Solutions to Exercises 3E :

1 Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Formally, a function T from V to W is a subset T of $V \times W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then this exercise could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \times W$.

① if T is a linear map :

1). $(0,0) = (0, T0) \in$ graph of T .

2). $\forall (v_1, Tv_1), (v_2, Tv_2) \in$ graph of T, $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2)$

$= (v_1 + v_2, T(v_1 + v_2)) \in$ graph of T.

3). $\forall (v, Tv) \in$ graph of T, $\forall a \in F$, $a(v, Tv) = (av, aTv) = (av, T(av)) \in$ graph of T.

② if the graph of T is a subspace of $V \times W$:

1). $\forall v_1, v_2 \in V$, $(v_1, Tv_1), (v_2, Tv_2) \in$ graph of T, and $(v_1, Tv_1) + (v_2, Tv_2) =$

$(v_1 + v_2, Tv_1 + Tv_2) \in$ graph of $T \Rightarrow T(v_1 + v_2) = Tv_1 + Tv_2$.

2). $\forall v \in V$, $\forall a \in F$, $(v, Tv) \in$ graph of T, and $a(v, Tv) = (av, aTv) \in$ graph of T

$\Rightarrow T(av) = aTv$.

2 Suppose that $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \ldots, m$.

If $\exists k \in \{1, \cdots, m\}$, V_k is not finite-dimensional : We can find a sequence $v_{k,1}, v_{k,2}, \cdots$ of elements of V_k, such that $v_{k,1}, \cdots, v_{k,n}$ is linearly independent for $\forall n \in N^+$. Then, consider the sequence $u_1, u_2, \cdots$ of elements of $V_1 \times \cdots \times V_m$, where $u_j = (0, \cdots, v_{k,j}, \cdots, 0)$, i.e., $v_{k,j}$ in the k^{th} slot and 0 in the other slots. It's easy to verify that $u_1, \cdots, u_n$ is linearly independent for $\forall n \in N^+$, which implies $V_1 \times \cdots \times V_m$ is infinite-dimensional.

$\therefore V_1 \times \cdots \times V_m$ is finite-dimensional $\Rightarrow V_k$ is finite-dimensional for $k = 1, \cdots, m$.

3 Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Define $\Psi : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$ by $\Psi((T_1, \cdots, T_m)) = S$, where $T_k \in \mathcal{L}(V_k, W)$ for $k = 1, \cdots, m$ and $S \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$ is defined by $S((v_1, \cdots, v_m)) = T_1v_1 + \cdots + T_mv_m$; it's easy to verify that S is indeed linear. Ψ is a linear map, as you should verify.

①. Ψ is injective :

Consider $\Psi((T_1, \cdots, T_m)) = 0 \Rightarrow \forall v_k \in V_k$, $k = 1, \cdots, m$, $T_1v_1 + \cdots + T_mv_m \equiv 0$; for each k, let $v_j = 0$ for all $j \neq k$, implying $T_kv_k \equiv 0$, hence $T_k = 0 \Rightarrow T_1 = 0, \cdots, T_m = 0$, implying null $\Psi = \{(0, \cdots, 0)\}$. Thus Ψ is injective.

②. Ψ is surjective :

$\forall S \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, define $T_k(v_k) = S((0, \cdots, v_k, \cdots, 0))$ for $\forall v_k \in V_k$, where

$(0, \cdots, v_k, \cdots, 0)$ has v_k in k^{th} slot and 0 in the other slots, $k = 1, \cdots, m$. It's easy to verify that $T_k \in \mathcal{L}(V_k, W)$. Then, $\forall (v_1, \cdots, v_m) \in V_1 \times \cdots \times V_m$, $S((v_1, \cdots, v_m)) = S(\sum_{k=1}^{m} (0, \cdots, v_k, \cdots, 0)) = \sum_{k=1}^{m} S((0, \cdots, v_k, \cdots, 0)) = T_1 v_1 + \cdots + T_m v_m$, implying $S = \psi((T_1, \cdots, T_m))$.

Thus ψ is surjective.

By ① and ②, $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic.

4 Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

Define $\phi : \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) \longrightarrow \mathcal{L}(V, W_1 \times \cdots \times W_m)$ by $\phi((T_1, \cdots, T_m)) = S$, where $T_k \in \mathcal{L}(V, W_k)$ for $k = 1, \cdots, m$ and $S \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$ is defined by $Sv = (T_1 v, \cdots, T_m v)$ for $\forall v \in V$; it's easy to verify that S is indeed linear. ϕ is a linear map, as you should verify.

① ϕ is injective:

Consider $\phi((T_1, \cdots, T_m)) = 0 \Rightarrow \forall v \in V$, $Sv = (T_1 v, \cdots, T_m v) \equiv 0$, implying $T_1 = 0, \cdots, T_m = 0 \Rightarrow \text{null } \phi = \{(0, \cdots, 0)\}$. Thus ϕ is injective.

②. ϕ is surjective:

$\forall S \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$, define $T_k \in \mathcal{L}(V, W_k)$ by let $T_k(v)$ equals the k^{th} slot of Sv for $\forall v \in V$, $k = 1, \cdots, m$. It's easy to verify that T_k is indeed linear. Then, $\forall v \in V$, $Sv = (T_1 v, \cdots, T_m v)$, implying $S = \phi((T_1, \cdots, T_m))$. Thus ϕ is surjective.

By ① and ②, $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic.

5 For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Prove that V^m and $\mathcal{L}(\mathbf{F}^m, V)$ are isomorphic vector spaces.

Define $\psi : V^m \rightarrow \mathcal{L}(F^m, V)$ by $\psi((v_1, \cdots, v_m)) = T$, where $T \in \mathcal{L}(F^m, V)$ is defined by $T((x_1, \cdots, x_m)) = x_1 v_1 + \cdots + x_m v_m$; it's easy to verify that T is indeed linear. ψ is a linear map, as you should verify.

①. ψ is injective:

Consider $\psi((v_1, \cdots, v_m)) = 0 \Rightarrow \forall (x_1, \cdots, x_m) \in F^m$, $x_1 v_1 + \cdots + x_m v_m \equiv 0$; for each $k = 1, \cdots, m$, let $x_j = 0$ for all $j \neq k$, implying $x_k v_k \equiv 0$, hence $v_k = 0 \Rightarrow v_1 = \cdots = v_m = 0$, implying $\text{null } \psi = \{(0, \cdots, 0)\}$. Thus ψ is injective.

②. ψ is surjective:

$\forall T \in \mathcal{L}(F^m, V)$, let $v_k = T(\vec{e_k})$, where $\vec{e_k}$ is the k^{th} element of the standard basis of F^m, $k = 1, \cdots, m$. Then, $\forall (x_1, \cdots, x_m) \in F^m$, $T((x_1, \cdots, x_m)) = T(\sum_{k=1}^{m} x_k \vec{e_k}) = \sum_{k=1}^{m} x_k T(\vec{e_k}) = x_1 v_1 + \cdots + x_m v_m$, implying $T = \psi((v_1, \cdots, v_m))$. Thus ψ is surjective.

By ① and ②, V^m and $\mathcal{L}(F^m, V)$ are isomorphic.

6 Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

As $v + U = x + W$ and $v = v + 0 \in v + U$, $\exists w \in W$, $v = x + w \Rightarrow v - x = w \in W$.

Then, $\forall u \in U$, $\exists w \in W$, $v + u = x + w \Rightarrow u = -(v - x) + w \in W \Rightarrow U \subseteq W$.

Similarly, $W \subseteq U$. $\therefore U = W$.

8 (a) Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $\{x \in V : Tx = c\}$ is either the empty set or is a translate of null T.
(b) Explain why the set of solutions to a system of linear equations such as 3.27 is either the empty set or is a translate of some subspace of $\mathbf{F}^n$.

(a). ①. If $c \notin \text{range } T$, $\{x \in V : Tx = c\} = \varnothing$.

②. If $c \in \text{range } T$, $\exists v \in V$, $c = Tv \Rightarrow v \in \{x \in V : Tx = c\}$. Then, $v + \text{null } T = \{x \in V : Tx = c\}$: $\forall u \in v + \text{null } T$, $\exists w \in \text{null } T$, $u = v + w \Rightarrow Tu = T(v+w) = Tv + Tw = c + 0 = 0$, implying $u \in \{x \in V : Tx = c\} \Rightarrow v + \text{null } T \subseteq \{x \in V : Tx = c\}$; $\forall x \in \{x \in V : Tx = c\}$, $Tx = Tv = c \Rightarrow T(x - v) = Tx - Tv = 0$, implying $x - v \in \text{null } T$ $\Rightarrow x = v + (x - v) \in v + \text{null } T \Rightarrow \{x \in V : Tx = c\} \subseteq v + \text{null } T$.

(b). A system of linear equations with n variables and m equations can be identified by a linear map from F^n to F^m and a vector in F^m, e.g., $Tx = c$, where $T \in \mathcal{L}(F^n, F^m)$ and $c \in F^m$ identify this system and $x \in F^n$ represents the variables. Then, the set of solutions is actually $\{x \in F^n : Tx = c\}$; by (a), it is either the empty set or a translate of null T (i.e., the set of solutions to the corresponding homogeneous system).

9 Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

① if nonempty subset A is affine (a translate of some subspace):

Suppose U is a subspace of V and $x \in V$, such that $A = x + U$. $\forall v, w \in A$, $\forall \lambda \in F$, $\exists u_1, u_2 \in U$, $v = x + u_1$, $w = x + u_2 \Longrightarrow \lambda v + (1-\lambda)w = \lambda(x + u_1) + (1-\lambda)(x + u_2) = x + (\lambda u_1 + (1-\lambda)u_2) \in x + U = A$.

② if $\forall v, w \in A$, $\forall \lambda \in F$, $\lambda v + (1-\lambda)w \in A$:

Choose $\forall x \in A$, let $U = (-x) + A$, hence $A = x + U$. Now we show U is a subspace:

1). $0 = (-x) + x \in (-x) + A = U$.

2). $\forall (-x) + v_1, (-x) + v_2 \in U$, where $v_1, v_2 \in A$, $((-x) + v_1) + ((-x) + v_2) = (-x) + (\frac{1}{2}(2v_1 - x) + \frac{1}{2}(2v_2 - x))$; by hypothesis, $2v_1 - x \in A$, $2v_2 - x \in A$, and further $\frac{1}{2}(2v_1 - x) + \frac{1}{2}(2v_2 - x) \in A$; hence $((-x) + v_1) + ((-x) + v_2) \in (-x) + A = U$.

3). $\forall (-x) + v \in U$, where $v \in A$, $\forall \lambda \in F$, $\lambda((-x) + v) = (-x) + (\lambda v + (1-\lambda)x) \in (-x) + A = U$.

10 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

If $A_1 \cap A_2 \neq \varnothing$: $\forall x, y \in A_1 \cap A_2$, $\exists u_1, u_3 \in U_1$ and $\exists u_2, u_4 \in U_2$, $x = v + u_1 = w + u_2$, $y = v + u_3 = w + u_4 \Longrightarrow \forall \lambda \in F$, $\lambda x + (1-\lambda)y = v + (\lambda u_1 + (1-\lambda)u_3) \in v + U_1 = A_1$, also $\lambda x + (1-\lambda)y = w + (\lambda u_2 + (1-\lambda)u_4) \in w + U_2 = A_2 \Longrightarrow \lambda x + (1-\lambda)y \in A_1 \cap A_2$. Then, by the result of exercise 3E 9, $A_1 \cap A_2$ is a translate of some subspace of V.

11 Suppose $U = \{(x_1, x_2, \ldots) \in \mathbf{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$.
(a) Show that U is a subspace of $\mathbf{F}^\infty$.
(b) Prove that $\mathbf{F}^\infty/U$ is infinite-dimensional.

(a). ①. $0 = (0, 0, \cdots) \in U$.

②. $\forall (x_1, x_2, \cdots), (y_1, y_2, \cdots) \in U$, suppose $x_k = 0$ for $\forall k \geq m$ and $y_j = 0$ for $\forall j \geq n$, where $m, n \in N^+$. Then, $(x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1+y_1, x_2+y_2, \cdots)$ satisfies that $x_k + y_k = 0$ for $\forall k \geq \max\{m, n\}$, implying $(x_1, x_2, \cdots) + (y_1, y_2, \cdots) \in U$.

③. $\forall (x_1, x_2, \cdots) \in U$, suppose $x_k = 0$ for $\forall k \geq m$, where $m \in N^+$. Then, $\forall \lambda \in F$, $\lambda(x_1, x_2, \cdots) = (\lambda x_1, \lambda x_2, \cdots)$ satisfies that $\lambda x_k = 0$ for $\forall k \geq m$, implying $\lambda(x_1, x_2, \cdots) \in U$.

(b). Let v_k be the element of F^∞ that has a 0 in the k^{th} slot and 1 in all other slots, $k = 1, 2, \cdots$. It's easy to verify that $v_k \notin U$ and $v_1, \cdots, v_m$ is linearly independent for $\forall m \in N^+$. Then, consider the sequence $v_1 + U, v_2 + U, \cdots$ of elements of F^∞ / U, it's easy to show that $v_1 + U, \cdots, v_m + U$ is linearly independent for $\forall m \in N^+$, implying F^∞/U is infinite-dimensional.

12 Suppose $v_1, \ldots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \ldots, \lambda_m \in \mathbf{F} \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$

(a) Prove that A is a translate of some subspace of V.
(b) Prove that if B is a translate of some subspace of V and $\{v_1, \ldots, v_m\} \subseteq B$, then $A \subseteq B$.
(c) Prove that A is a translate of some subspace of V of dimension less than m.

(a). $\forall u_1, u_2 \in A$, suppose $u_1 = a_1 v_1 + \cdots + a_m v_m$ and $u_2 = b_1 v_1 + \cdots + b_m v_m$, where $a_1 + \cdots + a_m = b_1 + \cdots + b_m = 1$

$\Rightarrow \forall \lambda \in F$, $\lambda u_1 + (1-\lambda) u_2 = (\lambda a_1 + (1-\lambda) b_1) v_1 + \cdots + (\lambda a_m + (1-\lambda) b_m) v_m$ and $(\lambda a_1 + (1-\lambda) b_1) + \cdots + (\lambda a_m + (1-\lambda) b_m) = \lambda(a_1 + \cdots + a_m) + (1-\lambda)(b_1 + \cdots + b_m) = \lambda + (1-\lambda) = 1 \Rightarrow \lambda u_1 + (1-\lambda) u_2 \in A$.

By the result of exercise 3E 9, A is a translate of some subspace of V.

(b). Suppose $B = w + U$, $w \in V$ and U is a subspace of V. As $v_1, \cdots, v_m \in B$, $\exists u_1, \cdots, u_m \in U$, $v_k = w + u_k$ for $k = 1, \cdots, m$. Then, $\forall v = \lambda_1 v_1 + \cdots + \lambda_m v_m \in A$, $v = \lambda_1 (w + u_1) + \cdots + \lambda_m (w + u_m) = (\lambda_1 + \cdots + \lambda_m) w + (\lambda_1 u_1 + \cdots + \lambda_m u_m) = w + (\lambda_1 u_1 + \cdots + \lambda_m u_m) \in w + U = B \Rightarrow A \subseteq B$.

(c). By (a), $\exists w \in V$ and $\tilde{A}$ a subspace of V, $A = w + \tilde{A}$. As $v_1, \cdots, v_m \in \operatorname{span}(v_1, \cdots, v_m)$ and $\operatorname{span}(v_1, \cdots, v_m)$ is a translate to itself, by (b), $A = w + \tilde{A} \subseteq \operatorname{span}(v_1, \cdots, v_m)$. Then, $\forall v \in \tilde{A}$, $v = (w + v) + (-w) \in \operatorname{span}(v_1, \cdots, v_m) \Rightarrow \tilde{A} \subseteq \operatorname{span}(v_1, \cdots, v_m)$.

①. If $v_1, \cdots, v_m$ is linearly independent:

We can only write $0 = 0v_1 + \cdots + 0v_m \Rightarrow 0 \notin A$. Then we claim $w \notin \tilde{A}$, otherwise $0 = w + (-w) \in w + \tilde{A} = A$, a contradiction. However, $w \in \operatorname{span}(v_1, \cdots, v_m)$, this implies $\tilde{A} \neq \operatorname{span}(v_1, \cdots, v_m)$. Hence, $\dim \tilde{A} < \dim \operatorname{span}(v_1, \cdots, v_m) = m$.

②. If $v_1, \cdots, v_m$ is not linearly independent:

$\dim \tilde{A} \leq \dim \operatorname{span}(v_1, \cdots, v_m) < m$.

13 Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

Let $W_1, \cdots, W_m$ be a basis of V/U and suppose $W_k = v_k + U$ for $k = 1, \cdots, m$. Then, $v_1, \cdots, v_m$ is linearly independent, as you should verify. Now we show that $V = U \oplus \operatorname{span}(v_1, \cdots, v_m)$:

①. $U \cap \operatorname{span}(v_1, \cdots, v_m) = \{0\}$:

$\forall u \in U \cap \operatorname{span}(v_1, \cdots, v_m)$, suppose $u = \lambda_1 v_1 + \cdots + \lambda_m v_m$, where $\lambda_1, \cdots, \lambda_m \in F$. By result 3.101, $u + U = 0 + U$. Also, $u + U = (\lambda_1 v_1 + \cdots + \lambda_m v_m) + U = \lambda_1 W_1 + \cdots + \lambda_m W_m$. As $W_1, \cdots, W_m$ is a basis, this implies $\lambda_1 = \cdots = \lambda_m = 0 \Rightarrow u = 0 \Rightarrow U \cap \operatorname{span}(v_1, \cdots, v_m) = \{0\}$.

②. $V = U + \operatorname{span}(v_1, \cdots, v_m)$:

$\forall v \in V$, $\exists \lambda_1, \cdots, \lambda_m \in F$. $v + U = \lambda_1 W_1 + \cdots + \lambda_m W_m = (\lambda_1 v_1 + \cdots + \lambda_m v_m) + U$; by result 3.101, $v - (\lambda_1 v_1 + \cdots + \lambda_m v_m) \in U \Rightarrow \exists u \in U$, $v = u + (\lambda_1 v_1 + \cdots + \lambda_m v_m) \Rightarrow V = U + \operatorname{span}(v_1, \cdots, v_m)$.

Define $T: V \to U \times (V/U)$ by $T(v) = (u, \lambda_1 W_1 + \cdots + \lambda_m W_m)$ for $\forall v \in V$, where we can write $v = u + \lambda_1 v_1 + \cdots + \lambda_m v_m$ uniquely. T is linear, as you should verify. Now we show T is an isomorphism:

(1). T is injective:

$\forall v \in \operatorname{null} T$ and $v = u + \lambda_1 v_1 + \cdots + \lambda_m v_m$. $Tv = (u, \lambda_1 W_1 + \cdots + \lambda_m W_m) = (0, 0 + U) \Rightarrow u = 0$ and $\lambda_1 = \cdots = \lambda_m = 0$, implying $v = 0 \Rightarrow \operatorname{null} T = \{0\}$. Thus T is injective.

(2). T is surjective:

$\forall (u, v+U) \in U \times (V/U)$, $\exists \lambda_1, \cdots, \lambda_m \in F$, $v + U = \lambda_1 W_1 + \cdots + \lambda_m W_m \Rightarrow T(u + \lambda_1 v_1 + \cdots + \lambda_m v_m) = (u, \lambda_1 W_1 + \cdots + \lambda_m W_m) = (u, v + U)$. Thus T is surjective.

From above, V and $U \times (V/U)$ are isomorphic.

14 Suppose U and W are subspaces of V and $V = U \oplus W$. Suppose $w_1, \ldots, w_m$ is a basis of W. Prove that $w_1 + U, \ldots, w_m + U$ is a basis of V/U.

①. $w_1 + U, \cdots, w_m + U$ is linearly independent:

Let $0 + U = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U) = (\lambda_1 w_1 + \cdots + \lambda_m w_m) + U \Rightarrow \lambda_1 w_1 + \cdots + \lambda_m w_m - 0 \in U$

$\Rightarrow \lambda_1 = \cdots = \lambda_m = 0$, because $U \cap W = \{0\} \Rightarrow w_1 + U, \cdots, w_m + U$ is linearly independent.

②. $w_1 + U, \cdots, w_m + U$ spans V/U:

$\forall v + U \in V/U$, we can write $v = u + \lambda_1 w_1 + \cdots + \lambda_m w_m$ uniquely, where $u \in U$ and $\lambda_1, \cdots, \lambda_m \in F$.

$\Rightarrow v - (\lambda_1 w_1 + \cdots + \lambda_m w_m) = u \in U \Rightarrow v + U = (\lambda_1 w_1 + \cdots + \lambda_m w_m) + U = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U)$.

This implies $w_1 + U, \cdots, w_m + U$ spans V/U.

15 Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.

By the result of exercise 3E 13, V is isomorphic to $U \times (V/U) \Rightarrow \dim V = \dim(U \times (V/U)) = \dim U + \dim V/U = n + m$. Let $0 = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n \Rightarrow a_1 v_1 + \cdots + a_m v_m = -b_1 u_1 + \cdots + (-b_n u_n)$, implying $a_1 v_1 + \cdots + a_m v_m \in U \Rightarrow 0 + U = (a_1 v_1 + \cdots + a_m v_m) + U = a_1(v_1 + U) + \cdots + a_m(v_m + U) \Rightarrow a_1 = \cdots = a_m = 0$ (because $v_1 + U, \cdots, v_m + U$ is a basis of V/U), then $b_1 u_1 + \cdots + b_n u_n = 0 \Rightarrow b_1 = \cdots = b_n = 0$ (because $u_1, \cdots, u_n$ is a basis of U). Thus, $v_1, \cdots, v_m, u_1, \cdots, u_n$ is linearly independent $\Rightarrow$ is a basis (because $\dim V = n + m$).

16 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\operatorname{null} \varphi) = 1$.

$\varphi \neq 0 \Rightarrow \dim \operatorname{range} \varphi > 0$. $\operatorname{range} \varphi \subseteq F \Rightarrow \dim \operatorname{range} \varphi \leq \dim F = 1$. Thus $\dim \operatorname{range} \varphi = 1$

By result 3.107, $V/(\operatorname{null} \varphi)$ and $\operatorname{range} \varphi$ are isomorphic $\Rightarrow \dim V/(\operatorname{null} \varphi) = \dim \operatorname{range} \varphi = 1$.

17 Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathcal{L}(V, \mathbf{F})$ such that $\operatorname{null} \varphi = U$.

As $\dim V/U = \dim F = 1$, there exists an isomorphism S from V/U onto F. Let π be the quotient map: $V \to V/U$, defined by $\pi(v) = v + U$ for $\forall v \in V$. Define $\varphi = S \circ \pi \in \mathcal{L}(V, F)$.

$\forall u \in U$, $\varphi(u) = S\pi(u) = S(u+U) = S(0+U) = 0 \Rightarrow u \in \operatorname{null}\varphi$, implying $U \subseteq \operatorname{null}\varphi$.

$\forall u \in \operatorname{null}\varphi$, $\varphi(u) = S\pi(u) = 0 \Rightarrow S^{-1}S\pi(u) = S^{-1}(0)$ (because S is invertible) $\Rightarrow \pi(u) = 0$, implying $u \in \operatorname{null}\pi = U \Rightarrow \operatorname{null}\varphi \subseteq U$. Thus, $\operatorname{null}\varphi = U$.

18 Suppose that U is a subspace of V such that V/U is finite-dimensional.

(a) Show that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

(b) Prove that there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

(a). Suppose $\dim V/U = m$ and let $v_1 + U, \cdots, v_m + U$ be a basis of V/U. Because $V = U + W$, we can suppose $v_k = u_k + w_k$, where $u_k \in U$ and $w_k \in W$, $k = 1, \cdots, m$. Then, $v_k - w_k = u_k \in U \Rightarrow v_k + U = w_k + U$. Thus $w_1 + U, \cdots, w_m + U$ is a basis of V/U, implying that $w_1, \cdots, w_m$ is linearly independent. Hence, $\dim W \geq m = \dim V/U$.

(b). Suppose $\dim V/U = m$ and let $w_1 + U, \cdots, w_m + U$ be a basis of $V/U \Rightarrow w_1, \cdots, w_m$ is linearly independent, as you should verify. Let $W = \operatorname{span}(w_1, \cdots, w_m)$, then $\dim W = m = \dim V/U$.

$\forall v \in U \cap W$, let $v = \lambda_1 w_1 + \cdots + \lambda_m w_m$, $v + U = (\lambda_1 w_1 + \cdots + \lambda_m w_m) = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U)$ and $v + U = 0 + U \Rightarrow \lambda_1 = \cdots = \lambda_m = 0$, hence $v = 0 \Rightarrow U \cap W = \{0\}$.

$\forall v \in V$, let $v + U = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U) = (\lambda_1 w_1 + \cdots + \lambda_m w_m) + U \Rightarrow \exists u \in U$, $v = u + w$, where $w = \lambda_1 w_1 + \cdots + \lambda_m w_m \in W$. Thus $V = U \oplus W$.

19 Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \operatorname{null} T$.

① if $\exists S \in \mathcal{L}(V/U, W)$, $T = S \circ \pi$:

$\forall u \in U$, $Tu = S(\pi(u)) = S(u+U) = S(0+U) = 0 \Rightarrow u \in \operatorname{null} T$, implying $U \subseteq \operatorname{null} T$

② if $U \subseteq \operatorname{null} T$:

Define $S \in \mathcal{L}(V/U, W)$ by $S(v+U) = Tv$ for $\forall v \in V$. To show that S is valid, let $v_1 + U = v_2 + U \Rightarrow v_1 - v_2 \in U \subseteq \operatorname{null} T \Rightarrow T(v_1 - v_2) = Tv_1 - Tv_2 = 0 \Rightarrow S(v_1 + U) = Tv_1 = Tv_2 = S(v_2 + U)$. Then $\forall v \in V$, $S \circ \pi(v) = S(v + U) = Tv \Rightarrow T = S \circ \pi$