Solutions to 3B:

1 Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

$T \in \mathcal{L}(F^5, F^2)$: $T(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2)$.

2 Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$.

$\forall v \in V$, $(ST)^2(v) = STST(v) = S(T(S(Tv)))$. As range $S \subseteq$ null T, $\forall u \in V$, $T(S(u)) = 0$.

Hence, there must be $T(S(Tv)) = 0$ for $\forall v \in V$. This implies $(ST)^2(v) = S(0) = 0$.

$\therefore (ST)^2 = 0$.

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

(a). If $v_1, \ldots, v_m$ spans V, $\forall v \in V$, $\exists z_1, \ldots, z_m \in F$, such that $v = z_1 v_1 + \cdots + z_m v_m$.

$\Rightarrow T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m = v \in$ range T. This implies that range $T = V$.

$\therefore T$ is surjective.

(b). If $v_1, \ldots, v_m$ is linearly independent, $0 = z_1 v_1 + \cdots + z_m v_m \Rightarrow z_1 = \cdots = z_m = 0$.

Hence, $T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m = 0 \Rightarrow (z_1, \ldots, z_m) = 0$. This implies that null $T = \{0\}$. $\therefore T$ is injective.

4 Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Consider $T_1, T_2 \in T$: $T_1(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$, $T_2(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4)$.

We have null $T_1 = \{(0, 0, x_3, x_4, x_5) \in R^5 : x_3, x_4, x_5 \in R\}$ and dim null $T_1 = 3 > 2$. Similarly, we have dim null $T_2 = 3 > 2$. However, $(T_1 + T_2)(x_1, \ldots, x_5) = T_1(x_1, \ldots, x_5) + T_2(x_1, \ldots, x_5) = (x_1, x_2, 0, 0) + (0, 0, x_3, x_4) = (x_1, x_2, x_3, x_4)$, implying dim range $(T_1 + T_2) = 4$ and dim null $(T_1 + T_2) = 1 < 2$. Thus $T_1 + T_2 \notin T$. $\therefore T$ is not a subspace of $\mathcal{L}(R^5, R^4)$.

7 Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Suppose $\dim V = n$ and $\dim W = m$, $2 \leq n \leq m$. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W. Define $T_1, T_2 \in \mathcal{L}(V, W)$: $T_1(a_1 v_1 + \cdots + a_n v_n) = a_2 w_2 + \cdots + a_n w_n$ (note that $2 \leq n \leq m$), $T_2(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_{n-1} w_{n-1}$. It's obvious that dim null T_1 = dim null $T_2 = 1$, implying T_1, T_2 are not injective. However, $(T_1 + T_2)(a_1 v_1 + \cdots + a_n v_n) = T_1(a_1 v_1 + \cdots + a_n v_n) + T_2(a_1 v_1 + \cdots + a_n v_n)$ $= a_1 w_1 + 2a_2 w_2 + \cdots + 2a_{n-1} w_{n-1} + a_n w_n$. It's easy to show that $(T_1 + T_2)(v) = 0 \Rightarrow v = 0$, implying $T_1 + T_2$ is injective. Hence, $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Let $0 = a_1 T v_1 + \cdots + a_n T v_n$, where $a_1, \ldots, a_n \in F$.

$\Rightarrow 0 = T(a_1v_1+\cdots+a_nv_n)$. Because T is injective, this implies $a_1v_1+\cdots+a_nv_n = 0$.

Furthermore, because $v_1,\cdots,v_n$ is linearly independent, we derive $a_1=\cdots=a_n=0$.

Hence, $Tv_1,\cdots,Tv_n$ is linearly independent.

10 Suppose $v_1,\dots,v_n$ spans V and $T \in \mathcal{L}(V,W)$. Show that $Tv_1,\dots,Tv_n$ spans range T.

$\forall w \in \text{range } T$, $\exists v \in V$, $w = Tv$. As $v_1,\cdots,v_n$ spans V, $\exists a_1,\cdots,a_n \in F$,

$v = a_1v_1+\cdots+a_nv_n$. $\Rightarrow w = Tv = T(a_1v_1+\cdots+a_nv_n) = a_1Tv_1+\cdots+a_nTv_n$.

$\therefore Tv_1,\cdots,Tv_n$ spans range T.

11 Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subspace U of V such that

$$U \cap \text{null } T = \{0\} \quad \text{and} \quad \text{range } T = \{Tu : u \in U\}.$$

Suppose $v_1,\cdots,v_n$ is a basis of null T. We can extend $v_1,\cdots,v_n$ to a basis of V:

$v_1,\cdots,v_n,u_1,\cdots,u_m$. We define $U = \text{span}(u_1,\cdots,u_m)$. Consider $\forall v \in U \cap \text{null } T$, we can

write $v = a_1v_1+\cdots+a_nv_n = b_1u_1+\cdots+b_mu_m \Rightarrow a_1v_1+\cdots+a_nv_n+(-b_1)u_1+\cdots+(-b_m)u_m = 0$.

As $v_1,\cdots,v_n,u_1,\cdots,u_m$ is a basis, hence linearly independent, this implies $a_1=\cdots=a_n=b_1=\cdots=b_m=0$

$\Rightarrow v=0 \Rightarrow U\cap\text{null } T = \{0\}$. Consider $\forall w \in \text{range } T$, $\exists v_0 \in V$, $w = Tv_0$. And we

can also write $v_0 = (a_1v_1+\cdots+a_nv_n)+(b_1u_1+\cdots+b_mu_m) = v_1+u_1$, where $v_1 \in \text{null } T$ and $u_1 \in U$.

Hence $Tv_0 = T(v_1+u_1) = Tv_1+Tu_1 = 0+Tu_1 = Tu_1 \Rightarrow w = Tu_1$, implying

range $T \subseteq \{Tu\}$. And we also have $\{Tu\} \subseteq \{Tv\} = \text{range } T$, thus range $T = \{Tu\}$.

$\therefore$ we construct a such U satisfying these two conditions.

13 Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that null $T = U$. Prove that T is surjective.

$\dim R^8 = \dim\text{null } T + \dim\text{range } T$, null $T = U$. U is three-dimensional

$\Rightarrow \dim\text{range } T = \dim R^8 - \dim\text{null } T = 8-3 = 5$

$\therefore \dim\text{range } T = \dim R^5$. Also, range T is a subspace of R^5.

$\Rightarrow \text{range } T = R^5 \Rightarrow T$ is surjective.

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

① if $\exists T \in \mathcal{L}(V,W)$, T is injective:

T is injective $\Rightarrow \dim\text{null } T = 0$. $\therefore \dim V = \dim\text{null } T + \dim\text{range } T = \dim\text{range } T$.

Also, range $T \subseteq W \Rightarrow \dim\text{range } T \leq \dim W$. $\therefore \dim V \leq \dim W$

② if $\dim V \leq \dim W$:

Let $v_1,\cdots,v_n$ be a basis of V, $w_1,\cdots,w_m$ be a basis of W, $n \leq m$.

Define $T \in \mathcal{L}(V,W)$: $T(a_1v_1+\cdots+a_nv_n) = a_1w_1+\cdots+a_nw_n$. If $T(v) = 0 \Rightarrow$

$a_1w_1+\cdots+a_nw_n = 0 \Rightarrow a_1=\cdots=a_n=0$, as $w_1,\cdots,w_n$ is linearly independent $\Rightarrow$

$v = a_1v_1+\cdots+a_nv_n = 0 \Rightarrow \text{null } T = \{0\} \Rightarrow T$ is injective.

18 Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if $\dim U \geq \dim V - \dim W$.

① if $\exists T \in \mathcal{L}(V,W)$, null $T = U$:

If $\dim U < \dim V - \dim W \Rightarrow \dim \operatorname{null} T + \dim W < \dim V$.

Also, range $T \subseteq W \Rightarrow \dim \operatorname{range} T \leq \dim W \Rightarrow \dim \operatorname{null} T + \dim \operatorname{range} T < \dim V$.

It's impossible. $\therefore \dim U \geq \dim V - \dim W$.

② if $\dim U \geq \dim V - \dim W$:

Let $w_1, \cdots, w_m$ be a basis of W, $u_1, \cdots, u_n$ be a basis of U, $u_1, \cdots, u_n, v_1, \cdots, v_k$ be a basis of V.

$\dim U \geq \dim V - \dim W \Rightarrow n \geq (n+k) - m \Rightarrow m \geq k$.

Define $T \in \mathcal{L}(V,W)$: $T(a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_kv_k) = b_1w_1 + \cdots + b_kw_k$

$\forall a_1u_1 + \cdots + a_nu_n \in U$, $T(a_1u_1 + \cdots + a_nu_n) = 0$. $\Rightarrow U \subseteq \operatorname{null} T$.

$\forall v = a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_kv_k \in \operatorname{null} T$, $T(v) = b_1w_1 + \cdots + b_kw_k = 0 \Rightarrow b_1 = \cdots = b_k = 0$,

as $w_1, \cdots, w_k$ is linearly independent. $\Rightarrow v = a_1u_1 + \cdots + a_nu_n$, meaning $v \in U \Rightarrow \operatorname{null} T \subseteq U$.

Hence, null $T = U$.

19 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

① if T is injective:

$\forall w \in \operatorname{range} T$, $\exists v \in V$, uniquely, $w = Tv$. Define $S' \in \mathcal{L}(\operatorname{range} T, V)$:

$S'(w) = v$, where $w = Tv$. S' is a linear map: $\forall w_1, w_2 \in \operatorname{range} T$, suppose $w_1 = Tv_1$,

$w_2 = Tv_2$, hence $w_1 + w_2 = Tv_1 + Tv_2 = T(v_1 + v_2) \Rightarrow S'(w_1 + w_2) = v_1 + v_2 = S'w_1 + S'w_2$;

$\forall w \in \operatorname{range} T$, $\forall a \in F$, suppose $w = Tv$, hence $aw = aTv = T(av) \Rightarrow S'(aw) = av = aS'w$.

As range T is a subspace of W, by the result of exercise 3A 13, S' can be extended

to $S \in \mathcal{L}(W,V)$ such that $Sw = S'w$ for $\forall w \in \operatorname{range} T$. Now consider $ST \in \mathcal{L}(V)$:

$\forall v \in V$, $(ST)(v) = S(Tv) = S'(Tv) = v \Rightarrow ST$ is the identity operator on V.

② if $\exists S \in \mathcal{L}(W,V)$, $ST = I$:

If T is not injective, $\exists v_1, v_2 \in V$, $v_1 \neq v_2$, $Tv_1 = Tv_2 = w$. Then, $STv_1 = S(Tv_1)$

$= Sw = S(Tv_2) = STv_2$. Also, $ST = I$, hence $STv_1 = v_1 \neq v_2 = STv_2$. This raises

a contradiction. $\therefore$ T is injective.

20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

① if T is surjective:

range $T = W$. By the result of exercise 3B 11, we can find a subspace U of V, such

that $U \cap \operatorname{null} T = \{0\}$ and range $T = \{Tu : u \in U\}$. Define $T' \in \mathcal{L}(U, W)$: $T'u = Tu$.

$\Rightarrow$ range $T' = \{Tu : u \in U\} = \operatorname{range} T = W$. We claim T' is injective. Otherwise, $\exists u_1, u_2 \in U$,

$u_1 \neq u_2$, $T'u_1 = T'u_2 \Rightarrow T'(u_1 - u_2) = 0 \Rightarrow T(u_1 - u_2) = 0 \Rightarrow u_1 - u_2 \in U \cap \operatorname{null} T = \{0\}$

$\Rightarrow u_1 - u_2 = 0$, i.e. $u_1 = u_2$, a contradiction. Define $S \in \mathcal{L}(W, V)$: $S(w) = v$, where $T'v = w$. The definition of S is valid : $\forall w \in W$, $\exists v \in U$, uniquely, $T'v = w$, as $T' \in \mathcal{L}(U, W)$ is both injective and surjective. And it's easy to verify that S is a linear map. Now consider $TS \in \mathcal{L}(W)$: $\forall w \in W$, $TSw = T(Sw) = T'(Sw)$ $= w \Rightarrow TS$ is a identity operator on W.

② if $\exists S \in \mathcal{L}(W, V)$, $TS = I$:

If T is not surjective, $\exists w \in W$, $w \notin \text{range } T$. Hence, consider $TSw = T(Sw)$, as $w \notin \text{range } T$, there must be $TSw \neq w$, implying $TS \neq I$, a contradiction.

$\therefore T$ is surjective.

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

Denote $S = \{v \in V : Tv \in U\}$. S is a subspace : ① U is a subspace of W, hence $T0 = 0 \in U \Rightarrow 0 \in S$; ② $\forall s_1, s_2 \in S$, $Ts_1, Ts_2 \in U \Rightarrow Ts_1 + Ts_2 = T(s_1 + s_2) \in U \Rightarrow$ $s_1 + s_2 \in S$; ③ $\forall s \in S$, $\forall a \in F$, $Ts \in U \Rightarrow aTs = T(as) \in U \Rightarrow as \in S$.

Define $T' \in \mathcal{L}(S, U)$: $\forall s \in S$, $T's = Ts$. By the definition of S, $\text{range } T' = U \cap \text{range } T$.

Now we show that $\text{null } T' = \text{null } T$: $\text{null } T' \subseteq \text{null } T$, obviously ; $\forall v \in \text{null } T$, $Tv = 0 \in U$ $\Rightarrow v \in S \Rightarrow T'v$ makes sense and $T'v = Tv = 0 \Rightarrow v \in \text{null } T'$, thus $\text{null } T \subseteq \text{null } T'$.

$\therefore \dim S = \dim \text{null } T' + \dim \text{range } T' \Rightarrow \dim\{v \in V : Tv \in U\} = \dim \text{null } T + \dim(U \cap \text{range } T)$

22 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

$\text{null } ST = \{u \in U : Tu \in \text{null } S\}$: $\forall u$, such that $Tu \in \text{null } S$, $STu = S(Tu) = 0 \Rightarrow u \in \text{null } ST$ $\Rightarrow \{u \in U : Tu \in \text{null } S\} \subseteq \text{null } ST$; $\forall u \in \text{null } ST$, $STu = S(Tu) = 0 \Rightarrow Tu \in \text{null } S \Rightarrow$ $u \in \{u \in U : Tu \in \text{null } S\} \Rightarrow \text{null } ST \subseteq \{u \in U : Tu \in \text{null } S\}$.

Then, by the result of exercise 3B 21, we have

$\dim \text{null } ST = \dim\{u \in U : Tu \in \text{null } S\} = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T)$.

Also, $\text{null } S \cap \text{range } T \subseteq \text{null } S \Rightarrow \dim(\text{null } S \cap \text{range } T) \leq \dim \text{null } S$.

Hence, $\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$.

23 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

① $\dim \text{range } ST \leq \dim \text{range } S$:

$\forall w \in \text{range } ST$, $\exists u \in U$, $w = STu \Rightarrow w = S(Tu)$, i.e., $w \in \text{range } S$.

$\Rightarrow \text{range } ST \subseteq \text{range } S \Rightarrow \dim \text{range } ST \leq \dim \text{range } S$.

② $\dim \text{range } ST \leq \dim \text{range } T$:

$\dim U = \dim \text{null } T + \dim \text{range } T = \dim \text{null } ST + \dim \text{range } ST$.

$\forall u \in \text{null } T$, $Tu = 0 \Rightarrow STu = S(Tu) = S(0) = 0 \Rightarrow u \in \text{null } ST$, implying $\text{null } T \subseteq \text{null } ST$

$\Rightarrow \dim \text{null } T \leq \dim \text{null } ST$. $\therefore \dim \text{range } T \geq \dim \text{range } ST$.

Combine ① and ② $\Rightarrow \dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$

24 (a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \text{range } TS \leq 2$.
(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \text{range } TS = 2$.

(a). $ST = 0 \Rightarrow \text{null } ST = V \Rightarrow \dim \text{null } ST = \dim V = 5$.

By the result of exercise 3B 22, $\dim \text{null } ST = 5 \leq \dim \text{null } S + \dim \text{null } T$.

Also, $\dim V = 5 = \dim \text{null } S + \dim \text{range } S = \dim \text{null } T + \dim \text{range } T \Rightarrow$

$\dim \text{range } T + \dim \text{range } S = 10 - (\dim \text{null } S + \dim \text{null } T) \leq 5 \Rightarrow$

$\min\{\dim \text{range } T, \dim \text{range } S\} \leq 2$. Finally, be the result of exercise 3B 23,

$\dim \text{range } TS \leq \min\{\dim \text{range } T, \dim \text{range } S\} \leq 2$.

(b). $T(x_1, \cdots, x_5) = (x_4, x_5, 0, 0, 0)$. $S(x_1, \cdots, x_5) = (0, 0, 0, x_4, x_5)$.

Then, $ST(x_1, \cdots, x_5) = S(x_4, x_5, 0, 0, 0) = (0, 0, 0, 0, 0) \Rightarrow ST = 0$;

$TS(x_1, \cdots, x_5) = T(0, 0, 0, x_4, x_5) = (x_4, x_5, 0, 0, 0) \Rightarrow \dim \text{range } TS = 2$.

25 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{null } S \subseteq \text{null } T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

① if $\exists E \in \mathcal{L}(W)$, $T = ES$:

$\forall v \in \text{null } S$, $Sv = 0 \Rightarrow Tv = ESv = E(Sv) = E(0) = 0 \Rightarrow v \in \text{null } T$.

Hence, $\text{null } S \subseteq \text{null } T$.

② if $\text{null } S \subseteq \text{null } T$:

W is finite-dimensional, $\text{range } S \subseteq W \Rightarrow \text{range } S$ is finite-dimensional. $S \in \mathcal{L}(V, \text{range } S)$ is surjective, by the result of exercise 3B 20, $\exists G \in \mathcal{L}(\text{range } S, V)$, such that $SG = I \in \mathcal{L}(\text{range } S)$. Define $\tilde{E} = TG \in \mathcal{L}(\text{range } S, W)$. Now we show $T = \tilde{E}S$:

if $T \neq \tilde{E}S$, $\exists v \in V$, $Tv \neq \tilde{E}Sv = TGSv \Rightarrow T(v - GSv) \neq 0 \Rightarrow v - GSv \notin \text{null } T$;

however, consider $S(v - GSv) = Sv - (SG)Sv = Sv - I(Sv) = Sv - Sv = 0 \Rightarrow$

$v - GSv \in \text{null } S$, implying a contradiction to $\text{null } S \subseteq \text{null } T$. Thus $T = \tilde{E}S$.

As $\text{range } S \subseteq W$, $\tilde{E} \in \mathcal{L}(\text{range } S, W)$ can be extended to a $E \in \mathcal{L}(W)$, where $Ew = \tilde{E}w$ for $\forall w \in \text{range } S$. Hence, $T = \tilde{E}S = ES$.

26 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{range } S \subseteq \text{range } T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

① if $\exists E \in \mathcal{L}(V)$, $S = TE$:

$\forall w \in \text{range } S$, $\exists v \in V$, $w = Sv = TEv = T(Ev) \in \text{range } T$. $\therefore \text{range } S \subseteq \text{range } T$.

② if $\text{range } S \subseteq \text{range } T$:

V is finite-dimensional, $\dim V = \dim \text{null } T + \dim \text{range } T \Rightarrow \text{range } T$ is finite-dimensional.

And $T \in \mathcal{L}(V, \text{range } T)$ is surjective, by the result of exercise 3B 20, $\exists G \in \mathcal{L}(\text{range } T, V)$, such that $TG = I \in \mathcal{L}(\text{range } T)$. Define $E = GS \in \mathcal{L}(V)$, this product GS makes sense because range $S \subseteq$ range T. Then, $TE = T(GS) = (TG)S = IS = S$.

27 Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

$\forall v \in \operatorname{null} P \cap \operatorname{range} P$, $Pv = 0$ and $\exists u \in V$, $v = Pu$ $\Rightarrow$ $Pv = PPu = P^2u = Pu = 0$

$\Rightarrow v = Pu = 0 \Rightarrow \operatorname{null} P \cap \operatorname{range} P = \{0\} \Rightarrow \dim(\operatorname{null} P \cap \operatorname{range} P) = 0$.

$\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$, $\dim(\operatorname{null} P + \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P -$

$\dim(\operatorname{null} P \cap \operatorname{range} P) \Rightarrow \dim(\operatorname{null} P + \operatorname{range} P) = \dim V$. And $\operatorname{null} P + \operatorname{range} P \leq V$,

thus $V = \operatorname{null} P + \operatorname{range} P$. Also, $\operatorname{null} P \cap \operatorname{range} P = \{0\}$, hence $V = \operatorname{null} P \oplus \operatorname{range} P$.

28 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every non-constant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'.

If D is not surjective, we can find a $p \in P(R)$ that $p \notin$ range D, and $\forall q \notin$ range D, $\deg q \geq \deg p$.

Let $\deg p = m \geq 0$, we can write $p(x) = a_m x^m + \cdots + a_0 x^0$, where $a_m, \cdots, a_0 \in R$ and $a_m \neq 0$.

Assume $\exists q \in$ range D such that $\deg q = \deg p = m$, we can write $q(x) = b_m x^m + \cdots + b_0 x^0$, where

$b_m, \cdots, b_0 \in R$ and $b_m \neq 0$. Then, $\frac{a_m}{b_m} q(x) = a_m x^m + \cdots + \frac{a_m}{b_m} b_0 x^0 \in$ range D. As p has the least degree

among the polynomials not in range D, and degree $p = m$, $r(x) = a_{m-1} x^{m-1} + \cdots + a_0 x^0$ and $s(x) =$

$\frac{a_m}{b_m} b_{m-1} x^{m-1} + \cdots + \frac{a_m}{b_m} b_0 x^0 \in$ range D, because $\deg r = \deg s = m-1 < m$. Then, $\frac{a_m}{b_m} q - s + r$

$= p \in$ range D, Implying a contradiction. Hence, such q doesn't exist, meaning $\forall q$ that

$\deg q = \deg p = m$, $q \notin$ range D. However, consider $\forall p \in P(R)$ that $\deg p = m+1$, we have

$\deg Dp = \deg p - 1 = m$, and $Dp \in$ range D. This contradiction implies that D is surjective.

29 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.

Define $D \in \mathcal{L}(P(R))$: $Dp = p'$, and define $T = 5D^2 + 3D \in \mathcal{L}(P(R))$: $Tp = 5p'' + 3p'$.

It's obvious that $\deg Tp = \deg p - 1$ for every non-constant $p \in P(R)$, then by the result of

exercise 3B 28, T is surjective. Hence, $\forall p \in P(R)$, $\exists q \in P(R)$, $Tq = p$ $\Rightarrow$

$5q'' + 3q' = p$.

30 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

$\forall v \in \operatorname{null} \varphi \cap \{au : a \in F\}$, $\varphi v = 0$, $\exists a \in F$, $v = au$ $\Rightarrow \varphi v = \varphi au = a\varphi u = 0$. As $u \notin \operatorname{null} \varphi$,

$\varphi u \neq 0$, hence $a = 0$ $\Rightarrow v = au = 0$ $\Rightarrow \operatorname{null} \varphi \cap \{au : a \in F\} = \{0\}$.

Suppose $\varphi u = b \in F$, $b \neq 0$. $\forall v \in V$, $\varphi v \in F$, suppose $c = \varphi v$. Then, $\exists a \in F$, $c = ab$

$\Rightarrow \varphi v = a\varphi u = \varphi(au) \Rightarrow \varphi(v - au) = 0$, implying $v - au \in \operatorname{null} \varphi$. Hence, $\exists w \in \operatorname{null} \varphi$,

$v - au = w \Rightarrow v = w + au$, where $w \in \operatorname{null} \varphi$, $au \in \{au : a \in F\}$. This implies

$V = \operatorname{null} \varphi + \{au : a \in F\}$. Also, we already have $\operatorname{null} \varphi \cap \{au : a \in F\} = \{0\}$.

$\therefore V = \operatorname{null} \varphi \oplus \{au : a \in F\}$.

31 Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

① if $\exists T \in \mathcal{L}(V,W)$, $\operatorname{null} T = X$, $\operatorname{range} T = Y$:

By the fundamental theorem of linear maps, it must satisfy $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

$\Rightarrow \dim X + \dim Y = \dim V$.

② if $\dim X + \dim Y = \dim V$:

Let $x_1, \cdots, x_n$ be a basis of X, $y_1, \cdots, y_m$ be a basis of Y. $x_1, \cdots, x_n$ can be extended to a basis of V : $x_1, \cdots, x_n, u_1, \cdots, u_m$. Define $T \in \mathcal{L}(V,W)$: $T(a_1x_1 + \cdots + a_nx_n + b_1u_1 + \cdots + b_mu_m) = b_1y_1 + \cdots + b_my_m$

And it's easy to show that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$.

32 Suppose V is finite-dimensional with $\dim V > 1$. Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

Hint: The description of the two-sided ideals of $\mathcal{L}(V)$ given by Exercise 17 in Section 3A might be useful.

$\forall T \in \operatorname{null} \varphi$, $\varphi(T) = 0 \Rightarrow \forall S \in \mathcal{L}(V)$, $\varphi(ST) = \varphi(S)\varphi(T) = \varphi(S) \cdot 0 = 0$.

$\varphi(TS) = \varphi(T)\varphi(S) = 0 \cdot \varphi(S) = 0 \Rightarrow ST, TS \in \operatorname{null} \varphi$. Hence, $\operatorname{null} \varphi$ is a two-sided ideal of $\mathcal{L}(V)$. By the result of exercise 3A 17, $\operatorname{null} \varphi = \{0\}$ or $\mathcal{L}(V)$.

Consider $\varphi(ST) = \varphi(S)\varphi(T) = \varphi(T)\varphi(S) = \varphi(TS) \Rightarrow \varphi(ST) - \varphi(TS) = \varphi(ST - TS) = 0$

$\Rightarrow ST - TS \in \operatorname{null} \varphi$, $\forall S, T \in \mathcal{L}(V)$. As $\dim V > 1$, suppose $v_1, \cdots, v_n$ is a basis of V, $n \geq 2$

Define $S, T \in \mathcal{L}(V)$: $S(a_1v_1 + \cdots + a_nv_n) = a_1v_1$, $T(a_1v_1 + \cdots + a_nv_n) = a_nv_1 + \cdots + a_1v_n \Rightarrow$

$ST(a_1v_1 + \cdots + a_nv_n) = S(a_nv_1 + \cdots + a_1v_n) = a_nv_1$, $TS(a_1v_1 + \cdots + a_nv_n) = T(a_1v_1) = 0 \Rightarrow ST \neq TS$

$\Rightarrow ST - TS \neq 0$, $ST - TS \in \operatorname{null} \varphi \Rightarrow \operatorname{null} \varphi \neq \{0\}$. Hence, $\operatorname{null} \varphi = \mathcal{L}(V)$, implying $\varphi = 0$

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}} : V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

for all $u, v \in V$.

(a) Show that $T_{\mathbf{C}}$ is a (complex) linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.
(b) Show that $T_{\mathbf{C}}$ is injective if and only if T is injective.
(c) Show that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ if and only if $\operatorname{range} T = W$.

*See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$. The linear map $T_{\mathbf{C}}$ is called the **complexification** of the linear map T.*

(a). $T_c((u_1+iv_1)+(u_2+iv_2)) = T_c((u_1+u_2)+i(v_1+v_2)) = T(u_1+u_2) + iT(v_1+v_2) = (Tu_1 + iTv_1) + (Tu_2 + iTv_2)$

$= T_c(u_1+iv_1) + T_c(u_2+iv_2)$.

$T_c(a(u+iv)) = T(au + iav) = Tau + iTav = a(Tu + iTv) = aT_c(u+iv)$.

(b). If T_c is injective : if T is not injective, $\exists u_1, u_2 \in V$, $u_1 \neq u_2$, $Tu_1 = Tu_2 \Rightarrow$

$u_1 + i0 \neq u_2 + i0$, $T_c(u_1 + i0) = Tu_1 + iT0 = Tu_2 + iT0 = T_c(u_2 + i0)$, a contradiction.

If T is injective : if T_c is not injective, $\exists u_1 + iv_1, u_2 + iv_2 \in V_c$, $u_1 + iv_1 \neq u_2 + iv_2$,

$T_c(u_1+iv_1) = Tu_1 + iTv_1 = Tu_2 + iTv_2 = T_c(u_2+iv_2) \Rightarrow Tu_1 = Tu_2$, $Tv_1 = Tv_2$. However,

there must be $u_1 \neq u_2$ or $v_1 \neq v_2$, implying T is not injective, a contradiction.

(c). If $\operatorname{range} T_c = W_c$: if $\operatorname{range} T \neq W$, $\exists w \in W$, $\forall u \in V$, $Tu \neq w \Rightarrow \forall u + i0 \in V_c$,

$T_c(u+i0) = Tu + iT0 = Tu + i0 \neq w + i0 \in W_c$, a contradiction

If $\operatorname{range} T = W$: $\forall w_1 + iw_2 \in W_c$, $\exists u, v \in V$, $w_1 = Tu$, $w_2 = Tv \Rightarrow \exists u + iv \in V_c$,

$T_c(u+iv) = Tu + iTv = w_1 + iw_2$. Hence, $\operatorname{range} T_c = W_c$.