Solutions to 1A :

1 Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

$\forall \alpha, \beta \in C$, suppose $\alpha = a + bi$, $\beta = c + di$, where $a, b, c, d \in R$.

$$\begin{aligned} \alpha + \beta &= (a+bi) + (c+di) \\ &= (a+c) + (b+d)i & \text{(i)} \\ &= (c+a) + (d+b)i & \text{(ii)} \\ &= (c+di) + (a+bi) & \text{(iii)} \\ &= \beta + \alpha \end{aligned}$$

$\therefore \alpha + \beta = \beta + \alpha$ for $\forall \alpha, \beta \in C$

(i), (iii) are derived from the definition of addition on C, and (ii) holds for the Commutativity on R

4 Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

$\forall \lambda, \alpha, \beta \in C$, suppose $\alpha = a + bi$, $\beta = c + di$, $\lambda = e + fi$.
where $a, b, c, d, e, f \in R$.

$$\begin{aligned} \lambda(\alpha + \beta) &= \lambda((a+bi) + (c+di)) \\ &= \lambda((a+c) + (b+d)i) & \text{(i)} \\ &= (e+fi)((a+c) + (b+d)i) \\ &= (e(a+c) - f(b+d)) + (e(b+d) + f(a+c))i & \text{(ii)} \\ &= (ea + ec - fb - fd) + (eb + ed + fa + fc)i & \text{(iii)} \end{aligned}$$

and

$$\begin{aligned} \lambda\alpha + \lambda\beta &= (e+fi)(a+bi) + (e+fi)(c+di) \\ &= ((ea - fb) + (eb + fa)i) + ((ec - fd) + (ed + fc)i) & \text{(iv)} \\ &= (ea - fb + ec - fd) + (eb + fa + ed + fc)i & \text{(v)} \\ &= (ea + ec - fb - fd) + (eb + ed + fa + fc)i & \text{(vi)} \end{aligned}$$

$\therefore \lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$, for $\forall \lambda, \alpha, \beta \in C$

(i), (v) are derived from the definition of addition on C. (ii), (iv) are derived from the definition of multiplication on C, (iii) is derived from the distributed property on R, and (vi) is derived from the commutativity on R.

5 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Assume $\forall \alpha \in C$, $\exists \beta, \gamma \in C$ and $\beta \neq \gamma$, s.t. $\alpha + \beta = 0$ and $\alpha + \gamma = 0$.

suppose $\alpha = a + bi$, $\beta = c + di$, $\gamma = e + fi$.

$$\begin{aligned} &\alpha + \beta = 0 \\ \Rightarrow\ &(a+bi) + (c+di) = 0 \\ \Rightarrow\ &(a+c) + (b+d)i = 0 + 0i \end{aligned}$$

$\Rightarrow \quad a + c = 0 \quad \text{and} \quad b + d = 0$

$\Rightarrow \quad c = -a \quad \text{and} \quad d = -b$

Similarly, we derive $e = -a$ and $f = -b$ from $\alpha + \gamma = 0$

Because the additive inverse on R is unique, we have $c = e$ and $d = f$,

introducing a contradiction : $\beta = \gamma$

$\therefore$ the additive inverse on C is unique

8 Find two distinct square roots of *i*.

Denote any square root of i as $a + bi$, $a, b \in R$.

$$(a+bi)^2 = i$$

$$\Rightarrow (a^2 - b^2) + 2abi = i$$

$$\Rightarrow a^2 = b^2 \quad \text{and} \quad ab = \tfrac{1}{2} > 0$$

$\Rightarrow$ square roots of i have the form $a + ai$, where $a \in R$ and $a^2 = \frac{1}{2}$

$\therefore$ square roots of i are $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ and $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

15 Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

$\forall x \in F^n$, suppose $x = (x_1, \cdots, x_n)$, where $x_i \in F$, $i = 1, \cdots, n$

for $\forall a, b \in F$, we have

$$
\begin{aligned}
(a+b)x &= ((a+b)x_1, \cdots, (a+b)x_n) && \text{(i)} \\
&= (ax_1 + bx_1, \cdots, ax_n + bx_n) && \text{(ii)} \\
&= (ax_1, \cdots, ax_n) + (bx_1, \cdots, bx_n) && \text{(iii)} \\
&= ax + bx && \text{(iv)}
\end{aligned}
$$

(i) and (iv) are derived from the definition of scalar multiplication in F^n,

(ii) is derived from the distributed property on F.

(iii) is derived from the definition of addition in F^n.