Solutions to Exercises 3F :

1 Explain why each linear functional is surjective or is the zero map.

Let $\varphi \in V' = \mathcal{L}(V, F)$. $\text{range}\,\varphi \subseteq F \Rightarrow \dim \text{range}\,\varphi \le \dim F = 1 \Rightarrow$

Either $\dim \text{range}\,\varphi = 0$, hence $\varphi = 0$; or $\dim \text{range}\,\varphi = 1$, hence $\text{range}\,\varphi = F$, surjective.

2 Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

$\forall f \in \mathbf{R}^{[0,1]}$, define $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{L}(\mathbf{R}^{[0,1]}, \mathbf{R})$ by $\varphi_1(f) = f(0)$, $\varphi_2(f) = f(\frac{1}{2})$, $\varphi_3(f) = f(1)$.

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Let $V = \text{span}(v) \oplus W$. $\forall u \in V$, we can write $u = av + w$ uniquely, where $a \in F$ and $w \in W$;

define $\varphi \in V'$ by $\varphi(u) = \varphi(av + w) = a$. It's easy to verify that φ is linear and $\varphi(v) = 1$.

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

Let $V = U \oplus W$. $U \neq V$, hence $W \neq \{0\} \Rightarrow \exists \psi \in W'$, $\psi \neq 0$. $\forall v \in V$, write $v = u + w$

uniquely, where $u \in U$ and $w \in W$; define $\varphi \in V'$ by $\varphi(v) = \varphi(u+w) = \psi(w)$. It's easy to

verify that φ is indeed linear and $\varphi(u) = 0$ for $\forall u \in U$ but $\varphi \neq 0$.

5 Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \ldots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \ldots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \ldots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \ldots, \varphi_m$ is a linear functional on V.

① $\forall v_1, v_2 \in V$, $Tv_1 + Tv_2 = T(v_1 + v_2) \Rightarrow (\varphi_1(v_1)w_1 + \cdots + \varphi_m(v_1)w_m) + (\varphi_1(v_2)w_1 + \cdots + \varphi_m(v_2)w_m)$

$= \varphi_1(v_1+v_2)w_1 + \cdots + \varphi_m(v_1+v_2)w_m \Rightarrow \sum_{k=1}^{m} (\varphi_k(v_1+v_2) - (\varphi_k(v_1) + \varphi_k(v_2)))w_k = 0$.

As $w_1, \cdots, w_m$ is a basis, this implies $\varphi_k(v_1+v_2) = \varphi_k(v_1) + \varphi_k(v_2)$ for $k = 1, \cdots, m$.

② $\forall v \in V$, $\forall \lambda \in F$, $T(\lambda v) = \lambda Tv \Rightarrow \varphi_1(\lambda v)w_1 + \cdots + \varphi_m(\lambda v)w_m = \lambda\varphi_1(v)w_1 + \cdots + \lambda\varphi_m(v)w_m$

$\Rightarrow \sum_{k=1}^{m} (\lambda\varphi_k(v) - \varphi_k(\lambda v))w_k = 0$. As $w_1, \cdots, w_m$ is a basis, this implies $\varphi_k(\lambda v) = \lambda\varphi_k(v)$, $k = 1, \cdots, m$.

6 Suppose $\varphi, \beta \in V'$. Prove that null $\varphi \subseteq$ null β if and only if there exists $c \in \mathbf{F}$ such that $\beta = c\varphi$.

① if null $\varphi \subseteq$ null β :

By the result of Exercises 3B 25, $\exists T \in \mathcal{L}(F)$, $\beta = T \circ \varphi$; i.e., $\exists c \in F$, $\beta = c\varphi$.

② if $\exists c \in F$, $\beta = c\varphi$:

$\forall v \in \text{null}\,\varphi$, $\beta v = c\varphi v = c \cdot 0 = 0 \Rightarrow v \in \text{null}\,\beta \Rightarrow \text{null}\,\varphi \subseteq \text{null}\,\beta$.

7 Suppose that $V_1, \ldots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

By the result of Exercises 3E 3, $\mathcal{L}(V_1 \times \cdots \times V_m, F)$ and $\mathcal{L}(V_1, F) \times \cdots \times \mathcal{L}(V_m, F)$ are isomorphic,

which gives the result.

8 Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis of V'. Define $\Gamma: V \to \mathbf{F}^n$ and $\Lambda: \mathbf{F}^n \to V$ by

$$\Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)) \quad \text{and} \quad \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

Explain why Γ and Λ are inverses of each other.

① . $\forall (a_1, \cdots, a_n) \in \mathbf{F}^n$, $(\Gamma\Lambda)((a_1,\cdots,a_n)) = \Gamma(a_1v_1 + \cdots + a_nv_n) = (\varphi_1(a_1v_1+\cdots+a_nv_n), \cdots, \varphi_n(a_1v_1+\cdots+a_nv_n))$

$= (a_1, \cdots, a_n) \Rightarrow \Gamma\Lambda = I \in \mathcal{L}(\mathbf{F}^n)$.

② . $\forall v \in V$, $(\Lambda\Gamma)(v) = \Lambda((\varphi_1(v), \cdots, \varphi_n(v))) = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n = v \Rightarrow \Lambda\Gamma = I \in \mathcal{L}(V)$.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

If $j = k \in \{0, 1, \cdots, m\}$, $(x^j)^{(k)}|_{x=0} = (x^k)^{(k)}|_{x=0} = k!|_{x=0} = k!$, hence $\varphi_k(x^j) = \frac{k!}{k!} = 1$.

If $j > k \in \{0, 1, \cdots, m\}$, $(x^j)^{(k)}|_{x=0} = \frac{j!}{(j-k)!} x^{j-k}|_{x=0} = \frac{j!}{(j-k)!} \cdot 0 = 0$, hence $\varphi_k(x^j) = \frac{0}{k!} = 0$

If $j < k \in \{0, 1, \cdots, m\}$, $(x^j)^{(k)}|_{x=0} = 0|_{x=0} = 0$, hence $\varphi_k(x^j) = \frac{0}{k!} = 0$.

Thus, $\varphi_k(x^j) = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}$, implying that $\varphi_0, \varphi_1, \cdots, \varphi_m$ is the dual basis of $1, x, \cdots, x^m$.

11 Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

For $k = 1, \cdots, n$, $(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_k) = \psi(v_1)\varphi_1(v_k) + \cdots + \psi(v_n)\varphi_n(v_k) = \psi(v_k) \cdot 1 = \psi(v_k)$.

As $v_1, \cdots, v_n$ is a basis, this implies $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$.

12 Suppose $S, T \in \mathcal{L}(V, W)$.

(a) Prove that $(S + T)' = S' + T'$.

(b) Prove that $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$.

This exercise asks you to verify (a) and (b) in 3.120.

(a). $\forall \varphi \in W'$, $(S+T)'(\varphi) = \varphi(S+T) = \varphi S + \varphi T = S'(\varphi) + T'(\varphi) = (S'+T')(\varphi) \Rightarrow (S+T)' = S' + T'$.

(b). $\forall \varphi \in W'$, $(\lambda T)'(\varphi) = \varphi(\lambda T) = \lambda(\varphi T) = \lambda T'(\varphi) = (\lambda T')(\varphi) \Rightarrow (\lambda T)' = \lambda T'$.

13 Show that the dual map of the identity operator on V is the identity operator on V'.

$\forall \varphi \in V'$, $I'(\varphi) = \varphi I = \varphi \Rightarrow I'$ is the identity operator on V'.

15 Define $T: \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by

$$(Tp)(x) = x^2 p(x) + p''(x)$$

for each $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $(T'(\varphi))(x^3)$.

(a). $\forall p \in \mathcal{P}(\mathbf{R})$, $(T'(\varphi))(p) = \varphi T(p) = \varphi(x^2p(x) + p''(x)) = (x^2p(x) + p''(x))'|_{x=4} = 2xp(x) +$

$x^2p'(x) + p^{(3)}(x)|_{x=4} = 8p(4) + 16p'(4) + p^{(3)}(4)$.

(b). $(T'(\varphi))(x^3) = \varphi T(x^3) = \varphi(x^5 + 6x) = \int_0^1 (x^5 + 6x)\,dx = \frac{1}{6}x^6 + 3x^2 \Big|_0^1 = \frac{19}{6}$.

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

$T' = 0 \Rightarrow \dim \operatorname{range} T = \dim \operatorname{range} T' = 0 \Rightarrow T = 0$. $T = 0 \Rightarrow T' = 0$, similarly.

17 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

T is invertible $\Leftrightarrow$ T is injective ; T is surjective $\Leftrightarrow$ T' is surjective ; T' is injective $\Leftrightarrow$ T' is invertible.

18 Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

Define $\Phi : \mathcal{L}(V, W) \to \mathcal{L}(W', V')$ by $\Phi(T) = T'$. Φ is linear, as you should verify.

If $\Phi(T) = T' = 0 \Rightarrow$ dim range T = dim range $T' = 0 \Rightarrow T = 0$, implying null $\Phi = \{0\} \Rightarrow$

Φ is injective. Then, as $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V) = (\dim W)(\dim V)$, Φ is invertible.

$\therefore$ Φ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

19 Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

$\varphi(u) = 0$ for all $u \in U$ $\Leftrightarrow$ $U \subseteq \operatorname{null} \varphi$

20 Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

①. $\forall u \in U$, for $\forall \varphi \in U^0$, $\varphi(u) = 0$ $\Rightarrow$ $U \subseteq \{v \in V : \varphi(v) = 0 \text{ for } \forall \varphi \in U^0\}$.

②. As U is a subspace, $\exists \psi \in V'$, null $\psi = U$ $\Rightarrow$ $\psi \in U^0$. Then, for $\forall v \notin U$, $\psi v \neq 0$,

implying $v \notin \{v \in V : \varphi(v) = 0 \text{ for } \forall \varphi \in U^0\}$. This implies $\{v \in V : \varphi(v) = 0 \text{ for } \forall \varphi \in U^0\} \subseteq U$.

21 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.

(a). ①. If $U \subseteq W$: $\forall \varphi \in W^0$, null $\varphi \supseteq W \supseteq U$, implying $\varphi \in U^0$ $\Rightarrow$ $W^0 \subseteq U^0$.

②. If $W^0 \subseteq U^0$: As W is a subspace, $\exists \psi \in V'$, null $\psi = W$. Then, $\psi \in W^0 \Rightarrow \psi \in U^0$,

meaning null $\psi \supseteq U$ $\Rightarrow$ $U \subseteq W$.

(b). $W^0 = U^0 \Leftrightarrow W^0 \subseteq U^0$ and $U^0 \subseteq W^0 \overset{\text{by (a)}}{\Longleftrightarrow} U \subseteq W$ and $W \subseteq U \Leftrightarrow U = W$.

22 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Show that $(U + W)^0 = U^0 \cap W^0$.

(b) Show that $(U \cap W)^0 = U^0 + W^0$.

(a). ①. $\forall \varphi \in (U+W)^0$, $\forall u \in U$, as $u = u + 0 \in U + W$, $\varphi(u) = 0$, implying $\varphi \in U^0$; $\varphi \in W^0$, similarly.

Hence, $\varphi \in U^0 \cap W^0 \Rightarrow (U+W)^0 \subseteq U^0 \cap W^0$.

②. $\forall \varphi \in U^0 \cap W^0$, $\forall u + w \in U + W$, $\varphi(u+w) = \varphi(u) + \varphi(w) = 0 + 0 = 0$, implying $\varphi \in (U+W)^0$

$\Rightarrow U^0 \cap W^0 \subseteq (U+W)^0$.

(b). ①. Suppose $U = (U \cap W) \oplus \bar{U}$ and $W = (U \cap W) \oplus \bar{W}$. Thus $U + W = (U \cap W) \oplus \bar{U} \oplus \bar{W}$, as you should verify. Suppose $V = (U+W) \oplus \bar{V} = (U \cap W) \oplus \bar{U} \oplus \bar{W} \oplus \bar{V}$. $\forall v \in V$, we can write

$v = x + \bar{u} + \bar{w} + \bar{v}$ uniquely, where $x \in U \cap W$, $\bar{u} \in \bar{U}$, $\bar{w} \in \bar{W}$, $\bar{v} \in \bar{V}$; for $\forall \varphi \in (U \cap W)^0$,

define $\psi_1 \in U^0$ by $\psi_1(v) = \frac{1}{2}\varphi(\bar{v}) + \varphi(\bar{w})$, define $\psi_2 \in W^0$ by $\psi_2(v) = \frac{1}{2}\varphi(\bar{v}) + \varphi(\bar{u})$.

ψ_1, ψ_2 are indeed linear and $\varphi = \psi_1 + \psi_2$, as you should verify. Hence, $\varphi = \psi_1 + \psi_2 \in U^0 + W^0$,

implying $(U \cap W)^0 \subseteq U^0 + W^0$.

②. $\forall \psi_1+\psi_2 \in U^0+W^0$, where $\psi_1 \in U^0$ and $\psi_2 \in W^0$, $\forall v \in U\cap W$, $(\psi_1+\psi_2)(v) = \psi_1(v)+\psi_2(v) = 0+0=0$, implying $\psi_1+\psi_2 \in (U\cap W)^0 \Rightarrow U^0+W^0 \subseteq (U\cap W)^0$.

23 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.
(a) $\operatorname{span}(\varphi_1, \ldots, \varphi_m)$
(b) $((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m))^0$
(c) $\{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

(a) = (b):

By the result of Exercises 3F 22, $((\operatorname{null}\varphi_1)\cap\cdots\cap(\operatorname{null}\varphi_m))^0 = (\operatorname{null}\varphi_1)^0+\cdots+(\operatorname{null}\varphi_m)^0$. By the result of Exercises 3F 6, $\exists c \in F$, $\beta = c\varphi_k \in \operatorname{span}(\varphi_k) \Leftrightarrow \operatorname{null}\beta \supseteq \operatorname{null}\varphi_k \Leftrightarrow \beta \in (\operatorname{null}\varphi_k)^0$, implying $\operatorname{span}(\varphi_k) = (\operatorname{null}\varphi_k)^0$. Hence, $((\operatorname{null}\varphi_1)\cap\cdots\cap(\operatorname{null}\varphi_m))^0 = \operatorname{span}(\varphi_1)+\cdots+\operatorname{span}(\varphi_m) = \operatorname{span}(\varphi_1,\cdots,\varphi_m)$.

(b) = (c):

By the result of Exercises 3F 19.

24 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma : V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$.
(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.
(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

(a). ①. If $v_1,\cdots,v_m$ spans V: $\forall v\in V$, $\exists a_1,\cdots,a_m \in F$, $v = a_1v_1+\cdots+a_mv_m$. Consider $\forall \varphi \in \operatorname{null}\Gamma$, $\varphi(v) = (\varphi(v_1),\cdots,\varphi(v_m)) = 0 \Rightarrow \varphi(v_1) = \cdots = \varphi(v_m) = 0 \Rightarrow \varphi(v) = \varphi(a_1v_1+\cdots+a_mv_m) = a_1\varphi(v_1)+\cdots+a_m\varphi(v_m) = 0$ for $\forall v\in V$, implying $\varphi = 0 \Rightarrow \operatorname{null}\Gamma = \{0\}$. Thus Γ is injective.

②. If Γ is injective: Assume that $v_1,\cdots,v_m$ doesn't span V. Hence, there exists a subspace W, $\dim W > 0$, $V = \operatorname{span}(v_1,\cdots,v_m) \oplus W$. As $\dim W>0$, $\exists \psi \in W'$, $\psi \neq 0$ For $\forall v \in V$, we can write $v = \bar{v}+w$ uniquely, where $\bar{v} \in \operatorname{span}(v_1,\cdots,v_m)$ and $w \in W$; define $\varphi \in V'$ by $\varphi(v) = \psi(w)$ Then, $\varphi(v_k) = \psi(0) = 0$ for $k = 1,\cdots,m$, implying $\Gamma(\varphi) = 0$. However, $\varphi \neq 0$, contradicting the hypothesis that Γ is injective. Hence, only that $v_1,\cdots,v_m$ spans V.

(b). ①. If $v_1,\cdots,v_m$ is linearly independent: Then $v_1,\cdots,v_m$ is a basis of $\operatorname{span}(v_1,\cdots,v_m)$. $\forall (c_1,\cdots,c_m) \in F^m$, by linear map lemma, $\exists \psi \in \mathcal{L}(\operatorname{span}(v_1,\cdots,v_m), F)$, $\psi(v_k) = c_k$ for $k = 1,\cdots,m$. As $\operatorname{span}(v_1,\cdots,v_m)$ is a subspace of V, ψ can be extended to $\varphi \in V'$ such that $\varphi(v) = \psi(v)$ for $\forall v \in \operatorname{span}(v_1,\cdots,v_m)$. Then, $\Gamma(\varphi) = (\varphi(v_1),\cdots,\varphi(v_m)) = (\psi(v_1),\cdots,\psi(v_m)) = (c_1,\cdots,c_m)$, implying that Γ is surjective.

②. If Γ is surjective: Assume that $v_1,\cdots,v_m$ is not linearly independent. Hence, $\exists a_1,\cdots,a_m \in F$, not all 0, $0 = a_1v_1+\cdots+a_mv_m$. Define $\phi \in (F^m)'$ by $\phi(c_1,\cdots,c_m) = a_1c_1+\cdots+a_mc_m$ for $\forall (c_1,\cdots,c_m) \in F^m$. Then, consider $(\Gamma'(\phi))(\varphi) = \phi\Gamma(\varphi) = \phi(\varphi(v_1),\cdots,\varphi(v_m)) = a_1\varphi(v_1)+\cdots+a_m\varphi(v_m) = \varphi(a_1v_1+\cdots+a_mv_m) = \varphi(0) = 0$ for $\forall \varphi \in V'$, implying $\phi \in \operatorname{null}\Gamma'$. Also, $\phi \neq 0$ as $a_1,\cdots,a_m$ not all 0, implying $\operatorname{null}\Gamma' \neq \{0\} \Rightarrow \Gamma'$ is not injective $\Rightarrow \Gamma$ is not surjective, a contradiction to our hypothesis. Hence, only that $v_1,\cdots,v_m$ is linearly independent.

25 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m \in V'$. Define a linear map $\Gamma : V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v))$.
(a) Prove that $\varphi_1, \ldots, \varphi_m$ spans V' if and only if Γ is injective.
(b) Prove that $\varphi_1, \ldots, \varphi_m$ is linearly independent if and only if Γ is surjective.

Consider $\Gamma' : (F^m)' \to V'$. For $\forall \phi \in (F^m)'$, let $\mathcal{M}(\phi)$ denote the matrix of ϕ that corresponds to the standard bases of F^m and F, thus $\phi(c_1,\cdots,c_m) = \mathcal{M}(\phi)_{1,1}c_1+\cdots+\mathcal{M}(\phi)_{1,m}c_m$ for $\forall (c_1,\cdots,c_m) \in F^m$.

$(\Gamma'(\phi))(v) = \phi\Gamma(v) = \phi(\varphi_1(v), \cdots, \varphi_m(v)) = \mathcal{M}(\phi)_{1,1}\varphi_1(v) + \cdots + \mathcal{M}(\phi)_{1,m}\varphi_m(v) = (\mathcal{M}(\phi)_{1,1}\varphi_1 + \cdots + \mathcal{M}(\phi)_{1,m}\varphi_m)(v)$, meaning $\Gamma'(\phi) = \mathcal{M}(\phi)_{1,1}\varphi_1 + \cdots + \mathcal{M}(\phi)_{1,m}\varphi_m \in V'$ for $\forall \phi \in (F^m)'$.

(a). $\varphi_1, \cdots, \varphi_m$ spans $V' \iff \forall \psi \in V', \exists a_1, \cdots, a_m \in F, \psi = a_1\varphi_1 + \cdots + a_m\varphi_m \iff \forall \psi \in V',$
$\exists \phi \in (F^m)', \psi = \mathcal{M}(\phi)_{1,1}\varphi_1 + \cdots + \mathcal{M}(\phi)_{1,m}\varphi_m = \Gamma'(\phi) \iff \Gamma'$ is surjective $\iff \Gamma$ is injective.

(b). $\varphi_1, \cdots, \varphi_m$ is linearly independent $\iff$ the only solution to $0 = a_1\varphi_1 + \cdots + a_m\varphi_m$ is $a_1 = \cdots = a_m = 0$
$\iff$ the only $\phi \in (F^m)'$ that $0 = \mathcal{M}(\phi)_{1,1}\varphi_1 + \cdots + \mathcal{M}(\phi)_{1,m}\varphi_m$ is $\phi = 0 \iff \operatorname{null}\Gamma' = \{0\}$
$\iff \Gamma'$ is injective $\iff \Gamma$ is surjective.

26 Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0.$$

Denote $W = \{v \in V : \varphi(v) = 0$ for $\forall \varphi \in \Omega\}$. Suppose $\varphi_1, \cdots, \varphi_m$ is a basis of Ω. Now we show $W = (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$: ①. $\forall w \in W$, by the definition of W, $\varphi_k(w) = 0$, $k = 1, \cdots, m \implies w \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$; ②. $\forall w \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$, $\varphi_k(w) = 0$, $k = 1, \cdots, m \implies \forall \varphi = a_1\varphi_1 + \cdots + a_m\varphi_m \in \Omega$, $\varphi(w) = a_1\varphi_1(w) + \cdots + a_m\varphi_m(w) = 0 \implies w \in W$.

Then, by the result of Exercises 3F 23, $\Omega = \operatorname{span}(\varphi_1, \cdots, \varphi_m) = ((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m))^0 = W^0$.

28 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)) = (\dim V) - m.$$

By the result of Exercises 3F 23, $((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m))^0 = \operatorname{span}(\varphi_1, \cdots, \varphi_m)$. Also, $\varphi_1, \cdots, \varphi_m$ is linearly independent $\implies \dim\operatorname{span}(\varphi_1, \cdots, \varphi_m) = m$. Hence, $\dim((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)) = \dim V - \dim((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m))^0 = \dim V - \dim\operatorname{span}(\varphi_1, \cdots, \varphi_m) = \dim V - m$.

29 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) Prove that if $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, then $\operatorname{range} T = \operatorname{null}\varphi$.
(b) Prove that if $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$, then $\operatorname{null} T = \operatorname{null}\psi$.

(a). $\operatorname{null} T' = (\operatorname{range} T)^0 = \operatorname{span}(\varphi)$. Then: 1). $\forall w \in \operatorname{range} T$, as $\varphi \in \operatorname{span}(\varphi) = (\operatorname{range} T)^0$, $\varphi(w) = 0$
$\implies w \in \operatorname{null}\varphi \implies \operatorname{range} T \subseteq \operatorname{null}\varphi$. 2). As $\operatorname{range} T$ is a subspace of W, by the result of Exercises 3F 20, $\operatorname{range} T = \{w \in W : \varphi(w) = 0$ for $\forall \varphi \in (\operatorname{range} T)^0\}$; $\forall w \in \operatorname{null}\varphi$, for $\forall \lambda\varphi \in \operatorname{span}(\varphi) = (\operatorname{range} T)^0$, where $\lambda \in F$, $(\lambda\varphi)(w) = \lambda\varphi(w) = \lambda \cdot 0 = 0 \implies w \in \operatorname{range} T \implies \operatorname{null}\varphi \subseteq \operatorname{range} T$ $\therefore \operatorname{range} T = \operatorname{null}\varphi$.

(b). $\operatorname{range} T' = (\operatorname{null} T)^0 = \operatorname{span}(\psi)$. Then: 1). $\forall v \in \operatorname{null} T$, as $\psi \in \operatorname{span}(\psi) = (\operatorname{null} T)^0$, $\psi(v) = 0 \implies$
$v \in \operatorname{null}\psi \implies \operatorname{null} T \subseteq \operatorname{null}\psi$. 2). As $\operatorname{null} T$ is a subspace of V, by the result of Exercises 3F 20, $\operatorname{null} T = \{v \in V : \psi(v) = 0$ for $\forall \psi \in (\operatorname{null} T)^0\}$; $\forall v \in \operatorname{null}\psi$, for $\forall \lambda\psi \in \operatorname{span}(\psi) = (\operatorname{null} T)^0$, where $\lambda \in F$, $(\lambda\psi)(v) = \lambda\psi(v) = \lambda \cdot 0 = 0 \implies v \in \operatorname{null} T \implies \operatorname{null}\psi \subseteq \operatorname{null} T$. $\therefore \operatorname{null} T = \operatorname{null}\psi$.

30 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

$\varphi_1, \cdots, \varphi_n$ is a basis of $V' \implies \dim\operatorname{null}\varphi_k = \dim V - \dim\operatorname{range}\varphi_k = n - 1$. Let $V = \operatorname{null}\varphi_k \oplus W_k$, where $\dim W_k = 1$, $k = 1, \cdots, n$. For each $k = 1, \cdots, n$, $\exists v_k \in W_k$, $\varphi_k(v_k) = 1$, as you should verify.

Then, $v_1, \cdots, v_n$ is the basis of V, whose dual basis is $\varphi_1, \cdots, \varphi_n$:

①. $\varphi_j(v_k) = 0$ for $j \neq k$: If $\exists w \in W_k \cap W_j$, $w \neq 0$, then $W_k = W_j = \operatorname{span}(w) \implies \operatorname{null}\varphi_k = \operatorname{null}\varphi_j$.

Then, by the result of Exercises 3F 6, $\exists c \in \mathbf{F}$, $\varphi_k = c\varphi_j$, contradicting that $\varphi_1, \cdots, \varphi_n$ is a basis. Hence, $W_k \cap W_j = \{0\} \Rightarrow W_k \subseteq \operatorname{null} \varphi_j \Rightarrow \varphi_j(v_k) = 0$

②. $v_1, \cdots, v_n$ is linearly independent: Let $0 = a_1 v_1 + \cdots + a_n v_n \Rightarrow \varphi_k(0) = \varphi_k(a_1 v_1 + \cdots + a_n v_n) = \sum_{j=1}^{n} a_j \varphi_k(v_j)$ $= a_k$, as we already have $\varphi_k(v_j) = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}$. This implies $a_k = 0$ for $k = 1, \cdots, n$.

31 Suppose U is a subspace of V. Let $i: U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$.

(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.

(c) Prove that if V is finite-dimensional, then $\tilde{i}'$ is an isomorphism from V'/U^0 onto U'.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

(a). $\operatorname{range} i = U \Rightarrow \operatorname{null} i' = (\operatorname{range} i)^0 = U^0$.

(b). i is injective $\Rightarrow \operatorname{null} i = \{0\} \Rightarrow \operatorname{range} i' = (\operatorname{null} i)^0 = U'$.

(c). By (a) and (b), $V'/(\operatorname{null} i') = V'/U^0$, $\operatorname{range} i' = U'$. Then by result 3.107, we derive the result.

32 The *double dual space* of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda: V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and each $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

(a). ①. $(\Lambda(v_1+v_2))(\varphi) = \varphi(v_1+v_2) = \varphi(v_1) + \varphi(v_2) = (\Lambda v_1)(\varphi) + (\Lambda v_2)(\varphi) = (\Lambda v_1 + \Lambda v_2)(\varphi)$ for $\forall v_1, v_2 \in V$ and $\forall \varphi \in V' \Rightarrow \Lambda(v_1+v_2) = \Lambda v_1 + \Lambda v_2$.

②. $(\Lambda(\lambda v))(\varphi) = \varphi(\lambda v) = \lambda(\varphi v) = \lambda(\Lambda v)(\varphi)$ for $\forall \lambda \in \mathbf{F}$, $\forall v \in V$, and $\forall \varphi \in V' \Rightarrow$ $\Lambda(\lambda v) = \lambda(\Lambda v)$.

(b). $((T'' \circ \Lambda)(v))(\varphi) = (T''(\Lambda v))(\varphi) = ((\Lambda v)T')(\varphi) = (\Lambda v)(T'\varphi) = (\Lambda v)(\varphi T) = \varphi T(v)$.

$((\Lambda \circ T)(v))(\varphi) = (\Lambda(Tv))(\varphi) = \varphi(Tv) = \varphi T(v)$. $\Rightarrow T'' \circ \Lambda = \Lambda \circ T$.

(c). $\forall v \in \operatorname{null} \Lambda$, $\forall \varphi \in V'$, $(\Lambda v)(\varphi) = \varphi(v) = 0 \Rightarrow v = 0 \Rightarrow \operatorname{null} \Lambda = \{0\} \Rightarrow \Lambda$ is injective

As $\dim V = \dim V' = \dim V''$, this implies Λ is an isomorphism from V onto V''.

33 Suppose U is a subspace of V. Let $\pi: V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

(b) Show that $\operatorname{range} \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.

(a). By the definition of π, π is surjective $\Rightarrow \pi'$ is injective.

(b). $\operatorname{null} \pi = U \Rightarrow \operatorname{range} \pi' \subseteq (\operatorname{null} \pi)^0 = U^0$ (as there's no assumption of finite-dimension).

$\forall \varphi \in U^0$, $\operatorname{null} \varphi \supseteq U$, by the result of Exercises 3E 19, $\exists \psi \in (V/U)'$, $\varphi = \psi \circ \pi = \pi'(\psi) \Rightarrow$ $U^0 \subseteq \operatorname{range} \pi'$. Hence, $\operatorname{range} \pi' = U^0$.

(c). By (a), π' is injective; by (b), π' is surjective onto U^0.