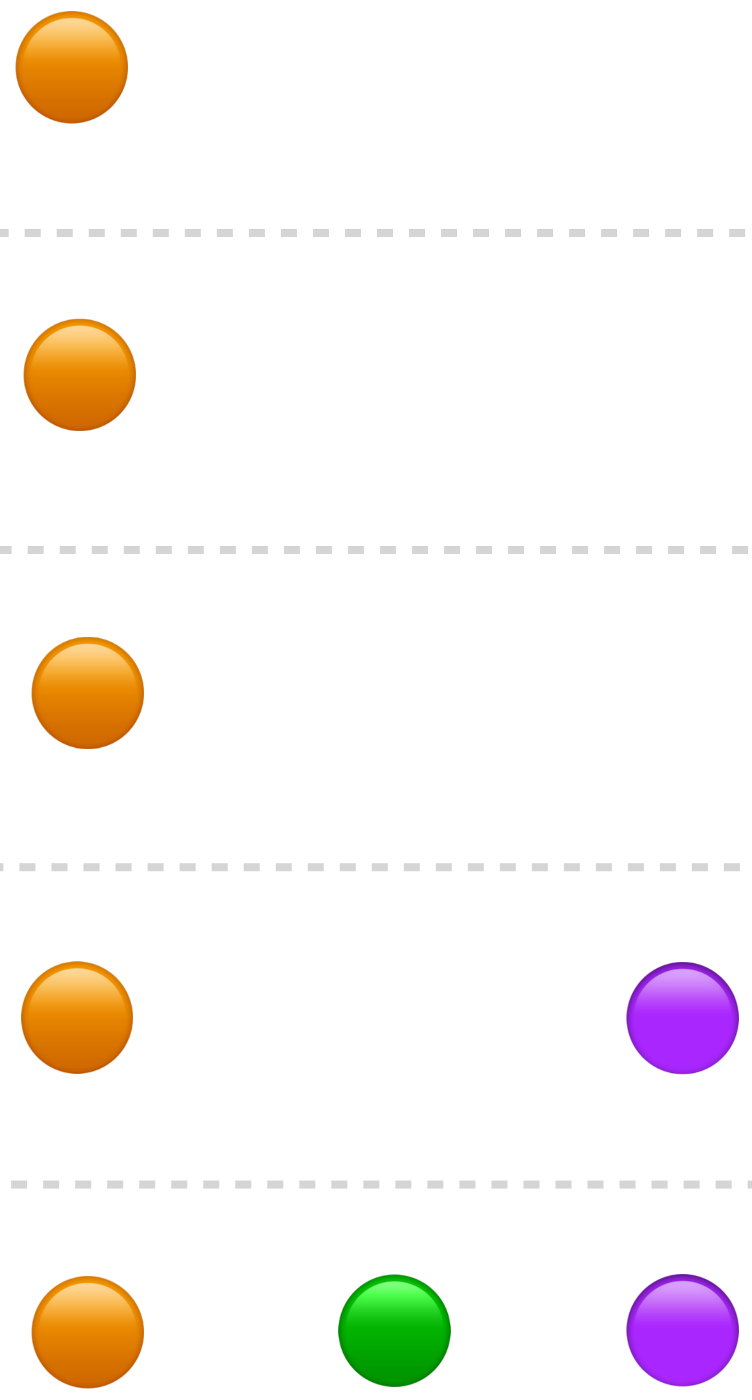
then it must be that

$$1 \star 5 \star 1 = 0.$$

SAD

(i.e. every move $\Rightarrow$ happy)

What is $1 \star 5$?



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SAD

(i.e. every move $\Rightarrow$ happy)

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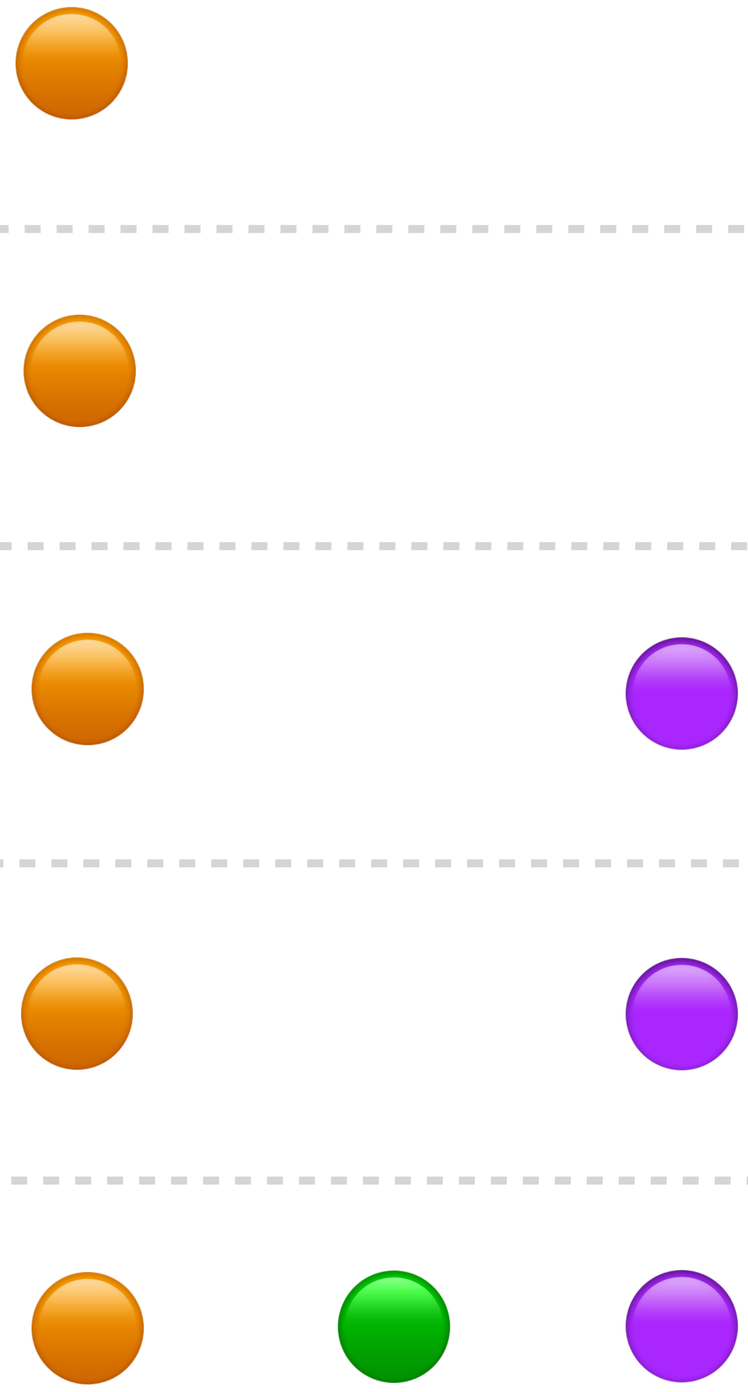

If we claim that

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$1 \star 5 \star 3 = 0$.

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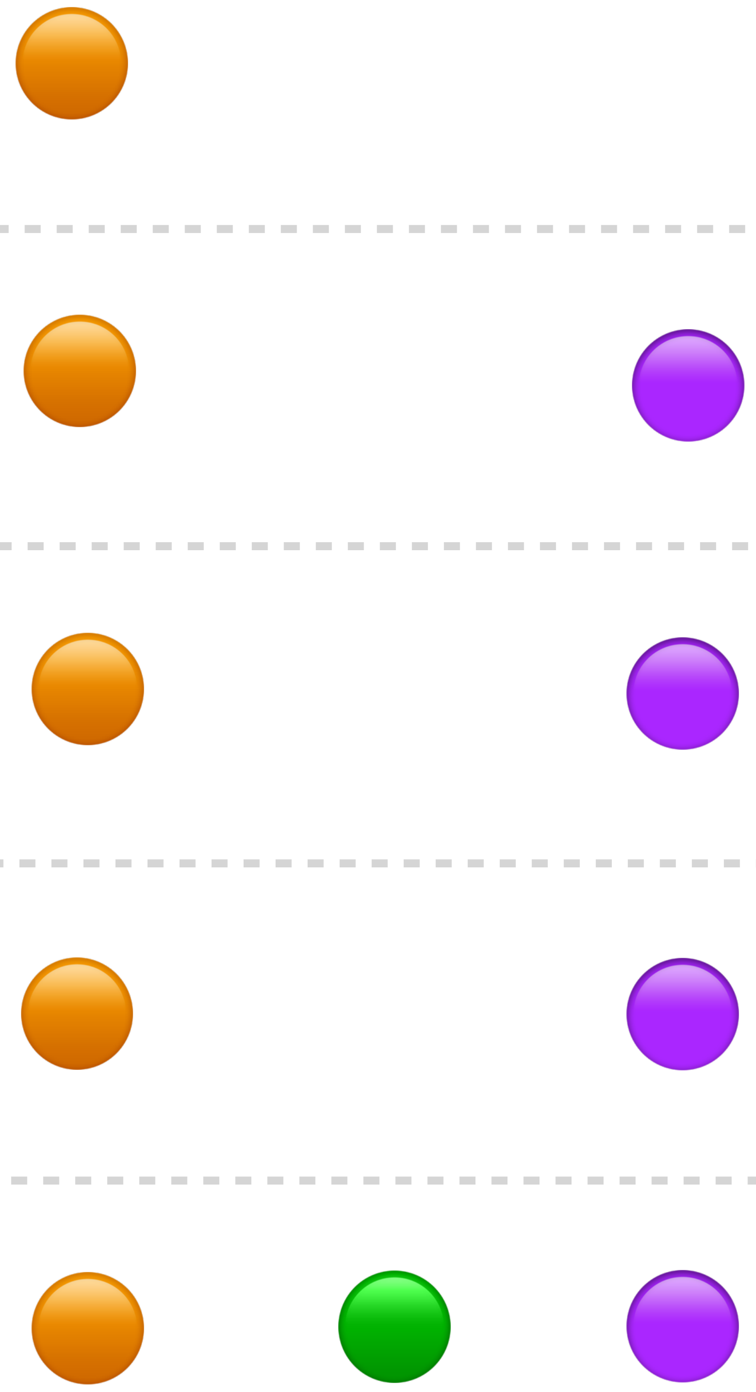
(ie. every move $\Rightarrow$ happy)

What is $1 \star 5$?



If we claim that

$$1 \star 5 = 3,$$

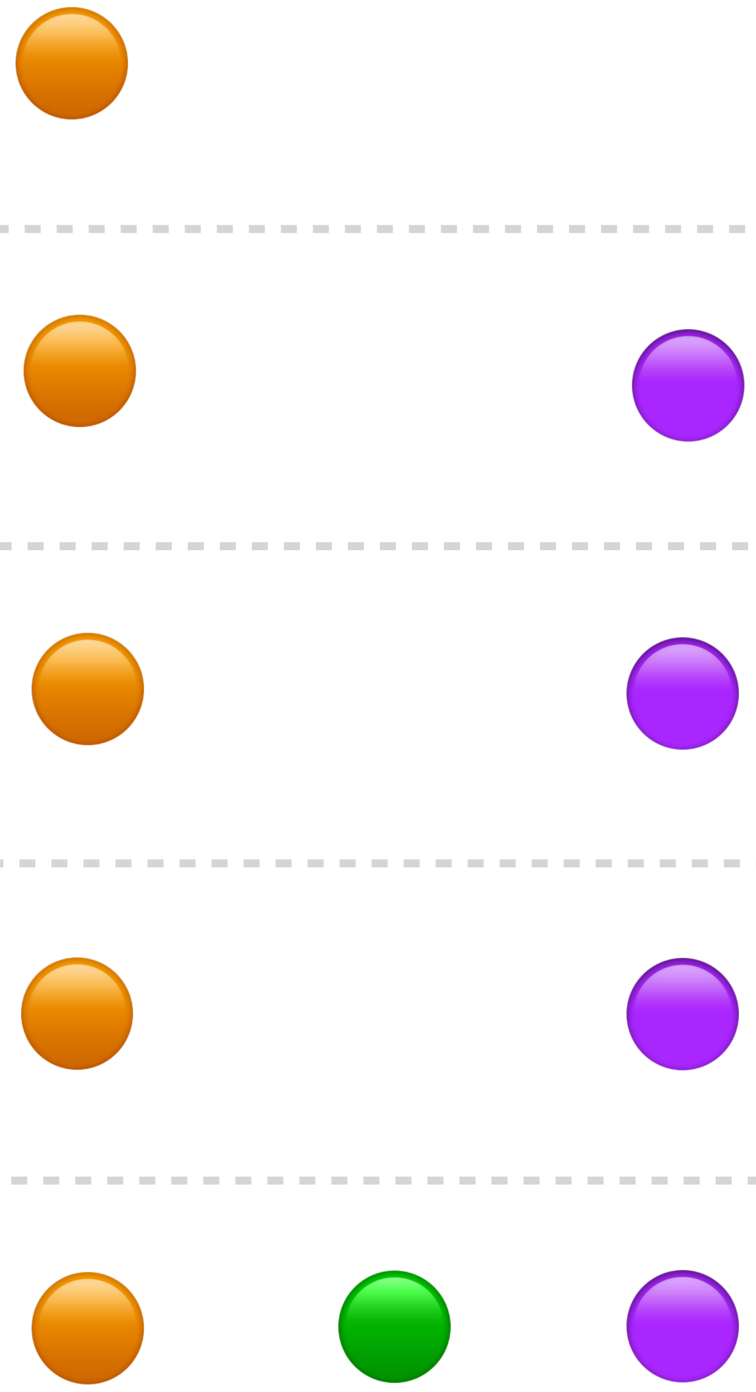
then it must be that

$$\underline{1 \star 5 \star 3 = 0.}$$

SAD

(ie. every move $\Rightarrow$ happy)

What is $1 \star 5$?



If we claim that

$$1 \star 5 = 4,$$

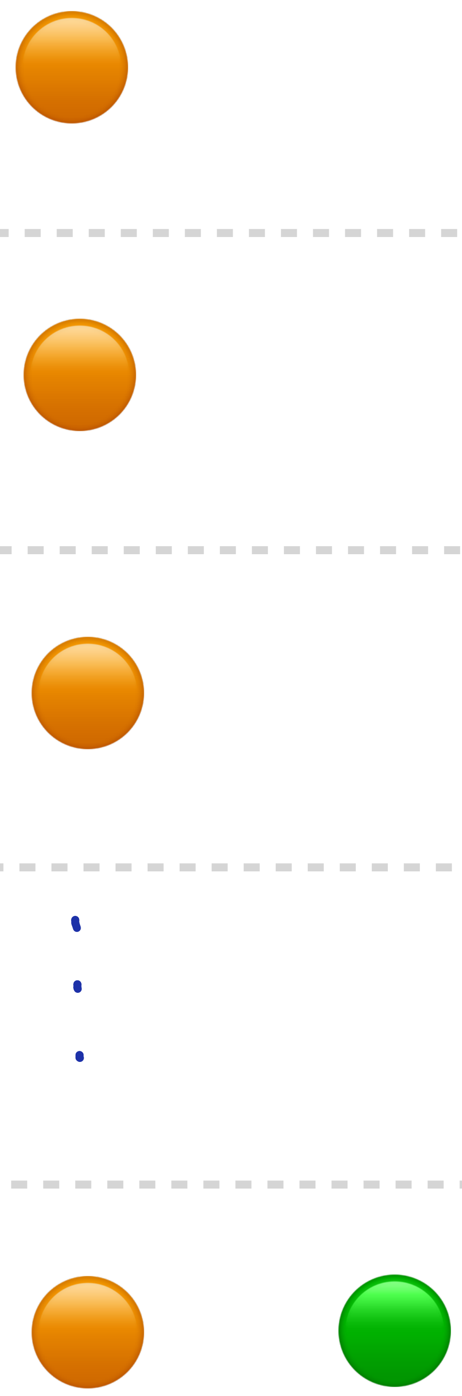
then it must be that

$$1 \star 5 \star 4 = 0.$$

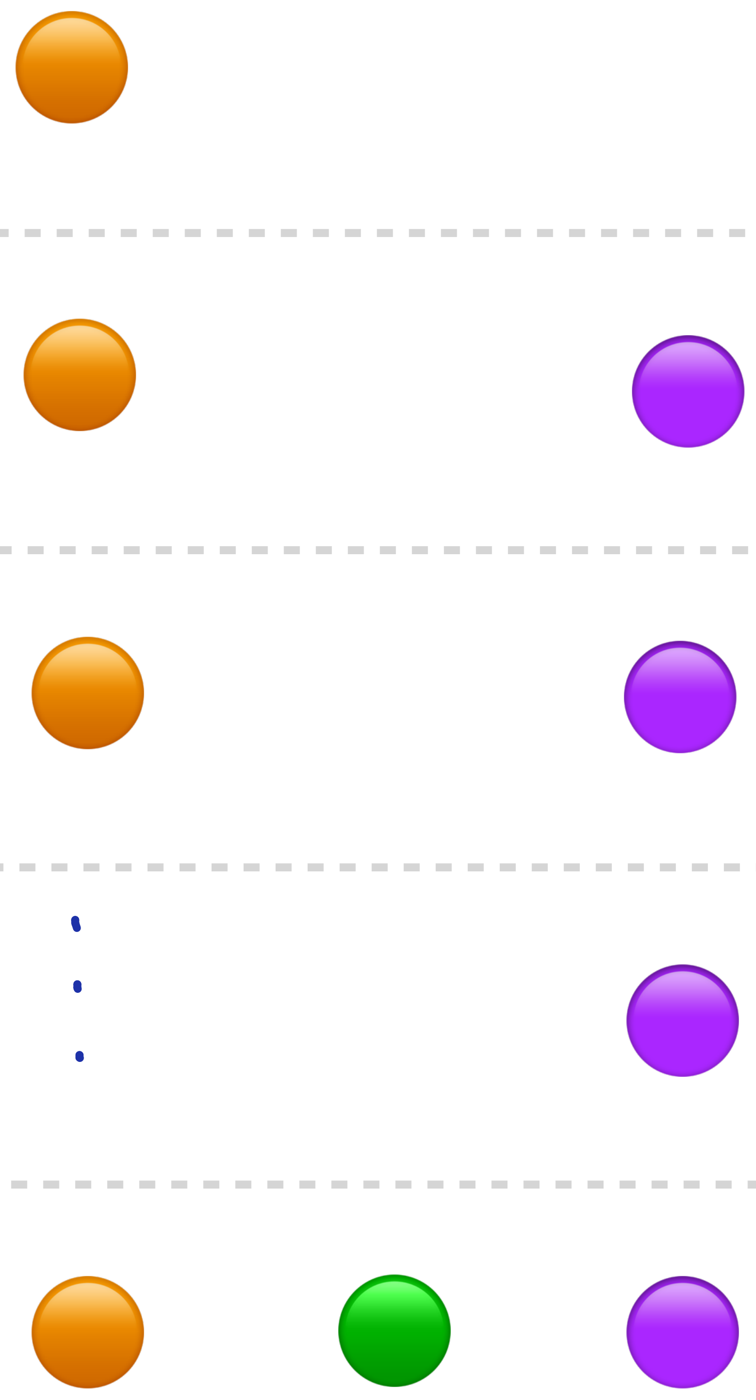
SAD

(ie. every move $\Rightarrow$ happy)

What is $1 \star k$?



What is $1 \star k$?



$$1 \star k = n$$

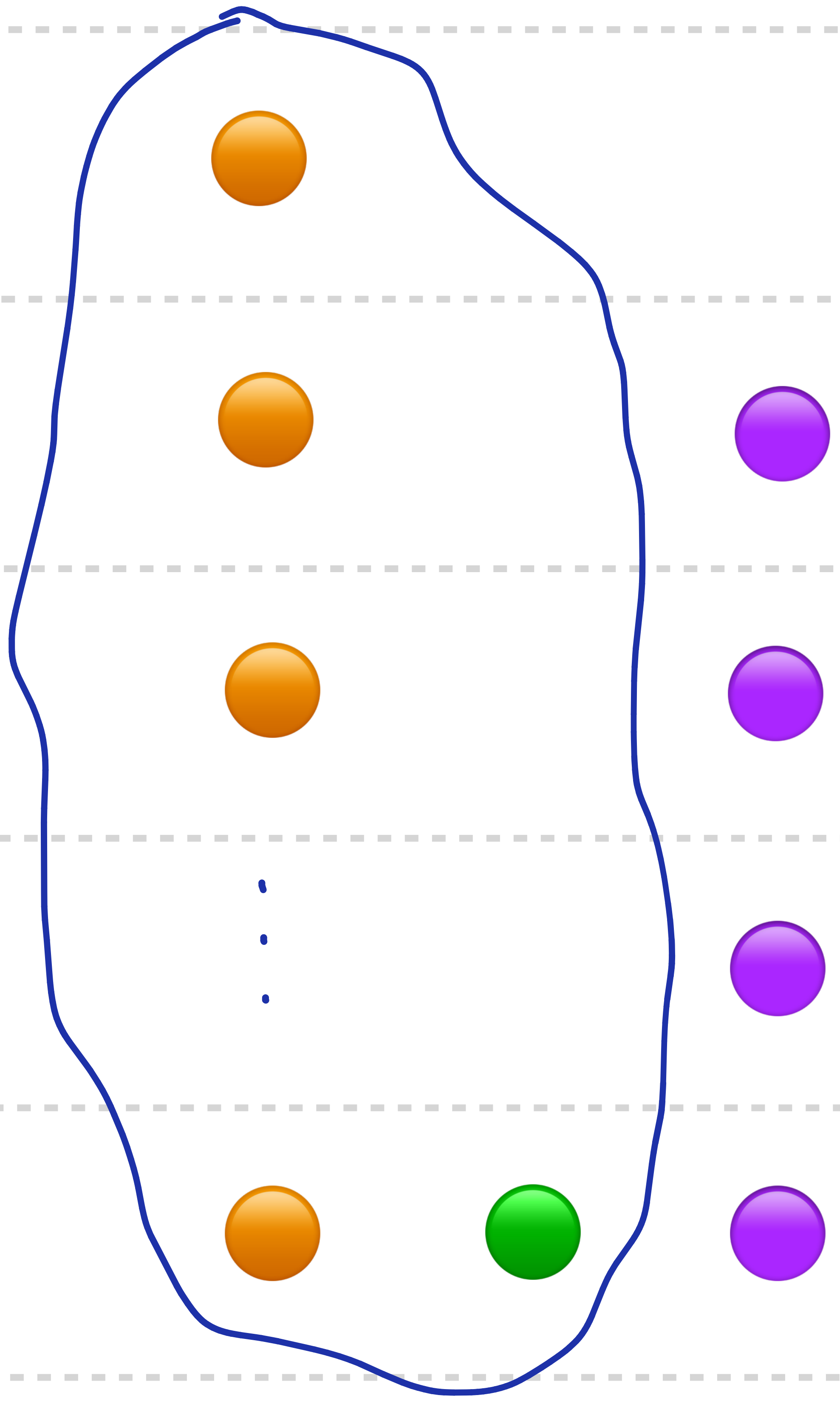
$$\equiv \underbrace{1 \star k \star n = 0}_{\text{SAD}}$$

What is $1 \star k$?

$$1 \star k = n$$

$$\equiv \underbrace{1 \star k \star n = 0}_{\text{SAD}}$$

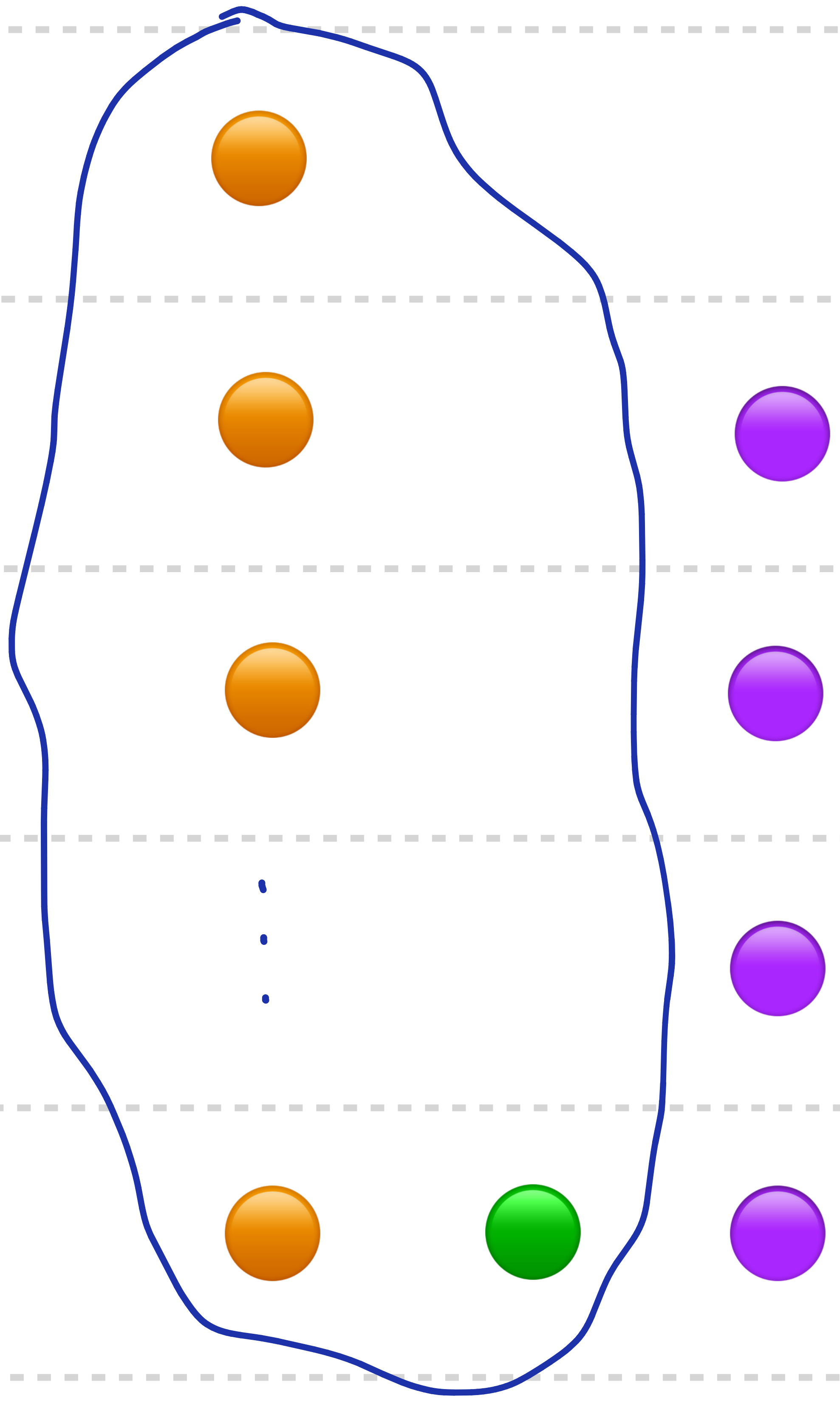
Suppose $\exists$ a move in this game leading to a game of value s



What is $1 \star k$?

$1 \star k = n$

$\equiv \underbrace{1 \star k \star n = 0}_{\text{SAD}}$



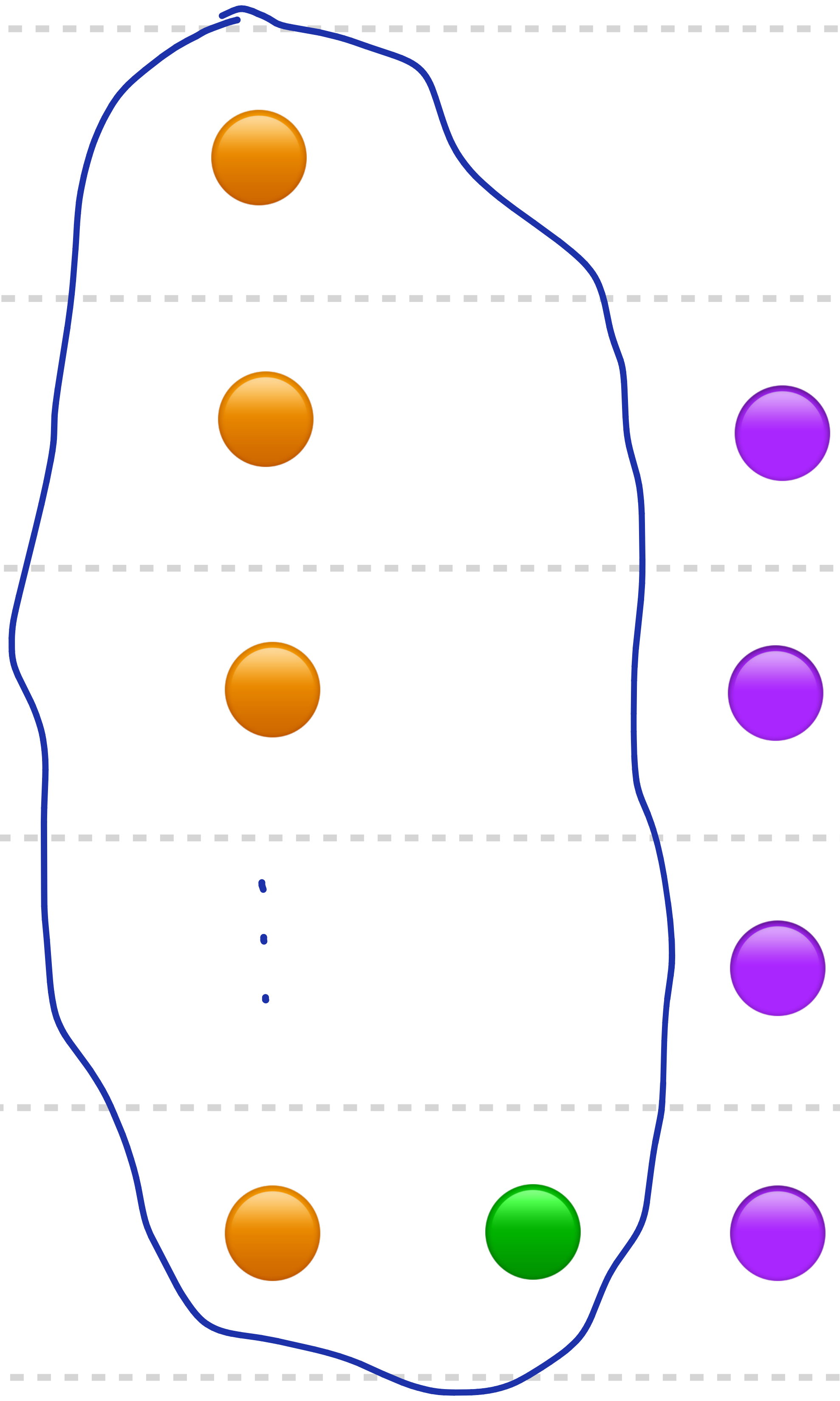
Suppose $\exists$ a move in this game leading to a game of value s

Then $n \neq s$, because $\exists$ a move to $\underbrace{s \star s}_{\text{SAD}}$

What is $1 \star k$?

$1 \star k = n$

$\equiv \underbrace{1 \star k \star n = 0}_{\text{SAD}}$



Suppose $\exists$ a move in this game leading to a game of value s

Then $n \neq s$, because $\exists$ a move to $\underbrace{s \star s}_{\text{SAD}}$

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 =$

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 =$ cannot be 0, 1, 2, 3, or 4

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

why?　　why?

$1 \star 4 =$ cannot be 0, 1, 2, 3, or 4

$$1 \star 0 = 1$$

$$1 \star 2 = 3$$

$$1 \star 3 = 2$$

why?

why?

$1 \star 4 =$ cannot be 0, 1, 2, 3, or 4

earliest possibility : 5

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

why? why?

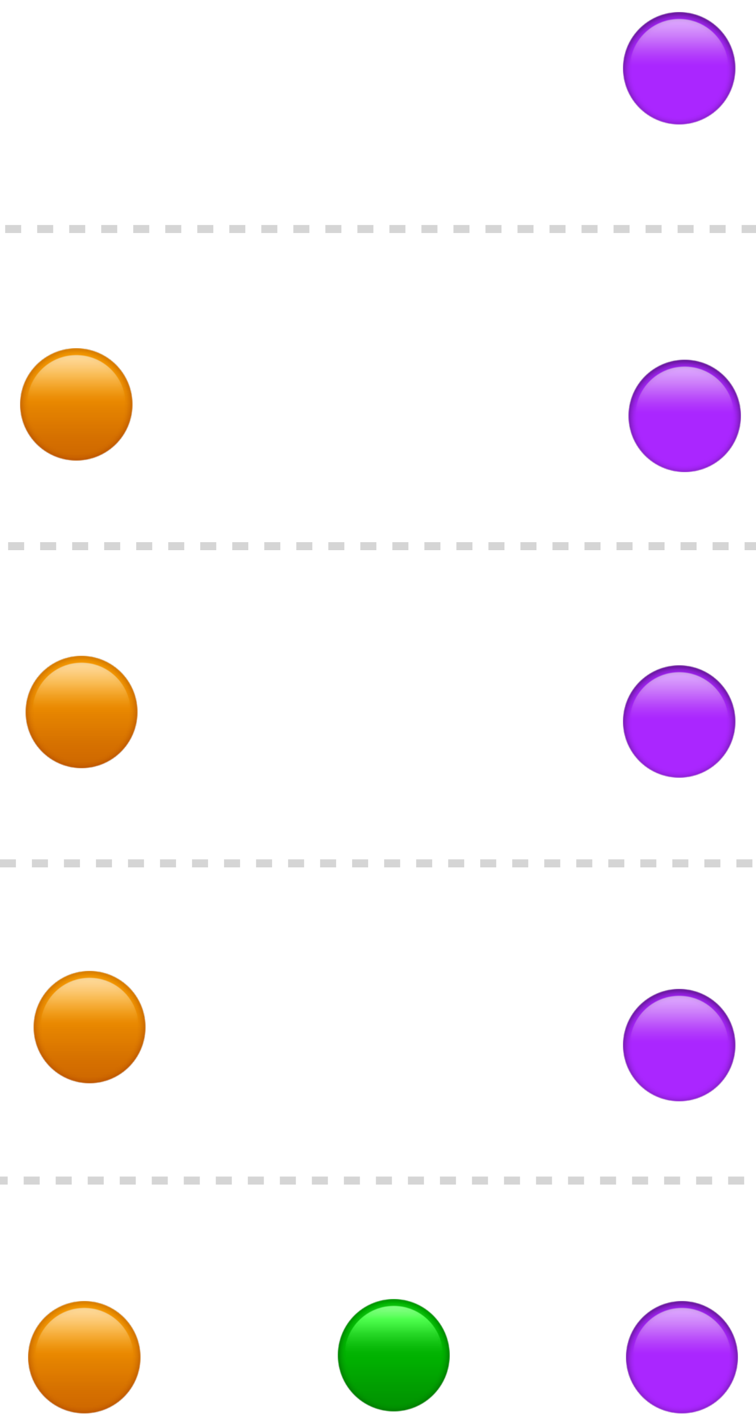
$1 \star 4 =$ cannot be 0, 1, 2, 3, or 4

earliest possibility: 5

Claim. $1 \star 4 = 5$.

$1 \star 4 \star 5 = 0$

WTS: every move is happy



by a case analysis into "smaller scenarios" that we understand

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 = 5$

$1 \star 5 = ?$

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 = 5$

$1 \star 5 = ?$ cannot be 0, 1, 2, 3, 5.

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 = 5$

$1 \star 5 = ?$ cannot be 0, 1, 2, 3, 5.

let's try ... 4?

$1 \star 0 = 1$

$1 \star 2 = 3$

$1 \star 3 = 2$

$1 \star 4 = 5$

$1 \star 5 = 4$

0	1	2	3	4	5	6	7
1	0	3	2	5	4		
2	3	0	1				
3	2	1	0				
4	5			0			
5	4				0		
6						0	
7							0

0	1	2	3	4	5	6	7	
1	0	3	2	5	4	7	6	. .
2	3	0	1					
3	2	1	0					
4	5			0				
5	4				0			
6	7					0		
7	6						0	