Solutions to 1C :

1 For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.
(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$
(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$
(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

(a). ① additive identity : $0+2\times0+3\times0=0$. hence $(0,0,0) \in U$. satisfy

② closed under addition :

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3) = (x_1+2x_2+3x_3)+(y_1+2y_2+3y_3) = 0+0=0$. satisfy

③ closed under scalar multiplication :

$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$.

$ax_1+2ax_2+3ax_3 = a(x_1+2x_2+3x_3) = a\cdot0 = 0$. satisfy

Hence, $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1+2x_2+3x_3 = 0\}$ is a subspace .

(b). $0 = (0,0,0) \notin U$ because $0+2\times0+3\times0 = 0 \neq 4$.

Hence, $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1+2x_2+3x_3 = 4\}$ doesn't include the additive identity.

It's not a subspace .

(c). $(0,1,1) \in U$, $(1,0,1) \in U$,

$(0,1,1)+(1,0,1) = (1,1,2) \notin U$, because $1\times1\times2 = 2 \neq 0$

Hence, $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$ is not closed under addition .

It's not a subspace .

(d). ① additive identity : $0 = 5\times0$. hence $(0,0,0) \in U$. satisfy

② closed under addition :

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

$x_1+y_1 = 5x_3+5y_3 = 5(x_3+y_3)$. satisfy

③ closed under scalar multiplication :

$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$

$ax_1 = a(5x_3) = 5(ax_3)$. satisfy

Hence, $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$ is a subspace .

4 Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

① If the set of such f is a subspace of $\mathbf{R}^{[0,1]}$, then the additive identity $0(x) = 0$, $x \in [0,1]$ should be in the subset . This gives $\int_0^1 0(x) = 0 = b$.

Hence, $b = 0$.

② In the other direction, if $b=0$, then the subset is the set of f on $[0,1]$ that satisfies $\int_0^1 f = 0$.

As $\int_0^1 0(x) = 0$, the subset includes the additive identity.

For any f, g in this subset, $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0+0=0$.

Hence, $f+g$ is also in this subset.

For any f in this subset and any real value a, $\int_0^1 af = a\int_0^1 f = a\cdot 0 = 0$.

Hence, af is also in this subset.

$\therefore$ this subset is a subspace of $R^{[0,1]}$.

From ① and ②, we proved the result.

5 Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

① additive identity:

The additive identity in C^2 is $(0+0i, 0+0i) = (0,0)$.

$(0,0) \in R^2$. This condition is satisfied.

② closed under addition:

For any $u, v \in R^2$, we can denote $u = (u_1+0i, u_2+0i)$ and $v = (v_1+0i, v_2+0i)$.

$u+v = (u_1+0i, u_2+0i) + (v_1+0i, v_2+0i) = (u_1+v_1, u_2+v_2) \in R^2$.

Hence, this is also satisfied.

③ closed under scalar multiplication:

For $\forall u = (u_1+0i, u_2+0i) \in R^2$, $a+bi \in C$,

$(a+bi)u = (a+bi)(u_1+0i, u_2+0i) = (au_1+bu_1 i, au_2+bu_2 i)$

If $bu_1 \neq 0$ or $bu_2 \neq 0$, $(a+bi)u \notin R^2$.

Hence, R^2 is not closed under scalar multiplication in C^2.

From above, we know that R^2 is not a subspace of C^2.

7 Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

A counterexample: $U = \{(x_1, x_2) \in R^2 : x_1, x_2 \in Z\}$.

It can be proved that U satisfies the suppose of the question.

However, $(1,1) \in U$, $\frac{1}{2}(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin U$, which means U is not closed under scalar multiplication. Hence, U is not a subspace of R^2.

8 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

$U = \{(x_1, x_2) \in R^2 : x_1 x_2 = 0\}$.

$\forall (x_1, x_2) \in U$, $\forall a \in R$, $a(x_1, x_2) = (ax_1, ax_2) \in U$ because $(ax_1)(ax_2) = a^2(x_1 x_2) = 0$.

However, $(1,0),(0,1)\in U$, $(1,0)+(0,1)=(1,1)\notin U$ because $|x|=1\neq 0$.

Hence, nonempty subset U is closed under scalar multiplication but not a subspace of $\mathbb{R}^2$.

9 A function $f\colon \mathbf{R}\to\mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x)=f(x+p)$ for all $x\in\mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

① additive identity: $0(x)=0$ is periodic. Thus satisfied.

② closed under addition:

Suppose f, g are periodic with least periods p, q respectively.

$(f+g)(x)=f(x)+g(x)=f(x+p)+g(x+q)=f(x+np)+g(x+mq)$. $n,m\in\mathbb{Z}^{++}$.

If we can find $n,m\in\mathbb{Z}^{++}$ such that $np=mq$, then $f(x+np)+g(x+mq)$

$=f(x+np)+g(x+np)=(f+g)(x+np)$. This gives that $f+g$ is also periodic.

However, we can not always find such solutions. For example, consider $p=1$ and

$q=\pi$, $n=\pi m$ has no solution for $n,m\in\mathbb{Z}^{++}$.

Hence, this condition is not satisfied.

③ closed under scalar multiplication:

Suppose f is periodic with period p, and a is an arbitrary real number.

$(af)(x)=af(x)=af(x+p)=(af)(x+p)$.

Hence, this condition is satisfied.

From ① to ③, we know that the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ is not a subspace of $\mathbb{R}^{\mathbb{R}}$, for it is not closed under addition.

11 Prove that the intersection of every collection of subspaces of V is a subspace of V.

For any collection of subspaces $\{V_k\}$, $k=1,2,\dots$

① additive identity: $0\in V_k$, $k=1,2,\dots$, $\Rightarrow$ $0\in\cap\{V_k\}$

② closed under addition:

$\forall u,v\in\cap\{V_k\}$, which gives $u,v\in V_k$, $k=1,2,\cdots$

we have $u+v\in V_k$, $k=1,2,\cdots$ $\Rightarrow$ $u+v\in\cap\{V_k\}$

③ closed under scalar multiplication:

$\forall u\in\cap\{V_k\}$, which gives $u\in V_k$, $k=1,2,\cdots$, and $\forall a\in F$

we have $au\in V_k$, $k=1,2,\cdots$ $\Rightarrow$ $au\in\cap\{V_k\}$

From ① to ③, we proved that $\cap\{V_k\}$ is a subspace of V.

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Suppose V_1, V_2 are two subspaces of V.

① Without loss of generality, we suppose $V_1 \subseteq V_2$, hence $V_1 \cup V_2 = V_2$.
As V_2 is a subspace, obviously $V_1 \cup V_2$ is a subspace of V.

② In the other direction, we suppose $V_1 \cup V_2$ is a subspace of V.
We assume that $V_1 \nsubseteq V_2$ and $V_2 \nsubseteq V_1$. Hence, $\exists v_1 \in V_1$ and $\exists v_2 \in V_2$, such that $v_1 \notin V_2$ and $v_2 \notin V_1$.
Because $v_1 \in V_1 \cup V_2$ and $v_2 \in V_1 \cup V_2$, with the condition that $V_1 \cup V_2$ is closed under addition as it's a subspace, we derive $v_1 + v_2 \in V_1 \cup V_2$.
However, we assert $v_1 + v_2 \notin V_1$. Otherwise, $(v_1 + v_2) + (-v_1) = v_2 \in V_1$, leading to a contradiction. Similarly, we can prove that $v_1 + v_2 \notin V_2$.
These give $v_1 + v_2 \notin V_1 \cup V_2$, a contradiction.
Hence, our assumption does not hold, there must be $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

From ① and ②, we proved that $V_1 \cup V_2$ is a subspace if and only if $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Suppose V_1, V_2, and V_3 are three subspaces of V.

① If one of the subspaces contains the other two, i.e., $V_1 \supseteq V_2$ and $V_1 \supseteq V_3$, then $V_1 \cup V_2 \cup V_3 = V_1$ is a subspace of V.

② In the other direction, we suppose $V_1 \cup V_2 \cup V_3$ is a subspace of V.
We assume there's no one of the subspaces containing the other two.
First, we consider the situation that there exists a V_i containing a V_j.
Without loss of generality, we suppose $V_1 \supseteq V_2$. Then we consider the containment relationships between V_1 and V_3: If $V_1 \supseteq V_3$, then $V_1 \supseteq (V_2 \cup V_3)$; If $V_3 \supseteq V_1$, then $V_3 \supseteq (V_1 \cup V_2)$. Both contradict our assumption. Hence, $V_1 \nsubseteq V_3$ and $V_3 \nsubseteq V_1$. This means, $\exists v_1 \in V_1$, $v_1 \notin V_3$, and $\exists v_3 \in V_3$, $v_3 \notin V_1$.
However, as $v_1, v_3 \in V_1 \cup V_2 \cup V_3$, and $V_1 \cup V_2 \cup V_3$ is a subspace, $v_1 + v_3 \in V_1 \cup V_2 \cup V_3$.
We claim that $v_1 + v_3 \notin V_1$, otherwise $(v_1 + v_3) + (-v_1) = v_3 \in V_1$, resulting in a contradiction.
Similarly, $v_1 + v_3 \notin V_3$. Hence, there must be $v_1 + v_3 \in V_2$ for $v_1 + v_3 \in V_1 \cup V_2 \cup V_3$.
However, we still have $V_1 \supseteq V_2$, such that $v_1 + v_3 \in V_1$, a contradiction.
From above, the first situation can not hold.
Second, we consider the situation that, no one subspace contains another, but there exist two subspaces among the three, their union contains the rest one.
Without loss of generality, we suppose $V_1 \cup V_2 \supseteq V_3$, while $V_i \nsubseteq V_j$, $\forall i \neq j$, $i, j \in \{1, 2, 3\}$. Hence, $V_1 \cap V_3 \neq \emptyset$. otherwise $(V_1 \cup V_2) \cap V_3 = (V_1 \cap V_3) \cup (V_2 \cap V_3)$
$= \emptyset \cup (V_2 \cap V_3) = V_2 \cap V_3 = V_3$ as $V_1 \cup V_2 \supseteq V_3$, leading to $V_2 \supseteq V_3$, a contradiction.

We claim that $\exists u \in V_1 \cap V_3$, such that $u \notin V_2$. Otherwise, we will derive $V_2 \supseteq V_1 \cap V_3$, then $V_2 \cap V_3 \supseteq (V_1 \cap V_3) \cap V_3 = V_1 \cap V_3$, leading to $(V_1 \cup V_2) \cap V_3 = (V_1 \cap V_3) \cup (V_2 \cap V_3) = V_2 \cap V_3$. Meanwhile, $(V_1 \cup V_2) \cap V_3 = V_3$ for $V_1 \cup V_2 \supseteq V_3$. Hence, $V_2 \cap V_3 = V_3$, gives $V_2 \supseteq V_3$, a contradiction. Similarly, we can find such $v \in V_2 \cap V_3$, and $v \notin V_1$. Consider $u+v$, if $u+v \in V_2$, then $u = (u+v) + (-v) \in V_2$, this is a contradiction. Hence, we have $u+v \notin V_2$ and $u+v \notin V_1$, similarly. Thus, there must be $u+v \in V_3$. However, this contradicts our suppose that $V_1 \cup V_2 \supseteq V_3$.

From above, the second situation can not hold.

Third and the last, we consider the rest of the situation. This implies, $\exists v_1 \in V_1$, $\exists v_2 \in V_2$, $\exists v_3 \in V_3$, such that $v_1 \notin V_2 \cup V_3$, $v_2 \notin V_1 \cup V_3$, $v_3 \notin V_1 \cup V_2$.

To see why this is true, assume that $\forall v_1 \in V_1$, $v_1 \in V_2 \cup V_3$, which means $V_2 \cup V_3 \supseteq V_1$, falling into previous situations. Now consider $v_1 + v_2$. If $v_1 + v_2 \in V_1$, then $v_2 = (v_1+v_2) + (-v_1) \in V_1$, resulting in contradiction. Hence, $v_1 + v_2 \notin V_1$ and $v_1 + v_2 \notin V_2$, similarly. Thus, there must be $v_1 + v_2 \in V_3$. Then let's consider $2v_1 + v_2$. Firstly, we have $2v_1 + v_2 \notin V_1$, otherwise $v_2 = (2v_1+v_2) + (-2v_1) \in V_1$. Similarly, we have $2v_1 + v_2 \notin V_2$. However, there must be $2v_1 + v_2 \notin V_3$, either. Otherwise, $v_1 = (2v_1 + v_2) + (-(v_1+v_2)) \in V_3$, as we already have $v_1 + v_2 \in V_3$. Thus, we result in an obvious contradiction: $2v_1 + v_2 \notin V_1 \cup V_2 \cup V_3$.

From above, the last situation can't hold.

Hence, the only possibility is that our assumption can not hold. This gives that there must be one of the subspaces containing the other two.

From ① and ②, we completed our proof of both directions of the result.

14 Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Consider a typical element $(a, -a, 2a)$ in U and a typical element $(b, b, 2b)$ in W.

$$(a, -a, 2a) + (b, b, 2b) = (a+b, b-a, 2(a+b)).$$

Thus, $U + W \subseteq \{(x, y, 2x) \in F^3 : x, y \in F\}$.

In the other direction, consider a typical element $(a, b, 2a)$ in $\{(x, y, 2x) \in F^3 : x, y \in F\}$.

$$(a, b, 2a) = \left(\tfrac{a-b}{2}, -\tfrac{a-b}{2}, a-b\right) + \left(\tfrac{a+b}{2}, \tfrac{a+b}{2}, a+b\right)$$

Thus, $U + W \supseteq \{(x, y, 2x) \in F^3 : x, y \in F\}$

Hence, $U + W = \{(x, y, 2x) \in F^3 : x, y \in F\}$. $U + W$ is the set of elements of F^3 whose third coordinate equals 2 times its first coordinate.

15 Suppose U is a subspace of V. What is $U+U$?

① $\forall u \in U$, $u = u + 0 \in U+U$, hence $U \subseteq U+U$

② $\forall u_1+u_2 \in U+U$, where $u_1 \in U$ and $u_2 \in U$, we have $u_1+u_2 \in U$,

hence $U \supseteq U+U$

From ① and ②, we derive $U+U=U$.

16 Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U+W=W+U$?

Consider a typical element $u+w \in U+W$, we have $u+w = w+u \in W+U$ because of the commutativity on V. Hence, $U+W \subseteq W+U$. Similarly, we can prove the other direction of inclusion. $\therefore U+W=W+U$

17 Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1+V_2)+V_3 = V_1+(V_2+V_3)?$$

Consider a typical element $(v_1+v_2)+v_3 \in (V_1+V_2)+V_3$, we have $(v_1+v_2)+v_3 =$ $v_1+(v_2+v_3) \in V_1+(V_2+V_3)$ for the associativity on V. Hence, $(V_1+V_2)+V_3 \subseteq V_1+(V_2+V_3)$. Similarly, we can prove the other direction of inclusion. $\therefore (V_1+V_2)+V_3 = V_1+(V_2+V_3)$.

18 Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

① additive identity:

The additive identity is $\{0\}$. Consider an arbitrary subspace V_1.

$V_1+\{0\} = \{v_1+v_2 : v_1 \in V_1, v_2 \in \{0\}\} = \{v_1+0 : v_1 \in V_1\} = V_1$

② additive inverse:

For any subspace $V_1 \neq \{0\}$, $\exists v_1 \in V_1$, $v_1 \neq 0$. Suppose W is a subspace.

$v_1 = v_1 + 0 \in V_1+W$, as $0 \in W$. This means $V_1+W \neq \{0\}$ for any W.

For subspace $\{0\}$, we have $\{0\}+\{0\} = \{0\}$.

Hence, a subspace has an additive inverse if and only if the subspace is $\{0\}$, and its additive inverse is $\{0\}$.

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1+U = V_2+U,$$

then $V_1 = V_2$.

A counterexample: $V_1 = \{(x,0) \in F^2 : x \in F\}$. $V_2 = \{(x,2x) \in F^2 : x \in F\}$.

$U = \{(0,x) \in F^2 : x \in F\}$. We have $V_1+U = V_2+U = F^2$.

However, $V_1 \neq V_2$.

20 Suppose
$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$
Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Let $W = \{(x, 0, x, 0) \in F^4 : x \in F\}$. Obviously, W is a subspace.

Consider $\forall (a, b, c, d) \in F^4$, we have

$$(a, b, c, d) = (b, b, d, d) + (a-b, 0, c-d, 0)$$

Thus, $F^4 \subseteq U + W$. And it's easy to show $F^4 \supseteq U + W$.

Hence, $F^4 = U + W$.

Consider $(a, a, b, b) \in U$ and $(c, 0, c, 0) \in W$. Let

$$(a, a, b, b) = (c, 0, c, 0) \Rightarrow a = b = c = 0.$$

Thus, $U \cap W = \{(0, 0, 0, 0)\}$.

Hence, $F^4 = U \oplus W$.

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that
$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$
then $V_1 = V_2$.

The counterexample is the same as that in exercise 1C 19:

$V = F^2$. $V_1 = \{(x, 0) \in F^2 : x \in F\}$. $V_2 = \{(x, 2x) \in F^2 : x \in F\}$.

$U = \{(0, x) \in F^2 : x \in F\}$.

It's easy to show that V_1, V_2, U are subspaces of V such that $V = V_1 \oplus U$ and $V = V_2 \oplus U$. However, $V_1 \neq V_2$.

24 A function $f: \mathbf{R} \to \mathbf{R}$ is called *even* if
$$f(-x) = f(x)$$
for all $x \in \mathbf{R}$. A function $f: \mathbf{R} \to \mathbf{R}$ is called *odd* if
$$f(-x) = -f(x)$$
for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and let V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

① $\forall f_e + f_o \in V_e + V_o$, obviously, $f_e + f_o \in \{f : R \to R\}$. Hence, $R^R \supseteq V_e + V_o$.

② $\forall f \in R^R$. We can decompose f as:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

And we find that $\frac{f(x)+f(-x)}{2}$ is an even function and $\frac{f(x)-f(-x)}{2}$ is an odd function.

This gives $f \in V_e + V_o$. Hence, $R^R \subseteq V_e + V_o$.

From ① and ②, we derive $R^R = V_e + V_o$.

Consider $f \in V_e \cap V_o$, which means $f(-x) = f(x) = -f(x)$, $\forall x \in R$.

Thus, there must be $f(x) \equiv 0$, giving that $V_e \cap V_o = \{0(x)\}$.

Hence, $R^R = V_e \oplus V_o$.