Solutions to 2B :

1 Find all vector spaces that have exactly one basis.

The vector space $\{0\}$ has exactly one basis : the empty list .

For any vector space except $\{0\}$, denoted by V , it's easy to show that any basis of V must not be empty . We suppose $v_1, \cdots, v_m$ is any basis of V , where $m \in \mathbb{Z}^{++}$

$\forall v \in V$, $\exists a_i \in F$, $v = a_1 v_1 + \cdots + a_m v_m = \frac{a_1}{2}(2v_1) + \cdots + \frac{a_m}{2}(2v_m)$. This implies that $2v_1, \cdots, 2v_m$ spans V.

Let $0 = a_1(2v_1) + \cdots + a_m(2v_m) = (2a_1)v_1 + \cdots + (2a_m)v_m$, where $a_i \in F$. We derive $2a_i = 0$ for $v_1, \cdots, v_m$ is linearly independent for it's a basis . Thus $a_i = 0$, implying $2v_1, \cdots, 2v_m$ is also linearly independent .

Hence, $2v_1, \cdots, 2v_m$ is a basis of V . It's easy to show that $2v_1, \cdots, 2v_m$ is different from $v_1, \cdots, v_m$, giving that V has more than one basis . $\therefore$ only $\{0\}$ has exactly one basis .

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

(a). $(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$

(b). Applying the procedure of the proof of result 2.32 , we derive the extended basis of $\mathbf{R}^5$

$(3,1,0,0,0)$, $(0,0,7,1,0)$, $(0,0,0,0,1)$, $(1,0,0,0,0)$, $(0,0,1,0,0)$

(c). From the proof of result 2.33 and (b) , we derive

$W = \operatorname{span}((1,0,0,0,0),(0,0,1,0,0)) = \{(x_1,0,x_3,0,0) \in \mathbf{R}^5 : x_1, x_3 \in \mathbf{R}\}$

such that $\mathbf{R}^5 = U \oplus W$.

5 Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Suppose $v_1, \cdots, v_m$ is a basis of V . As $V = U + W$, we suppose $v_i = u_i + w_i$, where $u_i \in U$ and $w_i \in W$. For $\forall v \in V$, $\exists a_i \in F$, such that

$$v = a_1 v_1 + \cdots + a_m v_m = a_1(u_1 + w_1) + \cdots + a_m(u_m + w_m) = a_1 u_1 + \cdots + a_m u_m + a_1 w_1 + \cdots + a_m w_m .$$

Hence, $u_1, \cdots, u_m, w_1, \cdots, w_m$ spans V . By result 2.30 , $u_1, \cdots, u_m, w_1, \cdots, w_m$ contains a basis of V , whose elements are in $U \cup W$.

7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

By result 2.28 , v_1, v_2, v_3, v_4 is a basis of V

$\Rightarrow$ $\forall v \in V$, $v = a_1 v_1 + \cdots + a_4 v_4$, where $a_i \in F$ is unique

$\Rightarrow$ $v = a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4$

This representation is also unique .

$\Rightarrow$ $v_1 + v_2$, $v_2 + v_3$, $v_3 + v_4$, v_4 is also a basis of V .

8 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

A counterexample:

$V = R^4$. $v_1 = (1,0,0,0)$. $v_2 = (0,1,0,0)$. $v_3 = (0,0,1,0)$. $v_4 = (0,0,0,1)$.

$U = \{(x_1, x_2, x_3, 2x_3) \in R^4 : x_1, x_2, x_3 \in R\}$. It's easy to prove that U is a subspace of V.

We see $v_1, v_2 \in U$ and $v_3, v_4 \notin U$.

However, v_1, v_2 is not a basis of U, because v_1, v_2 can't span U.

9 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \dots, v_m$ is a basis of V if and only if $w_1, \dots, w_m$ is a basis of V.

By result 2.28, $v_1, \dots, v_m$ is a basis of V

$\Leftrightarrow$ $\forall v \in V$, $\exists$ unique $a_i \in F$, $v = a_1 v_1 + \cdots + a_m v_m = b_1 w_1 + \cdots + b_m w_m$,

where $w_k = v_1 + \cdots + v_k$ and $b_k = \begin{cases} a_m & , k = m \\ a_k - a_{k+1} & , k = 1, \dots, m-1 \end{cases}$.

To show this representation is also unique, suppose $v = c_1 w_1 + \cdots + c_m w_m = \sum_{i=1}^{m} \left(\sum_{j=i}^{m} c_j\right) v_i$.

Because $v_1, \dots, v_m$ is a basis, $\sum_{j=i}^{m} c_j = a_i$, giving $c_i = \begin{cases} a_m & , i = m \\ a_i - a_{i+1} & , i = 1, \dots, m-1 \end{cases}$.

Hence, $c_k = b_k$, representation under $w_1, \dots, w_m$ is unique.

$\Leftrightarrow$ $w_1, \dots, w_m$ is a basis of V.

10 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

$\forall v \in V$, as $V = U \oplus W$, we can suppose $v = u + w = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$,

where $u \in U$ and $w \in W$, $a_i, b_j \in F$.

Assume that there exists a different representation:

$v = \tilde{u} + \tilde{w} = c_1 u_1 + \cdots + c_m u_m + d_1 w_1 + \cdots + d_n w_n$.

$\Rightarrow 0 = (u - \tilde{u}) + (w - \tilde{w}) = (a_1 - c_1) u_1 + \cdots + (a_m - c_m) u_m + (b_1 - d_1) w_1 + \cdots + (b_n - d_n) w_n$.

This gives that $(u - \tilde{u}) = (\tilde{w} - w) \in W$. Also, $(u - \tilde{u}) \in U$. Hence $(u - \tilde{u}) \in U \cap W$

$\Rightarrow u - \tilde{u} = 0$ for $U \cap W = \{0\}$ for $V = U \oplus W$.

$\Rightarrow 0 = (a_1 - c_1) u_1 + \cdots + (a_m - c_m) u_m \Rightarrow a_i = c_i$ for $u_1, \dots, u_m$ is a basis

Similarly, we derive $b_j = d_j$.

Hence, the representation of $\forall v \in V$ under $u_1, \dots, u_m, w_1, \dots, w_n$ is unique.

$\therefore$ $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V.

11 Suppose V is a real vector space. Show that if $v_1, \dots, v_n$ is a basis of V (as a real vector space), then $v_1, \dots, v_n$ is also a basis of the complexification V_C (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification V_C.

We only need to prove that $v_1, \dots, v_n$ spans V_C:

$\forall u + iv \in V_C$, where $u, v \in V$, $\exists a_i, b_i \in R$, such that

$u = a_1 v_1 + \cdots + a_n v_n$, $v = b_1 v_1 + \cdots + b_n v_n$.

$\Rightarrow \ u + iv = (a_1 + b_1 i) v_1 + \cdots + (a_n + b_n i) v_n$

Hence, $v_1, \cdots, v_n$ spans $V_{\mathbb{C}}$.

$\therefore$ $v_1, \cdots, v_n$ is also a basis of the complexification $V_{\mathbb{C}}$ (as a complex vector space) .