Solutions to 2C :

1 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

Suppose U is any subspace of R^2. By result 2.37, $0 \leq \dim U \leq \dim R^2 = 2$.

Hence, $\dim U \in \{0, 1, 2\}$.

① $\dim U = 0$: $U = \operatorname{span}() = \{0\}$

② $\dim U = 1$:

Thus, any basis of U only contains one element, denoted by $(x_1, x_2) \in R^2$.

The list (x_1, x_2) is linearly independent, hence $(x_1, x_2) \neq (0, 0)$.

$\forall (u_1, u_2) \in U$, $\exists a \in R$. $(u_1, u_2) = a(x_1, x_2)$. We also have $(0,0) \in U$.

$\therefore$ U is a line in R^2 containing the origin.

On the other hand, any line in R^2 containing the origin can be represented by

$U = \{a(x_1, x_2) \in R^2 : a, x_1, x_2 \in R, (x_1, x_2) \neq (0,0)\}$.

It's easy to show that such U is a subspace of R^2 and (x_1, x_2) is a basis of U, implying that $\dim U = 1$.

$\therefore$ subspaces of R^2 with dimension 1 are precisely all lines in R^2 containing the origin

③ $\dim U = 2$:

This implies $\dim U = \dim R^2$. By result 2.39, $U = R^2$.

$\therefore$ the subspaces of R^2 are precisely $\{0\}$, all lines in R^2 containing the origin, and R^2.

3 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

(a). $(z-6), z(z-6), z^2(z-6), z^3(z-6)$.

(b). $(z-6), z(z-6), z^2(z-6), z^3(z-6), 1$

(c). $W = \{p \in P_4(F) : p(z) \equiv a, a \in F\}$

7 (a) Let $U = \left\{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\right\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

(a). $x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}$

(b). $x, x^2 - \frac{1}{3}, x^3, x^4 - \frac{1}{5}, 1$

(c). $W = \{p \in P_4(R) : p(x) \equiv a, a \in R\}$

8 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Let $u_k = v_{k+1} - v_k$, $k = 1, \cdots, m-1$. As $u_k = v_{k+1} - v_k = (v_{k+1} + w) - (v_k + w)$,

$u_k \in \operatorname{span}(v_1 + w, \cdots, v_m + w)$. Consider $0 = a_1 u_1 + \cdots + a_{m-1} u_{m-1}$, $a_i \in F$. we have

$0 = a_1(v_2 - v_1) + \cdots + a_{m-1}(v_m - v_{m-1}) = (-a_1)v_1 + \sum_{i=1}^{m-2}(a_i - a_{i+1})v_{i+1} + a_{m-1}v_m$.

Because $v_1, \cdots, v_m$ is linearly independent, we derive $(-a_1) = 0$, $(a_i - a_{i+1}) = 0$ for $i = 1, \cdots, m-2$, $a_{m-1} = 0$. This implies that $a_i = 0$, $i = 1, \cdots, m-1$. Hence, $u_1, \cdots, u_{m-1}$ is linearly independent.

Thus, we find a linearly independent list of length $m-1$ in $\text{span}(v_1+w, \cdots, v_m+w)$.

$\therefore \dim \text{span}(v_1+w, \cdots, v_m+w) \geq m-1$

9 Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Suppose $0 = a_0p_0 + a_1p_1 + \cdots a_mp_m$, $a_i \in F$.

Because each p_k has degree k, we see the right side of the equation has degree m, which is contributed only by a_mp_m. Suppose the term with degree m in p_m is bz^m, where $b \in F$ and $b \neq 0$. Thus, the coefficient of z^m on the right side of the equation is a_mb. However, the left side has no z^m term. Hence, $a_mb = 0$, implying that $a_m = 0$. Thus, the equation is reduced to $0 = a_0p_0 + a_1p_1 + \cdots + a_{m-1}p_{m-1}$.

Similarly, we can derive $a_{m-1} = 0$. Eventually, we derive $a_i = 0$ for $i = 0, 1, \cdots, m$.

This means $p_0, p_1, \cdots, p_m$ is linearly independent. As $\dim P_m(F) = m+1$, the same as the length of $p_0, p_1, \cdots, p_m$, by result 2.38, $p_0, p_1, \cdots, p_m$ is a basis of $P_m(F)$.

10 Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k(1-x)^{m-k}.$$

Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

*The basis in this exercise leads to what are called **Bernstein polynomials**. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on* $[0, 1]$.

Suppose $0 = \sum_{i=0}^{m} a_ip_i(x) = \sum_{i=0}^{m} a_ix^i(1-x)^{m-i}$, where $a_i \in F$. (i)

We have $a_ix^i(1-x)^{m-i} = \sum_{k=i}^{m}(-1)^{k-i}a_ix^k$. Hence, the coefficient of x^i on the right side of the equation (i) is $\sum_{k=0}^{i}(-1)^{i-k}a_k$, where $i = 0, 1, \cdots, m$.

As the left side of the equation (i) is 0, we derive $\sum_{k=0}^{i}(-1)^{i-k}a_k = 0$ for $i = 0, 1, \cdots, m$.

This implies that $a_0 = 0$, $a_1 - a_0 = 0$, $a_2 - a_1 + a_0 = 0$, $\cdots$, $a_m - a_{m-1} + \cdots + (-1)^m a_0 = 0$

$\Rightarrow a_i = 0$, $i = 0, 1, \cdots, m$. Hence, $p_0, \cdots, p_m$ is linearly independent. Also, we have that length of $p_0, \cdots, p_m$ equals $\dim P_m(F)$. By result 2.38, $p_0, \cdots, p_m$ is a basis of $P_m(F)$.

11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

$C^6 \supseteq U + W \supseteq U \Rightarrow 6 = \dim C^6 \geq \dim(U+W) \geq \dim U = 4$, by result 2.37.

By result 2.43, $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$\Rightarrow \dim(U \cap W) = 8 - \dim(U+W) \in \{2, 3, 4\}$.

Hence, $\dim(U \cap W) \geq 2$, there exists a basis of $U \cap W$, whose length equal to or greater than 2.

Thus, the first two vectors of the basis is linearly independent, meaning neither of these vectors is a scalar multiple of the other.

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

By result 2.43, we have

$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3)$ ①

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ ②

From ① and ②, we derive

$\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim((V_1 \cap V_2) + V_3)$

By hypothesis $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$, we derive

$\dim(V_1 \cap V_2 \cap V_3) > (\dim V - \dim(V_1 + V_2)) + (\dim V - \dim((V_1 \cap V_2) + V_3))$

As $V_1 + V_2 \subseteq V$ and $(V_1 \cap V_2) + V_3 \subseteq V$, by result 2.37, we have

$\dim(V_1 + V_2) \leq \dim V$ and $\dim((V_1 \cap V_2) + V_3) \leq \dim V$.

Hence, $\dim(V_1 \cap V_2 \cap V_3) > 0 + 0 = 0 \Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}$.

16 Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

By result 2.39, $n - m > 0$ as $U \neq V$. Suppose $u_1, \cdots, u_m$ is a basis of U.

$u_1, \cdots, u_m$ can be extended to a basis of V: $u_1, \cdots, u_m, w_1, \cdots, w_{n-m}$, denoted as L.

Define $W_i = \operatorname{span}(\{L\} - \{w_i\})$, where $i = 1, \cdots, n-m$. Obviously, $\dim W_i = n-1$

As $u_1, \cdots, u_m \in W_i$, $u_1, \cdots, u_m \in \cap W_i$, implying $U = \operatorname{span}(u_1, \cdots, u_m) \subseteq \cap W_i$

Denote $\widetilde{W} = \operatorname{span}(w_1, \cdots, w_{n-m})$. Consider $\forall w \in \widetilde{W}$, $\exists a_i \in F$, $i = 1, \cdots, n-m$, such that

$w = a_1 w_1 + \cdots + a_{n-m} w_{n-m}$. Fix $i \in \{1, \cdots, n-m\}$, assume $w \in W_i$, $\exists b_j, c_k \in F$, where

$j = 1, \cdots, m$, $k = 1, \cdots, n-m$ but $k \neq i$, such that $w = \sum_{j=1}^{m} b_j u_j + \sum_{\substack{k=1 \\ k \neq i}}^{n-m} c_k w_k$.

$\Rightarrow a_1 w_1 + \cdots + a_{n-m} w_{n-m} = \sum_{j=1}^{m} b_j u_j + \sum_{\substack{k=1 \\ k \neq i}}^{n-m} c_k w_k$

$\Rightarrow 0 = b_1 u_1 + \cdots + b_m u_m + (-a_i) w_i + \sum_{\substack{k=1 \\ k \neq i}}^{n-m} (c_k - a_k) w_k$.

As $u_1, \cdots, u_m, w_1, \cdots, w_{n-m}$ is a basis of V, we derive $(-a_i) = 0 \Rightarrow a_i = 0$.

Hence, if $w \in \cap W_i$, meaning $w \in W_i$ for $i = 1, \cdots, n-m$, we will derive that $a_i = 0$ for $i = 1, \cdots, n-m$, implying that $w = 0$. Thus, $\widetilde{W} \cap (\cap W_i) = \{0\}$.

It's easy to prove that $\widetilde{W} + \cap W_i = V$, hence $\dim(\widetilde{W} + \cap W_i) = n$.

By result 2.43, $\dim(\widetilde{W} + \cap W_i) = \dim \widetilde{W} + \dim(\cap W_i) - \dim(\widetilde{W} \cap (\cap W_i))$

$\Rightarrow \dim(\cap W_i) = n + 0 - (n-m) = m = \dim U$.

As $U \subseteq \cap W_i$, by result 2.39, $U = \cap W_i$.

$\therefore$ We find $n-m$ subspaces, W_i, such that $\dim W_i = n-1$ and $\cap W_i = U$.

17 Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

① When $m = 1$, the inequality holds naturally.

② Assume the inequality holds for $m = k$, where $k \geq 1$, we will show that the inequality also holds for $m = k+1$:

Let $U_k = V_1 + \cdots + V_k$. From our hypothesis, we have $\dim U_k \le \dim V_1 + \cdots + \dim V_k$.

By result 2.43, $\dim(U_k + V_{k+1}) = \dim U_k + \dim V_{k+1} - \dim(U_k \cap V_{k+1})$.

As $\dim(U_k \cap V_{k+1}) \geq 0$, we derive $\dim(U_k + V_{k+1}) \le \dim U_k + \dim V_{k+1}$

$\Rightarrow \dim((V_1 + \cdots + V_k) + V_{k+1}) \le (\dim V_1 + \cdots + \dim V_k) + \dim V_{k+1}$.

Hence, the inequality holds for $m = k+1$.

From ① and ②, we proved that the inequality alway holds for positive integer m.

18 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

Suppose $v_1, \cdots, v_n$ is a basis of V. Let $V_i = \operatorname{span}(v_i)$.

① $V_i \cap V_j = \{0\}$, $\forall i, j \in \{1, \cdots, n\}$, $i \neq j$.

Consider $\forall u \in V_i \cap V_j$, $\exists a, b \in F$, such that $u = av_i = bv_j$.

$\Rightarrow av_i + (-b)v_j = 0$. As v_i, v_j is linearly independent, there must be $a = b = 0$,

implying $u = 0$. Hence, $V_i \cap V_j = \{0\}$.

② $V = V_1 + \cdots + V_n$:

As $v_1, \cdots, v_n$ is a basis of V, $\forall v \in V$, $\exists a_i \in F$, such that

$v = a_1v_1 + \cdots + a_nv_n = (a_1v_1) + \cdots + (a_nv_n) \in V_1 + \cdots + V_n$.

Thus, $V \subseteq V_1 + \cdots + V_n$. Also, we have $V_1 + \cdots + V_n \subseteq V$.

Hence, $V = V_1 + \cdots + V_n$.

From ① and ②, $V = V_1 \oplus \cdots \oplus V_n$, and obviously $\dim V_i = 1$.

20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}
&\dim(V_1 + V_2 + V_3) \\
&\quad = \dim V_1 + \dim V_2 + \dim V_3 \\
&\qquad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\
&\qquad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.
\end{aligned}$$

$\dim((V_1 + V_2) + V_3) = \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3)$. ①

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$. ②

Substitute ② into ①, implying

$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3)$ ③

Similarly, we derive the following equations from considering $\dim((V_1 + V_3) + V_2)$ and $\dim((V_2 + V_3) + V_1)$:

$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_3) - \dim((V_1 + V_3) \cap V_2)$ ④

$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 \cap V_3) - \dim((V_2 + V_3) \cap V_1)$ ⑤

Finally, $(③ + ④ + ⑤)/3$ giving the result.