Solutions to 2A :

2 Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

It's equivalent to prove $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

① $\forall v \in \operatorname{span}(v_1, v_2, v_3, v_4)$, $\exists a, b, c, d \in F$,

$$v = av_1 + bv_2 + cv_3 + dv_4 = a(v_1 - v_2) + (a+b)(v_2 - v_3) + (a+b+c)(v_3 - v_4) + (a+b+c+d)v_4$$

$$\in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$$

Hence, $\operatorname{span}(v_1, v_2, v_3, v_4) \subseteq \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$

② $\forall v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, $\exists a, b, c, d \in F$,

$$v = a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = av_1 + (b-a)v_2 + (c-b)v_3 + (d-c)v_4$$

$$\in \operatorname{span}(v_1, v_2, v_3, v_4)$$

Hence, $\operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) \subseteq \operatorname{span}(v_1, v_2, v_3, v_4)$

From ① and ②, if v_1, v_2, v_3, v_4 spans V, also the $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$.

7 (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1 + i, 1 - i$ is linearly independent.
(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1 + i, 1 - i$ is linearly dependent.

(a). Let $0 = a(1+i) + b(1-i)$, $a, b \in \mathbb{R}$.

$\Rightarrow a+b = 0$ and $a-b = 0$ $\Rightarrow$ $a = 0$ and $b = 0$

$\therefore$ the list $1+i, 1-i$ is linearly independent over $\mathbb{R}$

(b). $\because$ $0 = (-1+i)(1+i) + (1+i)(1-i)$

$\therefore$ the list $1+i, 1-i$ is linearly dependent over $\mathbb{C}$

8 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

For $\forall a, b, c, d \in F$, such that

$$0 = a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = av_1 + (b-a)v_2 + (c-b)v_3 + (d-c)v_4.$$

$\because v_1, v_2, v_3, v_4$ is linearly independent

$\therefore a = 0, b-a = 0, c-b = 0, d-c = 0$

$\Rightarrow a, b, c, d = 0$.

Hence, $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is also linearly independent

11 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

A counterexample: $(1,0), (0,1)$ and $(0,1), (1,0)$ are linearly independent lists, however,

$(1,0) + (0,1) = (1,1)$, $(0,1) + (1,0) = (1,1)$ is not linearly independent.

12 Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

If $v_1+w, \cdots, v_m+w$ is linearly dependent, from the linear dependence lemma, we have

$\exists k \in \{1, \cdots, m\}$, such that $v_k + w \in \operatorname{span}(v_1+w, \cdots, v_{k-1}+w)$

$\therefore \exists a_i \in F$, $i=1,\cdots,k-1$, such that $v_k + w = \sum_{i=1}^{k-1} a_i (v_i+w)$

$\Rightarrow (\sum_{i=1}^{k-1} a_i - 1) w = v_k - \sum_{i=1}^{k-1} a_i v_i$.

Assume that $\sum_{i=1}^{k-1} a_i = 1$, which gives $0 = v_k - \sum_{i=1}^{k-1} a_i v_i$. This contradicts our suppose that $v_1, \cdots, v_m$ is linearly independent. Hence, there must be $\sum_{i=1}^{k-1} a_i \neq 1$.

This implies that $w = (-\frac{a_1}{b}) v_1 + (-\frac{a_2}{b}) v_2 + \cdots + (-\frac{a_{k-1}}{b}) v_{k-1} + \frac{1}{b} v_k$, where $b = \sum_{i=1}^{k-1} a_i - 1 \neq 0$. $\therefore w \in \operatorname{span}(v_1, \cdots, v_k) \subseteq \operatorname{span}(v_1, \cdots, v_m)$

13 Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \dots, v_m, w \text{ is linearly independent } \iff w \notin \operatorname{span}(v_1, \dots, v_m).$$

① suppose $w \in \operatorname{span}(v_1, \cdots, v_m)$, hence $\exists a_i \in F$, $i=1,\cdots,m$, such that $w = \sum_{i=1}^{m} a_i v_i \Rightarrow 0 = (-1) w + \sum_{i=1}^{m} a_i v_i \Rightarrow v_1, \cdots, v_m, w$ is linearly dependent

$\therefore v_1, \cdots, v_m, w$ is linearly independent $\Rightarrow w \notin \operatorname{span}(v_1, \cdots, v_m)$.

② suppose $v_1, \cdots, v_m, w$ is linearly dependent, hence $\exists a_i, b \in F$, $i=1,\cdots,m$, not all 0, such that $0 = \sum_{i=1}^{m} a_i v_i + bw \Rightarrow (-b) w = a_1 v_1 + \cdots + a_m v_m$

Assume that $b=0$, which gives $0 = a_1 v_1 + \cdots + a_m v_m$. As $v_1, \cdots, v_m$ is linearly independent, this gives $a_i = 0$, $i=1,\cdots,m$. Hence a_i and b are all 0, contradicting our suppose. So there must be $b \neq 0$, giving $w = (-\frac{a_1}{b}) v_1 + \cdots + (-\frac{a_m}{b}) v_m$.

$\Rightarrow w \in \operatorname{span}(v_1, \cdots, v_m)$.

$\therefore w \notin \operatorname{span}(v_1, \cdots, v_m) \Rightarrow v_1, \cdots, v_m, w$ is linearly independent

From ① and ②, both directions of the result are proved.

15 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

$P_4(F) = \operatorname{span}(1, z, z^2, z^3, z^4)$, has a spanning list of length 5.

From result 2.22, we know that any linearly independent list in $P_4(F)$ will not have a length greater than 5.

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $v_1, \dots, v_m$ is linearly independent for every positive integer m.

① Suppose V is finite-dimensional, such that there exists some list which spans V.

Assume the list has length m, denote it as $v_1, \cdots, v_m$. Thus, $V = \operatorname{span}(v_1, \cdots, v_m)$.

$\Rightarrow \forall v \in V$, $\exists a_i \in F$, $i=1,\cdots,m$, such that $v = a_1 v_1 + \cdots + a_m v_m$.

This gives $0 = (-1) v + a_1 v_1 + \cdots + a_m v_m$, meaning $v, v_1, \cdots, v_m$ is not linearly independent for any v. Hence, there doesn't exist a linearly independent list of length $m+1$.

$\therefore$ there exists a linearly independent list of arbitrary length

$\Rightarrow V$ is infinite-dimensional.

② Suppose such sequence doesn't exist. Now we construct a list L through the following steps:

Step 1: Select an arbitrary vector $v_1 \in V$ into L.

Step $i = 2$ to m: Select a vector $v_i \in V$ into L, such that the expanded list L is linearly independent. If such v_i doesn't exist, terminate the process.

We claim that this process will eventually terminate, otherwise we will derive a sequence that contradicts our suppose. Assume the list L we derive has length m, we denote it as $v_1, \cdots, v_m$. Hence, $\forall v \in V$, $v_1, \cdots, v_m, v$ is not linearly independent.

$\Rightarrow \exists a_i, b \in F$, not all 0, such that $0 = \sum_{i=1}^{m} a_i v_i + bv$.

If $b = 0$, implying $0 = a_1 v_1 + \cdots + a_m v_m \Rightarrow a_i = 0$ for $v_1, \cdots, v_m$ is linearly independent

$\Rightarrow a_i$ and b are all 0, contradicting our suppose.

Thus, $b \neq 0$. This gives $v = (-\frac{a_1}{b}) v_1 + \cdots + (-\frac{a_m}{b}) v_m$, meaning $v \in \operatorname{span}(v_1, \cdots, v_m)$.

Hence, $V \subseteq \operatorname{span}(v_1, \cdots, v_m)$. And obviously, we have $\operatorname{span}(v_1, \cdots, v_m) \subseteq V$.

$\therefore V = \operatorname{span}(v_1, \cdots, v_m)$. $\Rightarrow V$ is finite-dimensional.

$\therefore V$ is infinite-dimensional $\Rightarrow$ there exists a linearly independent list of arbitrary length.

From ① and ②, both directions of the result are proved.

18 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Consider sequence $v_1, v_2, \cdots$, where $v_i \in F^\infty$ has 1 in i^{th} slot and 0 in all other slots.

Obviously, for $\forall m \in Z^{++}$, $v_1, \cdots, v_m$ is linearly independent.

Hence, from the result of exercise 2A 17, F^∞ is infinite-dimensional.

19 Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Consider sequence $x^0, x^1, x^2, \cdots$, where x^i denotes the function $f(x) = x^i$ for $\forall x \in [0,1]$, which is continuous real-valued. Obviously, for $\forall m \in Z^{++}$, $x^0, \cdots, x^m$ is linearly independent.

Hence, from the result of exercise 2A 17, the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

20 Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Denote $V = \{p : p \in \mathcal{P}_m(F), p(2) = 0\}$

① V is a subspace of $\mathcal{P}_m(F)$:

1). $0(2) \in V$ for $0(2) = 0$

2). $\forall p_1, p_2 \in V$, $(p_1 + p_2)(2) = p_1(2) + p_2(2) = 0 + 0 = 0 \Rightarrow p_1 + p_2 \in V$

3). $\forall p \in V$, $\forall a \in F$, $(ap)(2) = a(p(2)) = a \cdot 0 = 0 \Rightarrow ap \in V$

② $V = \operatorname{span}(z-2, z(z-2), \cdots, z^{m-1}(z-2))$

1). $\forall\, u = \sum_{i=0}^{m-1} a_i z^i (z-2) \in \operatorname{span}(z-2, z(z-2), \cdots, z^{m-1}(z-2))$, where $a_i \in \mathbb{F}$,

obviously $u(2) = 0$. Hence, $\operatorname{span}(z-2, z(z-2), \cdots, z^{m-1}(z-2)) \subseteq V$

2). $\forall\, v \in V$, because v is a polynomial and $v(2) = 0$, we can denote

$v(z) = (z-2)(a_0 + a_1 z + \cdots + a_{m-1} z^{m-1})$.

$\Rightarrow\ v(z) = a_0(z-2) + a_1 z(z-2) + \cdots + a_{m-1} z^{m-1}(z-2) \in \operatorname{span}(z-2, z(z-2), \cdots, z^{m-1}(z-2))$.

Hence, $V \subseteq \operatorname{span}(z-2, z(z-2), \cdots, z^{m-1}(z-2))$

From ②, we know that V has a spanning list of length m. And according to result 2.22, any linearly independent list in V will not have length greater than m.

$\therefore\ p_0, p_1, \cdots, p_m$ is not linearly independent.