Solutions to 3C :

1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Assume there exists a choice of bases of V and W, denoted by $v_1, \cdots, v_n$ and $w_1, \cdots, w_m$, such that $A = \mathcal{M}(T)$ has less than $\dim \operatorname{range} T$ nonzero entries. Without loss of generality, suppose only first r columns of A, i.e., $A_{\cdot,1}, \cdots, A_{\cdot,r}$ have nonzero entries, and obviously $r < \dim \operatorname{range} T$. This implies $Tv_{r+1} = \cdots = Tv_n = 0$. Then, $\forall w \in \operatorname{range} T$, $\exists a_1, \cdots, a_n \in \mathbb{F}$, $w = T(a_1 v_1 + \cdots + a_n v_n) = a_1 Tv_1 + \cdots + a_n Tv_n = a_1 Tv_1 + \cdots + a_r Tv_r \in \operatorname{span}(Tv_1, \cdots, Tv_r)$.
$\Rightarrow \operatorname{range} T \subseteq \operatorname{span}(Tv_1, \cdots, Tv_r) \Rightarrow \dim \operatorname{range} T \leq r$, a contradiction. Hence, such choice of bases doesn't exist, implying $\mathcal{M}(T)$ has at least $\dim \operatorname{range} T$ nonzero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

① if $\dim \operatorname{range} T = 1$:

Select $w \in \operatorname{range} T$, $w \neq 0$. Let $u_1, \cdots, u_m$ be a basis of W. Suppose $w = a_1 u_1 + \cdots + a_m u_m = \sum_{i \in I} a_i u_i$, $I \subseteq \{1, \cdots, m\}$, $I \neq \emptyset$ as $w \neq 0$, $a_i \neq 0$ for $i \in I$. Suppose $|I| = r$.
Fix $j \in I$, let : $w_j = a_j u_j - (m-r) u_j$; $w_k = a_k u_k$, $k \in I$ and $k \neq j$; $w_k = u_k + u_j$, $k \notin I$. It's easy to verify that $w_1, \cdots, w_m$ is also a basis of W, and $w = w_1 + \cdots + w_m$.
Suppose $\dim V = n$. Then $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = n - 1$. As $w \in \operatorname{range} T$ and $w \neq 0$, $\exists v_1 \in V$ and $v_1 \neq 0$, $w = Tv_1$. Let $x_2, \cdots, x_n$ be a basis of null T, and let $v_k = x_k + v_1$, where $k = 2, \cdots, n$. It's easy to verify that $v_1, \cdots, v_n$ is a basis of V, and $Tv_k = Tv_1 = w = w_1 + \cdots + w_m$, $k = 1, \cdots, n$. Hence, we construct $w_1, \cdots, w_m$ and $v_1, \cdots, v_n$, such that the corresponding $\mathcal{M}(T)$'s all entries are 1.

② if $\exists \mathcal{M}(T, (v_1, \cdots, v_n), (w_1, \cdots, w_m)) \in \mathbb{F}^{m,n}$, whose entries are all 1 :

$Tv_1 = \cdots = Tv_n = w_1 + \cdots + w_m$. $\forall y_1, y_2 \in \operatorname{range} T$, $\exists a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{F}$,
$y_1 = T(a_1 v_1 + \cdots + a_n v_n) = a_1 Tv_1 + \cdots + a_n Tv_n = a_1 (w_1 + \cdots + w_m) + \cdots + a_n (w_1 + \cdots + w_m) = (a_1 + \cdots + a_n)(w_1 + \cdots + w_m)$, $y_2 = (b_1 + \cdots + b_n)(w_1 + \cdots + w_m)$, similarly. If any of y_1, y_2 is 0, then y_1, y_2 is linearly independent. If $y_1, y_2 \neq 0$, implying $a_1 + \cdots + a_n \neq 0$ and $b_1 + \cdots + b_n \neq 0$, then $y_1 = \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} y_2 \Rightarrow y_1, y_2$ is linearly independent. Thus $\dim \operatorname{range} T < 2$.
Also, $Tv_1 = w_1 + \cdots + w_m \neq 0$ and $Tv_1 \in \operatorname{range} T$, implying $\operatorname{range} T \neq \{0\} \Rightarrow \dim \operatorname{range} T > 0$.
Hence, $\dim \operatorname{range} T = 1$.

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.

Suppose $\dim V = n$, $\dim W = m$, $\dim \operatorname{range} T = r$. Let $w_1, \cdots, w_r$ be a basis of range T, then it can be extended to a basis of W : $w_1, \cdots, w_m$. As $w_i \in \operatorname{range} T$, where $i \in \{1, \cdots, r\}$, $\exists v_i \in V$, such that $w_i = Tv_i$. $v_1, \cdots, v_r$ is linearly independent. otherwise, $\exists a_i \in F$, not all 0, $0 = a_1v_1 + \cdots + a_rv_r \Rightarrow 0 = T0 = T(a_1v_1 + \cdots + a_rv_r)$ $= a_1Tv_1 + \cdots + a_rTv_r = a_1w_1 + \cdots + a_rw_r$, a contradiction to that $w_1, \cdots, w_r$ is linearly independent as it's a basis. Then $\dim \operatorname{span}(v_1, \cdots, v_r) = r = \dim \operatorname{range} T$. Consider $\forall v \in \operatorname{null} T \wedge \operatorname{span}(v_1, \cdots, v_r)$, $Tv = T(a_1v_1 + \cdots + a_rv_r) = a_1w_1 + \cdots + a_rw_r = 0 \Rightarrow a_1 = \cdots = a_r = 0$, implying $v = 0 \Rightarrow \operatorname{null} T \cap \operatorname{span}(v_1, \cdots, v_r)$ $= \{0\}$. Then, $\dim(\operatorname{null} T + \operatorname{span}(v_1, \cdots, v_r)) = \dim \operatorname{null} T + \dim \operatorname{span}(v_1, \cdots, v_r) - \dim(\operatorname{null} T \cap \operatorname{span}(v_1, \cdots, v_r))$ $= \dim \operatorname{null} T + \dim \operatorname{range} T - 0 = \dim V$, implying $V = \operatorname{null} T \oplus \operatorname{span}(v_1, \cdots, v_r)$. Hence, let $v_{r+1}, \cdots, v_n$ be a basis of $\operatorname{null} T$, $v_1, \cdots, v_r, v_{r+1}, \cdots, v_n$ is a basis of V, and $Tv_{r+1} = \cdots = Tv_n = 0$. Thus, we construct $\mathcal{M}(T, (v_1, \cdots, v_n), (w_1, \cdots, w_m))$ that all entries are 0 except that the entries in row k, column k, equal 1 if $1 \le k \le \dim \operatorname{range} T$.

6 Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V.

①. If $Tv_1 = 0$, then select an arbitrary basis of W, denoted by $w_1, \cdots, w_n$. We have $Tv_1 = 0 = 0w_1 + \cdots + 0w_n$, implying all entries in the first column of $\mathcal{M}(T)$ are 0.

②. If $Tv_1 \neq 0$, let $w_1 = Tv_1$. As w_1 is linearly independent for $w_1 \neq 0$, we can extend it to a basis of W, $w_1, \cdots, w_n$. Then, as $Tv_1 = w_1$, all entries in the first column of $\mathcal{M}(T)$ are 0 except for a 1 in the first row, first column.

7 Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of W instead of being able to choose a basis of W.

① if $\operatorname{range} T \subseteq \operatorname{span}(w_2, \cdots, w_n)$:

Select any basis $v_1, \cdots, v_m$ of V, $Tv_k = a_{2,k}w_2 + \cdots + a_{n,k}w_n = 0w_1 + a_{2,k}w_2 + \cdots + a_{n,k}w_n$, where $k = 1, \cdots, m$. This implies the corresponding $\mathcal{M}(T)$'s first row is all 0.

② if $\operatorname{range} T \nsubseteq \operatorname{span}(w_2, \cdots, w_n)$:

Then, $\exists x_1 \in V$, such that $Tx_1 = a_{1,1}w_1 + \cdots + a_{n,1}w_n$, where $a_{1,1} \neq 0$. Let $v_1 = \frac{x_1}{a_{1,1}}$, and we can write $Tv_1 = w_1 + b_2w_2 + \cdots + b_nw_n$. Obviously $v_1 \neq 0$, then v_1 is linearly independent. Hence v_1 can be extended to $v_1, x_2, \cdots, x_m$, a basis of V. Consider $Tx_k = a_{1,k}w_1 + \cdots + a_{n,k}w_n$, where $k = 2, \cdots, m$, let $v_k = x_k - a_{1,k}v_1$. It's easy to verify that $v_1, v_2, \cdots, v_m$ is also a basis of V, and $Tv_k = 0w_1 + (a_{2,k} - a_{1,k}b_2)w_2 + \cdots + (a_{n,k} - a_{1,k}b_n)w_n$ for $k = 2, \cdots, m$. Hence, the corresponding $\mathcal{M}(T)$'s first row is all 0 except for a 1 in the first column.

11 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B + C) = AB + AC \quad \text{and} \quad (D + E)F = DF + EF.$$

As $A(B+C)$ and $(D+E)F$ make sense, we can suppose $A, D, E \in F^{m,n}$, $B, C, F \in F^{n,p}$ without loss of generality. Thus, AB, AC make sense and $AB, AC \in F^{m,p}$, hence $AB+AC$ makes sense; similarly, $DF, EF \in F^{m,p}$ and $DF+EF$ makes sense.

① $A(B+C) = AB+AC$:

$A(B+C),\ AB+AC \in F^{m,p}$.

$$(A(B+C))_{j,k} = \sum_{r=1}^{n} A_{j,r}(B+C)_{r,k} = \sum_{r=1}^{n} A_{j,r}(B_{r,k}+C_{r,k}) = \sum_{r=1}^{n}(A_{j,r}B_{r,k}+A_{j,r}C_{r,k})$$
$$= \sum_{r=1}^{n} A_{j,r}B_{r,k} + \sum_{r=1}^{n} A_{j,r}C_{r,k} = (AB)_{j,k}+(AC)_{j,k} = (AB+AC)_{j,k}$$

② $(D+E)F = DF+EF$:

$(D+E)F,\ DF+EF \in F^{m,p}$.

$$((D+E)F)_{j,k} = \sum_{r=1}^{n}(D+E)_{j,r}F_{r,k} = \sum_{r=1}^{n}(D_{j,r}+E_{j,r})F_{r,k} = \sum_{r=1}^{n}(D_{j,r}F_{r,k}+E_{j,r}F_{r,k})$$
$$= \sum_{r=1}^{n} D_{j,r}F_{r,k} + \sum_{r=1}^{n} E_{j,r}F_{r,k} = (DF)_{j,k}+(EF)_{j,k} = (DF+EF)_{j,k}$$

12 Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

Let $A = \mathcal{M}(T_1)$, $B = \mathcal{M}(T_2)$, $C = \mathcal{M}(T_3)$, where T_1, T_2, T_3 are linear maps.

As $(AB)C$ makes sense, $(AB)C = (\mathcal{M}(T_1)\mathcal{M}(T_2))\mathcal{M}(T_3) = \mathcal{M}(T_1T_2)\mathcal{M}(T_3) = \mathcal{M}((T_1T_2)T_3)$, implying that $(T_1T_2)T_3$ makes sense. By the associativity of products of linear maps, $(T_1T_2)T_3 = T_1(T_2T_3)$. Hence, $(AB)C = \mathcal{M}((T_1T_2)T_3) = \mathcal{M}(T_1(T_2T_3)) = \mathcal{M}(T_1)\mathcal{M}(T_2T_3)$
$= \mathcal{M}(T_1)(\mathcal{M}(T_2)\mathcal{M}(T_3)) = A(BC)$.

13 Suppose A is an n-by-n matrix and $1 \le j,k \le n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

$$(A^3)_{j,k} = ((AA)A)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}\Big(\sum_{p=1}^{n}A_{j,p}A_{p,r}\Big)A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n}A_{j,p}A_{p,r}A_{r,k}.$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^{\mathrm{t}}$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Let $T: F^{m,n} \to F^{n,m}$, $T(A) = A^t$.

① additivity:

$\forall A_1, A_2 \in F^{m,n}$, $(A_1+A_2)^t_{j,k} = (A_1+A_2)_{k,j} = A_{1\,k,j} + A_{2\,k,j} = A_1{}^t_{j,k} + A_2{}^t_{j,k}$
$= (A_1^t + A_2^t)_{j,k}$, implying $(A_1+A_2)^t = A_1^t + A_2^t$.

Hence, $T(A_1+A_2) = (A_1+A_2)^t = A_1^t + A_2^t = T(A_1)+T(A_2)$

② homogeneity:

$\forall A \in F^{m,n}$, $\forall \lambda \in F$, $(\lambda A)^t_{j,k} = (\lambda A)_{k,j} = \lambda A_{k,j} = \lambda A^t_{j,k}$, implying
$(\lambda A)^t = \lambda A^t$. Hence, $T(\lambda A) = (\lambda A)^t = \lambda A^t = \lambda T(A)$.

Thus, $T: A \mapsto A^t \in \mathcal{L}(F^{m,n}, F^{n,m})$.

15 Prove that if A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{t} = C^{t}A^{t}.$$

$A \in F^{m,n}$, $C \in F^{n,p}$ $\Rightarrow$ $AC \in F^{m,p}$ $\Rightarrow$ $(AC)^t \in F^{p,m}$. Also, $C^t \in F^{p,n}$, $A^t \in F^{n,m}$ $\Rightarrow$ $C^tA^t \in F^{p,m}$. Thus $(AC)^t$ and C^tA^t make sense and have the same size.

$(AC)^t_{j,k} = (AC)_{k,j} = \sum_{r=1}^{n} A_{k,r} C_{r,j} = \sum_{r=1}^{n} C^t_{j,r} A^t_{r,k} = (C^tA^t)_{j,k}$. $\therefore (AC)^t = C^tA^t$.

16 Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

① if rank $A = 1$:

Let $A = CR$ be the column-row factorization of A. rank $A = 1$ implies that $C \in F^{m,1}$ and $R \in F^{1,m}$. Then, let $(c_1, \cdots, c_m) \in F^m$ and $(d_1, \cdots, d_n) \in F^n$, where $c_j = C_{j,1}$ for $j = 1, \cdots, m$ and $d_k = R_{1,k}$ for $k = 1, \cdots, n$. Hence, $A_{j,k} = \sum_{r=1}^{1} C_{j,r} R_{r,k}$ $= C_{j,1} R_{1,k} = c_j d_k$.

② if $\exists c \in F^m$, $\exists d \in F^n$, $A_{j,k} = c_j d_k$:

Let $C \in F^{m,1}$, where $C_{j,1} = c_j$ for $j = 1, \cdots, m$. Then, $A_{\cdot,k} = d_k C$, implying that each column of A is a scalar multiplication of C. Hence, rank $A \leq 1$.

Also, $A \neq 0$, implying rank $A \neq 0$. $\therefore$ rank $A = 1$.

17 Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent.

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Let $A = \mathcal{M}(T) \in F^{n,n}$.

(a) $\Rightarrow$ (b):

We have T is injective. Assume the columns of A are not linearly independent in $F^{n,1}$, hence $\exists b_1, \cdots, b_n \in F$ not all 0, $0 = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}$. Also, $Tv_k = \sum_{j=1}^{n} A_{j,k} u_j$.

Thus, $T(b_1 v_1 + \cdots + b_n v_n) = b_1 T v_1 + \cdots + b_n T v_n = \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} A_{j,k} u_j \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} b_k A_{j,k} \right) u_j$ $= \sum_{j=1}^{n} 0 u_j = 0$. $\Rightarrow b_1 v_1 + \cdots + b_n v_n \in \operatorname{null} T$. As $v_1, \cdots, v_n$ is a basis and $b_1, \cdots, b_n$ not all 0, $b_1 v_1 + \cdots + b_n v_n \neq 0$, implying $\operatorname{null} T \neq \{0\}$ $\Rightarrow$ T is not injective, a contradiction. Hence, the columns of $\mathcal{M}(T)$ are linearly independent.

(b) $\Rightarrow$ (c):

The n columns of $\mathcal{M}(T)$ are linearly independent in $F^{n,1}$, and $\dim F^{n,1} = n$. Hence the columns of $\mathcal{M}(T)$ compose a basis of $F^{n,1}$, thus span $F^{n,1}$.

(c) $\Rightarrow$ (d):

The columns of $\mathcal{M}(T)$ span $F^{n,1}$, and $\dim F^{n,1} = n$ $\Rightarrow$ column rank of $\mathcal{M}(T)$ is n $\Rightarrow$ row rank of $\mathcal{M}(T)$ is also $n = \dim F^{1,n}$ $\Rightarrow$ the rows of $\mathcal{M}(T)$ span $F^{1,n}$.

$(d) \Rightarrow (e)$:

The n rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$, and $\dim \mathbf{F}^{1,n} = n$. Hence the rows of $\mathcal{M}(T)$ compose a basis of $\mathbf{F}^{1,n}$, thus are linearly independent in $\mathbf{F}^{1,n}$.

$(e) \Rightarrow (a)$:

We have that $A_{1,\cdot}, \cdots, A_{n,\cdot}$ is linearly independent. If T is not injective, $\exists v \in \operatorname{null} T$, $v \neq 0$. Let $v = b_1 v_1 + \cdots + b_n v_n$, where $b_1, \cdots, b_n \in \mathbf{F}$ not all 0. Also, $Tv_k = \sum_{j=1}^{n} A_{j,k} u_k$. Then $Tv = T(b_1 v_1 + \cdots + b_n v_n) = \sum_{k=1}^{n} b_k T v_k = \sum_{k=1}^{n} b_k \left(\sum_{j=1}^{n} A_{j,k} u_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} b_k A_{j,k} \right) u_k = 0$. As $u_1, \cdots, u_n$ is a basis, $\sum_{j=1}^{n} b_k A_{j,k} = 0$ for $k = 1, \cdots, n$, where b_k not all 0. This implies ~~that~~ $A_{1,\cdot}, \cdots, A_{n,\cdot}$ is not linearly independent, a contradiction. $\therefore$ T is injective.