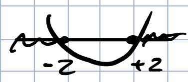
# Esercitazione

$f(x) = \frac{\quad}{D(x)}$ $\qquad D(x) \neq 0 \qquad \frac{\sqrt{12}}{7} \Rightarrow$ non va approssimato

$x^2 - 4 \geq 0$

$x^2 \geq 4$

$x \geq \pm 2$



- correzione del limite che avevo fatto giusto

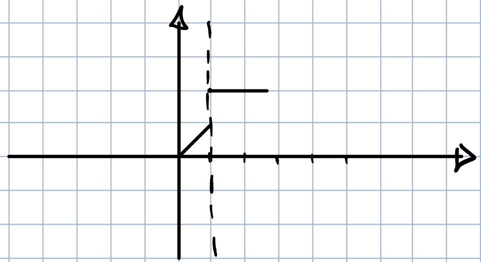
Esercizi su funzioni continue

$\lim_{x \to x_0} f(x) = f(x_0)$ $\qquad f(x) = \begin{cases} 5x - 1 & \text{se } x \geq 0 \\ 2x - a & \text{se } x < 0 \end{cases}$

$\lim_{x \to 0^+} (5x - 1) = -1$ $\qquad \lim_{x \to 0^-} (2x - a) = a$ $\qquad \lim^+ = \lim^-$

$-a = -1$

$f(x) = \begin{cases} x & \text{se } 0 < x < 1 \\ 2 & \text{se } 1 \leq x < 2 \end{cases}$ è def. nell'intervallo limitato $(0, 2)$



$\lim_{x \to 1^-} = 1$

$\lim_{x \to 1^+} = 2$ $\qquad$ salto di 1

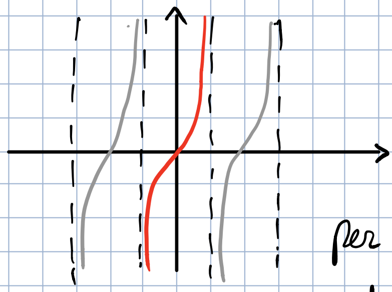
$\lim_{x \to 0^+} x^2 + 2ax + a = a$ $\qquad \lim_{x \to 0^-} \sqrt{x+2} = \sqrt{2}$

$a = \sqrt{2}$

$$f(x) = \begin{cases} \text{arctg } 1/x & \text{se } x \neq 0 \\ 0 \text{ se } & x = 0 \end{cases}$$

definita in tutto $\mathbb{R}$

$$\text{tg}\, x = \frac{\text{sen}\, x}{\cos x}$$



$\text{tg} : \mathbb{R} \to \mathbb{R} \setminus \{ \pi/2, +k\pi \}$

$\text{tg} \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$

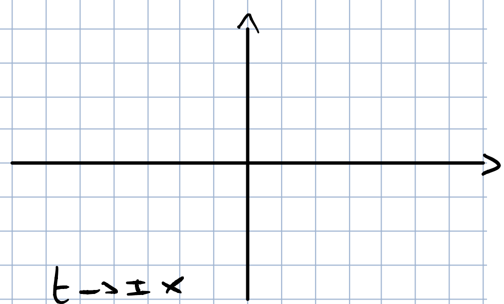
$x_0 =$ è singolarità di 1^ sp

per $x \to 0^-$: $\frac{1}{x} \to +\infty$

diciamo $\frac{1}{x} = t$

$$\lim_{t \to \infty} \text{arctg} = -\pi/2$$

$$f(x) = \begin{cases} x \text{ sen} (1/x) & \text{se } x \neq 0 \\ 1 \text{ se } & x = 0 \end{cases}$$



$$\lim_{x \to 0^{\pm}} x \text{ sen } \frac{1}{x} = \lim \frac{\text{sen } 1/x}{1/x} =$$

$$= \lim_{t \to \infty} \frac{\text{sen}\, t}{t} = \frac{n}{\infty} = 0$$

$t \to \pm \infty$

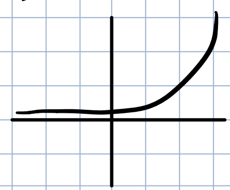
$t = \frac{1}{x}$

$$f(x) = \begin{cases} \text{sen}\, x \cdot 2^{1/x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

def. tutto $\mathbb{R}$, continua in $\mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0^+} \text{sen}\, x \cdot 2^{1/x} = \lim_{x \to 0^+} \left( \frac{\text{sen}}{x} \cdot \frac{2^{1/x}}{1/x} \right) = +(x)$$

$$\lim_{x \to 0^+} \frac{\text{sen}\, x}{x} \cdot \lim_{y \to 0^+} \frac{2^y}{y} = +\infty$$



$$f(x)=\begin{cases} \text{sen}\,x & \text{se } x \leq -\pi/2 \\ a\,\text{sen}\,x + b & \text{se } -\pi/2 < x < \pi/2 \\ \cos x & \text{se } \pi \geq \pi/2 \end{cases}$$

$a, b = ?$ in modo che $f(x)$ sia continua su tutto $\mathbb{R}$

$f(-\pi/2) = \text{sen}(-\pi/2) = -1$

$a\,\text{sen}\,x + b = -1$

$a(-1) + b = -1$

$f(\pi/2) = \cos \pi/2 = 0$

$a\,\text{sen}(\pi/2) = + b = 0$

$$\begin{cases} -a + b = -1 \\ +a + b = 0 \end{cases} \quad \begin{cases} +a + a = +1 \\ b = -a \end{cases}$$

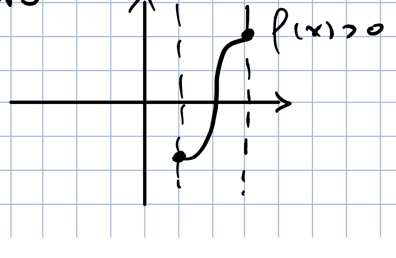
$2a = 1 \quad a = 1/2 \quad b = -1/2$

$$f(x)=\begin{cases} \dfrac{\log(1+x)}{x} & \text{se } -1 < x < 0 \\ \dfrac{e^x - 1}{x} & \text{se } 0 < x < 1 \end{cases}$$

è continua in $(-1, +1)$

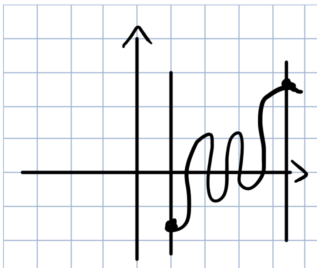
$$\lim_{x \to 0^-} \frac{\log(1+x)}{x} = 1 \qquad \lim_{x \to 0^+} \frac{e^x - 1}{x} = 1$$

Funzioni continue in intervallo $[a,b]$ chiuso e limitato




$\exists\, c \in (a,b)$ t.c.

$f(c) = 0$



$f(x) = [0,7] \to \mathbb{R}$

$x \to f(x) = \sqrt{x+2}$

$x+2 \geq 0 \quad x \geq -2$ quindi continua

non sono verificate le ip in $[0,7]$

• teorema zeri, trovare gli zeri della funzione

$f(x) = x + 2^x$ ha uno "zero" in $[-1,+1]$?

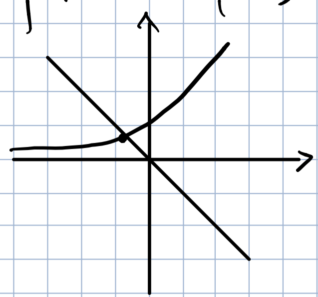
continua in tutto $\mathbb{R}$: somma di f continue in $\mathbb{R}$
quindi anche in $[-1,+1]$

$f(-1) = -1+2^{-1} = -1+\frac{1}{2} = -\frac{1}{2} < 0$

$f(+1) = +1+2 = 3 > 0$

$\to \exists$ un p.to c t.c: $f(c) = 0$ $\quad f(x) = c+2^c = 0$

$$\begin{cases} y = 2^x \\ y = -x \end{cases}$$



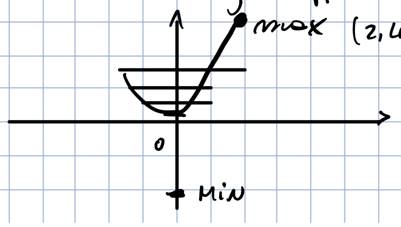
risolvo graficamente

si annulla fra $(-1,0)$

# Teorema di Weierstrass.

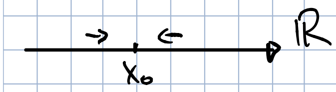
$f(x) = x^2$ in $[-1,2]$ chiuso e limitato

non arco gli opposti / ammette che vi siano min e max




teorema valori intermedi

# Teoria:

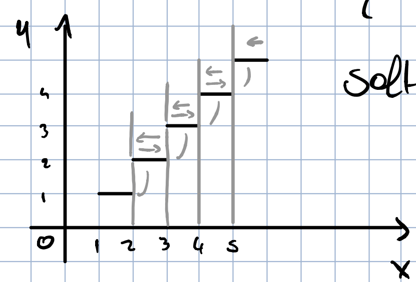


$f: X \to \mathbb{R}$

- $\lim_{x \to 0^+} f(x)$
- $\lim_{x \to 0^-} f(x)$

**Def:** Sia $f: [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$. $f$ si dice continua a destra in $x_0$ se $g: [x_0, b] \to \mathbb{R}$ è continua in $x_0$. $f$ si dice continua a sinistra in $x_0$ se $g: [a, x_0]$ è continua in $x_0$.

- $[x] = \underbrace{\lfloor x \rfloor}^{\text{IN BASSO}} = \max\{n \in \mathbb{Z} \mid n \leq x\}$



salti $\to$ discontinuità

## Prop delle funzioni continue

Siano $f, g: [a,b] \to \mathbb{R}$, $x_0 \in [a,b]$ se $f, g$ sono continue in $x_0$, allora:

$$f+g \text{ e } f \cdot g \text{ sono continue in } x_0$$

Sono f continue:

$\to$ le costanti

$\to x$ e $x^2$ $\to \operatorname{sen}(x)$ e $\cos(x)$

es: $f(x) = x^2 \operatorname{sen} x + 4x^2 \cos x$

intera $\forall$ f. continua

∘) In particolare i polinomi sono funzioni continue

## Principio di identità dei polinomi

Siano p e q due polinomi a coefficienti reali.
Se $p(x) = q(x)$ $\forall x \in \mathbb{R}$ allora p e q hanno gli stessi coefficienti.
Sia p un polinomio a coefficienti reali. Se $p(x)=0$ $\forall x \in \mathbb{R}$, allora tutti i coefficienti di p sono nulli.

$$\begin{cases} p(x) = ax^2+bx+c \\ p(x) = 0 \ \forall x \in \mathbb{R} \end{cases} \Rightarrow a=b=c=0 \ ?$$

con
$$\begin{cases} p(0) = c = 0 \\ p(1) = a+b+c=0 \\ p(2) = 4a+2b+c=0 \end{cases}$$
3 incognite
sistema di equazioni

$\Downarrow$ risoluzione

$$\begin{cases} c=0 \\ a=-b \\ -4b+2b=0 \end{cases} \qquad \begin{cases} c=0 \\ a=0 \\ b=0 \end{cases}$$
Devono essere nulli

• Ipotesi: $\begin{cases} p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ a_n \dots a_0, a_1 \in \mathbb{R} \\ \forall x \in \mathbb{R} \quad p(x)=0 \end{cases} \longrightarrow \forall x \in \mathbb{R}^* \ q(x)=0$

• Tesi: $a_n = a_{n-1} = a_1 = a_0 = 0$

$p(0) = 0 = a_0$

$p(x) = a_n x^n + \dots + a_1 x = x(a_n x^{n-1} + \dots + a_1) = x\,q(x)$

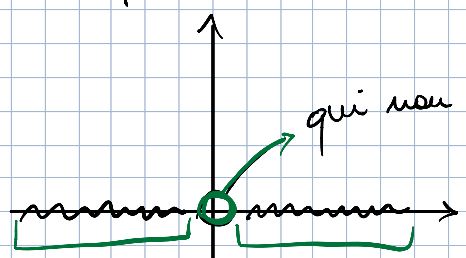
q è polinomio e dunque una funzione continua

$q(x) = a_n x_0 + a_{n-1} x^{n-2} + a_1$

oss: $p(x) = x$ $\forall x \in \mathbb{R}$
$q(x) = 0$

È vero che $q(0) = 0$?

qui non può essere altro che 0



$\forall x \in \mathbb{R}^*$ sappiamo che è continua

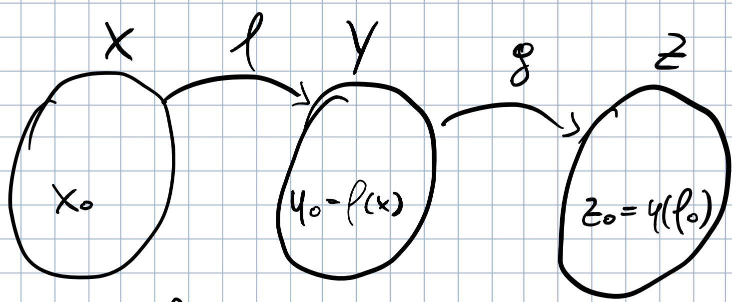
$f(x) = \operatorname{sen}^2(x)$

$g(x) = \operatorname{sen}(x^2)$

## Teorema di composizione:

Siano $f: X \to Y$, $g: Y \to Z$ $(X, Y, Z \in \mathbb{R})$ se $f$ è continua in $x_0 \in X$ e $g$ è continua in $y_0 = f(x_0) \in Y$ allora $g = f$ $X \to Z$ è continua in $x_0$

Dim:




$x_0$, $f(x_0) = y_0$

$z_0 = g(y) = g(f(x_0))$

$\forall \varepsilon > 0 \ \exists \ \delta > 0$ T.C. $\forall y \in Y$ se $|y - y_0| \leq \delta \Rightarrow |g(y) - g(y_0)| \leq \varepsilon$

$\forall \delta > 0 \ \exists \ \beta > 0$ T.C. $\forall x \in X$ se $|x - x_0| \leq \beta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon$

Dunque $\forall \varepsilon > 0 \ \exists \beta > 0$ T.C. $\forall x \in X$ se $x - x_0 \leq \beta \Rightarrow |f(x) - f(x_0)| \leq \delta$

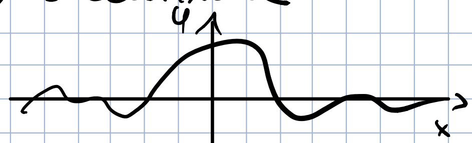
$\exists \delta > 0$ $\Rightarrow |g(x) - g(x_0)| \leq \varepsilon$

**Problema**: Data $f: (a, b) - \{c\} \to \mathbb{R}$ $(c < [a, b)$ è possibile trovare $\tilde{F}: (a, b) \to \mathbb{R}$ t.c. $\tilde{F}$ sia continua in $c$?

es: $\begin{cases} f: \mathbb{R}^* \to \mathbb{R} \\ f(x) = \dfrac{\operatorname{sen} x}{x} \end{cases}$ $\qquad \lim_{x \to 0} \dfrac{\operatorname{sen}(x)}{x} = 1$

$$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ \tilde{f}(x) = \begin{cases} \frac{\text{sen}(x)}{x} & \text{se } x \neq 0 \\ 1 & \text{se } x \neq 0 \end{cases} \end{cases}$$

$\tilde{f}$ è continua




**Def:** dato $f: (a,b) - \{x_0\} \to \mathbb{R}$ ($x_0 \in (a,b)$) si dice che $\lim_{x \to x_0} f(x) = L$

se $\tilde{f}: (a,b) \to \mathbb{R}$ $\quad \tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ L & \text{se } x = x_0 \end{cases}$

è continua in $x_0$

$\lim_{n \to \infty} a = b \quad \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(x) \in \mathbb{N}$ t.c. $\forall n \geq \bar{n}$ si ha $|a_n - L| < \varepsilon$

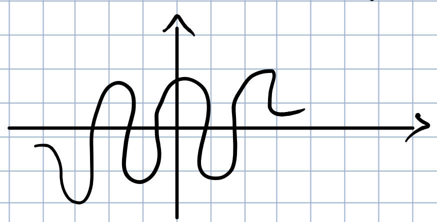
$\lim_{x \to x_0} f(x) = L \quad \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) \in \mathbb{R}^*$ t.c. $\forall x \in \frac{[x_0 - \delta, \ x_0 + \delta]}{|x - x_0| \leq \delta}$ $\boxed{|f(x) - L| \leq \varepsilon}$

**Teo:** si ha che $(x_n)_{n \in \mathbb{N}}$ è una succ. t.c.

$\lim_{n \to \infty} x_n = \bar{x}$ sia $f: D \to \mathbb{R}$ t.c. $\bar{x}, x_n \in D$ $\forall n \in \mathbb{N}$ se $f$ è continua in $\bar{x}$ allora $\lim_{n \to +\infty} f(x_n) = f(\bar{x}) = f\left(\lim_{n \to \infty} x_n\right)$

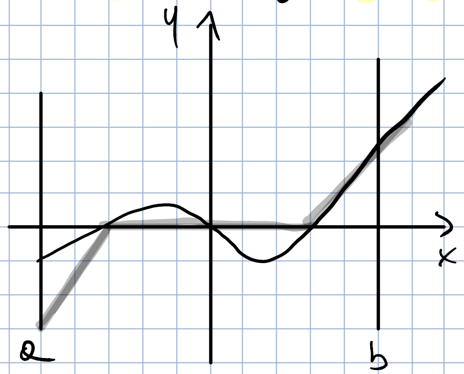
## Limiti di funzioni monotóne

Sia $f: (a,b) \to \mathbb{R}$ e sia $x_0 \in (a,b)$ allora esistono i limiti $\lim_{x \to x_0^-} f(x)$ $\quad \lim_{x \to x_0^+} f(x)$



$y = \text{sen}\left(\frac{1}{x}\right)$

# Teorema degli Zeri



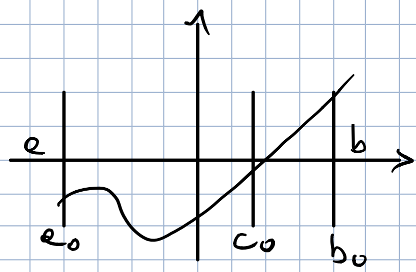

Sia $f: [a,b] \to \mathbb{R}$ continua

$\exists c \in (a,b)$ t.c. $f(c) = 0$

**Dim:** basta prendere

$c = \{x \in [a,b] \mid f(x) < 0\}$

$$\begin{cases} a_0 = a \\ b_0 = b \\ c_n = \dfrac{a_n + b_n}{2} \end{cases}$$




Primo passo:

- Se $f(c_0) = 0$ abbiamo concluso
- Se $f(c_0) < 0$ poniamo $\begin{cases} a = 0 \\ b_1 = b_0 \end{cases}$
- Se $f(c_0) > 0$ poniamo $\begin{cases} a = a_0 \\ b_1 = c_0 \end{cases}$

$f(a_1) < 0 < f(b_1)$

Secondo passo:

- Se $f(c_1) = 0$ abbiamo concluso
- Se $f(c_1) < 0$ poniamo $\begin{cases} a_2 = c_0 \\ b_2 = c_1 \end{cases}$

• Se $f(c_1)>0$ poniamo $\begin{cases} a_2 = a_1 \\ b_2 = c_1 \end{cases}$

$f(a_2)<0<f(b_2)$

Ad ogni passo l'intervallo dimezza

$$\begin{cases} b_0 - a_0 = b - a \\ b_1 - a_1 = \frac{1}{2}(b_0 - a_0) = \frac{b_0 - a_0}{2} \end{cases}$$

$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{b_1 - a_1}{4}$$

$$b_n - a_n = \frac{b_0 - a_0}{2^n} = \frac{b-a}{2^n}$$

Ci sono due possibilità:

① $\exists n \in \mathbb{N}$ t.c. $f(c_n)=0$
il teo è dimostrato con $c=c_n$

② $\forall n \in \mathbb{N}$ si ha $f(c_0) \neq 0$

OSS: $a = a_0 \leq a_1 \leq a_2 \leq$
$b = b_0 \geq b_1 \geq b_2 \geq$

Inoltre

$\lim\limits_{n \to +\infty} a_n = \lim\limits_{n \to \infty} b_0$

$\lim\limits_{n \to \infty} (b_n - a_n) =$

$\lim\limits_{n \to \infty} \frac{b-a}{2} = 0$

Chiamiamo c il valore comune dei due limiti

obiettivo: dimostrare che $f(c)=0$

$$\left.\begin{aligned} \lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right) = f(c) \leq 0 \\ \lim_{n \to \infty} f(b_n) = f\left(\lim_{n \to \infty} b_n\right) = f(c) \geq 0 \end{aligned}\right]$$ teorema della permanenza del segno

$$0 \leq f(c) \leq 0 \Rightarrow f(c)=0$$

$$\begin{cases} f: [0,1] \to \mathbb{R} \\ f(x) = 3\alpha - 1 \end{cases}$$

| | | | |
|---|---|---|---|
| $\begin{cases} a_0 = 0 \\ b_0 = 1 \end{cases}$ | $\begin{cases} a_1 = 0 \\ b_1 = \frac{1}{2} \end{cases}$ | $\begin{cases} a_2 = 1/4 \\ b_2 = 1/2 \end{cases}$ | $\begin{cases} a_3 = \frac{1}{4} \\ b_3 = \frac{3}{8} \end{cases}$ |
| $c_0 = \frac{1}{2}$ | $c_1 = \frac{1}{2}$ | $c_2 = 3/8$ | $c_3 = \frac{8}{10}$ |
| $f(c_0) = \frac{1}{2} > 0$ | $f(c_1) = \frac{1}{4}$ | $f(c_2) = 1/8$ | $f(c_3) = \frac{1}{10}$ |

$\forall c \in \mathbb{N} \quad c_n = \frac{1}{3}$

$c_n = \frac{n}{2^m} \quad m \in \mathbb{N}$

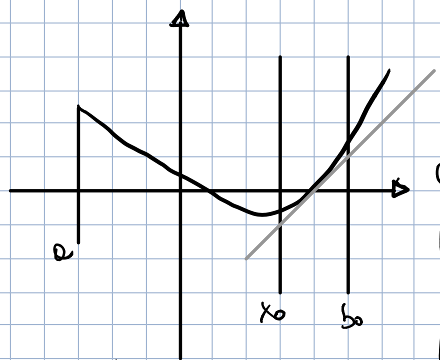
$\iff 2^{n+1} = 3m$

Può succedere che $\quad c_n = \frac{m}{2^m} = \frac{1}{2}$ ?

# Calcolo differenziale:

$f: [a,b] \to \mathbb{R}$

$$m = \frac{f(x_0+h) - f(x_0)}{h}$$




$P(x_0, f(x_0))$

$Q = (x_0+h, f(x_0+h)$

$h \in \mathbb{R}^*$

$y = mx + q$

$\begin{cases} f(x_0) = m x_0 + q \\ f(x_0+h) = m(x_0+h) + q \end{cases}$

$\begin{cases} q = f(x_0) = m x \\ f(x_0+h) = m(x_0+h) + f(x) - m x \end{cases}$

**Def:** chiamiamo rapporto di $f$ in $x_0$ la funzione

$$\rho f(x_1, x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

DEF: Si chiama derivata di $f$ nel punto $x_0$ il valore del limite $\lim_{h \to 0} \rho_f(x_0, h) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

Se questo limite esiste:
il valore si indica con $f'(x_0)$

◦) $f: \mathbb{R} \to \mathbb{R}$
costanti
$f(x) = c$

$\forall x_0 \in \mathbb{R}, \forall h \in \mathbb{R}^*: \rho_f(x_0, h) = \frac{f(x_0+h) - f(x_0)}{h} = 0$

$\forall x_0 \in \mathbb{R}, f'(x_0) = m$

◦) $f(x) = mx + q \quad \rho_f(x_0, h) = \frac{f\, m + (x_0+h) + q - (mx_0 - x)}{h} = m$

$\forall x_0 \in \mathbb{R} \quad f(x_0) = m$

◦) $f(x) = x^2$

$\rho_f(x_0, h) = \frac{(x_0+h)^2 - x_0^2}{h} = \frac{x_0^2 + 2hx_0 + h^2 - x_0^2}{h} = \frac{\cancel{h}(2x_0+h)}{\cancel{h}} =$

$= 2x_0 + h$

$\lim_{h \to 0} \rho_f(x_0, h) = \lim_{h \to 0} (2x_0 + h) = 2x_0$

◦) $f(x) = x^3$

$\rho(x, h) = \frac{(x_0+h)^3 - x_0^3}{h} = \frac{\cancel{x_0^3} + 3hx_0^2 + 3h^2x_0 + h^3 - \cancel{x_0^3}}{h} =$

$= 3x_0^2 + 3hx + h^2 \quad \Big| \quad \lim_{h \to 0} \rho_f(x_0, h) = 3x_0^2$

# Calcoli delle derivate | dimostrazione formule

- $F: \mathbb{R} \to \mathbb{R}$

costanti

$f(x) = c$

$\forall x_0 \in \mathbb{R}, \forall h \in \mathbb{R}^* \quad \rho_f(x_0, h) = \dfrac{f(x_0+h) - f(x_0)}{h} = 0$

$\forall x_0 \in \mathbb{R}, \quad f'(x_0) = 0$

- $f(x) = mx + q \to \rho_f(x_0, h) = \dfrac{f(m(x_0+h)+q) - (mx_0+q)}{h} = m$

- $f(x) = x^2$

$\rho_f(x_0, h) = \dfrac{(x_0+h)^2 - x_0^2}{h} = \dfrac{\cancel{x_0^2} + h^2 + 2x_0 h - \cancel{x_0^2}}{h} = \dfrac{\cancel{h} + 2x_0}{\cancel{h}}$

- $f(x) = x^3$

$\rho_f(x_0+h) = \dfrac{(x_0+h)^3 - x_0^3}{h} = \dfrac{x_0^3 + h^3 + 3x_0^2 h + 3x_0 h^2}{h} =$

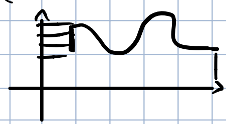
$= \dfrac{\cancel{h}(h^2 + 3x_0^2 + 3x_0 \cdot h)}{\cancel{h}} = 3x_0^2$

- $f(x) = e^x \to \rho_h(x_0+h) = \dfrac{e^{x_0+h} - e^{x_0}}{h} = \dfrac{e^{x_0} \cdot e^{h} - e^{x_0}}{h}$

$= \dfrac{e^{x_0}(e^h - 1)}{h} = e^{x_0} \cancel{\dfrac{(e^h - 1)}{h}}$ $\to 1$ per lim. not.

**Teorema:** se $f [a,b] \to \mathbb{R}$
è continua ed $f(a) < 0 < f(b)$, allora $\exists c \in (a,b)$ t.c.
$f(c) = 0$

**Teorema dei valori intermedi:**
Sia $f:[a,b] \to \mathbb{R}$ continua, allora $f$ assume tutti i valori compresi fra $f(a)$ e $f(b)$

Disegno/non ci vedo 

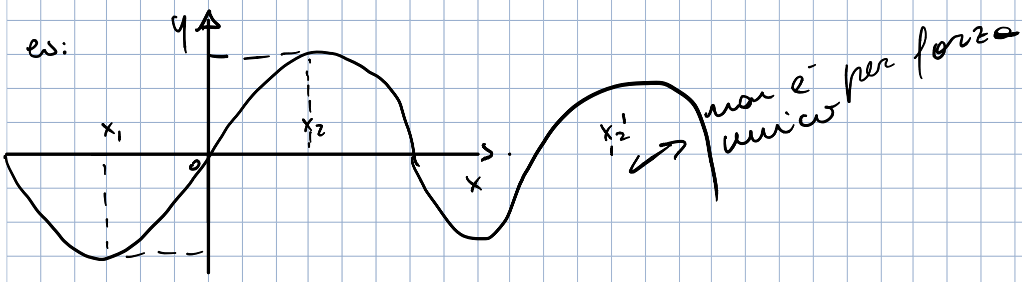
**Dim:** Supponiamo che $f(a) < f(b)$ sia $y_0 \in [f(a), f(b)]$
sia: $g:[a,b] \to \mathbb{R}$, $g(a) = f(x) - y_0$.
<u>OSS</u>: $g$ è continua e inoltre $g(a) = f(a) - y_0 \leq 0 \leq$
$\leq g(b) = f(b) - y_0$. Dunque $\exists c \in [a,b]$ t.c. $g(c) = 0$ cioè
$g(c) = 0 \iff f(c) = y_0$

**Teorema di inversione:**
Sia $f:[a,b] \to \mathbb{R}$ continua e strettamente crescente:
allora $f^{-1}:[f(a), f(b)] \to [a,b]$ è continua e strettamente crescente.

**Teorema del massimo (Bolzano - Weierstrass)**
Sia $f:[a,b] \to \mathbb{R}$ continua $\exists x_1, x_2 \in [a,b]$ t.c.
$\forall x \in [a,b]$ si ha $f(x_1) \leq f(x) \leq f(x_2)$.
$x_1$ si dice punto di minimo (non necessariamente unico)
$x_2$ si dice punto di massimo (non necessariamente unico)
$f(x_1)$ si dice valore minimo (unico)

$f(x_2)$ si dice valore massimo (unico)
L'immagine di $f$ è $[(f x_1), (f x_2)]$

es:




**Dim:** $M = \sup\{f(x) \mid x \in (a,b)\}$

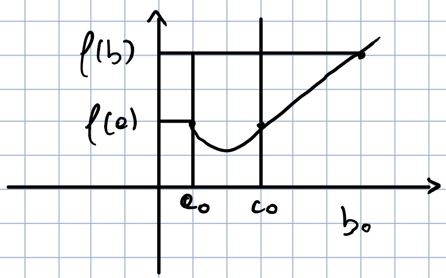
$$\begin{cases} a_0 = a \\ b_0 = b \\ c_n = \frac{a_n + b_n}{2} \end{cases}$$

$M_1' = \sup\{f(x) \mid x \in [a_0, c_0]\}$

$M_0'' = \sup\{f(x) \mid x \in [c_0, b_0]\}$

① $\begin{cases} a_1 = a_0 \\ b_1 = c_0 \end{cases}$ se $M_0' = M$

② $\begin{cases} a_1 = c_0 \\ b_1 = b_0 \end{cases}$ se $M_0'' = M$




$M_1' = \sup\{f(x) \mid x \in [a_1, c_1]\}$

$M_1'' = \sup\{f(x) \mid x \in [c_1, b_1]\}$

$a_0 \le a_1 \le a_2 \le \dots$

$b_0 \ge b_1 \ge b_2 \ge \dots$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = c$ $\quad c$ è il candidato punto massimo

Dato che $f$ è continua in $c$, dato $\varepsilon > 0$

$\exists \delta \in \mathbb{R}^+$ t.c.

$\forall x \in [a,b] \cap [c-\delta, c+\delta]$ si ha $|f(x) - f(c)| \leq \varepsilon$

cioè $f(x) \leq f(c) + \varepsilon$

Dato questo segue che $M < +\infty$

- Inoltre ricordando che per definizione di $M$ si ha $f(c) \leq M$ osserviamo anche che $\exists \bar{n} = \bar{n}(\varepsilon)$ t.c. $[a_n, b_n] \subseteq [c-\delta, c+\delta]$ e da questo ne segue $M = \sup\{f(x) / x \in [a_{\bar{n}}, b_{\bar{n}}]\} \leq f(c) + \varepsilon$

In definitiva, $\forall \varepsilon > 0$:

$$f(c) \leq M \leq f(c) + \varepsilon$$

$\Rightarrow f(c) = M$ cioè $(c = x_2, M = f(x_2))$

## Derivate:

Indicate come $f'$, o come $\frac{d}{dx} f$ o $Df$

- $k \in \mathbb{R}$ $\quad Dk = 0$
- $D(mx+q) = 2x$
- $D(x^2) = 2x$
- $D(x^3) = 3x^2$
- $D(e^x) = e^x$
- $D(x^n) = nx^{n-1}$ $\quad n \in \mathbb{N}$
- $D \log(x) = 1/x$
- $D(\sqrt{z}) = \frac{1}{2\sqrt{x}}$

$f(x) = x^n \quad x_0 \in \mathbb{R} \quad n \in \mathbb{N}$

$$\rho_f(x_0, h) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$\rho_f(x_0, h) = \frac{(x_0+h) - f(x_0)}{h}$$

$$= \frac{x_0^n + \binom{n}{1} x_0^{n+1} h + \binom{n}{2} x_0^{n+2} h^2 + \ldots + \binom{n}{n+1} x_0 h^{n-1} + h^{n-1}}{h}$$

$$= \frac{h\left(\binom{n}{1} x_0^{n+1} + \binom{n}{2} x_0^{n-2} h + \ldots + \binom{n}{n-1} x_0 h^{n-2} + h^{n-1}\right)}{h}$$

$$\lim_{h \to \infty} \rho_f(x_0, h) = \lim_{h \to \infty} \left( \binom{n}{1} x_0^{n-1} + \binom{n}{2} x_0^{n-2} h + \ldots + h^{n-1} \right) = \binom{n}{1} x_0^{n-1} = n x_0^{n-1}$$

$$f(x)=\sqrt{x} \qquad \rho_f(x_0,h)=\frac{\sqrt{x_0+h}-\sqrt{x_0}}{h}=\frac{\sqrt{x_0+h}+\sqrt{x_0}}{\sqrt{x_0+h}+\sqrt{x_0}}$$

$$=\frac{(x_0+h)-(x_0)}{h\left(\sqrt{x_0+h}+\sqrt{x_0}\right)}=\frac{1}{\sqrt{x_0+h}+\sqrt{x_0}}=\frac{1}{2\sqrt{x_0}}$$

$$f(x)=\log x$$

$$\rho_f(x_0,h)=\frac{\log(x_0+h)-\log(x_0)}{h}=\frac{\log\left(\frac{x_0+h}{x_0}\right)}{h}=\frac{\log\left(1+\frac{h}{x_0}\right)}{h}$$

$$\lim_{h\to 0}\rho_f(x_0,h)=\lim_{h\to 0}\frac{\log\left(1+\frac{h}{x_0}\right)}{h}\overset{(*)}{=}\lim_{y\to 0}\frac{\log(1+y)}{y}=1$$

$$(*)=\lim_{y\to 0}\frac{\log(1+y)}{x_0 y}=\frac{1}{x_0}\lim_{y\to 0}\frac{\log(1+y)}{y}=\frac{1}{x_0}$$

# Continuità

$f:[a,b]\to[f(a),f(b)]$ monotona strettamente cresc.
Se è continua, allora anche $f^{-1}:[f(a),f(b)]\to[a,b]$ è continua.

$\begin{cases} \text{sen } [-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1] \\ \text{arcsen } [-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}] \text{ è continua} \end{cases}$

$\begin{cases} \cos [0,\pi]\to[-1,1] \\ \text{arccos } [-1,1]\to[0,\pi] \text{ è continua} \end{cases}$

$\begin{cases} \text{tg } (-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R} \\ \text{arctg } :\mathbb{R}\to(-\frac{\pi}{2},\frac{\pi}{2}) \text{ è continua} \end{cases}$

$\exp_a:\mathbb{R}\to\mathbb{R}^+ \quad (a\in\mathbb{R}^+\setminus\{1\})$ è continua

Dobbiamo dimostrare che $\forall x_0\in\mathbb{R} \quad \lim_{x\to x_0} e^x = e^{x_0}$

$\lim_{x\to x_0}(e^{x-x_0}\cdot e^{x_0})$

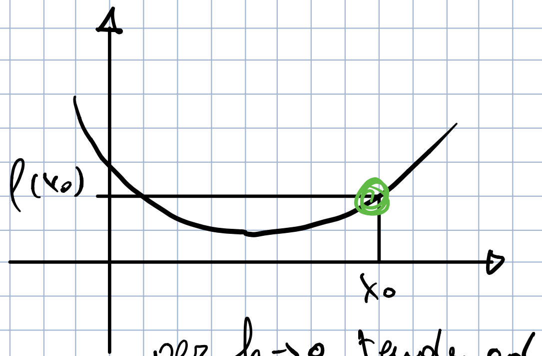
Dobbiamo dimostrare che $e^{x_0}\lim_{x\to x_0} e^x = e^{x_0}$

$\lim_{x\to x_0} e^{x-x_0} = 1 \qquad h = x-x_0 \qquad \Longleftrightarrow \quad \lim_{h\to 0} e^h = 1$

# Calcolo differenziale

$f:[a,b]\to\mathbb{R}$ $\quad x_0\in(a,b)$

$$\begin{cases}\rho_f\,(x,x_0)=\dfrac{f(x)-f(x_0)}{x-x_0}\\ \rho_f\,[a,b]-\{x_0\}\to\mathbb{R}\end{cases}$$



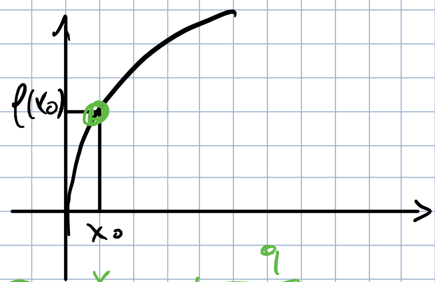

per $h\to0$ tende ad un valore

Dato:

$$f(x_0)=\lim_{h\to0}\frac{f(x_0+h)-f(x_0)}{h}$$

con es:

$$D\sqrt{x}=\frac{1}{2\sqrt{x}}\ \to \text{graficamente}$$




$y=mx+q$

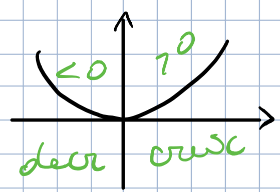
$$\begin{cases}f(x_0)=mx_0+q\\ f'(x_0)=m\end{cases}\longrightarrow\quad y=\overbrace{f'(x_0)}^{m}\overbrace{(x-x_0)}^{x}\overset{+}{+}\overbrace{f(x_0)}^{q}$$

quindi la retta passante per il punto $x_0$ fatidico

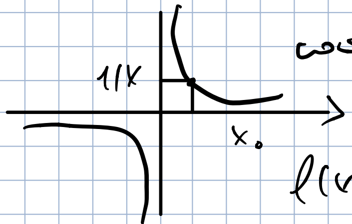
- la positività e negatività della funzione derivata indica la crescenza e la decrescenza

$f'(x_0)>0$

$f'(x_0)<0$




$$\begin{cases}f:\mathbb{R}^*\to\mathbb{R}\\ f(x)=\frac{1}{x}\end{cases}\qquad x_0\in\mathbb{R}^*$$




caso $\frac{1}{x}$

$f(x_0+h)$

So che:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\downarrow$$

$$\lim_{h \to 0} \frac{1}{h} \left( \frac{x_0 - (x_0+h)}{x_0(x_0+h)} \right) = \lim_{h \to 0} \frac{\cancel{x_0} - \cancel{x_0} - h}{h x_0 (x_0+h)} = \frac{1}{x_0^2}$$

$D \operatorname{sen}(x) = \cos(x) \quad x_0 \in \mathbb{R}$

$$\lim_{h \to 0} \frac{\operatorname{sen}(x_0+h) + \operatorname{sen}(x_0)}{h} =$$

$$\lim_{h \to 0} \frac{\operatorname{sen}(x_0)\cos(h) + \cos(x_0)\operatorname{sen}(h) - \operatorname{sen}(x_0)}{h}$$

$$\lim_{h \to 0} \left( \cos(x_0) \cdot \frac{\operatorname{sen}(h)}{h} + \operatorname{sen}(x_0) \frac{\cos(h) - 1}{h} \right)$$

$$\cos(x_0) + \operatorname{sen}(x_0) \lim_{h \to 0} \frac{\cos(h) - 1}{h} = \cos(x_0)$$

## Derivate

$D x^\alpha = x^{\alpha - 1} \qquad (\alpha \in \mathbb{N},\ \alpha = \frac{1}{2}, \alpha = -1)$

$D e^x = e^x$

$D \log(x) = \frac{1}{x}$

$D \operatorname{sen}(x) = \cos(x)$

$D \cos(x) = -\operatorname{sen}(x)$

### Regole di derivazione

$D(f+g) = Df + Dg$

$D(k f) = k Df \quad \forall k \in \mathbb{R}$

$D(f \cdot g) = (Df) g + f (Dg)$

$D\left(\frac{1}{g}\right) = -\frac{Dg}{g^2}$

$D\left(\frac{f}{g}\right) = \frac{(Df) \cdot g - f(Dg)}{g^2}$

$$\lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h)-f(x_0)\,g(x_0)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h)-f(x_0)\,g(x_0+h)+f(x_0)\,g(x_0+h)-f(x_0)\,g(x_0)}{h}$$

$$\lim_{h\to 0} \left( \frac{f(x_0+h)-f(x_0)}{h} \cdot g(x_0+h) + f(x_0)\, \frac{g(x_0+h)-g(x_0)}{h} \right)$$

$$= f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$

$$\lim_{h\to 0} \frac{\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}}{h} =$$

$$\lim_{h\to 0} \frac{1}{h} \left( \frac{g(x_0)-g(x_0+h)}{g(x_0)\,g(x_0+h)} \right) =$$

$$\lim_{h\to 0} \left( -\frac{1}{g(x_0)\,g(x_0+h)} \quad \frac{g(x_0+h)-g(x_0)}{h} \right) =$$

$$= -\frac{1}{g(x_0)^2}\, g'(x_0) = -\frac{g'(x_0)}{g^2(x_0)}$$

$$D\left(\frac{f}{g}\right) = D\left(f \cdot \frac{1}{g}\right) = (Df)\,\frac{1}{g} + f\, D\left(\frac{1}{g}\right) =$$

$$\frac{Df}{g} + f \cdot \left(-\frac{Dg}{g^2}\right) = \frac{(Df)\cdot g - f(Dg)}{g^2}$$

$$D x^{-n} = D\left(\frac{1}{x^n}\right) = -\frac{n x^{n-1} - \frac{1}{x_0}}{h}$$

$f(x) = x^n \quad (n \in \mathbb{N})$

$$= -\frac{n x^{n-1}}{x^{2n}}$$

$f'(x) = n x^{n-1}$

$$= -n \cdot x^{-n-1} \qquad (\alpha = -n)$$

$$D\, tg(x) = D\left(\frac{sen(x)}{cos(x)}\right) = \frac{(D\, sen(x)) \cdot cos(x) - sen(x)\, D(cos(x))}{cos^2 x}$$

$f(x) = sen(x)$

$g(x) = cos(x)$

$$= \frac{cos^2(x) - sen(x)(-sen(x))}{cos^2(x)}$$

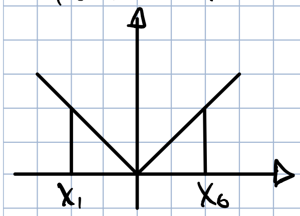
$$= \frac{cos^2(x) + sen^2(x)}{cos^2(x)} = \frac{1}{cos^2(x)} = 1 + tg^2(x)$$

$$f(x) = \frac{2x}{1+x^2} \qquad f'(x) = \frac{2(1+x^2) - 2x \cdot 2x}{(1+x^2)^2} =$$

$$\frac{2 + 2x^2 - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

TEORIA:

$$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = |x| \end{cases}$$




$\forall x > 0 \quad f'(x) = 1$

$\forall x_1 < 0 \quad f'(x_1) = 1$

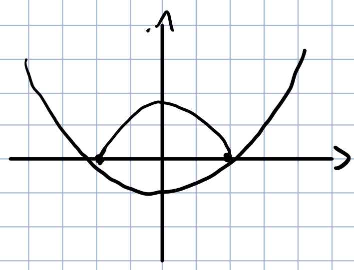
$$\rho_f(0,h) = \frac{|h| - |0|}{h} = \frac{|h|}{h}$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1$$

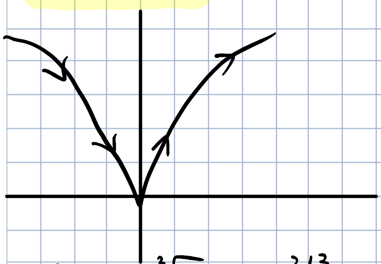
$$\lim_{h \to 0^-} \frac{|h|}{h} = -1$$

$f(x) = |x^2 - 1|$

$f(x) = |e^x|$



CUSPIDI:



$f(x) = \sqrt[3]{x^2} = x^{2/3}$

Teorema: Sia $f: [a,b] \to \mathbb{R}$, $x_0 \in (a,b)$. Se $f$ è derivabile in $x_0$, allora $f$ è continua in $x_0$

Dim: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

$\forall \varepsilon > 0 \; \exists \delta > 0$ t.c. $\forall h \in (-\delta, \delta) - \{0\}$ si ha $\left| f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \right| \leq \varepsilon$

$$-\varepsilon \leq f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h} \leq \varepsilon$$

$$-\varepsilon |h| \leq |h| f'(x_0) - (f(x_0+h) - f(x_0)) \frac{|h|}{h} \leq \varepsilon |h|$$

$$\underbrace{-\varepsilon|h| + |h| f'(x_0)}_{0} \leq \frac{|h|}{h} \underset{\overset{\downarrow}{0}}{(f(x_0+h) - f(x_0))} \leq \underbrace{\varepsilon|h| + |h| f'(x_0)}_{0}$$

$\lim_{h \to 0}$

Teorema del confronto

Se $f$ è derivabile in $x_0$, allora è differenziabile, cioè esiste una funzione lineare t.c. $f(x_0+h) - f(x_0) = h f'(x_0) + h r(h)$ dove $\lim_{h \to 0} r(h) = 0$

## Derivata della f. composta

$$D(g(f(x))) = f'(x) g'(f(x))$$

**Dim:** prendiamo $x_0 \in dom(f)$, $y_0 = f(x_0) \in dom(g)$, esistono $\alpha, \beta$ t.c.:

$$f(x_0+h) - f(x_0) = h f'(x_0) + h \alpha(h) \quad \text{con } \lim_{h \to 0} \alpha(h) = 0$$
$$g(y_0+k) - g(y_0) = k g'(y_0) + k \beta(k) \quad \lim_{k \to 0} \beta(k) = 0$$

prendiamo $k = f(x_0+h) - f(x_0)$

$$g(y+k) - g(y_0) = g(y_0 + f(x_0+h) - f(x_0)) - g(y_0) = g(f(x_0+h)) - g(f(x_0))$$

II Membro $\quad k = f(x_0+h) - f(x_0) \quad \lim_{h \to 0} k = 0$

$$= k g'(y_0) + k \beta(k)$$
$$= (f(x_0+h) - f(x_0)) g'(f(x_0)) + (f(x_0+h) - f(x_0)) \beta(f(x_0+h) - f(x_0))$$
$$= (h f'(x_0) + h\alpha(h)) g'(f(x_0)) + (h f'(x_0) + h\alpha(h)) \beta(h f'(x_0) + h\alpha(h))$$

$$= h((f'(x_0) + \alpha(h)) g'(f(x_0)) + (f'(x_0) + \alpha(h)) \beta(h f'(x_0) + h\alpha(h)))$$

$$\lim_{h \to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{h} =$$

$$\lim_{h \to 0} ((f'(x_0) + \alpha(h)) g'(f(x_0)) + (f'(x_0) + \alpha(h)) \beta(h f'(x_0) + h\alpha(h)))$$

$D\, sen^2(x)$ $\qquad D\, sen^2(x) = D(sen(x)\, sen(x))$

$D\, sen^2(x)$ $\qquad = \cos(x)\, sen(x) + sen(x)\cos(x)$

$= 2\, sen(x)\cos(x)$

$f(x) = sen(x)$ $\qquad D\, sen^2(x) = \cos(x)\; 2\, sen(x)$

$g(x) = x^2 \quad g'(x) = 2x$

$D\, sen(x^2) = 2x \cdot \cos(x^2)$

$f(x) = x^2 \qquad g(x) = sen(x)$

$f'(x) = 2x \qquad g'(x) = \cos(x)$

$Dg(f(x)) = f'(x)\, g'(f(x))$

$Dx^\alpha = \alpha x^{\alpha-1} \quad \forall x \in \mathbb{R}$

$x > 0 \qquad x^\alpha = \exp(\log(x^\alpha)) = \exp(\alpha \log(x))$

$f(x) = \alpha \log(x) \qquad g(x) = e^x$

$f'(x) = \frac{\alpha}{x} \qquad g'(x) = e^x$

$D\, x^\alpha = \frac{\alpha}{x} \exp(\alpha \log(x)) = \frac{\alpha}{x} \cdot x^\alpha$

$x > 0 \qquad x = \exp(\log(x))$ facciamo finta di <u>NON</u> conoscere

$D \log(x) \rightarrow Dx = D\exp(\log(x))$

$1 = (D\log(x)) \cdot \exp(\log(x)) = D(\log(x)) \cdot x$

$1 = (D\log(x)) \cdot x \quad = D\log(x) = \frac{1}{x}$

---

$x = tg(arctg(x))$ $\quad 1 = Dx = (D\, arctg(x))\, (arctg(x))$

$1 = (D\, arctg(x))(1+tg^2\, (arctg(x)) = (D\, arctg(x))(1+x^2)$

$x = sen(arcsen(x))$ $\quad x \in (-1,1)$ $\quad D\, arctg(x) = \frac{1}{1+x^2}$

$1 = (D\, arcsen(x))\, (D\, (sen(x))(arcsen(x))$

$1 = (D\, arcsen(x))\, cos(arcsen(x))$

$1 = (D\, arcsen(x))\sqrt{1 - sen^2(arcsen(x)}$

$1 = (D\, arcsen(x))\sqrt{1-x^2}$

$D\, sen(x) = cos(x) = 1 - \sqrt{sen^2(x)}$

$D\, arcsen(x) = \frac{1}{\sqrt{1-x^2}}$

**Teorema:**

Sia $f : [a,b] \to \mathbb{R}$, $x_0 \in (a,b)$ Se $x_0$ è un punto di max relativo (min rel), ed $f$ è derivabile in $x_0$, allora: $f'(x_0) = 0$

($x_0$ è un p.to di max rel se $\exists \delta > 0$ t.c. $\forall x \in (x_0 - \delta, x_0 + \delta)$ si ha $f(x) \leq f(x_0)$

$f'(x_0) = \lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h} \leq 0$

**DIM:**

$f'(x_0) = \lim_{h \to 0-} \frac{f(x_0+h) - f(x_0)}{h} \geq 0$ $\quad f(x) = x^3$

$\quad f'(x) = 3x^2$

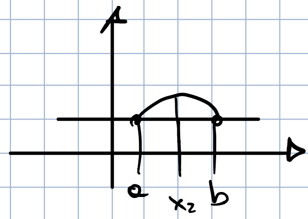
$0 \leq f'(x_0) \leq 0 \Rightarrow f'(x_0) = 0$

# Teorema di Rolle

sia $f:[a,b] \to \mathbb{R}$ una funzione continua, derivabile su $(a,b)$. Se $f(a)=f(b)$ allora $\exists c \in (a,b): f'(c)=0$

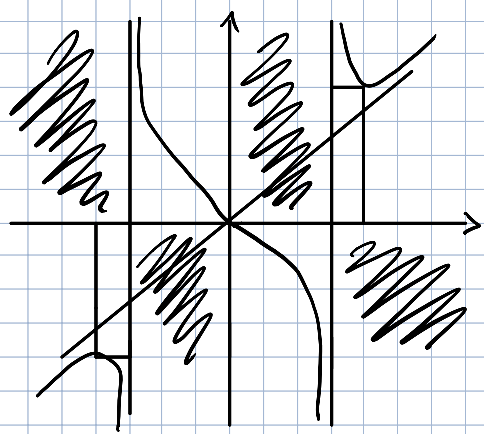
__Dim:__ per il teo di Bolzano-Weierstrass, $\exists x_1, x_2 \in [a,b]$ t.c. $\forall x \in [a,b]$ si ha $f(x_1) \leq f(x) \leq f(x_2)$

① $f(x_1)=f(x_2)$ Dunque $f$ è costante ed $f'(x_0)=0 \ \forall x_0 \in (a,b)$
② $f(x_1) < f(x_2) \Rightarrow \{x_1, x_2\} \neq \{a,b\}$ in altre parole almeno uno fra $x_1, x_2$ è interno ad $(a,b)$, e si usa il teo precedente.




$$f(x) = \frac{x^3+10x}{x^2-16}$$

$$= \frac{x(x^2+10)}{x^2-16}$$



$$D = \frac{h}{g} = \frac{h'g - hg'}{g^2}$$

$h(x) = x^3+10x$
$g(x) = x^2-16$
$h' = 3x^2+10$
$g' = 2x$

$$f'(x) = \frac{(3x^2+10)(x^2-16)-(x^3+10x)(2x)}{(x^2-16)^2} =$$

$$\frac{3x^4-48x^2+10x^2-160-2x^4-20x^2}{(x^2-16)^2} = \frac{x^4-58x^2-160}{(x^2-16)^2} \geq 0$$

$f'(x) \geq 0$ sse $x^4 - 58x^2 - 160 \geq 0$ (purché $x \in D$)

$$\begin{cases} t = x^2 \\ t^2 - 58t - 160 \geq 0 \end{cases}$$

$t_{1,2} = 29 \pm \sqrt{841 + 160} = 29 \pm \sqrt{1001}$

$t_1 = 29 - \sqrt{1001}$

$t_2 = 29 - \sqrt{1001}$

$t \leq t_1 \cup t > t_2$

$\cancel{x^2 \leq t_1} \cup x^2 \geq t_2 \longleftrightarrow x \leq \sqrt{t_1} \cup x \geq \sqrt{t_2}$

$f(x) = e^{1/x}$

$D = \mathbb{R}^*$ $f(x) \geq 0$ $\forall x \in D$

$\lim_{x \to \infty} f(x) = 1$ $\lim_{x \to 0^+} f(x) = +\infty$

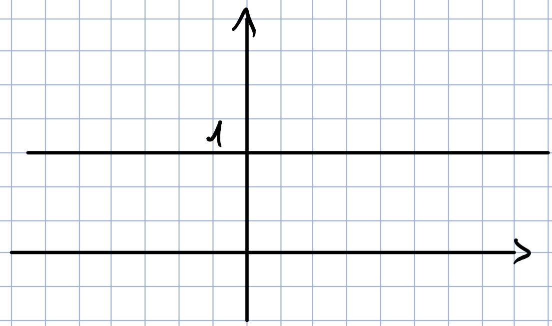
$\lim_{x \to -\infty} f(x) = 1$ $\lim_{x \to 0^-} f(x) = 0$

$f(x) = 1$

$e^{1/x} = 1 = e^0$

$\frac{1}{x} = 0$

non è possibile



$f(x) = \exp\left(\frac{1}{x}\right)$ $f'(x) = (-x^{-2}) \exp\left(\frac{1}{x}\right) = -\frac{1}{x^2} \exp\left(\frac{1}{x}\right)$

per quali $x \in D = \mathbb{R}^*$ si ha $f'(x) \geq 0$? $\rightarrow -\frac{1}{x^2} \geq 0$ per nessun $x \in D$

$\forall x \in D = \mathbb{R}^*$ si ha $f'(x) < 0$

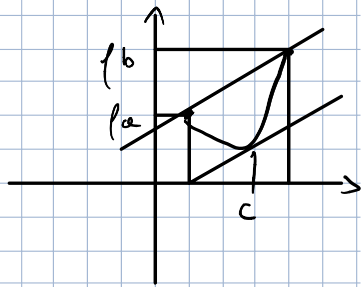
# Teorema di Rolle

Vedi 227 -> precedente

# Teorema di Lagrange

-> Il prof. sostiene sia il + facile da dimostrare

Sia $f:[a,b] \to \mathbb{R}$, continua, derivabile in $(a,b)$ allora $\exists c\ (a,b)$ t.c. $f'(c) = \frac{f(b)-f(a)}{b-a}$




Dim: $\begin{cases} F:[a,b] \to \mathbb{R} \\ F(x) = f(x) - kx \end{cases}$ $k \in \mathbb{R}$ opportuno

osserviamo che

- $F$ è continua in $[a,b]$ mi serve anche $\boxed{F(a) = F(b)}$
- $F$ è derivabile in $[a,b]$

$f(a) - ka = f(b) - kb$ $k(b-a) = f(b) - f(a)$

prendiamo $\boxed{k = \frac{f(b)-f(a)}{b-a}}$

Con questa scelta, $F$ soddisfa le 3 ipotesi del teorema di Rolle, dunque $\exists c\ (a,b)$ t.c. $F'(c) = 0$ $F'(x) = f'(x) - k$

$F'(c) = 0 = f'(c) - k$ cioè $\boxed{f'(c) = k}$

Applicazione:

Se $f[a,b] \to \mathbb{R}$ è continua e derivabile in $(a,b)$ con $f'(x) \geq 0$ $\forall x \in (a,b)$, allora $f$ è debolmente crescente.

Infatti, prendiamo $x_1 < x_2$ $x_1, x_2 \in (a,b)$

Applichiamo il teorema di Lagrange all'intervallo $[x_1, x_2]$ $\exists c \in (x_1, x_2)$ t.c. $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0 \iff f(x_2) - f(x_1) \geq 0 \iff f(x_1) \leq f(x_2)$

## Teorema di Cauchy

Siano $f, g : [a,b] \to \mathbb{R}$, continue e derivabili in $(a,b)$ t.c. $g'(x) \neq 0 \ \forall x \in (a,b)$ allora

- $g(a) \neq g(b)$
- $\exists c \in (a,b)$ t.c. $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

**Dim:** poniamo $F : [a,b] \to \mathbb{R}$

$F(x) = (g(b)-g(a))f(x) - (f(b)-f(a))g(x)$

- F è continua su $[a,b]$
- F è derivabile su $[a,b]$
- $F(a) = F(b)$ per verificare

$F(a) = (g(b)-g(a))f(a) - (f(b)-f(a))g(a) =$

$f(a)g(b) - \cancel{f(a)g(a)} - f(b)g(a) + \cancel{f(a)g(a)} =$

$f(a)g(b) - f(b)g(a)$

$F(b) = (g(b)-g(a))f(b) - (f(b)-f(a))g(b) =$

$\cancel{f(b)g(b)} - f(b)g(a) - \cancel{f(b)g(b)} + f(a)g(b) =$

$f(a)g(b) - f(b)g(a)$

Posso dunque applicare Rolle sulla F in $[a,b]$.

Dunque $\exists c \in (a,b)$ t.c. $F'(c) = 0$,

$F'(x) = (g(b)-g(a))f'(x) - (f(b)-f(a))g'(x)$

$F'(c) = 0 = (g(b)-g(a))f'(c) - (f(b)-f(a))g'(c)$

$(g(b)-g(a))\dfrac{f'(c)}{g'(c)} = f(b)-f(a)$

Resta da dimostrare che $g(a) \neq g(b)$

Per assurdo: se $g(a) = g(b)$ per il teorema di Rolle applicato a $g$ si avrebbe che $\exists c \in (a,b)$ t.c. $g'(c_1) = 0$.
(ASSURDO)

$\lim \frac{f(x)}{g(x)}$ → $f, g$, continue e derivabili in $(a,b)$
Applichiamo t.d.c in $[a,x]$ e facciamo tendere $x$ a $a^+$

## Teorema di de l'Hôpital

Siano $f, g : [a,b] \to \mathbb{R}$ continue tali che $f(a) = g(a) = 0$.
Siano inoltre $f, g$ derivabili in $(a,b)$ con $g(x), g'(x) \neq 0$
$\forall x \in (a,b)$ se $\exists \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in [-\infty, +\infty]$ allora
$\exists \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

NO!!! (porco dio)

$$\lim \frac{f(x)}{g(x)} \underset{H}{=} \lim \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$$

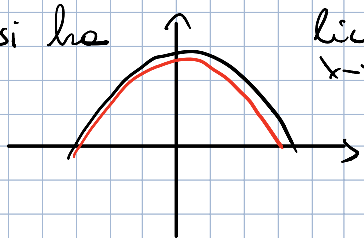
$$\lim_{x \to 0} \frac{\operatorname{sen}(x)}{2x} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

$$\lim_{x \to \infty} \frac{x + \operatorname{sen}(x)}{x} = \frac{1 + \frac{\operatorname{sen}(x)}{x}}{1} = 1$$

$$\lim_{x \to \infty} \frac{1 + \cos(x)}{1} \nexists$$

Se $x$ è "piccolo" si ha



$$\lim_{x \to 0} \frac{(1 - \frac{1}{2} x^2) - \cos(x)}{x^3}$$

$$\frac{1 - \cos(x)}{x^2} \approx \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\left(1 - \frac{1}{2}x^2 - \cos(x)\right)}{x^4} = -\frac{1}{24}$$

$$\lim_{x \to 0} \frac{\operatorname{sen}(x) - x}{4x^3} = -\frac{1}{24}$$

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x^2} = -\frac{1}{12} \lim_{x \to 0} \frac{1 - \cos(x)}{x} = -\frac{1}{24}$$

$$\cos(x) \sim 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{160}x^6 + \frac{1}{40560}x^8 - \dots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \dots$$

formula taylor McLaurin per il cos

[...] Esercitazione

## P.ti di max/min relativo e intervalli di monotonia

**max e min assoluto**

$f(x) = x\sqrt{1-x^2}$ in $x \in [0,1]$

D: $1-x^2 \geq 0$ $\quad x^2-1 \leq 0$ $\quad x^2 = 1$ $\quad x = \pm 1 \rightarrow$ D: $1 \leq x < 1$

in D la $f = f(x)$ è continua -> per il th. di Weierstrass ammette max e min.

$f(0) = 0 = f(1)$ -> quindi essendo $f(x) > 0$ in $(0,2)$ posso dire che $x=0$ e $x=1$ sono minimi assoluti

I p.ti di max: $f'(x) = \sqrt{1-x^2} + \frac{x}{\cancel{2}\sqrt{1-x^2}}(\cancel{2}x) =$

$$\frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}} \qquad f'(0) = \frac{1-2x^2}{\sqrt{1-x^2}} = 0$$

$$+\frac{2x^2}{2} = \frac{1}{2} \qquad x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} \begin{cases} -\frac{\sqrt{2}}{2} \notin I \\ +\frac{\sqrt{2}}{2} \in I \end{cases}$$

$f(x) = (x^2-8)\,e^x$ calcolare max/min relativo

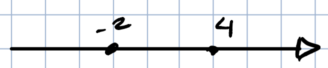
D: $\mathbb{R}$

$f'(x) = 2x \cdot e^x + (x^2-8)e^x = e^x(x^2+2x+8)$

p.ti di stazionarietà (dove si annulla la $f'(x)$)

$f'(x) = 0 \qquad e^x(x^2+2x-8) = 0$

$$x^2+2x-8=0 \qquad x_{1,2} = \frac{-2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 4}{2} \begin{cases} -2 \\ 4 \end{cases}$$



$f'(x) > 0$ $\quad f_1 > 0 \quad e^x > 0$

$f_2 > 0 \quad x^2+2x-8>0$

$x = -2$ p.to max. rel.

$x = 4$ p.to min. rel.

$f(x) = \dfrac{x}{1-x^2}$ $\quad D: x \neq \pm 1$ $\quad f'(x) = \dfrac{1-x^2-x(-2x)}{(1-x^2)^2} = \dfrac{x^2+1}{(1-x^2)^2}$

$D' = D$

intervalli di monotonia $f'(x) > 0$ $\quad N \geq 0: x^2+1 \geq 0 \ \forall x \in D$

$D \geq 0: (1-x^2)^2 > 0 \quad -1 < x < 1$

$(-1, +1)$

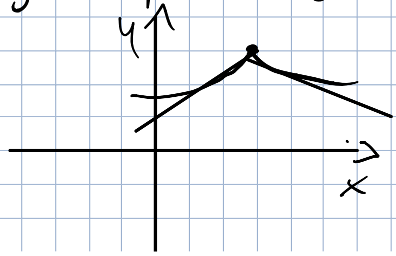
$(-\infty, -1) \cup (1, +\infty) \Rightarrow f'(x) < 0$

- $f(x) = \dfrac{2+x}{4+x^2}$
- $f(x) = e^{x^2}$
- $f(x) = \dfrac{e^x}{x}$
- $f(x) = x \cdot e^{-x}$
- $f(x) = \log x - x$

P.ti di non derivabilità $\quad$ Se $\nexists$ la derivata è perché:

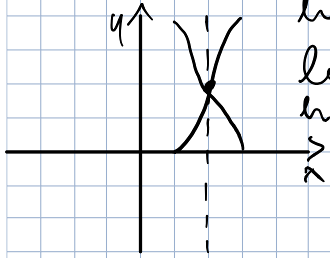
- $f(x)$ continua in $x_0$ ma non derivabile in $x_0$;

esistono $\left.\begin{array}{l} \lim\limits_{h\to 0^+} \dfrac{\Delta y}{\Delta x} \text{ finito} \\ \lim\limits_{h\to 0^-} \dfrac{\Delta y}{\Delta x} \text{ finito} \end{array}\right\}$ diversi $\Rightarrow x, x_0$ è un punto angoloso per la funz



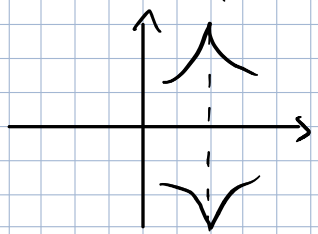
Se

• esistono $\left.\begin{array}{l}\lim_{h\to+\infty} \\ \lim_{h\to-\infty}\end{array}\right\} = \pm\infty \Rightarrow x, x_0$ è un punto a tangente verticale




Se

• esistono $f'_+(x_0)=\pm\infty$ e $f'_-(x_0)=\mp\infty \Rightarrow x, x_0$ è una cuspide



## Derivate per risolvere forme indeterminate (de l'Hôpital) ($\frac{0}{0}$)

$\lim_{x\to 0}\frac{f(x)}{g(x)} \overset{(H)}{=} \lim_{x\to 0}\frac{f'(x)}{g'(x)}$ NO forma sbagliata

$[\frac{0}{0}]$ opp $[\frac{\infty}{\infty}]$ - $g'(x)\neq 0$ $\quad \lim_{x\to 0}\frac{f'(x)}{g'(x)}$

es:) $\frac{x}{x+3}$ continua e derivabile in $x=2$ $(x\neq -3)$

$\lim_{x\to 2}\frac{x}{2+3}=\frac{3}{5}$ NO $\neq$ $\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\frac{1}{1}=1$ no $[\frac{0}{0}]$ opp $[\frac{\infty}{\infty}]$

Osservo che l'ipotesi che $\exists \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ è importante

$\lim_{x\to+\infty}\frac{x+\operatorname{sen} x}{x}=$ | $f(x)=x+\operatorname{sen} x \xrightarrow{+\infty} +\infty$

$$g(x) = x \xrightarrow{-\infty} -\infty$$

$$= \lim_{x \to +\infty} \left( \frac{1}{x} + \frac{\operatorname{sen} x}{x} \right) - 1 + 0 = 1$$

$$\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to +\infty} \left( 1 + \frac{\cos}{1} \right) = \nexists$$

- $\lim_{x \to 0} \frac{xe^{2x} - \operatorname{sen} x}{x^2} = \left[\frac{0}{0}\right]$

$f(x)$, $g(x)$ derivabili nell'intorno del punto 0 escluso 0 $= I(0) \setminus \{0\} \to$ sono verificate le ip di de l'Hôpital

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} =$$

$$= \lim_{x \to 0} \frac{e^{2x} + 2xe^{2x} - \cos x}{2} = \left[\frac{0}{0}\right]$$ (riapplico de l'Hôp)

$f''$ $g''$

$$\lim_{x \to 0} \frac{2e^{2x} + 2e^{2x} + 4xe^{2x} + \operatorname{sen} x}{2} = \frac{2+2}{2} = 2$$

Esercizi:

$$y = \frac{e^{2x} + 24}{e^x + 1}$$

① Dom f: $e^x + 1 \neq 0 \quad \forall x \in \mathbb{R} \Rightarrow D = \mathbb{R}$

② Intersez

$$\begin{cases} f(0) \\ f(x) = 0 \end{cases} \quad x_1 \begin{cases} \frac{e^{2x} + 24}{e^x + 1} = 0 \quad \emptyset \\ \frac{e^0 + 24}{e^0 + 1} = \frac{25}{2} \end{cases}$$

$I\left(0, \frac{25}{2}\right)$

③ Segno e 0 di f

$$\frac{e^{2x}+24}{e^x+1}>0 \quad (\forall x\in\mathbb{R}) \begin{matrix} N\geq 0 \\ D\geq 0 \end{matrix} \text{ sempre pos}$$

④ limiti

$$\lim_{x\to+\infty} \frac{e^{2x}+24}{e^x+1} = +\infty$$

$$\lim_{x\to-\infty} f(x) = 24$$ $y=24$ asintoto orizz.

⑤ Derivata

$$f'(x)\geq 0$$

$$y' = \frac{2e^{2x}(e^x+1)-e^x(e^{2x}+24)}{(e^x+1)^2} = \frac{e^x(e^{2x}+2e^x-24)}{(e^x+1)^2} \geq 0 =$$

$= N\geq 0 \quad \forall x\in\mathbb{R} \qquad e^{2x}+2e^x-24\geq 0 \qquad e^x=t$

$\quad D\geq 0 \quad \forall x\in\mathbb{R} \qquad t^2+t-24\geq 0 \qquad t_{1,2}=-1\pm 5 \left\langle \begin{matrix} -6 \,|\, e^x=-6 \text{ NO} \\ 4 \,|\, e^x=4 \text{ ok} \end{matrix} \right.$

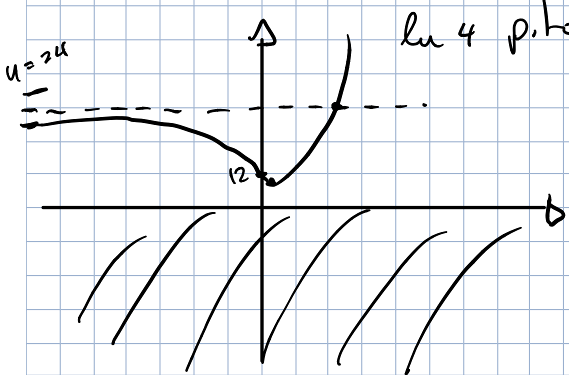
$x=\ln 4$

+ − +

−6 ↘ 4 ↗

$x>\ln 4$ p.to staz.

$\ln 4$ p.to min. rel.



Teoria:

## Funzioni crescenti e decrescenti

**Teorema:** Sia $f:[a,b] \to \mathbb{R}$ una funzione continua derivabile in $[a,b]$. Condizione necessaria e sufficiente affinché $f$ sia deb. crescente in $[a,b]$ è che $\forall x \in (a,b)$ $f'(x) \geq 0$

**Dim:** siano $a \leq x_1 \leq x_2 \leq b$ per il teorema di Lagrange si ha che $\exists c \in (x_1, x_2)$ t.c. $f'(c) = \frac{f(x_1) - f(x_2)}{x_2 - x_1}$ se $f'(x) \geq 0$ $\forall x \in (a,b)$ $\big((a,b) \supseteq (x_1, x_2)\big)$
allora $f'(c) \geq 0 \Rightarrow f(x_2) \geq f(x_1)$

Questo dimostra che la condizione data è sufficiente.
Il viceversa si dimostra studiando il segno del rapporto incrementale.

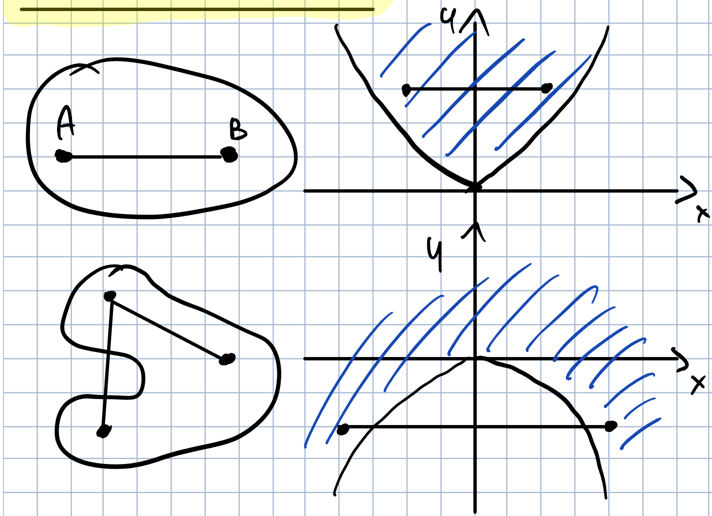
**Teorema:** Sia $f:[a,b] \to \mathbb{R}$, $f$ continua e derivabile in $(a,b)$ t.c. $f'(x) = 0$ $\forall x \in (a,b)$ allora $f$ è costante.

**Dim:** come sopra, usando il teo di Lagrange

**Derivate successive:** $f:[a,b] \to \mathbb{R}$, derivabile in $(a,b)$ se $f':(a,b) \to \mathbb{R}$ è derivabile la sua derivata si dice derivata seconda di $f$ e si indica con:

$$\frac{d^2 f}{d^2 x}, \quad f''(x), \quad D^2 f(x), \quad \left(\text{derivata n-esima } f^{(n)}(x)\right)$$

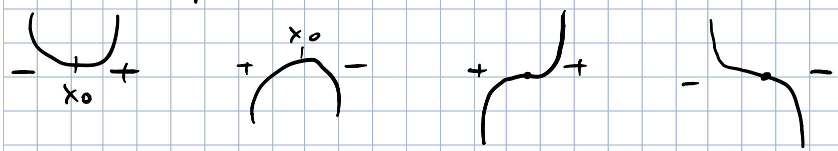
## Insiemi Convessi:




epigrafico = $\{(x,y)\in\mathbb{R}^2 \mid x\in[a,b] \wedge y\geq f(x)\}$

**Teorema:** Sia $f:[a,b]\to\mathbb{R}$ continua, derivabile 2 volte su $(a,b)$. Se $\forall x\in(a,b)$, $f''(x)\geq 0$ allora $f$ è convessa.

$f:[a,b]\to\mathbb{R}$ continua, derivabile in $(a,b)$ $x_0\in(a,b)$ si dice punto stazionario se $f'(x_0)=0$




- Nel primo caso abbiamo $f'(x_0)=0 \wedge f''(x_0)>0$ quindi $x_0$ è un p.to di minimo.
- Nel secondo caso abbiamo $f'(x_0)=0 \wedge f''(x_0)<0$ quindi $x_0$ è un p.to di massimo.
- Nel terzo caso abbiamo $f'(x_0)=0 \wedge f''(x_0)=0$ non possiamo concludere.

# Approssimazioni Polinomiali:

formula di Taylor - MacLaurin

Sia $f:[a,b] \longrightarrow \mathbb{R}$ continua, derivabile $n$ volte su $(a,b)$

Sia $x_0 \in (a,b)$: Allora $\forall x \in [a,b]$ $\exists c \in [a,b]$ t.c.

$|x_0 - c| < |x - c|$ t.c. $\displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k)}(c)}{n!}(x-x_0)^k$

cioè $f(x) = f(x_0) + \dfrac{f'(x_0)}{1}(x-x_0) + \dfrac{f''(x_0)}{2!}(x-x_0)^2 + \dots +$

$+ \dfrac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{(n-1)} + \dfrac{f^{(n)}(c)}{n!}(x-x_0)^n$

## Formula di Taylor - Maclaurin

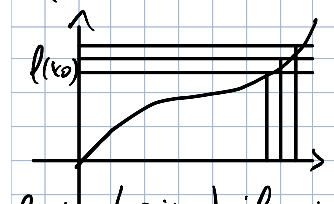
$f:[a,b]\to\mathbb{R}$ continua, derivabile $n$ volte in $(a,b)$

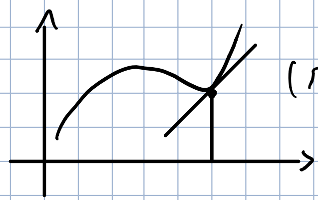
sia $x_0\in(a,b)$ allora $\forall x\in[a,b]\ \exists c\in(a,b)$ t.c.

$|x_0-c|<|x-x_0|$ $\quad f(x)=\underbrace{\sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Polinomio di grado}}+\underbrace{\frac{f^{(n)}c}{n!}(x-x_0)^n}_{\text{resto o errore}}$

$P_n(x,x_0)\leq n-1$

$f$ continua in $x_0$

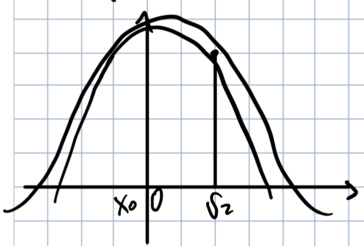


(ricordarsi la differenza)

$f$ è derivabile in $x_0$

Se $f$ è derivabile più volte




$$y=f(0)+\frac{f'(0)}{1}x+\frac{f''(x_0)}{2!}x^2=1-\frac{1}{2}x^2$$

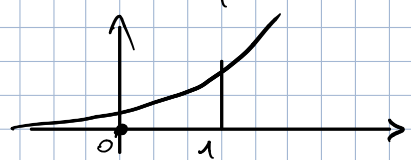
$$\lim_{x\to x_0}\frac{f(x)-P_{n-1}(x;x_0)}{x-x_0}$$

---

$f(x)=e^x,\quad x_0=0\qquad n$ arbitrario

$$e^x=\sum_{k=0}^{n-1}\frac{1}{k!}x^k+\frac{e^c}{n!}x^n$$

c sta fra 0 ed x




$0\leq x\leq 1$

$$0\leq\frac{e^c}{n!}x^n\leq\frac{e\,x^n}{n!}<\frac{1}{10^{10}}$$

$n!\geq e\cdot 10^{10}$

$\forall x \in \mathbb{R} \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$

tutto funziona perché

$\lim_{n \to +\infty} \frac{x^n}{n!} = 0 \quad \forall x \in \mathbb{R}$

serie di taylor per $e^x$

$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + \dots$

$e^x = \cancel{0} + 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \dots =$

$= 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + \dots$

$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6}x^6 + \dots$

$\operatorname{sen}(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots$

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\operatorname{sen}(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f'''(x) = \operatorname{sen}(x) \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$

(ma sta roba ha senso?)

$\log(1+x) = x - \frac{1}{2}x^2 \frac{1}{3}x^3 - \frac{1}{4}x^4 \dots$

$|x| < 1$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \dots$

se $|q| < 1$ allora

$1 + q + q^2 + q^3 + \dots = \frac{1}{1-q}$

$\left(\frac{\operatorname{sen} x}{x}\right)$

$\operatorname{arctg}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

$q = -x^2$

Formula di Leibniz:

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \qquad q = -x^2$

$\frac{\pi}{4} = \operatorname{arctg}\left(\frac{1}{2}\right) + \operatorname{arctg}\left(\frac{1}{3}\right)$

$|x| < 1 \qquad (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \binom{\alpha}{n} = \begin{cases} 1 & n = 0 \\ \frac{\alpha(x-1)\cdot(\alpha-n+1)}{n!} & n \in \mathbb{N}^* \end{cases}$

# Studio di Funzione

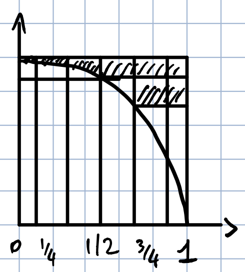
1) Dominio   2) Segno   3) Continuità
4) Limiti agli estremi   5) Derivabilità
6) Segno della derivata   7) P.ti di max/min/flesso
8) Derivata seconda   9) Immagine

---

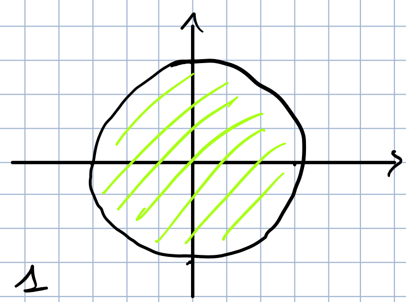
- Problema delle aree
- Problema inverso della derivazione (Integrali)

$f$, $D_f$

$f'(x) = \frac{x}{1+x^2}$   $f(x) = ?$




$y = \sqrt{1-x^2}$



$0 \le A \le 1$

$\frac{1}{2}\frac{\sqrt{3}}{2} \le A \le \frac{1}{2} + \frac{1}{2}\frac{\sqrt{3}}{2}$

$\le \frac{1}{4} + \frac{1}{4}\frac{\sqrt{15}}{4} + \frac{1}{4}\frac{\sqrt{3}}{2} + \frac{1}{4}\frac{\sqrt{7}}{4}$

Il punto è: con $f:[a,b] \to \mathbb{R}$ f. limitata, cioè $\exists\, m, M \in \mathbb{R}$ t.c. $\forall x \in [a,b]$ si ha $m \le f(x) \le M$

**DEF:** Un sottoinsieme finito $\sigma = \{x_0 < x_1 < x_2 < .. < x_n\} \subseteq [a,b]$ con $x_0 = a$, $x_n = b$, si dice decomposizione di $[a,b]$

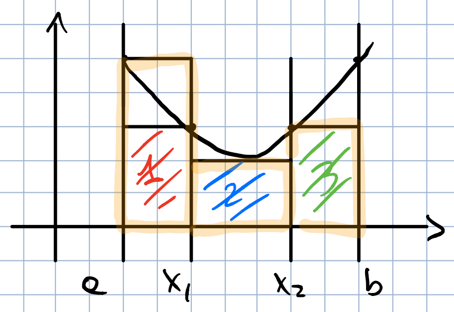
## Somma inferiore:

$$S'(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \inf f(x) \qquad x\in[x_i, x_{i+1}]$$

## Somma Superiore:

$$S''(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup f(x) \qquad x\in[x_i, x_{i+1}]$$

**$m \geq 0$** cioè supponiamo che $f(x) \geq 0 \ \forall x \in [a,b]$




$$(x_1-x_0)\inf_{x\in[x_0,x_1]} f(x) + (x_2-x_1)\inf_{x\in[x_1,x_2]} f(x) + (x_3-x_2)\inf_{x\in[x_2,x_3]} f(x) = S'(f,\{x_0,x_1,x_2,x_3\})$$

$$(x_1-x_0)\sup f(x) + (x_2-x_1)\sup f(x) + (x_3-x_2)\sup f(x) = S''(f,\{x_0,x_1,x_2,x_3\})$$

## Proprietà:

$\forall \sigma$ si ha

$$m(b-a) \leq S'(f,\sigma) \leq S''(f,\sigma) \leq M(b-a)$$

**Def:** $f:[a,b]\to\mathbb{R}$, limitata, si dice integrabile

$$\sup S''(f,\sigma) = \inf S''(f,\sigma)$$

In questo caso, questi numeri si indicano $\int_a^b f(x)\,dx$

$f : [0, A] \to \mathbb{R}$ $\quad \sigma_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1 \right\}$

$f(x) = x$

$\left[ \frac{n-1}{n}, \frac{n}{n} \right]$

$$S'(f, \sigma_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n}\right) =$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \dots + (n-1)\right)$$

$$= \frac{1}{n^2} \frac{n(n-1)}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

$$S'' = (f, \sigma_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n}\right)$$

$$= \frac{1}{n^2}\left(1 + 2 + \dots + n\right)$$

$$= \frac{1}{n^3}\left(1 + \frac{1}{n}\right) \qquad \frac{1}{2}\left(1 - \frac{1}{n}\right) \leq A \leq \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$f(x) = x^2$

$$S'(f, \sigma_n) = \frac{1}{n}\left(0 + \frac{1}{n^2} + \frac{4}{n^2} + \dots + \frac{(n-1)^2}{n^2}\right)$$

$$= \frac{1}{n^3}\left(1 + 2^2 + 3^2 + \dots + (n-1)^2\right)$$

$$= \frac{1}{n^3} \frac{(n-1) \cdot n(2n-1)}{6} = \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)$$

$$S''(f, \sigma_n) = \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) \to \frac{1}{3}$$

$$S'''(f, \sigma_n) = \frac{1}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) \to \frac{1}{3}$$

# Lo studio di funzione:

## 1) Dominio

→ Denom $\neq$
→ Radici pari $\geq 0$
→ argomenti log $> 0$
→ Se la $f(x)$ è tipo $f(x)^{g(x)}$, $f(x) > 0$
→ argomenti di arcsen e arccos sono tra $-1, 1$

## 2) Simmetria / Periodicità

$f(-x) =$
→ $f(x)$ Pari — simm asse y
→ $-f(x)$ Dispari — simm origine
→ $\neq$ Nulla

Le uniche funzioni periodiche sono tipo sen e cos.

## 3) Segno e intersezione assi.

$f(x) > 0$ Per segno

$f(0)$ intersez y.
$f(x) = 0$ intersez x.

$f(x) \geq 0$ per fare tutto
dove la f si annulla
ci sono le intersezioni
dove i val di x per cui $f(x) = 0$

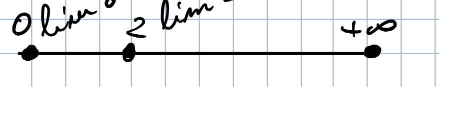
## 4) limiti estremi

$\lim\limits_{x \to \infty} f(x)$
$x \to -\infty$

e lim verticali dove dom $\neq \mathbb{R}$
tipo Dom $f \neq 0$ allora $\lim\limits_{x \to 0^+} f(x)$
$x \to 0^-$

$\boxed{\text{lim obliquo } y = mx + q}$

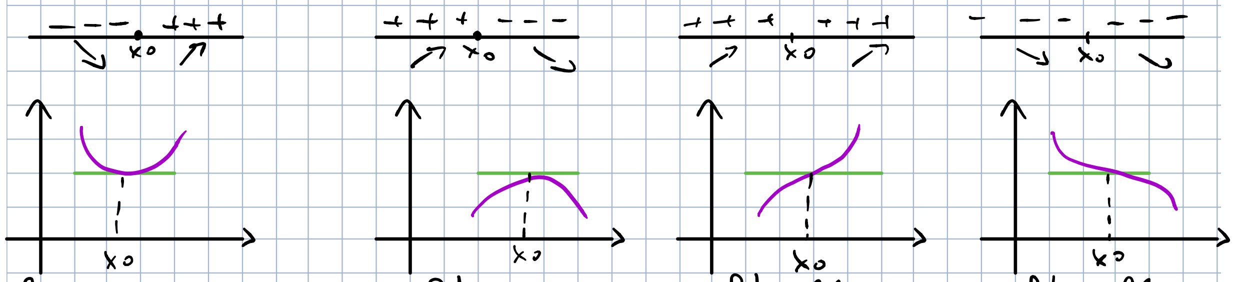
es: $y = \frac{\ln x}{2}$ D $\{x \in \mathbb{R} / x > 0 \wedge x \neq 2\}$ →



## 5) Derivata prima, andamento funzione

Negli intervalli dove $f(x)'$ è $>0 \to f(x)$ cresce
$<0 \to f(x)$ decresce

Se la $f'(x)$ si annulla in $(x_0) \to f'(x_0) = 0 \to$ 4 casi:




P.to min relativo

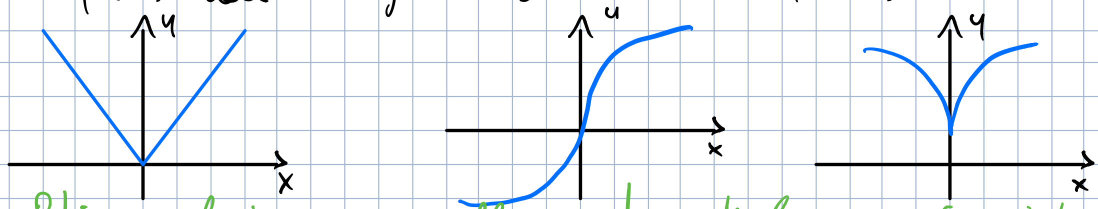
P.to max relativo

P.to flesso a tangente orizz. ascendente

P.to flesso a tange. orizz. discend.

## 5b) Segno della derivata prima

Se $f'(x)$ non è def. in $x_0$ ovvero $\nexists f'(x_0)$ 3 casi:




P.ti angolosi

$\lim_{x \to x_0^-} f'(x) \neq \lim_{x \to x_0^+} f'(x)$

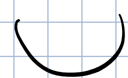
Flessi a tg. verticale

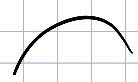
$\lim_{x \to x_0^-} f'(x)$ e $\lim_{x \to x_0^+} f'(x)$ sono entrambi a $+\infty$ o $-\infty$

Cuspidi

$\lim_{x \to x_0^-} f'(x)$ e $\lim_{x \to x_0^+} f'(x)$ fanno uno $+\infty$ e l'altro $-\infty$

## 6) Studio della derivata seconda

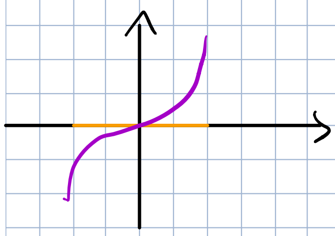
Negli intervalli dove la $f''(x)$ è pos $\to$ convessa 

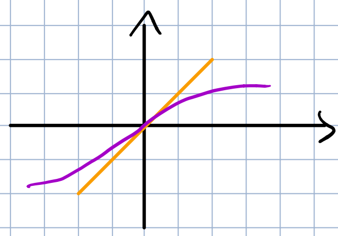
è neg $\to$ concava 

Se a cavallo di un certo p.to $x_0$ si annulla e cambia segno la derivata seconda è p.to di flesso

+ + + • − − −
$x_0$

− − − • + + +
$x_0$





Esercitazione

# Le Primitive

| Funzione | Primitive |
| --- | --- |
| $y=x^{2}$ $(\alpha\neq 1)$ | $\int x^{\alpha}\,dx = \frac{x}{\alpha+1}+c$ |
| $y=1$ $(\alpha=0)$ | $\int dx = x+c$ |
| $y=\frac{1}{x}$ | $\int \frac{1}{x}\,dx = \log\lvert x\rvert + c$ |
| $y=\cos x$ | $\int \cos x\,dx = \operatorname{sen} x + c$ |
| $y=\operatorname{sen} x$ | $\int \operatorname{sen} x\,dx = -\cos x + c$ |
| $y=\frac{1}{\cos^2 x}$ | $= \operatorname{tg} x + c$ |
| $y=\frac{1}{\operatorname{sen}^2 x}$ | $= -\operatorname{cotg} x + c$ |
| $y=\frac{1}{\sqrt{1-x^2}}$ | $= \operatorname{arcsen} x + c$ |
| $y=\frac{1}{1+x^2}$ | $= \operatorname{arctg} x + c$ |
| $y=e^{x}$ | $= e^{x}+c$ |
| $y=a^{x}$ | $= \frac{a^{x}}{\log} + c$ |

$$\int \frac{\sqrt[3]{x}+x^3-2x}{3x^2}\,dx = \frac{1}{3}\left[\int \frac{\sqrt[3]{x}}{x}\,dx + \int \frac{\cancel{x^3}}{\cancel{x^2}}\,dx - 2\int \frac{\cancel{x}}{x^{\cancel{2}}}\,dx\right.$$

$$= \frac{1}{3}\left(\int x^{-\frac{2}{3}}\,dx + \int x\,dx - 2\int \frac{1}{2}\,dx\right)$$

$$x^{\frac{1}{3}-2} = x^{-\frac{2}{3}} = \frac{1}{3} \qquad - \frac{1}{3}\left(\frac{x^{-\frac{2}{3}}}{-\frac{2}{3}} + \frac{x^2}{2} - 2\log\lvert x\rvert\right) + c$$

$$-\frac{5}{3}+1 = -\frac{2}{3} \qquad = \frac{1}{3}\left(-\frac{2}{3\sqrt[3]{x}} + \frac{1}{2}x^2 - 2\log\lvert x\rvert\right) + c$$

## Generalizziamo

$$y=[f(x)]^{\alpha} \longrightarrow \int [f(x)]^{\alpha}\cdot f'(x)\,dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + c$$

$$y=\frac{1}{f(x)} \longrightarrow \int \frac{1}{f(x)}\,f'(x)\,dx = \log\lvert f(x)\rvert + c$$

$$\int \underbrace{\text{sen}^2 x}_{[f(x)]^2} \cdot \underbrace{\cos x}_{f'(x)}\, dx = \frac{(\text{sen}\, x)^3}{3} + c = \frac{\text{sen}^3 x}{3} + c$$

$$\int tg\, x\, dx = \int \frac{\text{sen}\, x}{\cos x}\, dx = -\log|\cos x| + c$$

$$\int (\underbrace{x^2+1}_{f(x) \to f'(x)=2x})^4\, dx = \frac{1}{2}\int (x^2+1)\, 2x\, dx = \frac{(x^2+1)^5}{2 \cdot 5} +$$

$$\int \frac{2x+1}{(x^2+x)} = \log|x^2+x| + c$$

## Primitive delle funzioni goniometriche

$$\int \cos f(x) \quad f'(x)\, dx = \text{sen}\, f(x) + c$$

$$\int \text{sen}\, f(x) \quad f'(x\, dx = -\cos f(x) + c$$

$$\int \frac{1}{\cos^2 x} \quad f'(x)\, dx = tg(f(x)) + c$$

$$\frac{1}{10}\int 10x\, \text{sen}(5x^2+3)\, dx = -\frac{1}{10}\cos(5x^2+3) + c$$

$$\int e^x \cdot \cos e^x\, dx = \text{sen}\, e^x + c$$

$$\int \left(\frac{3x^2}{\cos^2 x} - \frac{e^x}{\text{sen}^2 e^x}\right) dx = \int \frac{3x^2\, dx}{\cos^2 x} - \int \frac{e^x}{\text{sen}^2 e^x}\, dx =$$

$$= tg\, x^3 - cotg\, e^x + c$$

Primitive delle funzioni goniom. inverse → generalizz.

$$\int \frac{1}{\sqrt{1-[f(x)]^2}} f'(x)\,dx = \arcsin f(x)+c$$

$$\int \frac{1}{1+[f(x)]^2} f'(x)\,dx = \operatorname{arctg} f(x)+c$$

$$\int\left(\frac{2}{\sqrt{1-x^2}} - \frac{1}{\underset{2(x^2+1)}{\underset{\parallel}{2x^2+2}}}\right)dx = 2\int\frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$= 2\arcsin x - \frac{1}{2}\operatorname{arctg} x + c \qquad \int\frac{2}{x} \rightarrow 2\int\frac{1}{x}$$

$$\int\frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{2x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\arcsin x^2+c$$

$$\frac{1}{\sqrt{2}}\int\frac{\sqrt{2}\,dx}{1+2x^2} = \frac{1}{2}\operatorname{arctg}(\sqrt{2}x)+c$$

$$\int\frac{1}{9+x^2}\,dx = \int\frac{1}{9\left(1+\frac{x^2}{3}\right)}\,dx = \frac{1}{9}\int\frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx =$$

$$= \frac{1}{3}\int\frac{\frac{1}{3}}{1+\left(\frac{x}{3}\right)^2}\,dx = \frac{1}{3}\operatorname{arctg}\left(\frac{x}{3}\right)+c \qquad \frac{1}{3}\cdot\frac{1}{3}$$

## Primitive delle f. esponenziali → Generalizz.

- $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$
- $\int a^{f(x)} f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$

## Integrazione per parti

- $D[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $f'(x) \cdot g(x) = D[f(x) \cdot g(x) - f(x) \cdot g'(x)]$
- $\int f'(x) \cdot g(x)\,dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\,dx$

- $\int x \cdot e^x dx$

$f = x \cdot g = e^x$

$f' = 1 \quad g' = e^x$

$e^x \cdot x - \int e^x dx = e^x x - e^x + c$

- $\int \ln x \cdot 1\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - x + c$

$\int e^x \operatorname{sen} x\,dx = e^x \operatorname{sen} x - \int e^x \cos x\,dx$

$= e^x \cdot \operatorname{sen} x - e^x \cos x - \int e^x \operatorname{sen} x\,dx = (???)$

- $\int e^x \cdot \cos x$ da fare solo

↳ $\overset{f'(x)}{e^x} \overset{g(x)}{\cos x} + \int e^x \cdot \operatorname{sen} x\,dx = e^x \cos x + e^x \operatorname{sen} x - \int e^x \cos x\,dx$

$$2\int e^x \cos x \, dx = \frac{e^x(\cos x + \operatorname{sen} x)}{2} + c$$

$$\int \underset{f'(x)}{x} \cdot \underset{g}{\log^2 x} \, dx = \frac{x^2}{2} \log^2 x - \int \frac{x^{\cancel{2}}}{\cancel{2}} \cancel{2} \log x \cdot \frac{1}{\cancel{x}} \, dx =$$

$$\frac{x^2}{2} \log^2 x - \int x \log x \, dx = \frac{x^2}{2} \log x - \left[\frac{x^2}{2} \log x - \frac{1}{2}\int x^2 \cdot \frac{1}{x} dx\right]$$

$$\frac{x^2}{2} \log^2 x - \frac{x^2}{2} \log x + \frac{1}{2} \cdot \frac{x^2}{2} + c =$$

$$\frac{x^2}{2}\left(\log^2 x - \log x + \frac{1}{2}\right) + c$$

$$\frac{1}{2}\int x \cdot 2 \operatorname{sen} x \cos x \, dx = \frac{1}{2}\int x \operatorname{sen}(2x) \, dx =$$

$$\frac{1}{2}\,\frac{1}{2}\int x \cdot \operatorname{sen}(2x) \cdot 2 \, dx = \frac{1}{2}\left[\frac{1}{2}\left(-\cos 2x \cdot x + \frac{1}{2}\int \cos 2x \, dx\right.\right.$$

$$-\frac{1}{4} x \cos 2x$$

x compito: $\int \operatorname{arctg} \frac{1}{x-1} \, dx$ $\quad \int x \cdot e^{3x} dx$

$\int \sqrt{x} \log x \, dx$ $\quad \int x \cdot e^{-x} dx$

# Teoria:

## Calcolo Integrale

$f: [a,b] \to \mathbb{R}$ limitata

$\exists m, M \in \mathbb{R}$ t.c. $\forall x \in [a,b]$ si ha $m \leq f(x) \leq M$

$\sigma$ Decomposizione di $[a,b]$

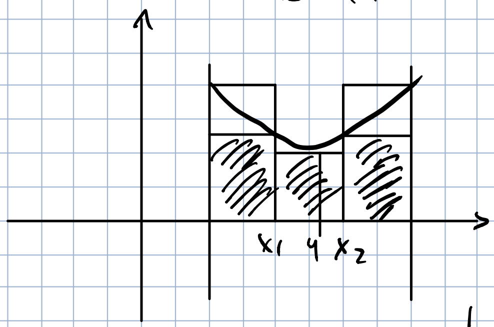
$\sigma = \{x_0 < x_1 < x_2 < \dots < x_n = b\}$

sigma $\sigma$

$$S'(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \inf f(x)$$

$$S''(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup f(x)$$

$$m(b-a) \leq S'(f,\sigma) \leq S''(f,\sigma) \leq (b-a)M$$



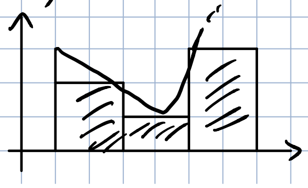

**Def**: $f: [a,b] \to \mathbb{R}$, limitata, è integrabile se

$$\sup S'(f,\sigma) = \inf_{\sigma} S''(f,\sigma)$$

In questo caso il valore comune si indica con $\int_a^b f(x)\,dx$

**OSS**: $\sigma \leq \tau$ allora

$$S'(f,\sigma) \leq S'(f,\tau) \leq S''(f,\tau) \leq S''(f,\sigma)$$



$\tau = \sigma \cup \{y\}$ $y \in (x_1, x_2)$

$(x_2 - x_1) \inf f(x) \leq (y - x_1) \inf f(x) + (x_2 - y) \inf f(x)$

**Teorema:**

Se $f:[a,b] \to \mathbb{R}$ è continua, allora è integrabile

**teorema:**

Se $f:[a,b] \to \mathbb{R}$ è monotona, allora è integrabile

**Dim:** sia $f$ crescente. Questo vuole dire che

$f(x_i) \leq \inf f(x), \quad f(x_{i+1}) \geq \sup f(x)$

prendiamo $n \in \mathbb{N}^*$, $\sigma = \{x_0 = a < x_1 < \dots < x_n = b\}$ dove

$x_i = a + (b-a)\frac{i}{n}$

$$S'(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \inf f(x) \geq \sum_{i=0}^{n-1} \frac{b-a}{n} f(x_i) = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i)$$

Analogamente

$$S''(f,\sigma) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup f(x) \leq \sum_{i=0}^{n-1} \frac{b-a}{n} f(x_{i+1}) =$$

$$= \frac{b-a}{n} \cdot \sum_{i=0}^{n-1} f(x_{i+1})$$

$$S'(f,\sigma) \geq \frac{b-a}{n} \left( f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) \right)$$

$$S''(f,\sigma) \leq \frac{b-a}{n} \left( f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n) \right)$$

$$S''(f,\sigma) - S'(f,\sigma) \leq \frac{b-a}{n} \left( f(x_n) - f(x_0) \right) = \frac{(b-a)}{n} \left( f(b) - f(a) \right)$$

# Proprietà degli integrali:

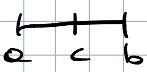
Sia $f,g:[a,b]\to\mathbb{R}$ integrabili, $k\in\mathbb{R}$.

① Linearità: $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

② Linearità II: $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

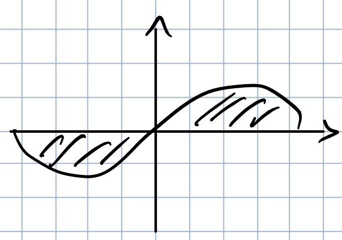
③ Additività: se $c\in(a,b)$ allora

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$




④ Monotonia: se $f(x)\le g(x)$, $\forall x\in[a,b]$

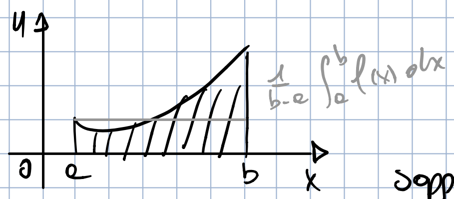
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



## Teorema della media integrale:

Sia $f:[a,b] \to \mathbb{R}$, integrabile. Allora posto $m = \inf f(x)$, $M = \sup f(x)$ si ha: $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Se $f$ è continua, allora $\exists\, c \in [a,b]$ t.c. $\frac{1}{b-a} \int_a^b f(x)\,dx = f(c)$.

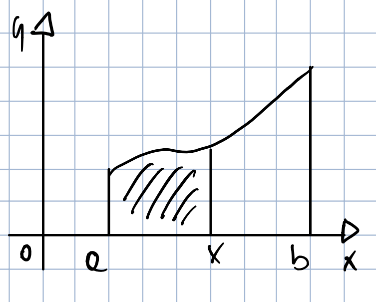



Dal teorema di Weierstrass: sappiamo che se $f$ è continua, $\forall y_0 \in [m,M]\ \exists\, c \in [a,b] : f(c) = y_0$

Per concludere, basta prendere $y_0 = \frac{1}{b-a}\int_a^b f(x)\,dx$

## Funzioni Integrali

$f:[a,b] \to \mathbb{R}$, continua. Definiamo funzione integrale di $f$ la funzione $\begin{cases} F:[a,b] \to \mathbb{R} \\ F(x) = \int_a^x f(t)\,dt \end{cases}$




## Teorema fondamentale del calcolo integrale:

date $f, F$ come sopra. Allora $\forall x \in (a,b)$, $F$ è derivabile e inoltre $F'(x) = f(x)$. In $a,b$ $F$ è derivabile a destra, sinistra rispettivamente.

**Dim.:** Sia $x_0 \in (a,b)$ consideriamo il rapporto incrementale per F in $x_0$.

$$\frac{F(x_0+h)-F(x_0)}{h} = \frac{1}{h}\left(\int_a^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt\right) =$$

$$\frac{1}{h}\left(\int_a^{x_0} f(t)\,dt + \int_{x_0}^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt\right) = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt$$

media integr. di $f$ su $[x_0, x_0+h]$

Per il teorema della media integrale $\exists c = c(h) \in [x_0, x_0+h]$ t.c. $\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt = f(c(h))$

$$F'(x_0) = \lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\to 0} f(c(h))$$

$$= f\left(\lim_{h\to 0} c(h)\right) = f(x_0)$$

Ricordiamo che ogni funzione derivabile G t.c. $G'(x) = f(x)$ $\forall x \in (a,b)$, si dice primitiva di $f$.
Il teo. fondamentale del calc. int. dice che la funz. integrale è una primitiva di $f$.

$$\int f(t)\,dt = \{ G \mid G'(x) = f(x) \ \forall x \in (a,b)\}$$

Fra tutte le primitive di $f$, F è l'unica per cui $F(a)=0$
F è una primitiva di $f$. $G_k(x) = F(x) + K$ è una primitiva di $f$, $\forall k \in \mathbb{R}$

**OSS.:** Siano G, H due primitive di una $f$ allora $\exists k \in \mathbb{R}$ t.c. $H(x) = G(x) + K$ $\forall x \in [a,b]$

Infatti $H'(x) - G'(x) = f(x) - f(x) = 0$

Dunque $\exists k \in \mathbb{R}$ t.c. $H(x) - G(x) = k \quad \forall x \in [a,b]$
Per il teorema di Lagrange.

$f : [a,b] \to$ Continua

$\begin{cases} F : [a,b] \to \mathbb{R} \\ F(x) = \int_a^x f(t)\, dt \end{cases}$

$\int f(t)\, dt = \{ G : [a,b] \to \mathbb{R}$ derivabile $G'(x) = f(x) \ \forall x \in [a,b] \}$

Se conosciamo una qualsiasi primitiva $G$ di $f$, allora $\exists k$ t.c. $G(x) = F(x) + k \quad \forall x \in [a,b]$

$x = a \quad G(a) = F(a) + k$

$x = b \quad G(b) = F(b) + k$

Quindi $F(b) = G(b) - G(a)$

$$F(b) = \underbrace{\int_a^b f(t)\, dt}_{= [G(t)]_a^b} = G(b) - G(a)$$

1. Integrali immediati
2. Integrali per parti
3. Integrali per sostituzione
4. Integrazioni di f. razionali

$f(x) = \dfrac{A(x)}{B(x)}$ $A, B$ polinomi

a) Divisione fra polinomi $\exists Q, R$ polinomi t.c.
$A(x) = Q(x)\, B(x) + R(x)$ e inoltre il grado di $R$ è $<$ del grado di $B$.

Dunque $f(x) = \frac{A(x)}{B(x)} = \underbrace{Q(x)}_{\text{Polinomio}} + \frac{R(x)}{B(x)}$ } f. razionale

$$\int f(x)\,dx = \underbrace{\int Q(x)\,dx}_{\text{immediato}} + \int \frac{R(x)}{B(x)}\,dx$$

② Grado di $B = 1$

$$\int \frac{a}{bx+c}\,dx = \frac{a}{b}\int \frac{b}{bx+c}\,dx \qquad (b \neq 0)$$

$$= \frac{a}{b}\log(|bx+c|) + k \qquad D\log(g(x)) = \frac{g'(x)}{g(x)} \qquad \begin{array}{l} g(x) = bx+c \\ g'(x) = b \end{array}$$

③ Grado di $B = 2$

$$\int \frac{\alpha x + \beta}{ax^2+bx+c}\,dx \qquad a \neq 0,\ \Delta = b^2-4ac$$

3a) $\Delta > 0$ esempio: $\int \frac{3x+4}{\underset{(x+1)(x+4)}{\underset{\shortparallel}{x^2+5x+4}}}\,dx$

Dobbiamo trovare $A, B \in \mathbb{R}$ t.c. $\frac{A}{x+1} + \frac{B}{x+4} = \frac{3x+4}{x^2+5x+4} =$

$$= \frac{A(x+4) + B(x+1)}{(x+1)(x+4)} = \frac{(A+B)x + (4A+B)}{x^2+5x+4}$$

Cerco $A, B \in \mathbb{R}$ t.c. $\begin{cases} A+B = 3 \\ 4A+B = 4 \end{cases}$ $\begin{cases} B = 3-A \\ 4A+B = 4 \end{cases}$ $\begin{cases} 3A = 1 = \frac{1}{3} \\ B = \frac{8}{3} \end{cases}$

$$\frac{3x+4}{x^2+5x+4} = \frac{1/3}{x+1} + \frac{8/3}{x+4} \qquad \text{Verificare}$$

$$\int \frac{3x+4}{x^2+5x+4}\,dx = \int \left(\frac{1/3}{x+1} + \frac{8/3}{x+4}\right)dx = \frac{1}{3}\int \frac{dx}{x+1}\ \frac{8}{3}\int \frac{dx}{x+4}$$

$\frac{1}{3}\log(|x+1|)+\frac{8}{3}\log(|x+4|)+k$

Se $\Delta>0$ $\quad ax^2+bx+c = a(x-x_1)(x-x_2)$

3b) $\Delta=0$

$ax^2+bx+c = a(x-x_1)^2$

$$\frac{\alpha x+\beta}{a(x-x_1)^2} = \frac{A}{x-x_1}+\frac{B}{(x-x_1)^2}$$

$$\int \frac{3x+4}{x^2+4x+4}\,dx$$

$x^2+4x+4=(x+2)^2$ $\quad y=x+2$, $x=y-2$ $\quad dx=dy$

$$\circledast \doteq \int \frac{3(y-2)+4}{y^2}\,dy = \int \frac{3y-2}{y^2}\,dy = \int\frac{3y}{y^2}dy+\int\frac{-2}{y^2}dy$$

$$=3\int\frac{dy}{y}-2\int\frac{dy}{y^2} = 3\log(|y|)-2(-y^{-1})+k$$

---

$$\int t^\alpha dt \begin{cases} \frac{1}{\alpha+1}t^{\alpha+1}+k & \text{se } \alpha\neq 1 \\ \log(|t|)+k & \text{se } \alpha=-1\end{cases}$$

$\alpha=-2$

---

$$=3\log(|y|)+\frac{2}{y}+k \qquad = 3\log(|x+2|)+\frac{2}{x+2}+k$$

3c) $\Delta<0$ $\quad$ obiettivo: ricondursi a $\int\frac{dx}{1+x^2}=\operatorname{arctg}(x)+k$;

$$\int\frac{2x}{1+x^2}\,dx = \log(1+x^2)+k$$

$$\int\frac{5x+2}{4x^2+12x+10}\,dx \qquad \Delta=-16 \qquad x=ay+b$$

$4x^2+12x+10 = 4(ay+b)^2+12(ay+b)+10 =$

$4a^2y^2+8aby+4b^2+12ay+12b+10 =$

$4a^2y^2+(8\cancel{ab+12a})y+4b^2+12b+10 =$

Scegliamo b in modo che $8ab+12a=0$

$b=-\frac{3}{2}$

Scegliamo $a$ in modo che $4a^2=1$ $a>0$ $a=\frac{1}{2}$

$x=\frac{1}{2}y-\frac{3}{2}$ quindi $4x^2+12x+10=y^2+1$

$$\int \frac{5\left(\frac{1}{2}y-\frac{3}{2}\right)+2}{y^2+1}\,\frac{1}{2}\,dy = \frac{1}{2}\int \frac{\frac{5}{2}y-\frac{11}{2}}{y^2+1}\,dy$$

$$= \frac{5}{4}\int \frac{y}{y^2+1}\,dy - \frac{11}{4}\int \frac{dy}{y^2+1} = \frac{5}{8}\log(y^2+1) - \frac{11}{4}\operatorname{arctg}(y)+k$$

$$= \frac{5}{8}\log(4x^2+12x+10) - \frac{11}{4}\operatorname{arctg}(2x+3)+k$$

$$\int \frac{dx}{x^3-x}$$ $x^3-x = x(x^2-1) = x(x-1)(x+1)$

cerchiamo $A, B, C \in \mathbb{R}$ t.c.

$$\frac{1}{x^3-x} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x^3-x}$$

$1 = Ax^2-A+Bx^2+Bx+Cx^2-Cx$

$1 = (A+B+C)x^2+(B-C)x-A$

$$\begin{cases} -A=1 \\ B-C=0 \\ A+B+C=0 \end{cases} \quad \begin{cases} A=-1 \\ B=C \\ 2B=-1 \end{cases}$$

$$\begin{cases} A=-1 \\ B=C=\frac{1}{2} \end{cases}$$

$$\int \frac{dx}{x^3+x}$$ $x(x^2+1)$ cerchiamo $A, B, C \in \mathbb{R}$ t.c.

$$\frac{1}{x^3+x} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$\frac{A(x^2+1)+Bx^2+Cx}{x^3+x}$$

$1 = Ax^2+A+Bx^2+Cx$

$$\begin{cases} A=1 \\ C=0 \\ A+B=0 \end{cases} \quad \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

# Integrazione per sostituzione

$$\frac{d}{dx} f(g(x)) = g'(x) f'(g(x))$$

$$f(g(x)) = \int g'(x) f'(g(x))\, dx + k$$

---

$$\int \cos(4x)\, dx = \frac{1}{4} \operatorname{sen}(4x) + k$$

$y = 4x$

$x = \frac{1}{4} y \quad dx = \frac{1}{4} dy$

$$\int \cos(4x)\, dy =$$

$y = 4x$

$$\int \cos(y) \cdot \frac{1}{4} dy$$

$$\frac{1}{4} \int \cos(y)\, dy$$

$$\frac{1}{4} \operatorname{sen} y + k$$

$$\frac{1}{4} \operatorname{sen}(4x) + k$$

---

$$\int \operatorname{sen}(x) \cos(x)\, dx = \int y\, dy \Big|_{y = \operatorname{sen}(x)} = \frac{1}{2} y^2 + k = \frac{1}{2} \operatorname{sen}^2(x) + k$$

$y = \operatorname{sen}(x)$

$dy = \cos(x)\, dx$

$$\int f'(x) f(x)^{n-1}\, dx = \int y^{n-1} dy \Big|_{y = f(x)} = \frac{1}{n} f(x)^n + k \quad n \neq 0$$

$y = f(x) \quad dy = f'(x)\, dx$

---

$$\int \frac{f'(x)}{f(x)}\, dx = \log(|f(x)|) + k$$

$f(x) = bx + e$

$f(x) = x^2 + 1$

$$\int \frac{2x}{x^2+1}\, dx$$

$\int x\sqrt{1-x}\,dx =$ Ci sono + modi per risolverlo

① $y = 1-x \quad x = 1-y$ $\quad -\int (1-y)\sqrt{y}\,dy = \int (y-1)\sqrt{y}\,dy =$

$dx = -dy \quad = \int \left(y^{\frac{3}{2}} - y^{\frac{1}{2}}\right) dy = \dots$

$\alpha \neq -1$

$\int y^{\alpha}\,dy = \frac{1}{\alpha+1} y^{\alpha+1} + k$

applicare due volte

---

$\int x\sqrt{1-x}\,dx =$

② $y = \sqrt{1-x} \quad x = 1-y^2 \quad \int (1-y^2)(-2y)\,dy = \int y^2(y^2-1)\,dy =$

$dy = -2y\,dy \quad = 2\int (y^4 - y^2)\,dy$

---

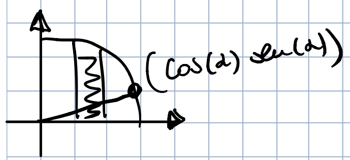
$dy = y'\,dx$

$\int 2x e^{x^2} dx \quad y = x^2 \quad dy = 2x\,dx$ + breve $\rightarrow \int e^y dy = e^y + k = e^{x^2} + k$

$x = \sqrt{y} \quad dx = \frac{dy}{2\sqrt{y}} \rightarrow \int 2\cancel{\sqrt{y}}\, e^y \frac{dy}{2\cancel{\sqrt{y}}} = \int e^y dy$''

---

$\int \sqrt{1-x^2}\,dx \quad x = \cos(\alpha) \quad dx = -\text{sen}(\alpha)\,d\alpha$




$\int \sqrt{1-\cos^2(\alpha)} \cdot (-\text{sen}(\alpha))\,d\alpha = -\int \text{sen}^2(\alpha)\,d\alpha$

(finire per parti)

$\cos^2(\alpha) = 1 - \text{sen}^2(\alpha)$

$\text{sen}^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$

$= -\int \frac{1-\cos(2\alpha)}{2}\,d\alpha = -\frac{1}{2}\int dx + \frac{1}{2}\int \cos(2\alpha)\,d\alpha$

$= -\frac{1}{2}\alpha + \frac{1}{2}\text{sen}(\alpha)\cos(\alpha) + k = \frac{1}{2}\arccos(x) + \frac{1}{2}\sqrt{1-x^2}\,x + k$

---

Es dell'esame

$\lim_{x\to\infty} \left(x(\log(x))^n\right)^{-1} \int_1^x (\log t)^n\,dt =$

$$\lim_{x\to\infty} \frac{\int_1^x (\log t)^n \, dt}{x(\log(x))^n} = \lim_{x\to\infty} \frac{F(x)}{G(x)} = 1$$

per il $)^{-1}$ posso portare al denominatore la parte prima del $\int$

$F(x) = \int_1^x (\log t)^n \, dt, \to \infty$

$G(x) = x(\log(x))^n \longrightarrow \infty$

Quindi posso applicare l'Hôpital

$$f'(x) = (\log(x))^n$$

$$G'(x) = (\log(x))^n + n \cdot x \frac{1}{x} (\log(x))^{n-1}$$

$$\lim_{x\to\infty} \frac{f'(x)}{G'(x)} = \frac{(\log(x))^n}{(\log(x))^n + n(\log(x))^{n-1}} = 1$$

---

$$\int x \operatorname{arctg}\left(\frac{1}{x}\right) dx = \quad y = \frac{1}{x} \quad x = \frac{1}{y} \quad dx = -\frac{1}{y^2} dy$$

$$\int \frac{1}{y} \operatorname{arctg}(y) \cdot \left(-\frac{1}{y^2} dy\right) = -\int \frac{1}{y^3} \operatorname{arctg}(y)$$

$$\int F'(y) G(y) \, dy = F(y) G(y) - \int F(y) G'(y) \, dy$$

$f'(y) = \frac{1}{y^3}$ $\quad$ $F(y) = -\frac{1}{2y^2}$

$G(y) = \operatorname{arctg}(y)$ $\quad$ $G'(y) = \frac{1}{1+y^2}$

(Da finire)

---

Sempre da esame:

$$\int \frac{1}{17\operatorname{sen}(x) + 6\cos(x) + 18} dx$$

$t = 2\operatorname{arctg}(x)$

$dx = \frac{2}{1+t^2} dt$

$t = tg(x/2)$

$\operatorname{sen}(x) = \frac{2t}{1+t^2}$

$\cos(x) = \frac{1-t^2}{1+t^2}$

$$\int \frac{1}{17 \cdot \frac{2t}{1+t^2} + 6 \frac{1-t^2}{1+t^2} + 18} \cdot \frac{2}{1+t^2} dt$$

$$\int \frac{2}{34t + 6(1-t^2) + 18(1+t^2)}\,dt$$

$$\int \frac{2}{12t^2 + 34t + 24}\,dt = \int \frac{dt}{6t^2 + 17t + 12}$$

$\Delta = 17^2 - 4 \cdot 6 \cdot 12 = 1$

$t_{1,2} = \frac{-17 \pm 1}{12} = \begin{array}{l} -\frac{18}{12} = -\frac{3}{2} \\ -\frac{16}{12} = -\frac{4}{3} \end{array}$

$$6t^2 + 17t + 12 = 6\left(t + \frac{3}{2}\right)\left(t + \frac{4}{3}\right)$$

$A, B \in \mathbb{R}$ t.c. $\frac{1}{6t^2 + 17t + 12} = \frac{A}{t + \frac{3}{2}} + \frac{B}{t + \frac{4}{3}} =$

$$= \frac{A\left(t + \frac{4}{3}\right) + B\left(t + \frac{3}{2}\right)}{\left(t + \frac{3}{2}\right)\left(t + \frac{4}{3}\right)} = \frac{(A+B)\,t + \frac{4}{3}A + \frac{3}{2}B}{\frac{1}{6}(6t^2 + 17t + 12)}$$

Continua lezione 34

# Esercitazione

## Integrazione funzioni fratte

| | | |
|---|---|---|
| $\bullet \displaystyle\int \frac{A}{x+B}\,dx \longrightarrow$ | $A \log\lvert x+B\rvert + c$ | caso derivata |
| $\bullet \displaystyle\int \frac{A}{(x-\alpha)^n}\,dx \longrightarrow$ | $\dfrac{A(x-\alpha)^{-n+1}}{-n+1} + c$ | caso semplice esponente |

## Razionali fratte e uso metodo

① $y = \dfrac{N(x)}{D(x)}$ $\quad\bullet$ grado $N(x) \geq D(x) \longrightarrow$ $\boxed{\text{divisione}}$

$$\begin{array}{rrrrr|l} x^4 & +0x^3 & +0x^2 & +0x & +1 & x^2+1 \\ -x^4 & & -x^2 & & & \underbrace{x^2-1}_{Q} \\ \hline & & -x^2 & & +1 & \\ & & x^2 & & +1 & \\ \hline & & & & \underbrace{2}_{R} & \end{array}$$

$$= \int Q\,dx + \int \frac{R}{D(x)}\,dx$$

② $y = \dfrac{q}{ax^2+bx+c}$ e $y = \dfrac{px+q}{ax^2+bx+c}$

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$\bullet\ \Delta > 0$ $\quad \displaystyle\int \frac{A}{(x-x_1)}\,dx + \int \frac{B}{(x-x_2)}\,dx$

Per trovare A e B applico il principio di identità dei polinomi e impongo un sistema:

$\boxed{\text{es!}}$ $\quad \dfrac{2x-7}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} = \dfrac{Ax - 2A + Bx - B}{(x-1)\,(x-2)}$

$$\begin{cases} A+B=2 \\ -2A-B=-7 \end{cases} \qquad \begin{cases} A=-2B \\ 4+2B-B=-7 \end{cases} \qquad \begin{cases} A=5 \\ B=3 \end{cases}$$

sostituisco

• $\Delta = 0$ $\quad \frac{A}{(x+a)} + \frac{B}{(x+a)^2}$ $\quad$ Da: $ax^2+bx+c = a(x+a)^2$

• $\int \frac{p}{a(x-x_1)^2}\,dx = \frac{p}{a}\int (x-x_1)^{2}\,dx \rightarrow$ ricondotto al caso potenza

• $\int \frac{px+q}{a(x-x_1)^2}\,dx = \int \frac{A}{a(x-x_1)}\,dx + \int \frac{B}{(x-x_1)^2}\,dx$

• $\Delta < 0$ $\quad$ non ho soluzioni reali! $\quad c = \frac{b}{2a}$

$ax^2+bx+c = a\left[(x+c)^2+m^2\right]$ $\quad m = \frac{-\Delta}{4a}$

• es: $\int \frac{1}{\underbrace{x^2+x+1}}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+1-\frac{1}{4}}\,dx$

$(x+\frac{1}{2})$ $\quad$ potrebbe esser ordinato aggiungendo il resto

$\hookrightarrow x^2+x+\frac{1}{4} = -\frac{1}{4}$

$= \int \frac{1}{\frac{3}{4}+(x+\frac{1}{2})^2} = \int \frac{1}{\frac{3}{4}\left(1+\left(\frac{x+1/2}{3/4}\right)\right)^2} =$

$= \frac{4}{3}\int \frac{1}{1+\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2}\,dx = \frac{\sqrt{3}}{2}\,\frac{4}{3}\int \frac{\frac{2}{\sqrt{3}}}{1+\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2}\,dx =$

$dx = \frac{2}{\sqrt{3}}$

$= \boxed{\frac{2\sqrt{3}}{3}\,\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x+\frac{\sqrt{3}}{3}\right)+c}$

# Integrazione per sostituzione

• $\int e^{3x-1}\,dx = \int e^{y}\,\frac{1}{3}\,dy = \frac{1}{3}\int e^{y}dy$ $\qquad y = 3x-1$

$\qquad dy = 3\,dx$

$= \frac{1}{3}\,e^{y} + c = \frac{1}{3}\,e^{3x+1} + c$

• $\int \frac{e^x}{e^x+2}\,dx = \int \frac{1}{y+2}\,dy =$ $\qquad y = e^x$

$\qquad dy = e^x dx$

$\ln|y+2| + c = \ln|e^x+2| + c$

• $\int \frac{\log x}{\cancel{x}}\,dx = \int \frac{y}{\cancel{e^y}}\,\cancel{e^y}\,dy$ $\qquad y = \log x$

$\qquad dy = e^y dx$

$\qquad dx = e^y dy$

$= \int y\,dy = \frac{y^2}{2} + c = \frac{\log x^2}{2} + c$

• $\int \cos x \cdot \frac{\sin^3 x - 2\sin x}{\sin^2 x - \sin x + 6}\,dx =$ $\qquad y = \sin x$

$\qquad dy = \cos x\,dx$

$= \int \frac{y^3 - 2y}{y^2 - y + 6}\,dy = \int -7y + 6\,dx - \int \frac{+y+6}{y^2-y+6}$

$$\begin{array}{l|l}
y^3 - 2y & y^2 - y + 6 \\ \hline
-y^3 + y^2 - 6y & y + 4 \\ \hline
/ \;\; +4y^2 - 8y & \\
-4y^2 - 4y - 24 & \\ \hline
/ \;\; -7y - 6 &
\end{array}$$

$\frac{7y+6}{(y-3)(y+2)} = \frac{A}{y-3} + \frac{B}{y+2} = \frac{Ay + 2A + By - 3B}{(y-3)(y+2)} =$

$= \frac{(A+B)y + 2A - 3B}{(y-3)(y+2)} \underset{sist}{=}$ $\begin{cases} A + B = 7 \\ 2A - 3B = 6 \end{cases}$

$\begin{cases} A = 7 - B \\ 14 - 2B - 3B = 6 \end{cases}$ $\qquad A = 7 - 8/5 = \frac{7}{6}$

$\qquad B = 8/5$

... e così via

• $\int \frac{1}{3\sin x - \cos x + 1} dx$ = formule parametriche

$$= \int \frac{1}{\frac{6t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2}{1+t^2} dt =$$

$\sin x = \frac{2t}{1+t^2}$ $\quad \frac{x}{2} = \operatorname{arctg} t$

$\cos x = \frac{1-t^2}{1+t^2}$ $\quad x = 2\operatorname{arctg} t$

$$= \int \frac{1}{\frac{6t - 1 + t^2 + 1 + t^2}{\cancel{1+t^2}}} \frac{2}{\cancel{1+t^2}} dt$$

$dx = 2 \frac{1}{1+t^2} dt$

$$= \int \frac{1}{2t^2+6t} \cdot 2\, dt = \int \frac{1}{t^2+3t} dt = \frac{1}{t(t+3)} = \frac{A}{t} + \frac{B}{t+3} =$$

$$= \frac{At + 3A + Bt}{t(t+3)} = \frac{(A+B)t + 3A}{t(t+3)} =$$

$\begin{cases} A+B=0 \\ 3A=1 \end{cases}$

$\begin{cases} B=-A=-\frac{1}{3} \\ A=\frac{1}{3} \end{cases}$

$$= \int \frac{1/3}{t} dt = \frac{1}{3} \int \frac{1}{t+3} dt =$$

$$\frac{1}{3} \ln|t| - \frac{1}{3} \ln|t+3| + c =$$

$$\frac{1}{3} \ln\left|\frac{t}{t+3}\right| + c = \frac{1}{3} \ln \frac{|\operatorname{tg} \frac{x}{2}|}{|\operatorname{tg}\frac{x}{2} + 3|} + c$$

• $\lim_{x \to 0} \frac{\log(1+5\operatorname{tg}^2 x)}{x^2} = \left[\frac{0}{0}\right]$ ind

$$\lim_{x\to 0} \frac{f'(x)}{g(x)} = \lim_{x\to 0} \frac{\frac{1}{1+5\operatorname{tg}^2 x} \cancel{10}^{5} \operatorname{tg} x \cdot \frac{1}{\cos^2 x}}{\cancel{2}x} =$$

$$= \lim_{x\to 0} \frac{5\operatorname{tg} x}{x\cos^2 x\,(1+5\operatorname{tg}^2 x)} = \lim_{x\to 0} 5\, \frac{\operatorname{tg} x}{x} =$$

$$5 \lim_{x\to 0} \left(\frac{\sin x}{\cos x} \frac{1}{x}\right) = 5 \cdot 1 = 5$$

- $f(x) = \dfrac{e^x}{e^{2x}+1}$ $\qquad e^{2x}+1 \neq 0 \quad \forall x \in \mathbb{R}$

- Segno $f(x) \geq 0 \rightarrow \dfrac{e^x}{e^{2x}+1} > 0$ $\qquad \begin{cases} e^x > 0 & \forall x \in \mathbb{R} \\ e^{2x} > 0 & \forall x \in \mathbb{R} \end{cases}$

$(-\infty, 0) \cup (0, +\infty)$
pos $\qquad$ pos

- limiti

$$\lim_{x \to \infty} \frac{e^x}{e^{2x}+1} = \frac{e^x}{e^{2x}\left(1+\frac{1}{e^{2x}}\right)} = 0$$

$$\lim_{x \to -\infty} = 0$$

$$f(x) = \frac{e^x(e^{2x}+1) - 2e^{2x} \cdot e^x}{(e^{2x}+1)^2} = \frac{e^{3x}+e^x-2e^{3x}}{(e^{2x}+1)^2} = \frac{-e^{3x}+e^x}{(e^{2x}+1)} =$$

$$= \frac{e^x(1-e^{2x})}{(e^{2x}+1)^2}$$

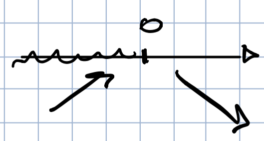
$$f'(x) \geq 0 \qquad \frac{e^x(1-e^{2x})}{(e^{2x}-1)^2} \geq 0 \rightarrow 1-e^{2x} \geq 0$$

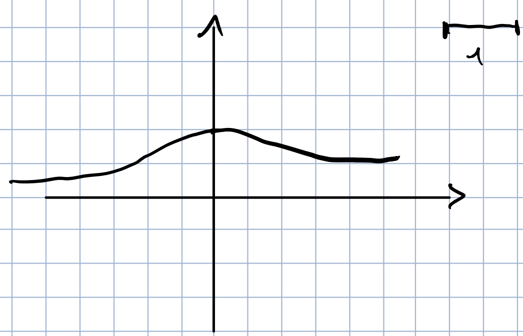
$$1 \geq e^{2x}$$

$$e^0 \geq e^{2x}$$

$$x \leq 0$$

$$f(0) = \frac{e^0}{e^0+1} = \frac{1}{2}$$





# Teoria

$$\int \frac{1}{17\operatorname{sen}(x)+6\cos(x)+18}\,dx = \dots = \frac{1}{2}\int \frac{dt}{6t^2+17t+12}$$

$t = tgx$

$\operatorname{sen}(x) = \frac{2t}{1+t^2}$ $\quad x = 2\operatorname{arctg}(t)$ $\quad \Delta = 1$

$6t^2+17t+12 = 6\left(t+\frac{4}{3}\right)\left(t+\frac{3}{2}\right)$

$\cos(x) = \frac{1-t^2}{1+t^4}$ $\quad dx = \frac{2}{1+t^2}\,dt$

$$\frac{1}{6t^2-17t+12} = \frac{A}{t+\frac{4}{3}} + \frac{B}{t+\frac{3}{2}}$$

$$= \frac{A\left(t+\frac{3}{2}\right)+B\left(t+\frac{4}{3}\right)}{\frac{1}{6}\left(6t^2+17t+12\right)} = \frac{6A\left(t+\frac{3}{2}\right)+6B\left(t+\frac{4}{3}\right)}{6t^2+17t+12}$$

$$1 = (6A+6B)t + 9A + 8B$$

$\begin{cases} 9A+8B=1 \\ 6A+6B=0 \end{cases}$

Sostituendo $\int \frac{dt}{6t^2+17t+12} = \int \frac{1}{t+\frac{4}{3}}\,dt + $

$\begin{cases} B=-A \\ A=1 \end{cases}$ $\begin{cases} A=1 \\ B=1 \end{cases}$

$$-\int \frac{1}{t+\frac{3}{2}}\,dt =$$

$$= \log\left(\left|t+\frac{4}{3}\right|\right) + k \qquad k \in \mathbb{R}$$

- $\Gamma(x) = \lim_{N\to+\infty} \int_0^N t^{x-1} e^{-t}\,dt = \int_0^{+\infty} t^{x-1} e^{-t}\,dt$

funzione trascendente di ordine superiore

$$\Gamma(1) = \lim_{N\to+\infty} \int_0^N e^{-t}\,dt = \lim_{N\to+\infty} \left[-e^{-t}\right]_0^N = \lim_{N\to\infty}\left(-e^{-N}+1\right) = 1$$

$$\Gamma(x+1) = \lim_{N\to\infty} \int_0^N t^x e^{-t}\,dt = \lim_{N\to+\infty} \left(-N^x e^{-N} + x\int_0^N t^{x-1} e^{-t}\,dt\right.$$

$$\int_0^N t^x \cdot e^{-t}\,dt = \quad F'(t) = e^{-t} \quad F(t) = -e^{-t}$$

$$G(t) = t^x \quad G'(t) = x t^{x-1}$$

$$= \left[t^x e^{-t}\right]_0^N + \int_0^N x t^{x-1} e^{-t}\,dt = -N^x e^{-N} + x\int_0^N t^{x-1} e^{-t}\,dt$$

$$\Gamma(x+1) = x \lim_{N\to\infty} \int_0^N t^{x-1} e^{-t}\,dt \qquad \boxed{\Gamma(x+1) = x\Gamma(x)}$$

$$\Gamma(1) = 1$$

$$\Gamma(6) = 5\Gamma(5) = 5\cdot 4\cdot \overset{x=4}{\Gamma(4)} = 5\cdot 4\cdot 3\,\overset{x=3}{\Gamma(3)} = 5\cdot 4\cdot 3\cdot 2\,\overset{x=2}{\Gamma(2)} =$$

$$5\cdot 4\cdot 3\cdot 2\cdot \overset{x=1}{1}\cdot \Gamma(1) = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5!$$

- $\displaystyle\int \frac{x^3}{x^4+3x^2+2}\,dx \qquad x^4+3x^2+2 =$

$$y = x^2 \quad y^2+3y+2=0$$

$$(y+2)(y+1)=0$$

Donque $x^4+3x^2+2 = (x^2+2)(x^2+1)$

$$x^4+3x^2+2=0 \qquad \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1} = \frac{1}{x^4+3x^2+2}$$

$$y=0$$

$$y^2+3y+2=0$$

$$(y+2)(y+1)=0$$

$$\int \frac{x^3\,dx}{x^4\,3x^2+2} = \int \frac{(\sqrt{y})^3}{y^2+3y+2}\,\frac{dy}{2\sqrt{y}} \qquad y=x^2 \quad dx = \frac{1}{2\sqrt{y}}\,dy$$

$$x = \sqrt{y}$$

$$\frac{1}{2}\int \frac{y\cancel{\sqrt{y}}}{y^2+3y+2}\,\frac{dy}{\cancel{\sqrt{y}}} \qquad A, B \in \mathbb{R} \text{ t.c. } \frac{y}{y^2+3y+2} = \frac{A}{y+2} + \frac{B}{y+1}$$

es:

$$\int \frac{x^3}{x^4+3x^2+2}\,dx = \frac{y\cancel{\sqrt{y}}}{y^2+3y+2}\cdot\frac{1}{2\cancel{\sqrt{y}}}\,dy = \frac{1}{2}\int\frac{y}{y^2+3y+2}\,dy =$$

$y = x^2$ $\quad dx = \frac{1}{2\sqrt{y}}\,dy$ $\quad$ Cerchiamo $A, B \in \mathbb{R}$ t.c.:

$x = \sqrt{y}$

$$\frac{y}{y^2+3y+2} = \frac{A}{(y+2)} + \frac{B}{(y+1)}$$

$$\frac{-3 \pm \sqrt{9-8}}{2}$$

mcm

$$= \frac{Ay + A + By + 2B}{(y+2)\ (y+1)}$$

$$\begin{cases} A+B=1 \\ A+2B=0 \end{cases} \quad \begin{cases} A = 1-B \\ 1-B+2B=0 \end{cases} \quad \begin{cases} A=2 \\ B=-1 \end{cases}$$

Sistema coefficienti

$$= \int\frac{2}{y+2}\,dy - \int\frac{1}{y+1}\,dy = \frac{1}{2}\left(2\ln|y+2| - \ln(y+1)\right) + c$$

$$= \cancel{2}\ln|x^2+2| - \frac{\ln(x^2+1)}{\cancel{2}} + c$$

es

$\int x \operatorname{arctg}\frac{1}{x}\,dx$ → generalmente per parti: $\int F' G\,dx = FG - \int F G'\,dx$

$G(x) = \operatorname{arctg}\frac{1}{x}$ $\qquad G'(x) = \frac{-1}{\left(1+\frac{1}{x^2}\right)x^2}$

$F'(x) = x$ $\qquad F(x) = \frac{x^2}{2}$

$$G'(x) = -\frac{1}{1+x^2} \quad = \frac{x^2}{2}\operatorname{arctg}\left(\frac{1}{x}\right) + \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\operatorname{arctg}\left(\frac{1}{x}\right) + \frac{1}{2}\int\frac{x^2}{x^2+1}\,dx = \frac{x^2}{2}\operatorname{arctg}\left(\frac{1}{x}\right) + \frac{1}{2}\left[\int\frac{x^2+1}{x^2+1}\,dx + \int\frac{-1}{1+x^2}\,dx\right]$$

$$= \frac{x^2}{2}\operatorname{arctg}\left(\frac{1}{x}\right) + \frac{1}{2}x - \frac{1}{2}\operatorname{arctg}(x) + C$$

2017

$$\int (8x+18)\log(x^2+7)\,dx = \qquad f' = 8x+18 \quad f = 4x^2+18x \quad g = \log(x^2+7) \quad g' = \frac{2x}{x^2+7}$$

$$(4x^2+18x)\log(x^2+7) - \int (4x^2+18x)\frac{2x}{x^2+7}\,dx$$

$$\int \frac{8x^3+36x^2}{x^2+7}\,dx$$

$$\Downarrow$$

$$\int \left(8x+36 - \frac{56x+252}{x^2+7}\,dx\right)$$

$$\begin{array}{rr|l} 8x^3 + 36x^2 & & x^2+7 \\ \hline -8x^3 & -56x & 8x+36 \\ \hline & 36x - 56x & \\ -36x^2 & -252 & \\ \hline & -56x-252 & \end{array}$$

$$8x^3+36x^2 = (8x^3+36)(x^2+7) + (-56x-252)$$

Resta da calcolare

$$\int \frac{56x+252}{x^2+7}\,dx = 28\int \frac{2x}{x^2+7}\,dx + 252\int \frac{dx}{x^2+7}$$

$$= 28\log(x^2+7) + 252\int \frac{dx}{x^2+7} =$$

$$\begin{cases} a^2 = 7 \\ a > 0 \end{cases}$$

$$x = ay$$

$$x^2+7 = a^2y^2+7$$

$$x = y\sqrt{7}$$

$$dx = \sqrt{7}\,dy$$

$$\int \frac{dx}{x^2+7} = \int \frac{\sqrt{7}\,dy}{7y^2+7} = \frac{\sqrt{7}}{7}\int \frac{dy}{y^2+1} = \frac{1}{\sqrt{7}}\operatorname{arctg}(y) \qquad x = y\sqrt{7}$$

$$= \frac{1}{\sqrt{7}}\operatorname{arctg}\left(\frac{x}{\sqrt{7}}\right) + c$$

$$\int \frac{\alpha x + \beta}{\gamma x^2 + \delta x + \varepsilon}$$

$$\Delta < 0$$

$$x = ay + b$$

$$\gamma x^2 + \delta x + \varepsilon = \gamma(ay+b)^2 + \delta(ay+b) + \varepsilon$$

$$(4x^2+18x)\log(x^2+7) - \frac{1}{\sqrt{7}}\operatorname{arctg}\left(\frac{x}{\sqrt{7}}\right) + c$$

2016

$$\lim_{x\to 0} \frac{x\,\text{sen}(x)}{\text{sen}(e^{3x}-1)-3x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\left(\frac{\text{sen}(x)}{x}\cdot\frac{x^2}{\text{sen}(e^{3x}-1)-3x}\right)$$

$$x \cdot x \cdot \frac{\text{sen}\,\cancel{x}}{\cancel{x}} \cdot \frac{1}{//}$$

$$= \lim_{x\to 0}\frac{x^2}{\text{sen}(e^{3x}-1)-3x} \underset{\text{Hôpital}}{=}$$

$$= \lim_{x\to 0}\frac{2x}{3e^{3x}\cos(e^{3x}-1)-3} = \frac{2}{3}\lim_{x\to 0}\frac{x}{e^{3x}\cos(e^{3x}-1)-1} =$$

$$= \frac{2}{3}\lim_{x\to 0}\frac{1}{3e^{3x}\cos(e^{3x}-1)-e^{3x}\cdot 3e^{3x}\,\text{sen}(e^{3x}-1)} = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

---

$$\int\frac{dx}{x\sqrt{x^2-1}} = \int y\frac{1}{\sqrt{\frac{1}{y^2}-1}}\left(-\frac{1}{y^2}\,dy\right) = -\int\frac{dy}{y\sqrt{\frac{1}{y^2}-1}} =$$

$y=\frac{1}{x}$ $\quad x=\frac{1}{y}$

$dx = -\frac{1}{y^2}\,dy$

$$-\int\frac{dy}{\sqrt{1-y^2}} = -\text{arcsen}(y)+k =$$

$$= -\text{arcsen}\left(\frac{1}{x}\right)+c =$$

Seconda soluzione

$y=\sqrt{x^2-1}$

$y^2 = x^2-1$

$x^2 = y^2+1$

$x = \sqrt{y^2+1}$

$dx = \frac{2y}{2\sqrt{y^2+1}}\,dy$

$$\int\frac{dx}{x\sqrt{x^2-1}} = \int\frac{1}{\sqrt{y^2+1}}\cdot\frac{1}{y}\cdot\frac{y\,dy}{\sqrt{y^2+1}}$$

$$= \int\frac{dy}{y^2+1} = \text{arctg}(y)+k$$

$$= \text{arctg}\left(\sqrt{x^2-1}\right)+k$$

Formula di taylor

$$1+t+t^2+t^3\dots=\frac{1}{1-t}\qquad |t|<1$$

$$\int_0^x (1+t+t^2+t^3+\dots)\,dt=\int_0^x\frac{dt}{1-t}\qquad |x|<1$$

$$=\left[t+\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{1}{4}t^4+\dots\right]_0^x=\left[-\log(|1-t|)\right]_0^x$$

$$x+\frac{1}{2}x^2+\frac{1}{3}x^3+\dots=-\log(1-x)$$

---

$$1-t^2+t^4-t^6+t^8-t^{10}\dots=\frac{1}{1+t^2}\qquad |x|<1$$

$$\int_0^x (1-t^2+t^4\dots)\,dt=\int_0^x\frac{dt}{1+t^2}\qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\dots=\frac{\pi}{4}$$

$$\left[t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\dots\right]_0^x=\left[\operatorname{arctg}(t)\right]_0^x\qquad \frac{\pi}{4}=\operatorname{arctg}\left(\frac{1}{2}\right)+\operatorname{arctg}\left(\frac{1}{3}\right)$$

$$x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\dots=\operatorname{arctg}(x)$$

---

$$\lim_{x\to 0}\frac{1}{x}\int_0^x\log(e^t+5)\,dt=\lim_{x\to 0}\frac{\int_0^x\log(e^t+5)\,dt}{x}$$

$$F(x)=\int_0^x\log(e^t+5)\,dt\qquad \lim_{x\to 0}\frac{F'(x)}{1}=\lim_{x\to 0}(e^x+5)=\log(6)$$

$$F'(x)=\log(e^x+5)$$

---

$$\lim_{x\to+\infty}\int_0^x\frac{\log(e^t+5)\,dt}{x}=\lim_{x\to\infty}\frac{\log(e^x+5)}{1}=+\infty$$

---

2017

$$\int\frac{x^3}{x^2-26x+144}\,dx=\qquad A(x)=Q(x)B(x)+R(x)$$

$$\frac{A(x)}{B(x)}=Q(x)+\frac{R(x)}{B(x)}$$

$$\begin{array}{l|l} x^3 & x^2-26x+144 \\ -x^3+26x^2-144x & x+26 \\ \hline \quad 26x^2-144x & \\ \quad -26x^2+676x-3744 & \\ \hline \qquad\qquad 532x-3744 & \end{array}$$

$$= \int \left( x+26+\frac{532x-3744}{x^2-26x+144} \right) dx$$

$$= \frac{1}{2}x^2+26x+\int \frac{532x-3744}{x^2-26x+144}\, dx$$

$\Delta = 100$

Cerchiamo $A, B \in \mathbb{R}$ t.c.

$$\frac{532x-3744}{x^2-26x+144} = \frac{A}{x-8}+\frac{B}{x-18} = \frac{A(x-18)+B(x-8)}{(x-8)(x-18)}$$

$$\Longleftrightarrow 532x-3744 = (A+B)x-18A-8B$$

$$\begin{cases} A+B = 532 \\ -18A+8B = -3744 \end{cases} \quad \begin{cases} A+B = 532 \\ 18A+8B = 3744 \end{cases} \quad \begin{cases} B = 532-A \\ 18A+8(532) = 3744 \end{cases}$$

e così via...

**END**

① SDF

$-x = +2x - 3x$

$f(x) = \dfrac{x-1}{x^2-x-6}$ Dom f $x^2 - x - 6 \neq 0$

$x^2 + 2x - 3x - 6 \neq 0$

1 - Dom $f \in \mathbb{R} \setminus \{-2, 3\}$ $x(x+2) - 3(x+2)$

$(x+2) - (x-3)$

$x+2 = 0 \longrightarrow x = -2$

2 - P/D $x - 3 = 0 \longrightarrow x = 3$

$f(-x) = \dfrac{-x-1}{-x^2-x-6}$ dispari?

3 - Intersez.

(y) $f(0) \rightarrow \dfrac{-1}{-6} = \dfrac{1}{6}$ Interseca $(1, \frac{1}{6})$

(x) $f(x) = 0$ $\dfrac{x-1}{x^2-x-6} = 0$ $x - 1 = 0$ $x = 1$

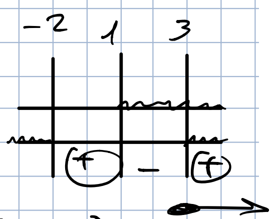
$x \neq -2, x \neq 3$

4 - Segno

$\dfrac{x-1}{x^2-x-6} > 0$ $x - 1 > 0$ $x > 1$

$x^2 - x - 6 > 0$ $x < -2 \vee x > 3$

$(-\infty, -2) \cup (3, +\infty)$




$S = (-2, 1) \cup (3, +\infty)$

5 - limiti

$\lim\limits_{x \to \infty} \dfrac{x-1}{x^2-x-6} = \dfrac{\cancel{x} - 1}{\cancel{x}(1 - \frac{1}{x} - \frac{6}{x^2})} = 0$

$\lim\limits_{x \to -\infty} = 0$

$$\lim_{x \to -2^+} \frac{x-1}{x^2\ x-6} = \infty$$

$$\lim_{x \to -2^-} \frac{x-1}{x^2-x-6} = -\infty$$

$$\lim_{x \to 3^+} \frac{x-1}{x^2-x-6} = \frac{2}{9-3-6} = \frac{2}{0^+} = +\infty$$

$$\lim_{x \to 3^-} \frac{x-1}{x^2-x-6} = \frac{2^-}{9-3-6} = -\infty$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{2g}$$

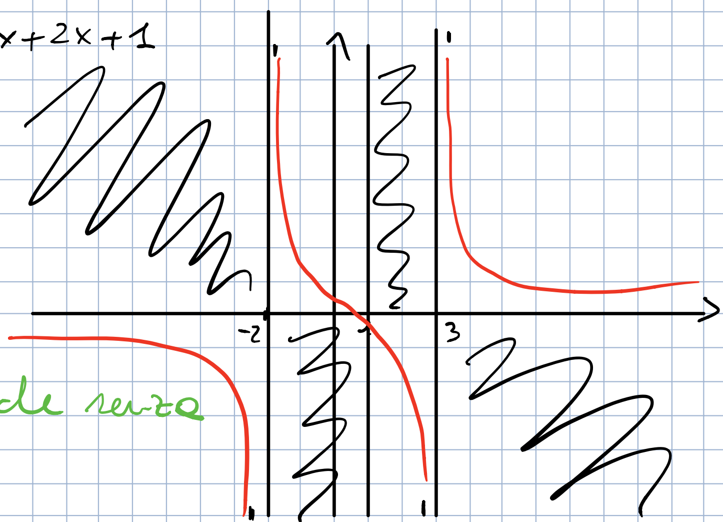
6 - Derivata

$$f(x) = \frac{x-1}{x^2-x-6} \qquad f'(x) = \frac{1 \cdot (x^2-x-6) - (x-1)(2x-1)}{(x^2-x-6)^2}$$

$$x^2-x-6 - 2x^2-1x+2x+1$$

$$\frac{-x^2+2x-7}{(x^2-x-6)^2}$$

Mancano $f'(x)>0$
ma il grafico
era intuibile anche senza



② SDF

$f(x) = e^{-x} - e^{-3x} \rightarrow e^{-x}(1 - e^{-2x})$

$e^{-x} > 0 \ \forall x \in R$

① Dom $f \in R \setminus \{0\}$     e $f(x) = 0$ sse $(1 - e^{-2x}) = 0$

ossia $x = 0$

② Assi

$f(0) \overset{y}{=} 1 - 1 = 0$     (interse 0,0)

$f(x) \overset{x}{=} 0 = e^{-x} - e^{-3x} = 0 \rightarrow 0$

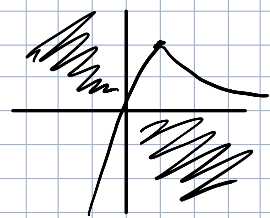
③ Segno

$e^{-x} - e^{-3x} > 0 \quad e^{-x} > e^{-3x} \quad -x > -3x$

$-x + 3x > 0 \quad 2x > 0 \quad x > \frac{0}{2} \quad x > 0$

la f è pos da 0 in poi

neg da 0 in dietro



④ lim

$\lim_{x \to \infty} e^{-x} - e^{-3x} = 0$    lim oriz in 0 a ∞

$\lim_{x \to -\infty} e^{-x} - e^{-3x} = -\infty$

⑤ Derivata

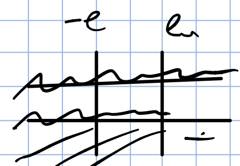
$y = e^{-x} - e^{-3x} = -e^{-x} - (-3e^{-x}) = -e^{-x} + 3e^{-3x}$

$y' > 0 \rightarrow -e^{-x} + 3e^{-3x} > 0 \rightarrow -e^{-x}(1 + 3e^{-2x}) > 0$

$-e^{-x}+3e^{-3x}>0 \rightarrow -e^{-3x}(e^{2x}-3)>0$ )

$$\begin{cases} -e^{-3x}>0 & x \in 0 \\ e^{2x}-3>0 & e^{2x}>3 \rightarrow 2x>\ln(3) \quad x>\dfrac{\ln(3)}{2} \end{cases}$$

$$\begin{cases} -e^{-3x}<0 & \forall x \in \mathbb{R} \\ e^{2x}-3<0 & e^{2x}<3 \quad 2x<\ln(3) \quad x<\dfrac{\ln(3)}{2} \end{cases}$$




$x<\dfrac{\ln(3)}{2}$

quindi cresce da $-\infty$ a $\dfrac{\ln(3)}{2}$

decresce da $\dfrac{\ln(3)}{2}$ a $+\infty$

⑥ Derivata''

$-e^{-x}+3e^{-3x} \Longrightarrow e^{-x}-9e^{-3x} = e^{-x}(1-9e^{-2x})$

$e^{-x}>0 \quad \forall x \in \mathbb{R}$

$1-9e^{-2x}>0 \quad 9e^{-2x}>1$

$e^{-2x}>\frac{1}{9}$

$-2x>\ln\frac{1}{9}\,(3^{-2})$

$x>\dfrac{\ln 3^{-2}}{2}$

$x>\dfrac{\cancel{2}\ln 3}{\cancel{2}}$

$x>\ln 3$

$1-9e^{-2x}=0$

$9e^{-2x}=1$

$e^{-2x}=\frac{1}{9}$

$-2x=\ln\frac{1}{9}$

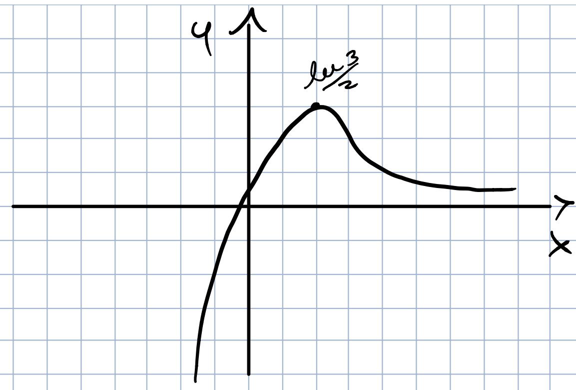
$-2x=\ln 3^{-2}$

$-2x=-2\ln(3)$

$x=\dfrac{\cancel{2}\ln 3}{\cancel{2}}$

$x=\ln 3$

concava $(-\infty, \ln 3)$ convessa $(\ln, +\infty)$




③ → Esame 24 Novembre 2016 – Parziale

① SDF $f(x) = \dfrac{x^2 - 5x + 6}{x - 4}$ $\quad x - 4 \neq 0$ $\quad x \neq 4$

1) $\text{Dom} f = x \in \mathbb{R} / \{4\}$

$(-\infty, 4) \cup (4, +\infty)$

2) P/D $f(-x) = \dfrac{-x^2 + 5x + 6}{-x - 4}$ no / P D

3) Intersezione assi

$f(0) = \dfrac{0 - 0 + 6}{0 - 4} = -\dfrac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}}$

$f(x) = 0 \quad \dfrac{x^2 - 5x + 6}{x - 4} = 0 \qquad x \neq 4$

$\Delta = 25 - 4 \cdot 1 \cdot 6 = 1 \qquad x_{1,2} = \dfrac{5 \pm 1}{2} \begin{cases} 3 \\ 2 \end{cases} \quad // \quad 0$

Interseca in $\left(0, -\dfrac{3}{2}\right)$

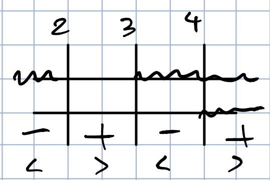
## 4) Segno

$$\frac{x^2-5x+6}{x-4} > 0$$

$x<2 \cup x>3$

$x^2-5x+6>0 \rightarrow (-\infty,2)\cup(3,+\infty)$

$x-4>0 \rightarrow x>4$



negative in in $(-\infty,2)$ $(3,4)$

positive in $(2,3)\cup(4,+\infty)$

## 5) limiti

$$\lim_{x\to+\infty} \frac{x^2-5x+6}{x-4} = \frac{\cancel{x^2}\left(1-\underbrace{\frac{5}{x}}_{\to 0}+\underbrace{\frac{6}{x^2}}_{\to 0}\right)}{\cancel{x}\left(1-\underbrace{\frac{4}{x}}_{\to 0}\right)} = +\infty$$

$$\lim_{x\to-\infty} \frac{x^2-5x+6}{x-4} = -\infty$$

$$\lim_{x\to4^+} \frac{x^2-5x+6}{x-4} = \frac{16-20+6}{4-4} = \frac{2}{0^+} = +\infty$$

$$\lim_{x\to4^-} \frac{x^2-5x+6}{x-4} = \frac{2}{0^-} = -\infty$$

## 6) Derivata 1ª

$$y=\frac{x^2-5x+6}{x-4} \qquad y'=\frac{(2x-5)\cdot(x-4)-(x^2-5x+6)}{(x-4)^2}=$$

$$\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{(g(x))^2}$$

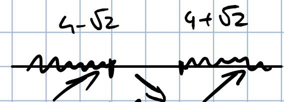
$$\frac{2x^2-8x-\cancel{5x}+20-x^2+\cancel{5x}-6}{(x-4)^2}=\boxed{\frac{x^2-8x+14}{(x-4)^2}}$$

$y'>0 \quad \frac{x^2-8x+14}{(x-4)^2}>0 \qquad x^2-8x+14>0$

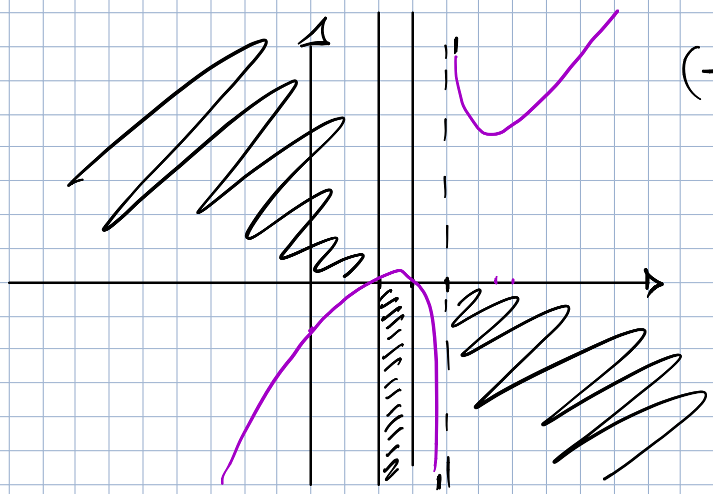
$(x-4)^2>0 \quad \forall x\in\mathbb{R}$

$\Delta = 64-4\cdot 1\cdot 14 = 8$

$\frac{8\pm 2\sqrt{2}}{2} < \begin{matrix} \frac{8+2\sqrt{2}}{2} = 4+\sqrt{2} \\ \frac{8-2\sqrt{2}}{2} = 4-\sqrt{2} \end{matrix} \qquad x<4-\sqrt{2} \vee x>4+\sqrt{2}$

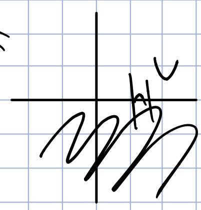



$(-\infty, 4-\sqrt{2})(4+\sqrt{2}, +\infty)$



Int $(0,-\frac{3}{2})$

$\sqrt{f(x)} \to \geq$



$(a, b, c, d)$
2 3 4 5

$\boxed{\log n, n, n^2, x^n, e^n, n!, n^n}$

$$\lim_{n\to+\infty} \frac{2^n+4n^2}{6^n+5} = \frac{2^n\left(1+\frac{4n^2}{2^n}\nearrow 0\right)}{6^n\left(1+\frac{5}{6^n}\nearrow 0\right)} = \frac{1\,\cancel{2^n}}{3\,\cancel{6^n}} = \frac{1}{3}$$

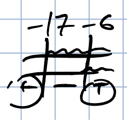
---

Esame 23/01/17

SDF

$f(x) = 1-\sqrt{\frac{x+17}{x+6}}$

Dom $f$: $\frac{x+17}{x+6}\geq 0 \qquad \begin{cases} x+17\geq 0 \\ x+6>0 \end{cases} \qquad \begin{cases} x\geq -17 \\ x>-6 \end{cases}$




$x\leq -17 \vee x>-6$

① Dom $f = (-\infty, -17] \cup ]-6, +\infty)$

② P/D $\rightarrow f(-x) = 1 - \sqrt{\frac{-x+17}{-x+6}}$ la $f$ è dispari(?)

③ Intersez.

$\sqrt{\frac{17}{6}} \rightarrow \frac{\sqrt{17}}{\sqrt{6}} \rightarrow \frac{\sqrt{17 \cdot 6}}{6} = \frac{\sqrt{102}}{6}$ whatever

$f(0) = 1 - \sqrt{\frac{17}{6}}$ interseca in $\left(0, 1 - \sqrt{\frac{17}{6}}\right)$

$f(x) = 0 \quad 1 - \sqrt{\frac{x+17}{x+6}} = 0 \rightarrow \emptyset$

④ Segno

c.e.: $x+17 \geq 0 \quad x \geq -17$
$x+6 \geq 0 \quad x \geq -6$

$1 - \sqrt{\frac{x+17}{x+6}} > 0 = -\sqrt{\frac{x+17}{x+6}} > 1 = \sqrt{\frac{x+17}{x+6}} < 1 =$

elevo tutto $^2 \rightarrow \frac{x+17}{x+6} < 1 = \frac{x+17}{x+6} - 1 < 0 = \frac{x+17-x-6}{x+6} < 0 =$

$\frac{11}{x+6} < 0 = x+6 < 0 = x < -6$

| | $-17$ | $-6$ |
|---|---|---|
| $-$ | $+$ | $+$ |
| $-$ | $-$ | $+$ |
| $+$ | $+$ | $-$ |
| $-$ | $+$ | $+$ |

$x \leq -17$

$(-\infty, -17)$ Pos

⑤ $\lim_{x \to +\infty} 1 - \sqrt{\frac{x+17}{x+6}} = \lim_{x \to +\infty} \underbrace{1}_{\to 1} - \lim \sqrt{\frac{x+17}{x+6}}$

regola $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ quindi $\sqrt{\lim_{x \to \infty} \frac{x+17}{x+6}}$

$\sqrt{\lim_{x \to \infty} \frac{x(1+\frac{17}{x})}{x(1+\frac{6}{x})}} = \sqrt{\lim_{x \to \infty} 1} = \sqrt{1}$ quindi $1 - \sqrt{1} = 0$

$\lim_{x \to -\infty} 1 - \sqrt{\frac{x+17}{x+6}} = 0$ As. orizz a 0.

As obliquo = $y = mx + q$ $\quad m = \lim_{x \to +\infty} \frac{f(x)}{x}$ $\quad q = \lim_{x \to \infty} f(x) - mx$

$$m = \lim_{x \to \infty} \frac{1 - \sqrt{\frac{x+17}{x+6}}}{x} \quad \begin{cases} \lim_{x \to \infty} 1 - \sqrt{\frac{x+17}{x+6}} = 0 \\ \lim_{x \to \infty} x = \infty \end{cases} \quad 0 \cdot \infty = 0$$

no asintoti obliqui

⑥ Derivata

$$y = 1 - \sqrt{\frac{x+17}{x+6}} \qquad \text{prop rad} \quad y' = 1 - \frac{\sqrt{x+17}}{\sqrt{x+6}} \qquad \text{prop der.} \quad (f+g)' = f' + g'$$

$$= \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{\sqrt{x+17}}{\sqrt{x+6}}\right) = 0 - \frac{\frac{d}{dx}(\sqrt{x+17}) \cdot \sqrt{x+6} - \sqrt{x+17} \cdot \frac{d}{dx}(\sqrt{x+6})}{(\sqrt{x+6})^2} =$$

$$- \frac{\left(\frac{1}{2\sqrt{x+17}}\right) \cdot (\sqrt{x+6}) - \sqrt{x+17} \cdot \frac{1}{2\sqrt{x+6}}}{x+6} = \frac{\frac{1}{2\sqrt{x+17}} \cdot (\sqrt{x+6}) - \frac{\sqrt{x+17}}{2\sqrt{x+6}}}{x+6} =$$

$$- \frac{\frac{x+6-x-17}{2\sqrt{(x+17) \cdot (x+6)}}}{x+6} = - \frac{\frac{-11}{2\sqrt{x^2+6x+17x+102}}}{x+6} = - \frac{\frac{-11}{2\sqrt{x^2+23x+102}}}{x+6}$$

$$\frac{11}{2\sqrt{x^2+23x+102} \cdot (x+6)}$$

Segno derivata

$$\frac{11}{2\sqrt{x^2+23x+102}\,(x+6)} > 0 \qquad 2\sqrt{x^2+23x+102}\,(x+6) > 0 \quad = \quad \frac{\text{tutto fratto}}{2}$$

$= x\sqrt{x^2+23x+102}\,(x+6) > 0$

a) $\begin{cases} x+6>0 \\ x\sqrt{x^2+23x+102} > 0 \end{cases}$ $\begin{cases} x>-6 \\ x\le -17 \vee x \ge -6 \end{cases}$

elevo $^2$

$\sqrt{x^2+23x+102} > 0$

$\Delta = 529 - 4\cdot 102 = 121$
$\quad\quad\ 408$

$121 \to \sqrt{\ } \to 11$

$x_{1,2} = \frac{-23 \pm 11}{2}$ $\quad -\frac{12}{2} = -6 \quad -\frac{34}{2} = -17$

$-6 \quad 0$

$(0, +\infty)$

$\begin{cases} x \le -17 \vee x \ge -6 \\ x > 0 \end{cases}$

$-17 \quad -6 \quad 0$

$(0, +\infty)$

b) $\begin{cases} x\sqrt{x^2+23x+102} < 0 \\ x < -6 \end{cases}$

$\begin{cases} x < 0 \\ x^2+23x+102 \end{cases}$

$\parallel$

$x \le -17 \vee x \ge -6$

$-17 \quad -6 \quad 0$

$(-\infty, -17)\ (-6, 0)$

$-17 \quad -6 \quad 0$

$(-\infty, -17)$

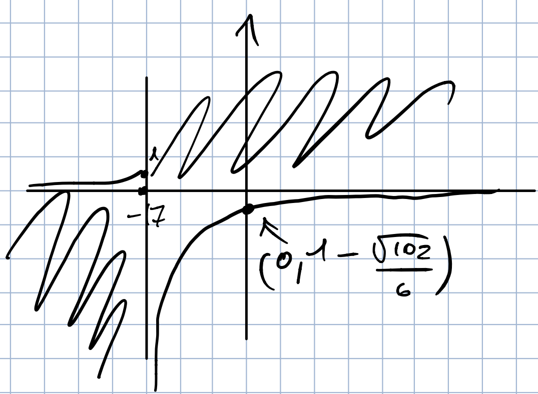
U:

$(-\infty, -17) \cup (0, +\infty)$

$-17 \quad 0$

$+ \quad - \quad +$

$f(-17) = 1$




2) Calcolare $\displaystyle\int \frac{1}{17\operatorname{sen}(x) + 6\cos(x) + 18}\,dx$

$$\int \frac{1}{17\,\mathrm{sen}(x)}\,dx + \int \frac{1}{6\cos(x)}\,dx + \int \frac{1}{18}\,dx$$

$$\frac{1}{17}\int \frac{1}{\mathrm{sen}(x)}\,dx + \boxed{\frac{1}{6}\int \frac{1}{\cos(x)} + \frac{1}{18}x}$$

$\mathrm{sen}^2(t) = 1 - \cos^2(t)$

$\cos^2(t) = 1 - \mathrm{sen}^2(t)$

$$\frac{1}{17}\int \frac{\mathrm{sen}(x)}{\mathrm{sen}^2(x)}\,dx + \frac{1}{6}\int \frac{\cos(x)}{\cos^2(x)}\,dx$$

$$\frac{1}{17}\int \frac{\mathrm{sen}(x)}{1-\cos^2(x)}$$

$y = \cos(x)$

$dy = -\mathrm{sen}(x)$

$dx = \frac{1}{t'}\,dt$

$1 - \mathrm{sen}^2(t) = \cos^2(x)$

$1 - \cos^2(x) = \mathrm{sen}^2(x)$

$$\frac{1}{17}\int \frac{\cancel{\mathrm{sen}(x)}}{1-\cos^2(x)} \cdot -\frac{1}{\cancel{\mathrm{sen}(x)}}\,dy = \frac{1}{17}\int -\frac{1}{1-\cos^2(x)}\,dy = \frac{1}{17}\int -\frac{1}{\mathrm{sen}^2(x)}\,dy$$

$$\frac{1}{17}\int -\frac{1}{1-\cos^2(x)}\,dy = \frac{1}{17}\int -\frac{1}{1-y^2}\,dy \to -\frac{1}{17}\int \frac{1}{1-y^2}\,dy$$

$$\int \frac{1}{x^2-a^2}\,dx$$

$$\frac{1}{2a}\cdot \ln\left(\left|\frac{x-a}{x+a}\right|\right)$$

$$\frac{1}{17}\cdot\frac{1}{2}\cdot \ln\left(\left|\frac{t-1}{t+1}\right|\right)$$

$$\frac{1}{34}\cdot \ln\left(\frac{\cos(x)-1}{\cos(x)+1}\right) + c$$

—

$$\frac{1}{6}\int \frac{1}{\cos(x)} = \frac{1}{6}\int \frac{\cos(x)}{\cos^2(x)} = \frac{1}{6}\int \frac{\cos(x)}{1-\mathrm{sen}^2(x)}$$

$t = \mathrm{sen}(x)$

$t' = \cos(x)$

$\frac{1}{\cos(x)}\,dy$

$$= \frac{1}{6}\int \frac{1}{1-t^2}\,dt = -\frac{1}{6}\cdot\frac{1}{2}\ln\left(\left|\frac{t-1}{t+1}\right|\right)$$

$t^2 - 1^2$

$$= -\frac{1}{12}\ln\left(\left|\frac{\mathrm{sen}(x)-1}{\mathrm{sen}(x)+1}\right|\right) + c$$

Quindi $\frac{1}{34}\ln\left(\left|\frac{\cos(x)-1}{\cos(x)+1}\right|\right) - \frac{1}{12}\ln\left(\left|\frac{\mathrm{sen}(x)-1}{\mathrm{sen}(x)+1}\right|\right) + \frac{1}{18}x + c$

① SDF $f(x) = \frac{x^2}{(x-3)(x-5)}$

$(x-3)(x-5) \neq 0$
$x \neq 3$
$x \neq 5$

Dom $f \in \mathbb{R} \setminus \{3, 5\}$
Dom $f = (-\infty, 3[ \cup ]3, 5[ \cup ]5, +\infty)$

P/D $= f(-x) = \frac{-x^2}{(-x-3)(-x-5)} \rightarrow$ dispari

Intersezioni

$y \; f(0) = \underline{0} \rightarrow 0$

$f(x) = 0 \rightarrow \frac{x^2}{(x-3)(x-5)} =$ $\quad x^2 = 0$ EQ (base $= 0$)

$x - 3 = 0 \quad x \neq +3$
$x - 5 = 0 \quad x \neq +5$

Interseca in $(0,0)$

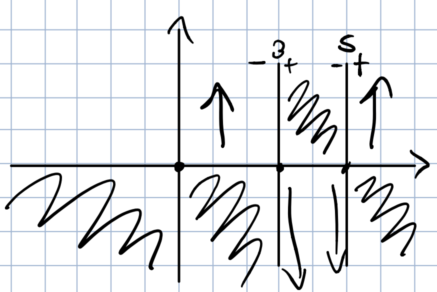
Segno

$\frac{x^2}{(x-3)(x-5)} > 0$

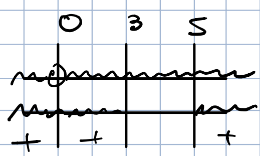
$\begin{cases} x^2 > 0 & \forall x \in \mathbb{R} \setminus \{0\} \\ (x-3)(x-5) > 0 \end{cases}$

$x > 3 \cup x > 5$

$(-\infty, 3) \cup (5, +\infty)$



$\begin{cases} \forall x \in \mathbb{R} / \{0\} \\ (-\infty, 3)(5 + \infty) \end{cases}$



pos in $(-\infty, 0)(0, 3)(5, +\infty)$

## limiti

$$\lim_{x \to \infty} \frac{x^2}{(x-3)(x-5)} = \frac{x^2}{x^2-3x-5x+15} = \frac{\cancel{x^2}}{\cancel{x^2}\left(1-\frac{8}{x}+\frac{15}{x^2}\right)} = 1$$

$$x^2-8x+15$$

$$\lim_{x \to -\infty} = 1$$ as or $+\infty$ a 1, $-\infty$ a 1

$$\lim_{x \to 3^+} = -\infty$$ as vert $3^+$ a $-\infty$

$$\lim_{x \to 3^-} = +\infty$$ as vert $3^-$ a $+\infty$

$$\lim_{x \to 5^+} = +\infty$$ as vert $5^+$ a $+\infty$

$$\lim_{x \to 5^-} = -\infty$$ as vert $5^-$ a $-\infty$

## derivata

$$y = \frac{x^2}{(x-3)(x-5)} \to \frac{x^2}{x^2-8x+15} \qquad \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$y' = \frac{2x \cdot (x^2-8x+15) - x^2 \cdot (2x-8)}{(x^2-8x+15)^2}$$

$$y' = \frac{\cancel{2x^3} - 16x^2 + 30x - \cancel{2x^3} + 8x^2}{(x^2-8x+15)^2} = \frac{-8x^2+30x}{(x^2-8x+15)^2}$$

## Segno derivata

$$\frac{-8x^2+30x}{(x^2-8x+15)^2}>0$$

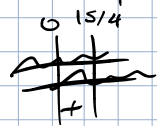
$CE = x \neq 5$
$x \neq 3$

Fattorizzo raccogliendo $-2x$

$$\frac{-2x(4x-15)}{(x^2-8x+15)^2}>0$$

$$\begin{cases} -2x(4x-15)>0 \\ (x^2-8x+15)^2>0 \end{cases}$$

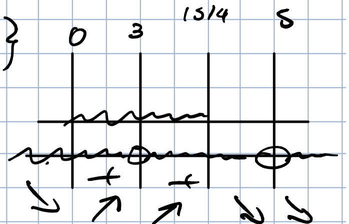
$$\begin{cases} -2x>0 \\ 4x-15>0 \end{cases} \qquad \begin{cases} x>0 \\ x<\frac{15}{4} \end{cases}$$





$\left(0, \frac{15}{4}\right)$

$$\begin{cases} \left(0, \frac{15}{4}\right) \\ \forall x \in \mathbb{R} \setminus \{5, 3\} \end{cases}$$

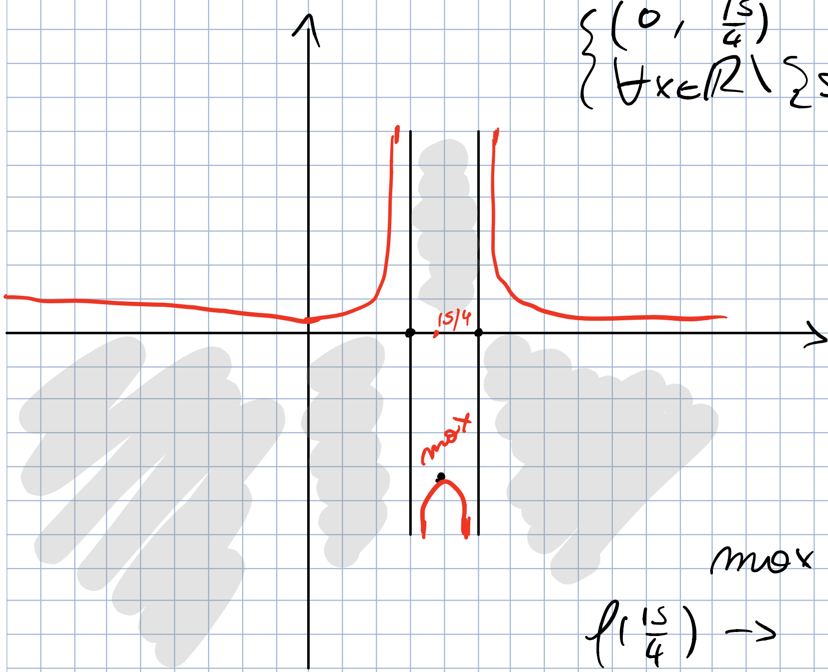




$(0,3)\left(3,\frac{15}{4}\right)$

$\|$

$\left(0,\frac{15}{4}\right) \setminus \{3\}$





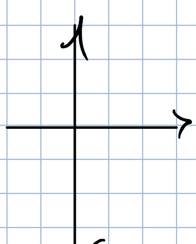
max in $15/4$

$$f\left(\frac{15}{4}\right) \to \frac{\left(\frac{15}{4}\right)^2}{\left(\frac{15}{4}-3\right)\left(\frac{15}{4}-5\right)} = \frac{225}{16}$$

$$\frac{3}{4} \cdot -\frac{5}{4} = -\frac{15}{16}$$

$$\frac{225}{\cancel{16}} \cdot -\frac{\cancel{16}}{15} = -15$$

max$\left(\frac{15}{4}, -15\right)$



$(-\infty,-15)(f(0),+\infty)$

$$f(x) = \frac{x}{x^2-24x+128}$$

1 SDF 16/17

$D: b^2-4ac = 64$

$$x_{1,2} = \frac{24 \pm 8}{2} \begin{cases} \cancel{\frac{32}{2}}\ 16 \\ \cancel{\frac{16}{2}}\ 8 \end{cases}$$

① Dom $\in \mathbb{R} \setminus \{8, 16\}$

② Intersezioni

$f(0) = 0$ (interseca $0,0$)

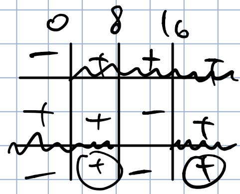
$f(x) = 0 \to 0$

③ Segno

$$\frac{x^3}{x^2-24x+128} > 0$$

$x^3 > 0 \quad x > 0$

$x < 8 \vee x > 16$

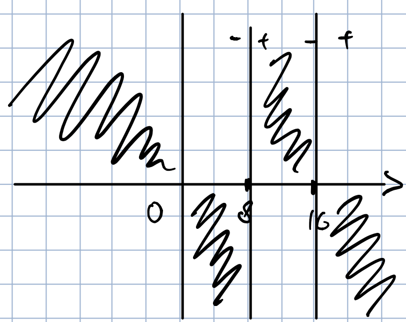


$(0,8) \cup (16, +\infty)$

④ limiti

$$\lim_{x\to\infty} \frac{x^3}{x^2\left(1-\frac{24}{x}+\frac{128}{x^2}\right)} = \frac{x}{1} = +\infty$$

$$\lim_{x\to-\infty} = -\infty$$



$$\lim_{x\to 8^+} = -\infty \qquad \lim_{x\to 8^-} = +\infty$$

$$\lim_{x\to 16^+} = +\infty \qquad \lim_{x\to 16^-} = -\infty$$

$y = mx + q \qquad m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad q = f(x) - mx$

$$m = \lim_{x\to\infty} \frac{x^3}{x^2-24x+128} \cdot \frac{1}{x} = \frac{x^3}{x^3-24x^2+128x} = 1$$

$$q = \lim_{x \to \infty} \frac{x^3}{x^2-24x+128} - x = \frac{x^3 - x(x^2-24x+128)}{x^2-24x+128} =$$

$$= \frac{\cancel{x^3} - \cancel{x^3} + 24x^2 - 128x}{x^2-24x+128} = \frac{\cancel{x^2}\left(24 - \frac{128}{x}\right)}{\cancel{x^2}\left(1 - \frac{24}{x} + \frac{128}{x^2}\right)} = 24$$

$$y = x + 24$$

⑤ Derivate

$$f = \frac{x^3}{x^2-24x+128} \qquad f' = \frac{3x^2 \cdot (x^2-24x+128) - x^3 \cdot (2x-24)}{(x^2-24x+128)^2}$$

$$= \frac{3x^4 - 72x^3 + 384x^2 - 2x^4 + 24x^3}{(x^2-24x+128)^2} = \frac{x^4 - 48x^3 + 384x^2}{(x^2-24x+128)^2}$$

$$\frac{x^4-48x^3+384x^2}{\underbrace{(x^2-24x+128)^2}_{>0}} > 0 \qquad \longrightarrow \forall x \in \mathbb{R}$$

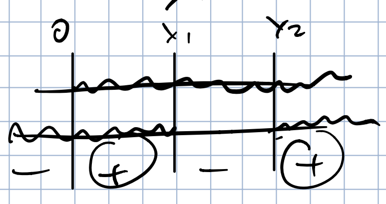
$$x^4 - 48x^3 + 384x^2 > 0 \qquad x^2(x^2 - 48x + 384) > 0$$

$$\begin{cases} x^2 > 0 & \forall x \in \mathbb{R} \\ x^2 - 48x + 384 > 0 \end{cases}$$

$$\Delta = 2304 - \underset{1536}{4 \cdot 384} = 768$$

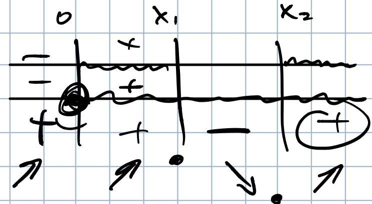
$$x_{1,2} = \frac{48 \pm \sqrt{768}}{2} = \frac{48 \pm 16\sqrt{3}}{2} = \qquad \sqrt{768} = \sqrt{16^2 \cdot 3} = \sqrt{16^2}\sqrt{3} = 16\sqrt{3}$$

$$\frac{\cancel{2}(24 \pm 16\sqrt{3})}{\cancel{2}} \quad \begin{cases} 24 + 16\sqrt{3} & x_2 \\ 24 - 16\sqrt{3} & x_1 \end{cases}$$




$$x < 24 - 16\sqrt{3} \ \cup \ x > 24 + 16\sqrt{3}$$

$(0, 24-16\sqrt{3}) \cup (24+16\sqrt{3}, +\infty)$



$(-\infty, 0)(0, 24-16\sqrt{3})$

$(0, x_1) \cup (x_2, +\infty)$

$(-\infty, 0)(0, 8)(8, 24-16\sqrt{3})(24+16\sqrt{3}, +\infty)$

$x+8$

$x \neq 16$

$max = 24-16\sqrt{3}$ , $y$?

$min = 24+16\sqrt{3}$ , $y$?

$f(max) =$ $\dfrac{x^3}{x^2-24+128}$

$f(min) =$

$$\frac{(24-16\sqrt{3})^3}{4-16\sqrt{3})^2-24(24-16\sqrt{3})+128} = x-51$$

---

## 2012-2013

SDF

$y = \dfrac{e^{2x}+8}{e^x+1}$

Dom $e^x+1 \neq 0$

$e^x \neq -1 \quad \forall x \in R$

Dom $\in R$

Intersez assi

$f(0) = \frac{9}{2}$

Interseca in $(0, \frac{9}{2})$

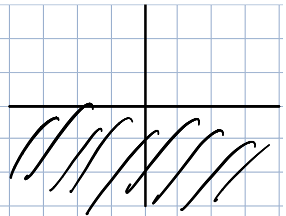
$f(x) = 0 \quad \dfrac{e^{2x}+8}{e^x+1} = 0 \in \emptyset$

$\uparrow$

Segno

$$\frac{e^{2x}+8}{e^x+1}>0 \qquad \begin{cases} e^{2x}+8>0 & \forall x\in\mathbb{R} \\ e^x+1>0 & \forall x\in\mathbb{R} \end{cases}$$

Sempre pos



Asintoti

$$\lim_{x\to\infty} \frac{e^{2x}+8}{e^x+1} = \frac{e^{2x}\left(1+\frac{8}{\cancel{e^{2x}}}\right)}{\cancel{e^x}\left(1+\frac{1}{\cancel{e^x}}\right)} = \infty$$

$$\lim_{x\to-\infty} = \frac{e^{2x}+8}{e^x+1} = \frac{0+8}{0\;\;1} = 8$$ As orizz a 8

$$y=mx+q \qquad m=\lim_{x\to\infty}\frac{f(x)}{x} \qquad q=\lim_{x\to\infty} f(x)-mx$$

$$m=\lim_{x\to\infty} \frac{e^{2x}+8}{e^x+1}\cdot x = \frac{x e^{2x}+8x}{e^x+1} = \frac{e^{2x}\left(x+\frac{8x}{e^{2x}}\right)}{\cancel{e^x}\left(1+\frac{1}{e^x}\right)} = +\infty$$

Derivata

$$y=\frac{e^{2x}+8}{e^x+1} \qquad y'=\frac{2e^{2x}\cdot(e^x+1)-(e^{2x}+8)\cdot e^x}{(e^x+1)^2}=$$

$$=\frac{2e^{3x}+2e^{2x}-e^{3x}-8e^x}{(e^x+1)^2}=\frac{e^{3x}+2e^{2x}-8e^x}{(e^x+1)^2}$$

Segno derivata $\begin{cases} e^{3x}+2e^{2x}-8e^x>0 \\ \forall x\in\mathbb{R} \end{cases}$

$$e^x(e^{2x}+2e^x-8)>0 = e^x(e^{2x}+4e^x-2e^x-8)>0$$

$2e^x = 4e^x-2e^x$

$$e^x(e^x(e^x+4)-2(e^x+4))>0$$

$$e^x((e^x+4)(e^x-2))>0 \to \begin{cases} e^x>0 & x\in\mathbb{R} \\ (e^x+4)(e^x-2)>0 \end{cases}$$

$e^x>-4 \quad \forall x\in\mathbb{R}$

$e^x>2 \to \ln(2)$

$-4 \quad 2$

$x$

inutile perché $\forall x\in\mathbb{R}$

$(\ln(2), +\infty)$

$$f(\ln(2)) = \quad y=\frac{e^{2x}+8}{e^x+1}$$

$$\frac{e^{2\ln(2)}+8}{e^{\ln(2)}+1} =$$

$$\frac{e^{\ln(2)^2}+8}{e^{\ln(2)}+1} =$$

$2$

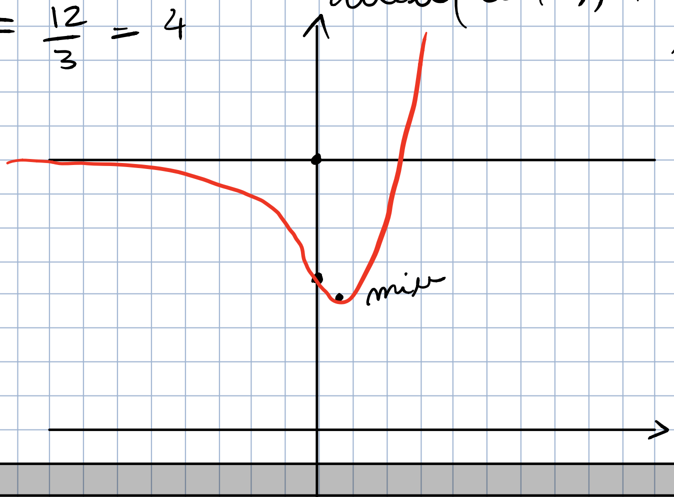
$$x\cdot\ln(a)=\ln(a^x)$$

$$2\ln(2)=\ln(2^2)$$

$$e^{\ln(x)}=x$$

$$\frac{e^{\ln(2)}+8}{3} = \frac{2^2+8}{3} = \frac{12}{3} = 4$$

$\min(\ln(2), 4)$




$$\int \cos(x)\cdot\frac{\text{sen}^3(x)-5\,\text{sen}(x)}{\text{sen}^2(x)-\text{sen}(x)-6}\,dx$$

$y=\text{sen}(x)$

$dy=\cos(x)\,dx$

$$\int\frac{y^3-5y}{y^2-y-6}\,dy$$

$y^3 \qquad -5y \qquad |\,y^2-y-$

$$\int y+1\,dy + \int \frac{2y+6}{y^2-y-6}\,dy \qquad \begin{array}{l|l} -y^3+y^2+6y & y+1 \\ \hline y^2+1y & \\ -y^2+y+6 & \\ \hline 2y+6 & \end{array}$$

$$\int y\,dy + \int 1\,dy +$$

$t = y^2-y-6$

$dt = 2y-1\,dy$

$$\int \frac{2y+7-1}{t}\,dt \qquad \int \frac{7}{t}\,dt \qquad 7\int \frac{1}{t}\,dt$$

$$\int y\,dy + \int 1\,dy + 7\ln|t|$$

$$\frac{y^2}{2} + y + 7\ln|y^2-y-6|$$

$$\frac{\operatorname{sen}^2 x}{2} + \operatorname{sen} x + 7\ln|\operatorname{sen}^2 x - \operatorname{sen} x - 6| + c$$

# 16-17 ②

$$f(x) = 1 - \sqrt{\frac{x+17}{x+6}}$$

$x+6 \neq 0 \quad x \neq -6$

Dom $f$ $(-\infty,-17)\cup]-6,+\infty)$

$\frac{x+17}{x+6} \geq 0 \quad x \geq -17$

$x > -6$

Intersez

$f(0) = 1 - \sqrt{\frac{17}{6}} = \approx 0,25$ Intersezio $(0, 0,25)$

$f(x)=0 \quad + \sqrt{\frac{x+17}{x+6}} = +1 \quad \frac{x+17}{x+6} = +1 \rightarrow 1 - \left(\frac{x+17}{x+6}\right) =$

$$\frac{x+6-x-17}{x+6} = \frac{-11}{x+6} = 0 \in \emptyset$$

Segno

$$1-\sqrt{\frac{x+17}{x+6}} > 0 \rightarrow \sqrt{\frac{x+17}{x+6}} < 1 \rightarrow \frac{x+17}{x+6} < 1$$

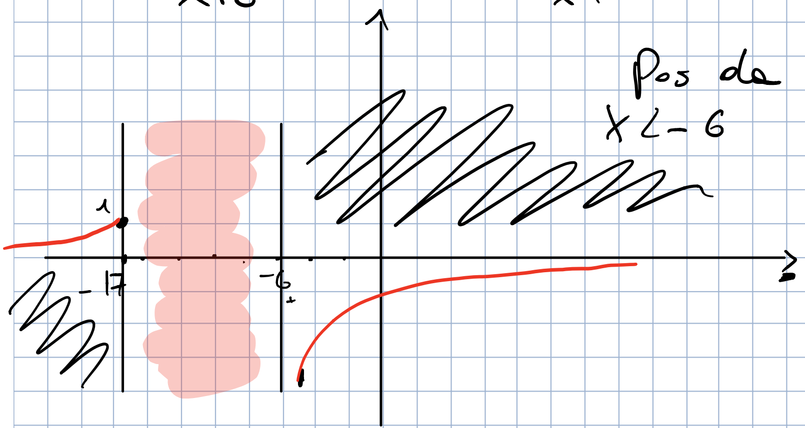
$x \neq -6$

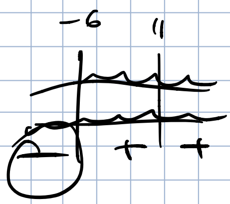
$$-1+\frac{x+17}{x+6} < 0 \rightarrow \frac{-\cancel{x}-6+\cancel{x}+17}{x+6} = \frac{11}{x+6} \geq 0$$

$11 > 0 \ \forall x \in \mathbb{R}$

$x+6 > 0$

$x > -6$




$$\lim_{x\to\infty} 1-\sqrt{\frac{x+17}{x+6}} = 1-\sqrt{\frac{x+17}{x+6}} \cdot \frac{1-\sqrt{\frac{x+17}{x+6}}}{1+\sqrt{\frac{x+17}{x+6}}}$$

$$\frac{1-\frac{x+17}{x+6}}{1+\sqrt{\frac{x+17}{x+6}}} \qquad \frac{\frac{x+6-x-17}{x+6}}{1+\sqrt{\frac{x+17}{x+6}}} =$$

$$\underbrace{\frac{\cancel{x}+6-\cancel{x}-17}{x+6}}_{\to -\frac{11}{x+6}} \cdot \frac{1}{1+\sqrt{\frac{x+17}{x+6}}} = \left(-\frac{11}{x+6}\right) \cdot \frac{1}{1+\frac{\sqrt{x+17}}{\sqrt{x+6}}}$$

$$\frac{1}{\frac{\sqrt{x+6}+\sqrt{x+17}}{\sqrt{x+6}}} = \frac{(\sqrt{x+6})^{1/2}}{\sqrt{x+6}+\sqrt{x+17}} \cdot \frac{-11}{(x+6)^{-1}}$$

$$= \frac{-11}{(\sqrt{x+6})(\sqrt{x+6}+\sqrt{x+17})} = \frac{-11}{(x+6)(\sqrt{x^2+23x+112})} = \frac{-11}{+\infty} = 0^-$$

$\lim_{x \to -\infty} f(x) = 0^+$ asintoti orizzontali

$\lim_{x \to -6^+} = -\infty$ $\quad 1 - \sqrt{\frac{-6+17}{-6^++6}} = 1 - \frac{11^+}{0^+} = -\infty$

$\to 1-(+\infty)$

$\lim_{x \to -6^-} =$ non c'è

$$\lim_{x \to -17^-} = 1 - \sqrt{\frac{-17^-+17}{-17^-+6}} = 1 - \sqrt{\frac{0^-}{-11}} = 1 - \sqrt{\frac{0^+}{11}} = 1$$

Derivata prima

$$y' = -\left[\frac{1}{2\sqrt{\frac{x+17}{x+6}}}\left(\frac{1\cdot(x+6)-(x+17)\cdot 1}{(x+6)^2}\right)\right]$$

$$= -\left[\frac{1}{2\sqrt{\frac{x+17}{x+6}}}\left(\frac{-11}{(x+6)^2}\right)\right] = \frac{(\sqrt{x+6})\cdot -11}{2\cdot\sqrt{x+17}\cdot(x+6)^2} =$$

$$\frac{11}{2}\,\frac{1}{(x+6)^2(\sqrt{x+6})(\sqrt{x+7})}$$

$-6$; $+ \;|\; +$

Limite concorso

$$\lim_{x \to \infty} x^{-3} \int_0^x \frac{17t^3+6x+18}{t+1}$$

Sono continue e derivabili e $t+1 \neq 0 \to t \neq -1$

$\downarrow$

Hôpital

$$\lim_{x \to \infty} \frac{\int_0^x \frac{17t^3+6x+18}{t+1}}{x^3}$$

$$\lim_{x\to\infty} \frac{\frac{17x^3+6x+8}{x+1}}{3x^2} = \frac{17x^3+6x+8}{x+1} \cdot \frac{1}{3x^2} =$$

$$\lim_{x\to\infty} \frac{17x^3+6x+8}{3x^3+3x^2} = \frac{x^3\left(17+\frac{6}{x^2}+\frac{8}{x^3}\right)}{x^3\left(3+\frac{3}{x}\right)} = \frac{17}{3}$$

---

Seconda prova 16/17

$$f(x) = \sqrt{8x^3-18x^2+3x}$$

$$\Delta = 324 - \underbrace{4 \cdot 3 \cdot 8}_{96} = 228 \quad \sqrt{228} \to 2\sqrt{57}$$

$$\frac{+18 \pm 2\sqrt{57}}{16}$$

$$x_1 = \frac{\cancel{2}(9+\sqrt{57})}{\cancel{16}} = \frac{9+\sqrt{57}}{8}$$

$$x_2 = \frac{9-\sqrt{57}}{8}$$

$$8x^3-18x^2+3x \geq 0$$

$$x(8x^2-18x+3) \geq 0$$

$$\begin{cases} x \geq 0 \\ 8x^2-18x+3>0 \end{cases}$$

$$x < \frac{9-\sqrt{57}}{8} \lor x > \frac{9+\sqrt{57}}{8}$$

$$x < 15 \cup x > 15$$

$$(0, 15) \cup (15, +\infty)$$

Intersezione

$$f(0) = \sqrt{0} = 0$$

$f(x)=0$ $\quad \sqrt{8x^3-18x^2+3x} = 0$ una radice può essere $=0$ solo $\sqrt{x}$ $\quad x=0$ $\quad 8x^3-18x^2+3x=0$

$$x=0$$

$$x_1 = \frac{9+\sqrt{57}}{8} \qquad x_2 = \frac{9-\sqrt{57}}{8} \qquad (0,0)$$

## Segno

$$\sqrt{8x^3-18x^2+3x} > 0$$

$$8x^3-18x^2+3x > 0$$

$$x(8x^2-18x+3) > 0$$

$$\begin{cases} x > 0 \\ 8x^2-18x+3 > 0 \end{cases}$$

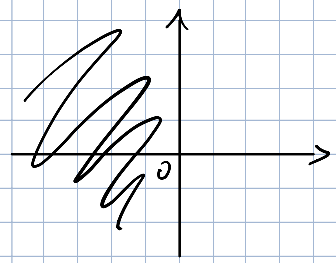
$\Downarrow$

$$x < x_2 \cup x > x_1$$

| | $x_2$ | | $0$ | | $x_1$ | |
|---|---|---|---|---|---|---|
| $-$ | | $-$ | | $+$ | | $+$ |
| $+$ | | $-$ | | $-$ | | $+$ |
| $-$ | | $+$ | | $-$ | | $+$ |

$$x_2 < x < 0 \cup x > x_1$$

## Limiti



$$\lim_{x\to\infty} \sqrt{8x^3-18x^2+3x} = +\infty$$

$$\lim_{x\to-\infty} \sqrt{8x^3-18x^2+3x} = \text{indefinito}$$

$$y = mx+q \qquad mx = \lim_{x\to\infty} \sqrt{8x^3-18x^2+3x} \cdot \frac{1}{x} = \frac{\sqrt{8x^3-18x^2+3x}}{x}$$

## Derivata

$$\frac{d}{dx}\sqrt{\underset{\uparrow g}{8x^3-18x^2+3x}} = \qquad = \frac{1}{2\sqrt{g}} \cdot (24x^2-24x+3) =$$

$$\frac{d}{dg}\sqrt{g} \cdot \frac{dx}{dx}(8x^3-18x^2+3x) =$$

$$= \frac{1}{2\sqrt{8x^3-18x^2+3x}} \cdot (24x^2-24x+3) = \frac{24x^2-24x+3}{2\sqrt{8x^3-18x^2+3x}}$$

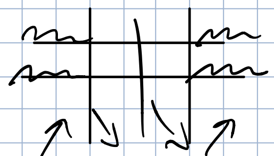
## Segno derivata

$$\frac{24x^3-24x+3}{2\sqrt{8x^3-18x^2+3x}} \geq 0$$

$$24x^2-24x+3 > 0$$

$$\begin{cases} x > 0 \\ 8x^2-18x+3 > 0 \end{cases} = x < x_1 \cup x > x_2$$

$$= 576 - \underbrace{4\cdot 24\cdot 3}_{288} = 288 \to 2\sqrt{67} =$$



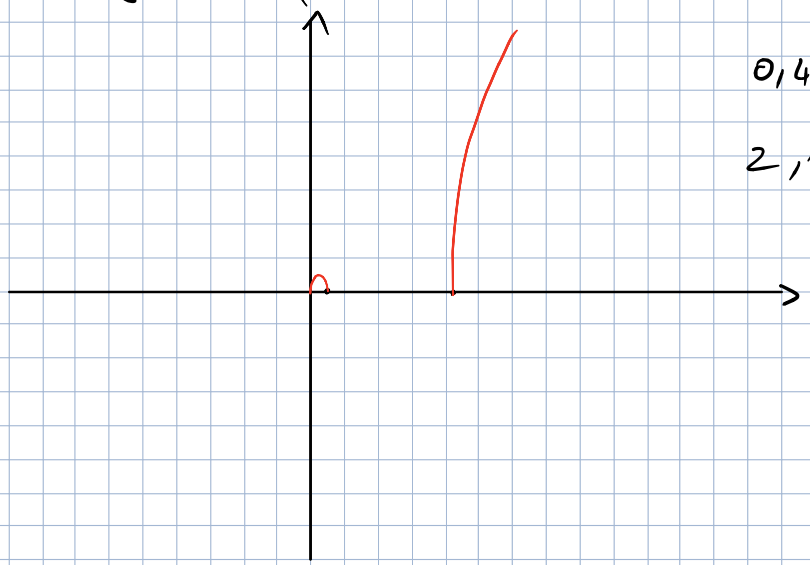
$$f\left(\frac{9-\sqrt{67}}{8}\right)$$

$$\downarrow$$

$$\sqrt{8\left(\frac{9-\sqrt{67}}{8}\right)^3 - 18\left(\frac{9-\sqrt{57}}{8}\right)^2 + 3\left(\frac{9-\sqrt{57}}{8}\right)}$$

max $\frac{9-\sqrt{67}}{8}$, 0

min $\frac{9+\sqrt{67}}{8}$, 0



0,4 , 0

2,0

1

---

$$\int \operatorname{arctg}(8x-18)\,dx$$

sost → $y = 8x-18$

$dy = 8\,dx$

aggiungi · 1

$$\frac{1}{8}\int \operatorname{arctg}(y)\,dy = \frac{1}{8}\int \underbrace{\operatorname{arctg}(y)}_{f}\cdot\underbrace{1}_{g'}\,dy$$

$f = \operatorname{arctg}(y)$ $f' = \frac{1}{1+y^2}$

$g' = 1$ → $g = y$

$$\frac{1}{8}\left(\operatorname{arctg}(y)\cdot y - \int y\cdot\frac{1}{1+y^2}\,dy\right)$$

$$\frac{1}{8}\left(\operatorname{arctg}(y)\cdot y - \int \frac{y}{1+y^2}\,dy\right) \xrightarrow{\text{sost}}$$

$t = 1+y^2$

$dt = 2y\,dy$

$$\frac{1}{8}\left(\operatorname{arctg}(4)\cdot 4-\frac{1}{2}\int \frac{1}{t}\,dt\right)$$

$$\int \frac{1}{x}\,dx = \ln(|x|)$$

$$\frac{1}{8}\left(\operatorname{arctg}(4)\cdot 4-\frac{1}{2}\cdot \ln(|t|)\right)=\frac{1}{8}\left(\operatorname{arctg}(4)\cdot 4-\frac{1}{2}\ln(|1+4^2|)\right)$$

$$=\frac{1}{8}\left(\operatorname{arctg}(8x-18)\cdot(8x-18)\right)-\frac{1}{2}\ln\left(|1+(8x-18)^2|\right)$$

moltiplico

$$=\frac{1}{8}\left(8\operatorname{arctg}(8x-18)\cdot x-18\operatorname{arctan}(8x-18)-\frac{1}{2}\ln\left(1+(8x-18)^2\right)\right)$$

rimuovo $\left(\frac{1}{8}\right)$

$$=\operatorname{arctg}(8x-18)\cdot x-\frac{9\operatorname{arctg}(8x-18)}{4}-\frac{1}{16}\ln\left(1+(8x-18)^2\right)$$

espando

$$=\operatorname{arctg}(8x-18)\cdot x-\frac{9\operatorname{arctg}(8x-18)}{4}-\frac{1}{16}\ln\left(\underbrace{1+64x^2-288x+324}_{\text{sommo}}\right)$$

$$=\operatorname{arctg}(8x-18)\cdot x-\frac{9\operatorname{arctg}(8x-18)}{4}-\frac{1}{16}\ln\left(64x^2-288x+325\right)+C$$

---

$$\lim_{x\to 0}\frac{\operatorname{sen}(8x^3-18x^2)}{1-\cos(3x)}=\left[\frac{0}{1}\right]=\text{Hôpital}$$

$$\lim_{x\to 0}\frac{\frac{d}{dx}\operatorname{sen}(8x^3-18x^2)}{\frac{d}{dx}\,1-\cos(3x)}$$ → composte $(f(g))'=f'(g)\cdot g'$

$$\lim_{x\to 0}\frac{\cos(8x^3-18x^2)\cdot(24x^2-36x)}{3\operatorname{sen}(3x)}=$$ raccolgo un 3 per semplificarlo

$$\lim_{x\to 0}\frac{\cos(8x^3-18x^2)\cdot\cancel{3}(8x^2-12x)}{\cancel{3}\operatorname{sen}(3x)}=$$ Poiché ottengo una forma ind. uso Hôp ancora

$$\lim_{x \to 0} \frac{\frac{d}{dx}\left(\cos(8x^3 - 18x^2) \cdot (8x^2 - 12x)\right)}{\frac{d}{dx}\left(\text{sen}(3x)\right)} \qquad f \cdot g = f' \cdot g + f \cdot g'$$

$$\lim_{x \to 0} \frac{\overbrace{-\text{sen}(8x^3 - 18x^2)(24x^2 - 36x)}^{f'} \cdot \overbrace{(8x^2 - 12x)}^{g} + \cos(8x^3 - 18x^2) \cdot (16x - 12)}{3\cos(3x)}$$

$$\lim_{x \to 0} \frac{-3(64x^4 - 192x^3 + 144x^2) \cdot \text{sen}(8x^3 - 18x^2) + \cos(8x^3 - 18x^2) \cdot (16x - 12)}{3\cos(3x)}$$

$\cos(0) = 1$

$$\frac{-3(0 - 0 + 0) \cdot \text{sen}(0 - 0) + \cos(0 - 0) \cdot (-12)}{3}$$

$$\frac{3 \cdot 0 \,\text{sen}(0) + \cos(0) \cdot (-12)}{3} = \frac{0 - 1 \cdot 12}{3} = \frac{-\cancel{12}}{\cancel{3}} = -4$$

---

SDF

$f(x) = \log(9x^2 + 12x + 17)$ $\qquad 9x^2 + 12x + 17 \neq 0$

$\Delta = 144 - 4 \cdot 9 \cdot 17 = -468 \quad \emptyset$

$\quad 612$

Dom $f \in \mathbb{R}$

Intersezioni:

$f(0) = \log(17)$ $\qquad$ Int $(0, \log(17))$

$\qquad 2{,}8 \to 3$

$f(x) = 0 \in \emptyset \to 0$

Segno

$\log(9x^2 + 12x + 17) = x \in \mathbb{R}$ sempre positiva

Limiti

$$\lim_{x \to \infty} \log(9x^2 + 12x + 17) = \infty$$

$\lim_{x\to-\infty} = +\infty$ perché $\log > 0$

$m = \frac{f(x)}{x} \quad \lim_{x\to\infty} \frac{\log(9x^2+12x+17)}{x} = +\infty$

# Derivata

$f(g(x)) = f'(g)\cdot g'$

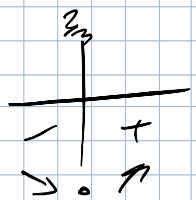
$\log(9x^2+12x+17) = \frac{1}{9x^2+12x+17} \cdot 18x+12 = \frac{18x+12}{9x^2+12x+17}$

Segno derivata

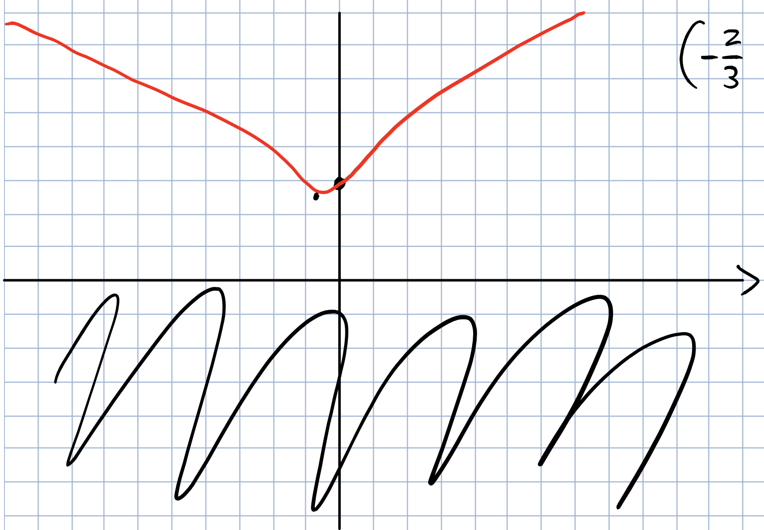
$\frac{18x+12}{9x^2+12x+17} \geq 0$ $\quad \begin{cases} 18x+12\geq 0 & x \geq -\frac{12}{18} \quad x\geq \frac{2}{3} \\ 9x^2+12+17>0 & \forall x\in\mathbb{R}\end{cases}$

$f(-\frac{2}{3}) \quad 9\left(-\frac{2}{3}\right)^2 + \cancel{12}^{3}\left(-\frac{2}{3}\right) + 17$

$4 - 6 + 17 = 15 \to \ln 15$




min $\frac{2}{3}$



$\left(-\frac{2}{3}, \ln(15)\right)$

# INTEGRALI PER SOSTITUZIONE

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} f(y)\, dy$$

$y = e^x$
$dy = e^x dx$

**esempio 1**

$$\int \operatorname{sen}(e^x)\, e^x dx = \int \operatorname{sen}(y)\, dy = -\cos(y) =$$

$y = e^x$
$dy = e^x dx$

$$= -\cos(e^x) + c$$

1) Sostituzione $y = \varphi(x)$
2) Nuovo differenziale $dy = \varphi'(x)\, dx$
3) Se definito: $a \to \varphi(a)$
$b \to \varphi(b)$

**esempio 2**

$$\int \cos x \, \operatorname{sen}(\operatorname{sen} x)\, dx =$$

$y = \operatorname{sen} x$
$dy = \cos x \, dx$

$$= \int \operatorname{sen}(y)\, dy = -\cos(y) + c =$$
$$= -\cos(\operatorname{sen}(x)) + c$$

**esempio 3**

$$\int \frac{e^x}{1+e^{2x}}\, dx = \int \frac{e^x}{1+(e^x)^2}\, dx = \int \frac{1}{1+y^2}\, dy$$

$y = e^x$
$dy = e^x dx$

$$= \arctan(y) + c = \arctan(e^x) + c$$

**esempio 4**

$$\int_0^{\sqrt{\pi}} x \cos(x^2)\, dx = \int_0^{\pi} \cos(y)\, \frac{dy}{2} =$$

$y = x^2$
$dy = 2x\, dx$

$0 \to \varphi(0) = 0^2 \to 0$
$\sqrt{\pi} \to \sqrt{\pi}^2 = \pi$

$$= \frac{1}{2}\int_0^{\pi} \cos(y)\, dy = \frac{1}{2}\left[\operatorname{sen} y\right]_0^{\pi} =$$

$$= \frac{1}{2}\left[\operatorname{sen}\pi - \operatorname{sen} 0\right] = 0$$

esempio 5

$$\int \frac{1}{x(\ln x+1)}\,dx = \int \frac{1}{\ln x+1}\cdot\frac{1}{x}\,dx =$$

$y = \ln x + 1$

$dy = \frac{1}{x}\,dx$

$$= \int \frac{1}{y}\,dy = \ln|y| + c = \ln|\ln x + 1| + c$$

esempio 6

$y = x^2 \quad x = \sqrt{x}$

$x = \sin y$

$dx = \cos y\,dy$

$$\int \sqrt{1 - x^2}\,dx = \int \sqrt{1-\sin^2 y}\cdot \cos y\,dy =$$

$$= \int \cos y \cdot \cos y\,dy = \int \cos^2 y\,dy = \ldots$$

$\to$ parti

$\to$ duplicaz coseno

pt.2

- meglio dx o dy?
- es. + complicati

$$\int \frac{e^x}{e^{2x}+1}\,dx$$

$\to$ $y = e^x$, $dy = e^x dx$: $\int \frac{dy}{y^2+1} = \arctan(y) + c = \arctan(e^x) + c$

$\to$ $y = e^x$, $x = \ln x$, $dx = \frac{1}{y}\,dy$: $\int \frac{y}{y^2+1}\,\frac{1}{y}\,dy = \int \frac{1}{y^2+1}\,dy = \arctan(e^x) + c$

Quando trovare dy è semplice, metodo 1

Se dy è nascosto, metodo 2

esempio 7

$$\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x \overset{1-\sin^2 x}{\cos^2 x}\cdot \cos x\,dx$$

$y = \sin x$

$dy = \cos x\,dx$

$$= \int y^4(1-y^2)\,dy = \int (y^4 - y^6)\,dy = \int y^4\,dy - \int y^6\,dy =$$

$$= \frac{y^5}{5} - \frac{y^7}{7} + c = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

## esempio 8

$$\int \frac{x-1}{1+\sqrt{x}}\,dx = \int \frac{y^2-1}{1+y}\,2y\,dy =$$

$y=\sqrt{x} \rightarrow x=y^2$

$dy = \frac{1}{2\sqrt{x}} \rightarrow dx = 2y\,dy$

$$= \int \frac{(\cancel{y+1})(y-1)}{\cancel{1+y}}\,2y\,dy = \int (y-1)\cdot 2y\,dy$$

m.2. perché il dy è mascosto bene.

$$\int 2y^2 - 2y\,dy = \int 2y^2 dy - \int 2y\,dy$$

$$= \frac{2}{3}y^3 - y^2 + c = \frac{2}{3}(\sqrt{x})^3 - x + c$$

dove ci sono le radici conviene usare il metodo 2

scorciatoia

$$\int \frac{x-1}{1+\sqrt{x}}\,dx = \int \frac{(\sqrt{x}-1)(\cancel{\sqrt{x}+1})}{\cancel{1+\sqrt{x}}}\,dx = \int (\sqrt{x}-1)\,dx = \int \sqrt{x}\,dx - \int dx$$

$$= \frac{2}{3}(\sqrt{x})^3 - x + c$$

## esempio 9

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int 2(y^2+1)\,dy =$$

$y=\sqrt{x-1} \rightarrow y^2 = x-1$

$x = y^2+1$

$dy = \frac{1}{2\sqrt{x+1}}\,dx$

$$= 2\int y^2+1\,dy = 2\int y^2 dy + 2\int 1\,dy =$$

$$= \frac{2}{3}y^3 + 2y + c = \frac{2}{3}(\sqrt{x+1})^3 + 2\sqrt{x-1} + c =$$

$$= 2\sqrt{x-1}\left[\frac{(x-1)}{3} + 1\right] + c = 2\sqrt{x-1}\cdot\frac{x+2}{3} + c$$

# Integrazione per parti

$$\int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx$$

esempio 1

$$\int \underset{f}{x}\,\underset{g'}{\cos x}\,dx = x\,\operatorname{sen} x - \int \operatorname{sen} x = \qquad f = x \quad f' = 1 \qquad g' = \cos x \quad g = \operatorname{sen} x$$

$$= x\,\operatorname{sen} x + \cos x + c$$

esempio 2

$$\int \underset{f}{x}\,\underset{g'}{e^x}\,dx = x e^x - \int e^x = \qquad f = x \quad f' = 1 \qquad g' = e^x \quad g = e^x$$

$$= x e^x - e^x + c$$

esempio 3

$$\int \underset{f}{x^2}\,\underset{g'}{e^x}\,dx = x^2 e^x - \int 2x e^x\,dx = \qquad f = x^2 \quad f' = 2x \qquad g' = e^x \quad g = e^x$$

$$= x^2 e^x - 2\int \boxed{x e^x\,dx =}$$

x parti di nuovo vedi es 2

$$= x^2 e^x - 2\,x e^x - e^x + c$$

$$\int x^\alpha e^x\,dx \qquad \int x^\alpha \cos x\,dx \qquad \int x^\alpha \operatorname{sen} x\,dx \qquad \alpha \in \mathbb{N}$$

Si procede più volte per parti

esempio 4

$$\int (x^4 + 3x^2 - 6)\,e^x\,dx \overset{\text{Spezzarlo}}{=} \int x^4 e^x\,dx + \int 3x^2 e^x\,dx + \int -6\,e^x\,dx =$$

$$= \underbrace{\int x^4 e^x\,dx}_{\text{parti}} + 3\underbrace{\int x^2 e^x\,dx}_{\text{parti}} - 6\underbrace{\int e^x\,dx}_{\text{imm.}} =$$

$\int p(x)\, e^x\, dx \qquad \int p(x) \cos x\, dx \qquad \int p(x) \operatorname{sen} x\, dx$

$p =$ polinomio, vedi def sopra

esempio 5

$$\int \underset{g'}{x}\, \underset{f}{\ln x}\, dx = \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\, dx = \qquad f = \ln x \quad f' = \frac{1}{x} \qquad g' = x \quad g = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int \underbrace{x\, dx}_{\frac{x^2}{2}} = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c \underset{\text{raccolt.}}{=}$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

## Moltiplicazione $\cdot 1$ e Integrali ciclici

esempio 1

$$\int \ln x\, dx = \int \underset{f}{\ln x} \cdot \underset{g'}{1}\, dx \qquad f = \ln x \quad f' = \frac{1}{x} \qquad g' = 1 \quad g = x$$

$$x \ln x - \int 1\, dx = x\ln x - x + c =$$

$$= x(\ln x - 1) + c$$

esempio 2

$$\int \operatorname{arctg}(x)\, dx = \int \underset{f}{\operatorname{arctg}(x)} \cdot \underset{g'}{1}\, dx \qquad f = \operatorname{arctg}(x) \quad f' = \frac{1}{1+x^2} \qquad g' = 1 \quad g = x$$

molt. dividi

$$= x \operatorname{arctg}(x) - \frac{1}{2}\int 2\frac{x}{1+x^2}\, dx =$$

$$= x \operatorname{arctg}(x) - \frac{1}{2}\ln|1+x^2| + c$$

esempio 3

$$\int x\, dx = \int x \cdot 1\, dx$$

Usando la stessa logica di prima

$f = x \quad f' = 1 \qquad g' = 1 \quad g = x$

$x^2 - \int x\,dx$ = conviene dire che $\frac{x^2}{2}$ è direttamente la funzione primitiva, perché così giro in tondo

ciclici!!!

Altrimenti

$\int x\,dx = x^2 - \int x\,dx$ l'integrale che volevamo calcolare è uguale a qualcosa meno l'integrale di partenza

quindi $\rightarrow 2\int x\,dx = x^2 \rightarrow \int x\,dx = \frac{x^2}{2} + c$

esempio 4

$f = \cos x \quad f' = -\operatorname{sen} x$

$g' = \cos x \quad g = \operatorname{sen} x$

$\int \cos^2 x\,dx = \int \overset{f}{\cos x} \cdot \overset{g'}{\cos x}\,dx =$

$\operatorname{sen} x \cos x - \int (-\operatorname{sen} x) \cdot \operatorname{sen} x\,dx =$

$= \operatorname{sen} x \cos x + \int \operatorname{sen}^2 x\,dx = \operatorname{sen} x \cos x + \int 1 - \cos^2 x\,dx$

$\operatorname{sen} x \cos x + x - \int \cos^2 x\,dx$ porta a sinistra

$= \int \cos^2 x\,dx + \int \cos^2 x\,dx = \operatorname{sen} x \cos x + x$

$= 2\int \cos^2 x\,dx = \operatorname{sen} x + \cos x + x \quad =$

$\int \cos^2 x\,dx = \frac{\operatorname{sen} x + \cos x + x}{2} + c$

esempio 5

ancora per parti

$f = e^x \quad f' = e^x$

$g' = \operatorname{sen} x \quad g = -\cos x$

$\int \overset{f}{e^x} \overset{g'}{\operatorname{sen} x}\,dx = -\cos x\, e^x - \int (-\overset{g'}{\cos x}) \overset{f}{e^x}\,dx$

$f = e^x \quad f' = e^x$

$g' = -\cos x \quad g = -\operatorname{sen} x$

$(-\cos x)e^x - \left[\operatorname{sen} x\, e^x - \int (-\operatorname{sen} x) e^x\,dx\right]$

$(-\cos x)e^x + \operatorname{sen} x\, e^x - \int \operatorname{sen} x\, e^x\,dx$

$\int e^x \operatorname{sen} x\,dx + \int e^x \operatorname{sen} x\,dx = e^x(\operatorname{sen} x - \cos x)$

$\int e^x \operatorname{sen} x\,dx = \frac{1}{2} e^x (\operatorname{sen} x - \cos x) + c$

# Divisione fra polinomi

$A(x) = B(x) \cdot Q(x) + R(x)$

dividendo — divisore — quoziente — resto

$7 : 2 = 3, \ 1$

D E Q R

esempio 1

$\dfrac{x^3 - 2x^2 + x - 3}{}$

$$\begin{array}{rrrr|l}
x^3 & -2x^2 & +x & -3 & x^2+1 \\ \hline
-x^3 & & -x & & x-2 \\ \hline
 & -2x^2 & & -3 & \\
 & +2x^2 & & +2 & \\ \hline
 & & & \boxed{-1} &
\end{array}$$

- Prendi il monomio di grado max, dividi per mon. di grado max.
- Scrivi risultato sotto
- molt per divisore e scrivi cambiato di segno.
- il resto scende, altro si elimina
- ripeti aggiungendo al •
- quando il grado del dividendo è < del grado del divisore, è fatta.

esempio 2

$\dfrac{x^3 - 4x - 2}{x+1}$

$$\begin{array}{rrrr|l}
x^3 & & -4x & -2 & x+1 \\
-x^3 & -x^2 & & & x^2 - x - 3 \\ \hline
 & -x^2 & -4x & -2 & \\
 & -x^2 & +x & & \\ \hline
 & & -3x & -2 & \\
 & & +3x & +1 & \\ \hline
 & & & -1 &
\end{array}$$