Solutions to 3A :

2 Suppose $b, c \in \mathbf{R}$. Define $T: \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

① If T is linear :

$$T(\lambda p) = (3(\lambda p)(4) + 5(\lambda p)'(6) + b(\lambda p)(1)(\lambda p)(2),\ \int_{-1}^{2} x^3(\lambda p)(x)dx + c\sin(\lambda p)(0))$$

$$= \lambda(3p(4) + 5p'(6) + bp(1)p(2),\ \int_{-1}^{2} x^3 p(x)dx + c\sin p(0))$$

$$+ (b(\lambda^2-\lambda)p(1)p(2),\ c(\sin\lambda p(0) - \lambda\sin p(0)))$$

$$= \lambda Tp + (b(\lambda^2-\lambda)p(1)p(2),\ c(\sin\lambda p(0) - \lambda\sin p(0)))$$

$\Rightarrow b = c = 0$

② If $b = c = 0$:

$$T(p+q) = (3(p+q)(4) + 5(p+q)'(6),\ \int_{-1}^{2} x^3(p+q)(x)dx)$$

$$= ((3p(4) + 5p'(6)) + (3q(4) + 5q'(6)),\ (\int_{-1}^{2} x^3 p(x)dx) + (\int_{-1}^{2} x^3 q(x)dx))$$

$$= Tp + Tq$$

$$T(\lambda p) = (3(\lambda p)(4) + 5(\lambda p)'(6),\ \int_{-1}^{2} x^3(\lambda p)(x)dx)$$

$$= \lambda(3p(4) + 5p'(6),\ \int_{-1}^{2} x^3 p(x)dx) = \lambda Tp$$

$\therefore T$ is linear

From ① and ②, the result holds.

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

Suppose $x_1, \cdots, x_n$ is the standard basis of F^n, and $Tx_i = (A_{1,i}, \cdots, A_{m,i}) \in F^m$, where $i = 1, \cdots, n$. For $\forall\, x = (c_1, \cdots, c_n) \in F^n$, where $c_i \in F$ for $i = 1, \cdots, n$, we have

$$x = (c_1, \cdots, c_n) = c_1(1,0,\cdots,0) + \cdots + c_n(0,\cdots,0,1) = c_1x_1 + \cdots + c_nx_n.$$

$\therefore$ $Tx = T(c_1x_1 + \cdots + c_nx_n) = T(c_1x_1) + \cdots + T(c_nx_n)$ (additivity)

$= c_1Tx_1 + \cdots + c_nTx_n$ (homogeneity)

$= c_1(A_{1,1}, \cdots, A_{m,1}) + \cdots + c_n(A_{1,n}, \cdots, A_{m,n})$

$= (A_{1,1}c_1 + \cdots + A_{1,n}c_n, \cdots, A_{m,1}c_1 + \cdots + A_{m,n}c_n)$

Hence, the result holds.

4 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

If $v_1, \cdots, v_m$ is not linearly independent, $\exists\, a_1, \cdots, a_m \in F$, not all 0, such that $0 = a_1v_1 + \cdots + a_mv_m$. $\Rightarrow$ $T(a_1v_1 + \cdots + a_mv_m) = T(a_1v_1) + \cdots + T(a_mv_m)$ $= a_1Tv_1 + \cdots + a_mTv_m = T(0) = 0$. (By definition of linear maps and result 3.10)

This implies that $Tv_1, \cdots, Tv_m$ is not linearly independent, which is contradictory to the hypothesis. Hence, $v_1, \cdots, v_m$ is linearly independent.

5 Prove that $\mathcal{L}(V, W)$ is a vector space, as was asserted in 3.6.

① Commutativity:

$(T_1 + T_2)(x) = T_1(x) + T_2(x) = T_2(x) + T_1(x) = (T_2 + T_1)(x)$

② associativity:

$((T_1+T_2)+T_3)(x) = (T_1+T_2)(x) + T_3(x) = (T_1(x) + T_2(x)) + T_3(x)$

$= T_1(x) + (T_2(x) + T_3(x)) = T_1(x) + (T_2 + T_3)(x) = (T_1 + (T_2+T_3))(x)$

$(\lambda_1(\lambda_2 T))(x) = \lambda_1((\lambda_2 T)(x)) = \lambda_1(\lambda_2(T(x))) = (\lambda_1\lambda_2)(T(x)) = ((\lambda_1\lambda_2)T)(x)$

③ additive identity:

The additive identity of $\mathcal{L}(V, W)$ is $0(x) = 0$ for $\forall x \in V$.

$(T+0)(x) = T(x) + 0(x) = T(x) + 0 = T(x)$

④ additive inverse:

For $\forall T \in \mathcal{L}(V, W)$, define $-T \in \mathcal{L}(V, W)$: $(-T)(x) = -(T(x))$.

$(T + (-T))(x) = T(x) + (-T)(x) = T(x) + (-(T(x))) = 0 = 0(x)$

⑤ multiplicative identity:

$(1T)(x) = 1(T(x)) = T(x)$

⑥ distributive properties:

$(\lambda(T_1+T_2))(x) = \lambda((T_1+T_2)(x)) = \lambda(T_1(x) + T_2(x)) = \lambda(T_1(x)) + \lambda(T_2(x))$

$= (\lambda T_1)(x) + (\lambda T_2)(x) = (\lambda T_1 + \lambda T_2)(x)$

$((\lambda_1+\lambda_2)T)(x) = (\lambda_1+\lambda_2)(T(x)) = \lambda_1(T(x)) + \lambda_2(T(x)) = (\lambda_1 T)(x) + (\lambda_2 T)(x)$

$= (\lambda_1 T + \lambda_2 T)(x)$

$\therefore$ $\mathcal{L}(V, W)$ is a vector space.

6 Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.

① associativity:

$((T_1 T_2) T_3)(w) = (T_1 T_2)(T_3(w)) = T_1(T_2(T_3(w))) = T_1((T_2 T_3)(w)) = (T_1(T_2 T_3))(w)$

② identity:

$(IT)(w) = I(T(w)) = T(w)$ and $(TI)(w) = T(I(w)) = T(w)$

$\Rightarrow IT = TI = T$

③ distributive properties:

$((S_1+S_2)T)(w) = (S_1+S_2)(T(w)) = S_1(T(w)) + S_2(T(w)) = (S_1 T)(w) + (S_2 T)(w)$

$= (S_1 T + S_2 T)(w)$

$(S(T_1+T_2))(w) = S((T_1+T_2)(w)) = S(T_1(w) + T_2(w)) = S(T_1(w)) + S(T_2(w))$

$= (ST_1)(w) + (ST_2)(w) = (ST_1 + ST_2)(w)$

7 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Suppose v_1 is a basis of V. Because $Tv_1 \in V$, $\exists \lambda \in F$, such that

$Tv_1 = \lambda v_1$. For $\forall v \in V$, $\exists a \in F$, such that $v = av_1$.

$\Rightarrow Tv = T(av_1) = aTv_1 = a(\lambda v_1) = \lambda(av_1) = \lambda v$.

8 Give an example of a function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

For $v = (v_1, v_2) \in R^2$, let $\varphi(v) = \begin{cases} \frac{v_1^2}{v_2}, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases}$.

It's easy to show that $\varphi(av) = a\varphi(v)$ for $\forall a \in R$ and $\forall v \in R^2$.

But obviously $\exists u, v \in R^2$, $\varphi(u+v) \neq \varphi(u) + \varphi(v)$. Thus φ is not linear.

9 Give an example of a function $\varphi\colon \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

Let $C = \{a+bi : a, b \in R\}$, i.e., all complex numbers.

For $\forall z = a+bi \in C$, define $\varphi(z) = a$.

Hence, $\forall w = a+bi, z = c+di \in C$, $\varphi(w+z) = \varphi((a+c)+(b+d)i) = a+c$ and

$\varphi(w) + \varphi(z) = a+c \quad \Rightarrow \quad \varphi(w+z) = \varphi(w) + \varphi(z)$.

However, $\varphi(i(1+i)) = \varphi(-1+i) = -1$ and $i\varphi(1+i) = i \cdot 1 = i$

$\Rightarrow \varphi(i(1+i)) \neq i\varphi(1+i) \quad \Rightarrow \quad \exists a \in C, \exists w \in C, \varphi(aw) \neq a\varphi(w)$

Thus φ is not linear.

11 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

① If $\exists a \in F$, $T = aI$:

$\forall S \in \mathcal{L}(V)$, $(ST)(x) = S(T(x)) = S(aI(x)) = S(ax) = aS(x)$

and $(TS)(x) = T(S(x)) = aI(S(x)) = aS(x)$. Thus $ST = TS$.

② If $\forall S \in \mathcal{L}(V)$, $ST = TS$:

Suppose $\dim V = n$ and $v_1, \cdots, v_n$ is a basis of V. Define $S_i \in \mathcal{L}(V)$:

$S_i(a_1v_1 + \cdots + a_nv_n) = a_iv_i$, where $a_i \in F$, $i = 1, \cdots, n$. Consider $S_iT(v_i) =$

$TS_i(v_i) \Rightarrow T(v_i) = S_i(T(v_i)) \Rightarrow \exists \lambda_i \in F$, $T(v_i) = \lambda_i v_i$.

Furthermore, define $S_{ij} \in \mathcal{L}(V)$: $S_{ij}(\cdots + a_iv_i + \cdots + a_jv_j + \cdots) = \cdots + a_jv_i + \cdots + a_iv_j + \cdots$,

where $a_i, a_j \in F$, $i, j \in \{1, \cdots, n\}$, $i \neq j$. Consider $S_{ij}T(a_iv_i + a_jv_j) = TS_{ij}(a_iv_i + a_jv_j)$

$\Rightarrow \lambda_j a_j v_i + \lambda_i a_i v_j = \lambda_i a_j v_i + \lambda_j a_i v_j \Rightarrow \lambda_i = \lambda_j$.

Hence, $\exists \lambda \in F$, $T(v_i) = \lambda v_i$ for $i = 1, \cdots, n$, implying that T is a scalar multiple of the identity.

From ① and ②, we proved the both directions of the result.

12 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T\colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Consider $\forall u \in U$ such that $Su \neq 0$, and $\forall v \in V$ such that $v \notin U$. We have $u+v \notin U$, otherwise $v = (u+v) + (-u) \in U$, a contradiction. Hence, $T(u+v) = 0$. However, $T(u) + T(v) = Su + 0 = Su$, implying $T(u+v) \neq Tu + Tv$.

$\therefore$ T is not a linear map on V.

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

There exists a subspace M of V, such that $V = U \oplus M$. $\forall v \in V$, $\exists u \in U$, $\exists m \in M$, uniquely, $v = u + m$. Suppose $R \in \mathcal{L}(M, W)$. For $\forall v = u + m \in V$, define $Tv = Su + Rm$.

We have $T(\lambda v) = T(\lambda u + \lambda m) = S(\lambda u) + R(\lambda m) = \lambda Su + \lambda Rm = \lambda(Su + Rm) = \lambda Tv$.

We also have $T(v_1 + v_2) = T((u_1+m_1) + (u_2+m_2)) = T((u_1+u_2) + (m_1+m_2)) = S(u_1+u_2) + R(m_1+m_2)$

$= Su_1 + Su_2 + Rm_1 + Rm_2 = (Su_1 + Rm_1) + (Su_2 + Rm_2) = Tv_1 + Tv_2$.

Hence, T is linear.

14 Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Suppose $\dim V = n$ and $v_1, \cdots, v_n$ is a basis of V. As W is infinite-dimensional, we can find a sequence of elements in W, denote by $w_1, w_2, \cdots$, such that $w_1, \cdots, w_m$ is linearly independent for any positive integer m. For $\forall v = a_1v_1 + \cdots + a_nv_n \in V$, where $a_i \in F$, define $T_k \in \mathcal{L}(V, W)$: $T_k(v) = a_1 w_k$, where $k = 1, 2, \cdots$. Consider $0(v) = \lambda_1 T_1(v) + \cdots + \lambda_m T_m(v) = a_1(\lambda_1 w_1 + \cdots + \lambda_m w_m)$

$\Rightarrow \lambda_1 = \cdots = \lambda_m = 0$, as $w_1, \cdots, w_m$ is linearly independent $\Rightarrow T_1, \cdots, T_m$ is linearly independent.

Hence, we find a sequence of elements in $\mathcal{L}(V, W)$, $T_1, T_2, \cdots$, such that $T_1, \cdots, T_m$ is linearly independent for any positive m. $\therefore$ $\mathcal{L}(V, W)$ is infinite-dimensional.

15 Suppose $v_1, \dots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \dots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.

$v_1, \cdots, v_m$ is linearly dependent $\Rightarrow \exists a_1, \cdots, a_m \in F$, not all 0, such that $0 = a_1v_1 + \cdots + a_mv_m$

As $W \neq \{0\}$ and $a_1, \cdots, a_m$ not all 0, we can find $w_1, \cdots, w_m$, such that $a_1w_1 + \cdots + a_mw_m \neq 0$.

$\Rightarrow \forall T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k$, $T(0) = T(a_1v_1 + \cdots + a_mv_m) = a_1Tv_1 + \cdots + a_mTv_m =$

$a_1w_1 + a_mw_m \neq 0$. This is impossible. Hence, the result is proved.

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

Suppose $v_1, \cdots, v_m$ is a basis of V, and $m \geq 2$ for $\dim V > 1$. For $\forall v = a_1v_1 + \cdots + a_mv_m \in V$, where $a_1, \cdots, a_m \in F$, define $S \in \mathcal{L}(V)$: $S(v) = S(a_1v_1 + a_2v_2 + \cdots) = a_2v_1 + a_1v_2 + \cdots$, i.e., switching the coefficients of v_1 and v_2; define $T \in \mathcal{L}(V)$: $T(v) = T(a_1v_1 + \cdots + a_mv_m) = a_1v_1$.

Then, $ST(v_1 + 2v_2) = S(T(v_1 + 2v_2)) = S(v_1) = v_2$, and $TS(v_1 + 2v_2) = T(S(v_1 + 2v_2)) = T(2v_1 + v_2)$

$= 2v_1 \Rightarrow ST \neq TS$. Hence, the result is proved.

17 Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

*A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a **two-sided ideal** of $\mathcal{L}(V)$ if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.*

It's easy to verify that $\{0\}$ and $\mathcal{L}(V)$ are two-sided ideals of $\mathcal{L}(V)$. Also, when $\dim V = 0$, the result is trivial. Suppose $\dim V = n \geq 1$, $v_1, \cdots, v_n$ is a basis of V. Define $T_{ij} \in \mathcal{L}(V)$: $T_{ij}(a_1v_1 + \cdots + a_nv_n) = a_j v_i$, $i, j \in \{1, \cdots, n\}$. Suppose $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$, and $\mathcal{E} \neq \{0\}$, $\mathcal{E} \neq \mathcal{L}(V)$. As $\mathcal{E} \neq \{0\}$, $\exists E \in \mathcal{E}$, $\exists i \in \{1, \cdots, n\}$, $E(v_i) \neq 0$. Suppose $E(v_i) = \sum_{j \in J} e_j v_j$, $J \subseteq \{1, \cdots, n\}$, $e_j \in F$, $e_j \neq 0$. Select a $j \in J$, $\forall k \in \{1, \cdots, n\}$, define $S_k = \frac{1}{e_j} T_{kj} E T_{ik}$. Then, for $\forall a_1v_1 + \cdots + a_nv_n \in V$, $S_k(a_1v_1 + \cdots + a_nv_n)$
$= (\frac{1}{e_j} T_{kj} E)(T_{ik}(a_1v_1 + \cdots + a_nv_n)) = (\frac{1}{e_j} T_{kj} E)(a_k v_i) = \frac{a_k}{e_j} T_{kj}(E(v_i)) = \frac{a_k}{e_j} T_{kj}(\sum_{j \in J} e_j v_j)$
$= \frac{a_k}{e_j} \cdot e_j v_k = a_k v_k$. By the property of two-sided ideal, $S_k \in \mathcal{E}$, implying that $I = S_1 + \cdots + S_n \in \mathcal{E}$. As $\mathcal{E} \neq \mathcal{L}(V)$, $\exists T \in \mathcal{L}(V)$, $T \notin \mathcal{E}$. However, $IT = T \in \mathcal{E}$ as $I \in \mathcal{E}$, leading to a contradiction. Hence, such $\mathcal{E}$ doesn't exist.

$\therefore$ the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.