Solutions to 3D:

1 Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

As T^{-1} is the inverse of T, we have $TT^{-1} = T^{-1}T = I$, which implies that conversely T is the inverse of T^{-1}. Thus $(T^{-1})^{-1} = T$.

2 Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

$T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible $\Rightarrow T^{-1} \in \mathcal{L}(V, U)$ and $S^{-1} \in \mathcal{L}(W, V)$ $\Rightarrow T^{-1}S^{-1}$ makes sense, and $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$, $(T^{-1}S^{-1})(ST) = I$ similarly. Hence, $ST \in \mathcal{L}(U, W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) T is invertible.

(b) $Tv_1, \dots, Tv_n$ is a basis of V for every basis $v_1, \dots, v_n$ of V.

(c) $Tv_1, \dots, Tv_n$ is a basis of V for some basis $v_1, \dots, v_n$ of V.

(a) $\Rightarrow$ (b): Let $0 = a_1Tv_1 + \cdots + a_nTv_n = T(a_1v_1 + \cdots + a_nv_n)$, where $a_1, \dots, a_n \in F$

T is invertible $\Rightarrow$ T is injective $\Rightarrow$ null $T = \{0\}$ $\Rightarrow a_1v_1 + \cdots + a_nv_n = 0$

$\Rightarrow a_1 = \cdots = a_n = 0$ as $v_1, \dots, v_n$ is a basis $\Rightarrow Tv_1, \dots, Tv_n$ is linearly independent.

Also, $\dim V$ = length of $v_1, \dots, v_n$ = length of $Tv_1, \dots, Tv_n$. Thus $Tv_1, \dots, Tv_n$ is a basis of V, where $v_1, \dots, v_n$ is any basis of V.

(b) $\Rightarrow$ (c): obviously

(c) $\Rightarrow$ (a): Suppose exists some basis $v_1, \dots, v_n$ of V, such that $Tv_1, \dots, Tv_n$ is also a basis of V. By the linear map lemma, there exists $S \in \mathcal{L}(V)$ such that $S(Tv_k) = v_k$ for $k = 1, \dots, n$. Then for $\forall v \in V$, written as $v = a_1v_1 + \cdots + a_nv_n$,

$(ST)(v) = (ST)(a_1v_1 + \cdots + a_nv_n) = a_1(S(Tv_1)) + \cdots + a_n(S(Tv_n)) = a_1v_1 + \cdots + a_nv_n = v$,

implying $ST = I$. By result 3.68, we also derive $TS = I$. Hence T is invertible.

4 Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Let $\Psi = \{T \in \mathcal{L}(V) : T \text{ is noninvertible}\}$. Let $v_1, \dots, v_n$ be a basis of V, where $n = \dim V > 1$.

Define $P_j \in \mathcal{L}(V)$: $\forall v \in V$, suppose $v = a_1v_1 + \cdots + a_nv_n$ where $a_1, \dots, a_n \in F$, $P_j(v) = a_jv_j$; $j = 1, \dots, n$.

Obviously P_j is not surjective, hence is not invertible, thus $P_j \in \Psi$. However, $P_1 + \cdots + P_n = I \notin \Psi$. $\therefore$ Ψ is not a subspace of $\mathcal{L}(V)$.

5 Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

① if $\exists T \in \mathcal{L}(V)$, T is invertible and $Tu = Su$ for $\forall u \in U$:

Assume that S is not injective $\Rightarrow$ null $S \neq \{0\}$ $\Rightarrow \exists u \in U, u \neq 0, Su = 0$

$\Rightarrow Tu = Su = 0$, implying null $T \neq \{0\}$ $\Rightarrow$ T is not injective, which is impossible as T is invertible. $\therefore$ S is injective.

② if S is injective:

1). If $U = V$, let $T = S \in \mathcal{L}(V)$. Then $Tu = Su$ for $\forall u \in U$. And by the result 3.65, as $T = S$ is injective, T is invertible.

2). If $U \neq V$, suppose $V = U \oplus W$ and $V = \operatorname{range} S \oplus \tilde{W}$. As $\dim V = \dim U + \dim W = \dim \operatorname{range} S + \dim \tilde{W}$, $\dim U = \dim \operatorname{range} S + \dim \operatorname{null} S$, and S is injective, it can be implied that $\dim W = \dim \tilde{W}$. Hence, there exists an isomorphism $\tilde{S} \in \mathcal{L}(W, \tilde{W})$, by the result 3.70. For $\forall v \in V$, we can write $v = u + w$ uniquely, where $u \in U$ and $w \in W$, and we define $T \in \mathcal{L}(V)$: $Tv = Su + \tilde{S}w$. It's easy to verify that T is linear. And obviously, $Tu = Su$ for $\forall u \in U$. Consider $\forall v \in \operatorname{null} T$, $Tv = Su + \tilde{S}w = 0 \Rightarrow Su = -\tilde{S}w$, implying $Su, \tilde{S}w \in \operatorname{range} S \cap \tilde{W} = \{0\} \Rightarrow u \in \operatorname{null} S = \{0\}$, as S is injective; $w \in \operatorname{null} \tilde{S} = \{0\}$, as $\tilde{S}$ is isomorphism, hence injective $\Rightarrow u = w = 0$, $v = u + w = 0 \Rightarrow \operatorname{null} T = \{0\}$, implying T is injective. Then, by the result 3.65, T is invertible.

6 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S = \operatorname{null} T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

① if $\operatorname{null} S = \operatorname{null} T$:

Then $\operatorname{null} T \subseteq \operatorname{null} S$, and by the result of exercise 3B 25, there exists $\tilde{E} \in \mathcal{L}(W)$ such that $S = \tilde{E}T$. We restrict $\tilde{E}$ to the domain $\operatorname{range} T$, getting $E' \in \mathcal{L}(\operatorname{range} T, W)$: $E'w = \tilde{E}w$ for $w \in \operatorname{range} T$. Consider $\forall w \in \operatorname{null} E'$, as $\operatorname{null} E' \subseteq \operatorname{range} T$, we can write $w = Tv$ for some $v \in V$; $E'w = 0$, and $E'w = \tilde{E}w = \tilde{E}Tv = Sv \Rightarrow Sv = 0$, implying $v \in \operatorname{null} S \Rightarrow v \in \operatorname{null} T$, as $\operatorname{null} T = \operatorname{null} S \Rightarrow w = Tv = 0 \Rightarrow \operatorname{null} E' = \{0\}$, implying E' is injective. Then, by the result of exercise 3D 5, there exists an invertible $E \in \mathcal{L}(W)$ such that $Ew = E'w$ for $w \in \operatorname{range} T \Rightarrow ET = E'T = \tilde{E}T = S$.

② if $\exists E \in \mathcal{L}(W)$, E is invertible, $S = ET$:

1). $\forall v \in \operatorname{null} T$, $Sv = ETv = E0 = 0 \Rightarrow v \in \operatorname{null} S$, implying $\operatorname{null} T \subseteq \operatorname{null} S$.

2). $\forall v \in \operatorname{null} S$, $Tv = ITv = E^{-1}ETv = E^{-1}Sv = E^{-1}0 = 0 \Rightarrow v \in \operatorname{null} T$, implying $\operatorname{null} S \subseteq \operatorname{null} T$.

Hence, $\operatorname{null} S = \operatorname{null} T$.

7 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

① if $\operatorname{range} S = \operatorname{range} T$:

Then $\operatorname{range} S \subseteq \operatorname{range} T$; by the result of exercise 3B 26, there exists $\tilde{E} \in \mathcal{L}(V)$ such that $S = T\tilde{E}$. Let U be a subspace of V that satisfies $V = \operatorname{null} S \oplus U$. We restrict $\tilde{E}$ to the domain U, getting $E' \in \mathcal{L}(U, V)$: $E'u = \tilde{E}u$, $\forall u \in U$. As $\operatorname{range} S = \operatorname{range} T$ and $\dim V = \dim \operatorname{range} S + \dim \operatorname{null} S = \dim \operatorname{range} T + \dim \operatorname{null} T$, we can derive $\dim \operatorname{null} S = \dim \operatorname{null} T$. Hence, there exists an isomorphism $\bar{E} \in \mathcal{L}(\operatorname{null} S, \operatorname{null} T)$.

Define $E \in \mathcal{L}(V)$: $\forall v \in V$, we can write $v = u + w$ uniquely, where $u \in U$ and $w \in \text{null } S$,

$Ev = E'u + \bar{E}w$. It's easy to verify that E is linear. Then, $TEv = T(E'u + \bar{E}w)$

$= T\tilde{E}u + T(\bar{E}w) = Su + 0 = Su + Sw = S(u+w) = Sv \Rightarrow S = TE$.

Finally, let's show that E is invertible :

1). $\text{null } E' = \{0\}$: $\forall u \in \text{null } E'$, $Su = T\tilde{E}u = TE'u = T0 = 0 \Rightarrow u \in \text{null } S \cap U$

$= \{0\}$, hence $u = 0 \Rightarrow \text{null } E' = \{0\}$.

2). $\text{range } E' \cap \text{range } \bar{E} = \{0\}$: As $\bar{E}$ is an isomorphism, $\text{range } \bar{E} = \text{null } T$. $\forall v \in \text{range } E' \cap$

$\text{range } \bar{E}$, $\exists u \in U$, $v = E'u$ and $Tv = 0 \Rightarrow Su = T\tilde{E}u = TE'u = Tv = 0$, meaning

$u \in \text{null } S \Rightarrow u \in \text{null } S \cap U = \{0\} \Rightarrow u = 0$, hence $v = E'u = 0$, implying

$\text{range } E' \cap \text{range } \bar{E} = \{0\}$.

3). E is invertible : $\forall v \in \text{null } E$, $Ev = E'u + \bar{E}w = 0 \Rightarrow E'u = -\bar{E}w$, implying

$E'u, \tilde{E}w \in \text{range } E' \cap \text{range } \bar{E} \Rightarrow$ by 2), $E'u = \tilde{E}w = 0$. Then, by 1), $u = 0$;

as $\tilde{E}$ is an isomorphism, $w = 0$. Thus $v = u + w = 0$, implying $\text{null } E = \{0\} \Rightarrow$

$E \in \mathcal{L}(V)$ is injective $\Rightarrow$ E is invertible.

② if $\exists E \in \mathcal{L}(V)$, E is invertible, $S = TE$:

1). $\forall w \in \text{range } S$, $\exists v \in V$, $w = Sv = TEv = T(Ev) \in \text{range } T \Rightarrow \text{range } S \subseteq \text{range } T$.

2). $\forall w \in \text{range } T$, $\exists v \in V$, $w = Tv = TIv = TEE^{-1}v = SE^{-1}v = S(E^{-1}v) \in \text{range } S$

$\Rightarrow \text{range } T \subseteq \text{range } S$.

$\therefore \text{range } S = \text{range } T$.

8 Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \text{null } S = \dim \text{null } T$.

① if $\dim \text{null } S = \dim \text{null } T$:

Then, there exists an isomorphism $\tilde{E}_1 \in \mathcal{L}(\text{null } S, \text{null } T)$. As $\tilde{E}_1 \in \mathcal{L}(\text{null } S, V)$ is injective,

by the result of exercise 3D 5, there exists an invertible $E_1 \in \mathcal{L}(V)$ such that $E_1 u = \tilde{E}_1 u$

for $\forall u \in \text{null } S$. Now we show $\text{null } TE_1 = \text{null } S$: $\forall u \in \text{null } S$, $TE_1 u = T(E_1 u)$

$= T(\tilde{E}_1 u) = 0$ (because $\text{range } \tilde{E}_1 = \text{null } T$), implying $\text{null } S \subseteq \text{null } TE_1$; $\forall u \in \text{null } TE_1$,

$TE_1 u = T(E_1 u) = 0 \Rightarrow E_1 u \in \text{null } T = \text{range } \tilde{E}_1$, hence $\exists v \in \text{null } S$, $E_1 u = \tilde{E}_1 v = E_1 v$,

this implies $u = v$ because E_1 is invertible, thus $u \in \text{null } S \Rightarrow \text{null } TE_1 \subseteq \text{null } S$.

By the results of exercise 3D 6, as $\text{null } S = \text{null } TE_1$, there exists an invertible $E_2 \in \mathcal{L}(W)$

such that $S = E_2 T E_1$.

② if exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$, $S = E_2 T E_1$:

By the result of exercise 3B 23, $\dim \text{range } S = \dim \text{range } E_2 T E_1 \leq \min\{\dim \text{range } E_2,$

$\dim \text{range } T, \dim \text{range } E_1\} \leq \dim \text{range } T$. Similarly, consider $T = E_2^{-1} S E_1^{-1}$, we derive

$\dim \text{range } T \leq \dim \text{range } S$. Thus $\dim \text{range } T = \dim \text{range } S$. Also, we have

$\dim V = \dim \text{range } S + \dim \text{null } S = \dim \text{range } T + \dim \text{null } T$. $\therefore \dim \text{null } S = \dim \text{null } T$.

9 Suppose V is finite-dimensional and $T: V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.

As $T \in \mathcal{L}(V, W)$ is surjective, by the result of exercise 3B 20, $\exists S \in \mathcal{L}(W, V)$ such that $TS = I \in \mathcal{L}(W)$. Let $U = \text{range } S$. Now we show that:

① $T|_U$ is injective: $\forall u \in \text{null } T|_U$, as $\text{null } T|_U \subseteq U = \text{range } S$, $\exists w \in W$, $u = Sw \Rightarrow$

$Tu = TSw = Iw = w$ and $Tu = T|_U u = 0 \Rightarrow w = 0$, hence $u = Sw = 0 \Rightarrow$

$\text{null } T|_U = \{0\}$, implying $T|_U$ is injective.

② $T|_U$ is surjective: $\forall w \in W$, $w = Iw = TSw = T(Sw) = T|_U(Sw) \in \text{range } T|_U$,

implying $W = \text{range } T|_U$, hence $T|_U$ is surjective.

By ① and ②, $T|_U$ is an isomorphism of U onto W.

10 Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}.$$

(a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.
(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Hint: Define $\Phi: \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is $\text{null } \Phi$? What is $\text{range } \Phi$?

(a). ①. As $\text{null } 0 = V \supseteq U$, $0 \in \mathcal{E}$.

②. $\forall T_1, T_2 \in \mathcal{E}$, $\forall u \in U$, $T_1 u = T_2 u = 0 \Rightarrow (T_1 + T_2)u = T_1 u + T_2 u = 0 \Rightarrow$

$u \in \text{null}(T_1 + T_2)$, implying $U \subseteq \text{null}(T_1 + T_2) \Rightarrow T_1 + T_2 \in \mathcal{E}$.

③. $\forall T \in \mathcal{E}$, $\forall \lambda \in F$, $\forall u \in U$, $Tu = 0 \Rightarrow (\lambda T)u = \lambda 0 = 0 \Rightarrow$

$u \in \text{null } \lambda T$, implying $U \subseteq \text{null } \lambda T \Rightarrow \lambda T \in \mathcal{E}$.

(b). Define $\Phi: \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. It's easy to verify that $\Phi \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$. Now we show that:

①. $\text{null } \Phi = \mathcal{E}$: 1). $\forall T \in \mathcal{E}$, $\text{null } T \supseteq U \Rightarrow \forall u \in U$, $T|_U u = Tu = 0 \Rightarrow$

$\Phi(T) = T|_U = 0 \in \mathcal{L}(U, W)$, implying $T \in \text{null } \Phi \Rightarrow \mathcal{E} \subseteq \text{null } \Phi$.

2). $\forall T \in \text{null } \Phi$, $\Phi(T) = T|_U = 0 \in \mathcal{L}(U, W) \Rightarrow \forall u \in U$, $Tu = T|_U u$

$= 0(u) = 0$, implying $u \in \text{null } T \Rightarrow U \subseteq \text{null } T$, implying $T \in \mathcal{E} \Rightarrow \text{null } \Phi \subseteq \mathcal{E}$.

②. Φ is surjective: $\forall S \in \mathcal{L}(U, W)$, as U is a subspace of V, $\exists T \in \mathcal{L}(V, W)$, $Tu = Su$ for $\forall u \in U \Rightarrow \Phi(T) = T|_U = S$.

By ①, $\dim \mathcal{E} = \dim \text{null } \Phi$. By ②, $\dim \text{range } \Phi = \dim \mathcal{L}(U, W) = \dim U \dim W$.

Additionally, we have $\dim \mathcal{L}(V, W) = \dim \text{null } \Phi + \dim \text{range } \Phi$ and $\dim \mathcal{L}(V, W) = \dim V \dim W$. $\therefore \dim \mathcal{E} = \dim W (\dim V - \dim U)$.

11 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

① if ST is invertible:

1). S is invertible: Otherwise, as $S \in \mathcal{L}(V)$, S is not surjective. Then $\exists u \in V$ $\forall v \in V$, $Sv \neq u \Rightarrow \forall v \in V$, $Tv \in V$, hence $STv = S(Tv) \neq u \Rightarrow$

ST is not surjective, which contradicts that ST is invertible.

2). T is invertible: Otherwise, as $T \in \mathcal{L}(V)$, T is not injective $\Rightarrow \exists v \in V$, $v \neq 0$, $Tv = 0 \Rightarrow STv = S(Tv) = S0 = 0$, where $v \neq 0$. This implies that ST is not injective either, a contradiction to that ST is invertible.

② if S and T are invertible:

$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$, $(T^{-1}S^{-1})(ST) = I$ similarly.

Hence, ST is also invertible, and $(ST)^{-1} = T^{-1}S^{-1}$.

12 Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

$S(TU) = STU = I$ is invertible, then by the result of exercise 3D 11, S and TU are invertible; and further, T and U are invertible. $STU = I \Rightarrow TU = S^{-1} \Rightarrow U = T^{-1}S^{-1} \Rightarrow US = T^{-1}$.

13 Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

Let $V = \mathbf{F}^\infty$, which is infinite-dimensional. Let T be the backward shift linear map: $T((x_1, x_2, \cdots)) = (x_2, \cdots)$. T is not injective, hence is not invertible. However, let $S = I \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ and define U by $U((x_1, x_2, \cdots)) = (0, x_1, x_2, \cdots)$, we have $STU = I$.

14 Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

Suppose RST is surjective, implying $\dim \operatorname{range} RST = \dim V$. If S is not injective, meaning $\dim \operatorname{null} S > 0$, implying that $\dim \operatorname{range} S < \dim V$. However, by the result of exercise 3B 23, $\dim \operatorname{range} RST \leq \min\{\dim \operatorname{range} R, \dim \operatorname{range} S, \dim \operatorname{range} T\} \leq \dim \operatorname{range} S \Rightarrow \dim V < \dim V$, which is impossible. Hence S must be injective.

15 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_m$ is a list in V such that $Tv_1, \dots, Tv_m$ spans V. Prove that $v_1, \dots, v_m$ spans V.

①. $\dim \operatorname{span}(Tv_1, \cdots, Tv_m) = \dim V$: Because $Tv_1, \cdots, Tv_m$ spans V and $Tv_1, \cdots, Tv_m \in V$.

②. $\dim \operatorname{span}(v_1, \cdots, v_m) \geq \dim \operatorname{span}(Tv_1, \cdots, Tv_m)$:

Reduce $Tv_1, \cdots, Tv_m$ to a basis of $\operatorname{span}(Tv_1, \cdots, Tv_m)$: $\{Tv_j\}$, $j \in J \subseteq \{1, \cdots, m\}$, and suppose $|J| = n \leq m$. Hence $\{Tv_j\}_{j \in J}$ is linearly independent. Then, by the result of exercise 3A 4, $\{v_j\}_{j \in J}$ is linearly independent. Thus, $\dim \operatorname{span}(v_1, \cdots, v_m) \geq |J| = n = \dim \operatorname{span}(Tv_1, \cdots, Tv_m)$.

By ① and ②, $\dim \operatorname{span}(v_1, \cdots, v_m) \geq \dim V$, implying $v_1, \cdots, v_m$ spans V.

16 Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbf{F}^{n,1}$.

Take the standard bases of $\mathbf{F}^{n,1}$ and $\mathbf{F}^{m,1}$, then the corresponding $\mathcal{M}(T)$ is the A that satisfies $Tx = Ax$ for $\forall x \in \mathbf{F}^{n,1}$, as you should verify.

17 Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.

(a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

(a). We show $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$: 1). $\forall T \in \operatorname{null} \mathcal{A}$, $\mathcal{A}(T) = ST = 0 \in \mathcal{L}(V) \Rightarrow$ $\forall v \in V$, $STv = 0$, implying $Tv \in \operatorname{null} S \Rightarrow \operatorname{range} T \subseteq \operatorname{null} S$, i.e., $T \in \mathcal{L}(V, \operatorname{null} S)$. $\Rightarrow \operatorname{null} \mathcal{A} \subseteq \mathcal{L}(V, \operatorname{null} S)$. 2). $\forall T \in \mathcal{L}(V, \operatorname{null} S)$, $\operatorname{range} T \subseteq \operatorname{null} S \Rightarrow \forall v \in V$, $STv = S(Tv) = 0$, implying $\mathcal{A}(T) = ST = 0 \in \mathcal{L}(V) \Rightarrow T \in \operatorname{null} \mathcal{A} \Rightarrow \mathcal{L}(V, \operatorname{null} S) \subseteq \operatorname{null} \mathcal{A}$.

Hence, $\dim \operatorname{null} \mathcal{A} = \dim \mathcal{L}(V, \operatorname{null} S) = (\dim V)(\dim \operatorname{null} S)$.

(b). By (a) and $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$ and $\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$, we derive $\dim \operatorname{range} \mathcal{A} = \dim \mathcal{L}(V) - \dim \operatorname{null} \mathcal{A} = (\dim V)^2 - (\dim V)(\dim \operatorname{null} S) = (\dim V)(\dim V - \dim \operatorname{null} S) = (\dim V)(\dim \operatorname{range} S)$.

18 Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

For $\forall v \in V$, define $T^v \in \mathcal{L}(F, V)$: $\forall \lambda \in F$, $T^v(\lambda) = \lambda v$. It's easy to verify that T^v is linear. Then, define $\mathcal{A} : V \to \mathcal{L}(F, V)$ by $\mathcal{A}(v) = T^v$ for $\forall v \in V$.

①. $\mathcal{A} \in \mathcal{L}(V, \mathcal{L}(F, V))$:

1). $\forall v_1, v_2 \in V$, $\mathcal{A}(v_1+v_2) = T^{v_1+v_2}$; $\forall \lambda \in F$, $T^{v_1+v_2}(\lambda) = \lambda(v_1+v_2) = \lambda v_1 + \lambda v_2$ and $(T^{v_1}+T^{v_2})(\lambda) = T^{v_1}(\lambda) + T^{v_2}(\lambda) = \lambda v_1 + \lambda v_2$, implying $T^{v_1+v_2} = T^{v_1} + T^{v_2}$; hence, $\mathcal{A}(v_1+v_2) = T^{v_1+v_2} = T^{v_1} + T^{v_2} = \mathcal{A}(v_1) + \mathcal{A}(v_2)$

2). $\forall v \in V$, $\forall a \in F$, $\mathcal{A}(av) = T^{av}$; $\forall \lambda \in F$, $T^{av}(\lambda) = \lambda(av) = a(\lambda v) = aT^v(\lambda) = (aT^v)(\lambda)$, implying $T^{av} = aT^v$; hence, $\mathcal{A}(av) = T^{av} = aT^v = a\mathcal{A}(v)$.

②. $\mathcal{A}$ is an isomorphism :

1). $\mathcal{A}$ is injective : $\forall v \in \operatorname{null} \mathcal{A}$, $\mathcal{A}(v) = T^v = 0 \in \mathcal{L}(F, V) \Rightarrow \forall \lambda \in F$, $T^v(\lambda) = \lambda v \equiv 0$, implying $v = 0 \Rightarrow \operatorname{null} \mathcal{A} = \{0\}$, hence $\mathcal{A}$ is injective.

2). $\mathcal{A}$ is surjective : $\forall S \in \mathcal{L}(F, V)$, let $S(1) = v \in V$. Then, $\forall \lambda \in F$, $S(\lambda) = \lambda S(1) = \lambda v = T^v(\lambda)$, implying $S = T^v = \mathcal{A}(v)$. Hence $\mathcal{A}$ is surjective.

By ① and ②, we know that V and $\mathcal{L}(F, V)$ are isomorphic.

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

① if T has the same matrix with respect to every basis of V :

Let $v_1, \cdots, v_n$ be a basis of V and $A = \mathcal{M}(T, (v_1, \cdots, v_n))$.

1). A is diagonal : Otherwise, $\exists A_{i,j} \neq 0$, $i \neq j$. Replace v_i with $2v_i$ in the list $v_1, \cdots, v_n$, getting a new basis of V ; we denote the new $\mathcal{M}(T)$ as $\tilde{A}$. Then, $Tv_j = A_{1,j}v_1 + \cdots + A_{i,j}v_i + \cdots + A_{n,j}v_j = \tilde{A}_{1,j}v_1 + \cdots + \tilde{A}_{i,j}(2v_i) + \cdots + \tilde{A}_{n,j}v_j \Rightarrow$ $(\tilde{A}_{1,j} - A_{1,j})v_1 + \cdots + (2\tilde{A}_{i,j} - A_{i,j})v_i + \cdots + (\tilde{A}_{n,j} - A_{n,j})v_n = 0$; by our hypothesis, $A = \tilde{A} \Rightarrow A_{k,j} = \tilde{A}_{k,j}$, $k = 1, \cdots, n$, implying $A_{i,j}v_i = 0 \Rightarrow A_{i,j} = 0$, a contradiction. Hence, A must be diagonal.

2). A's entries on the diagonal are the same: Otherwise, $\exists\, i \neq j$, $A_{i,i} \neq A_{j,j}$.

Swap v_i and v_j in the list $v_1, \cdots, v_n$, getting a new basis of V; we denote the new $\mathcal{M}(T)$ as $\tilde{A}$. By 1), we know A is diagonal, hence $Tv_j = A_{j,j} v_j$;

by our hypothesis, $\tilde{A} = A$, hence $Tv_j = \tilde{A}_{i,i} v_j = A_{i,i} v_j \Rightarrow A_{i,i} = A_{j,j}$,

a contradiction. Hence, A's entries on the diagonal are the same.

From above, we show that $\mathcal{M}(T) = A = \lambda I$, where $\lambda \in F$. This implies that T is a scalar multiple of the identity operator.

②. if $\exists \lambda \in F$, $T = \lambda I$:

$\mathcal{M}(T) = \mathcal{M}(\lambda I) = \lambda \mathcal{M}(I)$. For any basis of V, the $\mathcal{M}(I)$ is the same, i.e., the identity matrix. Hence, $\mathcal{M}(T) = \lambda \mathcal{M}(I)$ is the same with respect to every basis of V.

20 Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$.

Suppose $\deg q = m$. Define $T: \mathcal{P}_m(\mathbf{R}) \to \mathcal{P}_m(\mathbf{R})$ by $T(p(x)) = (x^2+x)p''(x) + 2xp'(x) + p(3)$ for $\forall p(x) \in \mathcal{P}_m(\mathbf{R})$. It's easy to show that $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{R}))$. It's easy to verify that $T(p(x))$ has a degree no less than $p(x)$, hence $\forall p(x) \not\equiv 0$, $T(p(x)) \not\equiv 0 \Rightarrow \operatorname{null} T = \{0\}$, implying $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{R}))$ is injective, then surjective. Hence, $\exists p(x) \in \mathcal{P}_m(\mathbf{R})$, $q(x) = T(p(x))$.

21 Suppose n is a positive integer and $A_{j,k} \in \mathbf{F}$ for all $j, k = 1, \ldots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables).

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = 0.$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = c_n.$$

Let $A \in F^{n,n}$ such that the entry in row j, column k equals $A_{j,k}$.

Let $x = (x_1 \cdots x_n)^t \in F^{n,1}$, $c = (c_1 \cdots c_n)^t \in F^{n,1}$.

We rephrase the conditions in matrix form:

(a): $x = 0$ is the only solution to $Ax = 0$ (b): $\forall c$, $\exists x$, $Ax = c$.

Define $T \in \mathcal{L}(F^{n,1})$: $T(x) = Ax$, $\forall x \in F^{n,1}$. It's easy to verify that T is linear.

Then, (a) actually means $\operatorname{null} T = \{0\} \Rightarrow T$ is injective, and (b) actually means $\operatorname{range} T = F^{n,1} \Rightarrow T$ is surjective. As T maps $F^{n,1}$ to itself, (a) and (b) are equivalent.

22 Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$

① if $\mathcal{M}(T)$ is invertible:

Let $A = (\mathcal{M}(T))^{-1} \in F^{n,n}$. As $\mathcal{M}$ is an isomorphism from $\mathcal{L}(V)$ to $F^{n,n}$, $\exists S \in \mathcal{L}(V)$, $A = \mathcal{M}(S)$. Then, $\forall v \in V$, $\mathcal{M}(v) = I\mathcal{M}(v) = \mathcal{M}(T)\mathcal{M}(S)\mathcal{M}(v) = \mathcal{M}(TSv) \Rightarrow$

$v = TSv$ (because $\mathcal{M}$ is injective) for $\forall v \in V \Rightarrow TS = I$. Then, by result 3.68, $ST = I$. Hence, T is invertible and $T^{-1} = S \Rightarrow \mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$.

② if T is invertible:

$\mathcal{M}(T^{-1})\mathcal{M}(T) = \mathcal{M}(T^{-1}T) = \mathcal{M}(I) = I$, and $\mathcal{M}(T)\mathcal{M}(T^{-1}) = I$ similarly.

Hence, $\mathcal{M}(T)$ is invertible and $(\mathcal{M}(T))^{-1} = \mathcal{M}(T^{-1})$.

23 Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

Because $Tv_k = u_k$ for $k = 1, \cdots, n$, $\mathcal{M}(T, (v_1, \cdots, v_n), (u_1, \cdots, u_n)) = I$, the identity matrix.

Then, $\mathcal{M}(T, (v_1, \cdots, v_n)) = \mathcal{M}(I, (u_1 \cdots, u_n), (v_1, \cdots, v_n))\mathcal{M}(T, (v_1, \cdots, v_n), (u_1, \cdots, u_n)) =$

$\mathcal{M}(I, (u_1 \cdots, u_n), (v_1, \cdots, v_n)) I = \mathcal{M}(I, (u_1, \cdots, u_n), (v_1, \cdots, v_n))$.

24 Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.

Suppose $A, B \in \mathbf{F}^{n,n}$. Let V be a vector space with $\dim V = n$, then $\mathbf{F}^{n,n}$ is isomorphic with $\mathcal{L}(V)$. Fix a basis of V, then the correspond $\mathcal{M} \in \mathcal{L}(\mathcal{L}(V), \mathbf{F}^{n,n})$ is an isomorphism.

Then, $\exists S, T \in \mathcal{L}(V)$, $A = \mathcal{M}(S)$, $B = \mathcal{M}(T)$, and $I = \mathcal{M}(I)$.

$AB = I \Rightarrow \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(ST) = \mathcal{M}(I)$, implying $ST = I$ (because $\mathcal{M}$ is injective).

By result 3.68, $TS = I \Rightarrow BA = \mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(TS) = \mathcal{M}(I) = I$.