Solutions to 1B:

1 Prove that $-(-v) = v$ for every $v \in V$.

To prove $-(-v) = v$, which means v is the additive inverse of $-v$, we only need to prove $(-v) + v = 0$. As $-v$ is the additive inverse of v, we have $v + (-v) = 0$. Because of the commutativity, we derive $(-v) + v = v + (-v) = 0$. $\therefore -(-v) = v$ for every $v \in V$.

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Assume $a \neq 0$ and $v \neq 0$. We have

$$
\begin{aligned}
v &= 1 \cdot v && (\text{multiplicative identity}) \\
&= (\tfrac{1}{a} \cdot a) v && (a \neq 0) \\
&= \tfrac{1}{a}(av) && (\text{associativity}) \\
&= \tfrac{1}{a} \cdot 0 && (av = 0) \\
&= 0 && (\text{result } 1.31)
\end{aligned}
$$

This is a contradiction. $\therefore$ $a = 0$ or $v = 0$

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

The empty set fails to satisfy only one of the requirements to be a vector space — additive identity. There doesn't exist an additive identity in the empty set, as it's empty.

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

① Assume V is any vector space under the definition 1.20.
From result 1.30, we have that V satisfies the changed definition.

② Assume V is any vector space under the changed definition.
For all $v \in V$, we have

$$
\begin{aligned}
v + (-1)v &= 1v + (-1)v && (\text{multiplicative identity}) \\
&= (1 + (-1))v && (\text{distributive properties}) \\
&= 0v \\
&= 0 && (\text{question condition})
\end{aligned}
$$

This gives that each v has a additive inverse $(-1)v$, so V satisfies the definition 1.20.

From ① and ②, we proved that these two definitions are equivalent.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

$R \cup \{\infty, -\infty\}$ is not a vector space over R. It doesn't satisfy the distributed properties. For example,

$$(2 + (-1))(-\infty) = 1(-\infty) = -\infty$$

however, $2(-\infty) + (-1)(-\infty) = (-\infty) + \infty = 0$

gives $(2+(-1))(-\infty) \neq 2(-\infty) + (-1)(-\infty)$

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Suppose V is any vector space over F. For $\forall f, g \in V^S$, $\forall a \in F$, we define

$$(f+g)(x) = f(x) + g(x), \quad \forall x \in S$$

$$(af)(x) = a(f(x)), \quad \forall x \in S$$

Note that $f, g \in V^S : S \to V$, so $f(x) + g(x) \in V$ and $a(f(x)) \in V$, which mean $f+g \in V^S$ and $af \in V^S$.

Now we prove V^S is a vector space over F:

① commutativity:

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

② associativity:

$$((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g(x) + h(x))$$

$$= f(x) + (g+h)(x) = (f+(g+h))(x)$$

$$((ab)f)(x) = (ab)f(x) = a(bf(x)) = a((bf)(x)) = (a(bf))(x)$$

③ additive identity:

The additive identity is $0(x) = 0$.

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x) = (f)(x)$$

④ additive inverse:

As V^S is the set of functions from S to V, for $\forall f \in V^S$ and $\forall x \in S$, there exists $g \in V^S$ such that $g(x)$ is the additive inverse of $f(x)$.

This gives: $\forall f \in V^S$, $\exists g \in V^S$, $(f+g)(x) = f(x) + g(x) = 0 = (0)(x)$

⑤ multiplicative identity:

$(1f)(x) = 1(f(x)) = f(x) = (f)(x)$

⑥ distributive properties:

$(a(f+g))(x) = a((f+g)(x)) = a(f(x)+g(x)) = af(x)+ag(x)$

$= (af)(x) + (ag)(x) = (af+ag)(x)$

$((a+b)f)(x) = (a+b)(f(x)) = af(x) + bf(x) = (af)(x) + (bf)(x)$

$= (af+bf)(x)$

From ① to ⑥, we proved that V^S is a vector space over F.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by V_C, equals $V \times V$. An element of V_C is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on V_C is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on V_C is defined by
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, V_C is a complex vector space.

① Commutativity:

$(u_1+iv_1) + (u_2+iv_2) = (u_1+u_2) + i(v_1+v_2) = (u_2+u_1) + i(v_2+v_1) = (u_2+iv_2) + (u_1+iv_1)$

② associativity:

$((u_1+iv_1)+(u_2+iv_2)) + (u_3+iv_3) = ((u_1+u_2)+i(v_1+v_2)) + (u_3+iv_3)$

$= ((u_1+u_2)+u_3) + i((v_1+v_2)+v_3) = (u_1+(u_2+u_3)) + i(v_1+(v_2+v_3))$

$= (u_1+iv_1) + ((u_2+u_3)+i(v_2+v_3)) = (u_1+iv_1) + ((u_2+iv_2)+(u_3+iv_3))$

$((a+bi)(c+di))(u+iv) = ((ac-bd)+(ad+bc)i)(u+iv)$

$= (acu-bdu-adv-bcv) + i(acv-bdv+adu+bcu)$

$(a+bi)((c+di)(u+iv)) = (a+bi)((cu-dv)+i(cv+du))$

$= (acu-adv-bcv-bdu) + i(acv+adu+bcu-bdv)$

$= (acu-bdu-adv-bcv) + i(acv-bdv+adu+bcu)$

$\therefore ((a+bi)(c+di))(u+iv) = (a+bi)((c+di)(u+iv))$

③ additive identity:

Suppose 0 is the additive identity in V, then the additive identity in V_C is $0+i0$.

$(u+iv) + (0+i0) = (u+0) + i(v+0) = u+iv$

④ additive inverse:

Suppose $-u, -v$ are the additive inverses of u, v, then the additive inverse of $u+iv$ is $(-u)+i(-v)$.

$(u+iv) + ((-u)+i(-v)) = (u+(-u)) + i(v+(-v)) = 0+i0$

⑤ multiplicative identity:

$$(1+0i)(u+iv) = (1u-0v)+i(1v+0u) = u+iv$$

⑥ distributed properties:

$$\begin{aligned}(a+bi)((u_1+iv_1)+(u_2+iv_2)) &= (a+bi)((u_1+u_2)+i(v_1+v_2)) \\ &= (au_1+au_2-bv_1-bv_2)+i(av_1+av_2+bu_1+bu_2) \\ &= ((au_1-bv_1)+(au_2-bv_2))+i((av_1+bu_1)+(av_2+bu_2)) \\ &= ((au_1-bv_1)+i(av_1+bu_1))+((au_2-bv_2)+i(av_2+bu_2)) \\ &= (a+bi)(u_1+iv_1)+(a+bi)(u_2+iv_2)\end{aligned}$$

$$\begin{aligned}((a+bi)+(c+di))(u+iv) &= ((a+c)+(b+d)i)(u+iv) \\ &= (au+cu-bv-dv)+i(av+cv+bu+du) \\ &= ((au-bv)+(cu-dv))+i((av+bu)+(cv+du)) \\ &= ((au-bv)+i(av-bu))+((cu-dv)+i(cv+du)) \\ &= (a+bi)(u+iv)+(c+di)(u+iv)\end{aligned}$$