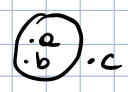
2/10/18 **Esame: lim, succ, integrali e derivate**

Insiemi: è una collezione di oggetti: A, B, C

- **relazioni di appartenenza** $a \in A$, $a \notin A$

$A = \{a, b\}$



$a \in A$
$b \in A$
$c \notin A$

A è dato per elencazione

$A = \{a,b\}\ \{b,a\}\ \{a,b,a\}\ \{a,a,b,b\} \to$ scomodo

$\hookrightarrow A = \{x \in B \mid p(x)\}$ p(x) prop. caratteristica

La diseq $\{x \in \mathbb{R} \mid 2x + 1 \geq 0\}$ ha sol $\emptyset$ insieme vuoto

[...]

X insieme; 2 funz $f, g : X \to \mathbb{R}$

trovo l'insieme di soluzioni S degli x, $f(x) < g(x)$ ($>$, $\leq$, $\geq$)

$S = \{x \in X : f(x) \leq g(x)\}$

- **Diseq. di primo grado**

$x - a > 0$ $(<, \geq, \leq)$ $\{x \in \mathbb{R} : x - a > 0\}$ o $\{x \in \mathbb{R} : x > a\}$ $(a, +\infty)$

- **Diseq. di secondo grado**

$ax^2 + bx + c > 0$ $(<, \leq, \geq)$

$a > 0$ se divido per a; $x^2 + \frac{b}{a}x + \frac{c}{a} > 0$

se $a < 0$ $ax^2 - bx - x < 0$

1) **Radici reali distinte** $\Delta > 0$ x_1, x_2

$\{x \in \mathbb{R} : x^2 + xb + c > 0\} = \{x : (x - x_1)(x - x_2) > 0\}$

$\{x : x - x > 0 \text{ e } x - x_2 > 0\} \cup \{x \in \mathbb{R} : (x - x) < 0 \text{ e } (x - x_2) < 0$

2) Radici reali e coincidenti $\Delta = 0$

3) Radici non reali $\Delta < 0$

$x^2 > 25 \quad x = \pm\sqrt{25} \quad x = \pm 5$

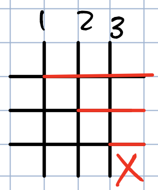
$3x^2 > 0 \quad \forall x \in \mathbb{R} / \{0\} \quad x^2 + 1 > 0 \quad \forall x \in \mathbb{R}$

Non ho sol. reali : $\Delta < 0$, $x_1, x_2, \in \mathbb{R}$
sono radici complesse

• Sistemi di disequazioni

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases} \qquad S = \left\{ x \in \mathbb{R} : \begin{array}{l} f(x) > 0 \text{ e } g(x) > 0 \\ f(x) > 0 \wedge g(x) > 0 \end{array} \right.$$

$$\begin{cases} x - 1 > 0 \\ x - 2 > 0 \\ x - 3 > 0 \end{cases} \qquad \begin{cases} x > 1 \\ x > 2 \\ x > 3 \end{cases}$$



$S = x > 3$

• Disequazioni frazionarie

$$\left\{ x : \frac{f(x)}{g(x)} \geq 0 \right\}$$

• Diseq. con valore assoluto

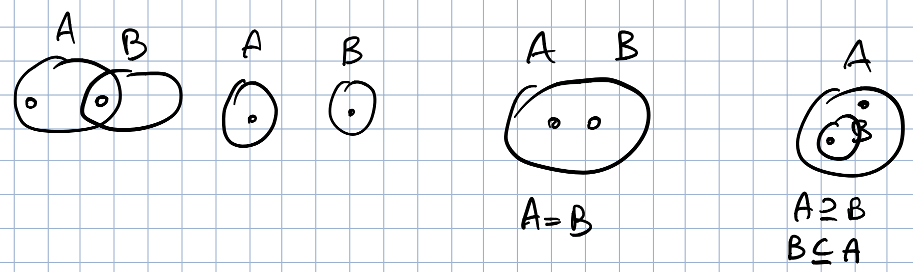
$$|A(x)| < k \qquad \begin{cases} A(x) \geq 0 \\ A(x) < k \end{cases} \cup \begin{cases} A(x) < 0 \\ -A(x) < k \end{cases}$$

Insiemi

L'insieme vuoto è $\emptyset$ $\quad \forall x,\ x \in \emptyset$

$A = \{0, 1\}$ e $B\ \{x \in \mathbb{R} \mid x^2 - x = 0\}$ è vero che $A = B$?

$\forall x \in B,\ x \in B \dots\ A \subseteq B$ (sottoinsieme)



• Insieme delle parti

$A = \emptyset \qquad P(A) = \{\emptyset\}$

$A = 1 \qquad P(A) = \{1\}\{\emptyset\}$

$A = \{1, 2\} \qquad P(A) = \{\emptyset\}\{1\}\{2\}\{1, 2\}$

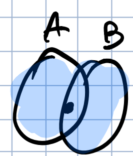
$A = \{1, 2, 3\} \qquad P(A) = \{\emptyset\}\{1\}\{2\}\{3\}\{1, 2\}\{3, 1\}\{3, 2\}$

• Operazioni sugli insiemi

su A, B

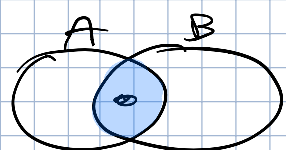
→ Unione

$A \cup B = \{x \mid x \in A \cup x \in B\}$



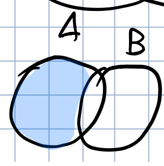
→ Intersezione

$A \cap B = \{x \mid x \in A \wedge x \in B\}$



→ Differenza

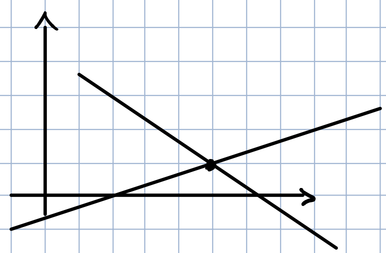
$A - B = \{x \mid x \in A \wedge x \notin B\}$



$$\{x\in\mathbb{R} \mid p(x)\cdot q(x)=0\} = \{x\in\mathbb{R} \mid p(x)=0 \lor q(x)=0\}$$
$$= \{x\in\mathbb{R} \mid p(x)=0\} \cup \{x\in\mathbb{R} \mid q(x)=0\}$$

$$\{(x,y)\in\mathbb{R}^2\} \mid x+y=5 \land x-y=1$$

$$\begin{cases} x+y=5 \\ x-y=1 \end{cases}$$



$$\left\{x\in\mathbb{R} \quad \frac{p(x)}{q(x)}=0\right\} = \{x\in\mathbb{R} \mid p(x)=0\} - \{x\in\mathbb{R} \mid q(x)=0\}$$

$A\cup B = B\cup A$ Se $A\cap B=\emptyset$, A,B si dicono disgiunti

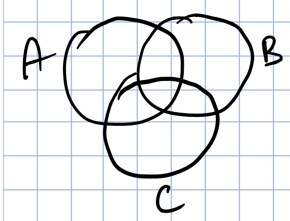
$A\cap B = B\cap A$ $(A-B)\cap(B-A)=\emptyset$

$A\cup B = A\cap A$ | $A\cup B = A$? $B\subseteq A$

$A-A=\emptyset$ | $A\cap B = A$? $A\subseteq B$ $\{1\}\cap\{1,2\}=\{1\}$

$A-B=A$? $A\cap B=\emptyset$

$$|A\cup B| = |A|+|B|-|A\cap B|$$



$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C| + |A\cap B\cap C|$$

• Proprietà caratteristica

$\{x \mid x \in X \wedge p(x)\} = \{x \in X \mid p(x)\}$

$\{x \mid x \notin X\}$ da un paradosso di Russel

• I numeri naturali

$\mathbb{N} = \{0, 1, 2, 3, 4, 5\}$ 5 assiomi:

$n^+ \quad +n$

$n^+ \quad n+1$

1. $0 \in \mathbb{N}$
2. $\forall n \in \mathbb{N} \ \exists n \in \mathbb{N}, n \neq n^+$
3. $\forall n, m \in \mathbb{N}$ se $n \neq m$ si ha $n^+ \neq m^+$
4. $\forall n \in \mathbb{N} \quad n^+ \neq 0$
5. Se $S \subseteq \mathbb{N}$ con $0 \in S \wedge \forall n \in \mathbb{N}$ $(n \in S \rightarrow n^+ \in S) \rightarrow S = \mathbb{N}$

• I numeri Interi

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$

$\{x \in \mathbb{Z} \mid x + a = b\} = \{b - a\}$

$\{x \in \mathbb{N} \mid x + a = b\}$

$\{x \in \mathbb{N} \mid x + 5 = 7\} = \{2\}$

$\{x \in \mathbb{N} \mid x + 7 = 5\} = \emptyset$

① Dimostrare che $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ per $n \in \mathbb{N}$

② Dimostrare che $\sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1 - q}$ per $q \in \mathbb{R} - \{1\}$

$$\sum_{i=0}^{n} x_i = x_0 + x_1 + x_2 + \dots + x_n \qquad 0+1+2+\dots+n$$

Applicazione del teorema dell'induzione

$$S = \{ n \in \mathbb{N} \mid 0+1+2+3+\dots+n = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

1) Base induttiva: è vero che $0 \in S$?

2) Passo induttivo: è vero che se $n \in S$ allora $n+1 \in S$?

$n \in S$ cioé $0+1+2+3+\dots+n = \dfrac{n(n+1)}{2}$ $\Downarrow$?

$n+1 \in S$ $\quad 0+1+2+\dots+n+n(n+1) = \dfrac{(n+1)(n+2)}{2}$

Sappiamo già quanto vale $0+1+2+\dots+n$

quindi $(0+1+2+\dots n)(n+1) = \dfrac{n(n+1)}{2} + (n+1) = (n+1)\left(\dfrac{n}{2}+1\right)\dfrac{n+2}{2}$

$= S = \mathbb{N}$ $\quad \sum_{i=0}^{n} i^2 = \left(\dfrac{n(n+1)(2n+1)}{2}\right) \quad \sum_{i=0}^{2n} i^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

$$\sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q}$$

$n=1$

$1+q = ?\quad \dfrac{1-q^2}{1-q} = \left(\dfrac{(1-q)(1+q)}{2}\right)$

$n=2$

$1+q+q^2 = \dfrac{1-q^3}{1-q} \qquad (1+q+q^2)(1-q) = 1-q$

$1-\cancel{q}+\cancel{q}-\cancel{q^2}+\cancel{q^2}-q^3$

$$S = \{ n \in \mathbb{N} \mid 1+q+q^2+\dots+q^n \quad \frac{1-q^{n+1}}{1-q}$$

1) base induttiva $0 \in S$?

$1 = \dfrac{1-q}{1-q}$

2) passo induttivo $n \in S$ cioé $1+q+q^2+\dots+q^n = \dfrac{1-q^{n-1}}{1-q}$

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Numeri Interi

$\{x \in \mathbb{N} \mid x + a = b\}$

$\{x \in \mathbb{Z} \mid x + a = b\} = \{b - a\}$

$\{x \in \mathbb{Z} \mid x + a + 1 = b + 1\} = \{b - a\}$

Numeri Razionali $\mathbb{Q}$

$\left\{\frac{p}{q} \mid p \in \mathbb{Z}, \ q \in \mathbb{Z} - \{0\}\right\}$

$\{x \in \mathbb{Z} \mid ax = b\}$

$\{x \in \mathbb{Z} \mid 2x = 1\} = \emptyset$

c'è bisogno di $\mathbb{Q}$

$\{x \in \mathbb{Q} \mid ax = b\} = \left\{\frac{b}{a}\right\}$

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$

$\sqrt{2} \notin \mathbb{R}$ $\quad \{x \in \mathbb{Q} \mid x^2 = 2\} = \emptyset$

Dim per assurdo

Se $\sqrt{2}$ è $\mathbb{R} \rightarrow \exists a, b \in \mathbb{N} - \{0\}$

T.R. $\sqrt{2} = \frac{a}{b}$ allora posso supporre che la fraz $\frac{a}{b}$ sia ridotta ai minimi termini dunque

$2 = \frac{a^2}{b^2}$ cioè $a^2 = 2b^2$ quindi a è pari $a = 2\alpha$

dove $\alpha \in \mathbb{N}$ $\quad 4\alpha^2 = 2b^2 \quad b^2 = 2\alpha^2$

quindi b è pari e questo è **assurdo**.

• I numeri reali $\mathbb{R}$

$\pm n . c_1 c_2 c_3 \dots c_n$

$q = \frac{1}{10}$ $\quad n = 0, 1, 2, 3, 4 \dots$

$n=0 \quad S_0 = 1$

$n=1 \quad S_1 = 1+q = \frac{1-q^2}{1-q} \dots$

$n=2 \quad S_2 = 1+q+q^2 = \frac{1-q^3}{1-q} \dots$

$\forall q \in \mathbb{R} - \{1\}, \forall n \in \mathbb{N}$ si ha

$$S_n = \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} = 1+q+q^2+ \dots + q^n$$

• Eq di 2° grado

Dati $a, b, c \in \mathbb{R}$ con $a \neq 0$ $\{x \in \mathbb{R} \mid ax^2+bx+c = 0\}$

$$\begin{cases} y = ax^2+bx+c \\ y = 0 \end{cases}$$

$ax^2+bx+c = 0$

$4a(ax^2+bx+c) = 0$

$4a^2x^2+4abx+4ac = 0$

$(4a^2x^2+4abx+b^2)+4ac-b^2 = 0$

$(2ax+b)^2 = b^2-4ac \qquad \Delta = b^2-4ac$

$\Delta < 0$ NO $S \in \mathbb{R}$

$\Delta = 0$ 1 $S \in \mathbb{R}$ $\quad x = -\frac{b}{2a}$

$\Delta > 0$ 2 $S \in \mathbb{R}$ distinte

Principio di induzione: Esercitazione

Sia $S \subset \mathbb{N}$ tale che:

① $0 \in S$ ② $\forall n \in \mathbb{N},\ n \in S \mapsto n+1 \in S$

allora $S = \mathbb{N}$

Esercizi sui valori assoluti:

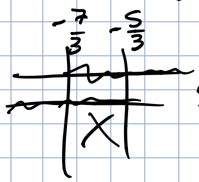
$|3x+7| < 2$

Se la quantità all'interno tolgo il val ass e non cambio di segno. Altrimenti cambio segno x averlo positivo.

$|3x+7| < 2$

$$\begin{cases} 3x+7 \geq 0 \\ 3x+7 < 2 \end{cases} \cup \begin{cases} 3x+7 < 0 \\ -3x+7 < 2 \end{cases}$$

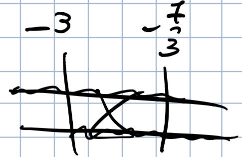
$$\begin{cases} x \geq -\frac{7}{3} \\ \frac{3x}{3} < -\frac{5}{3} \end{cases}$$



$S_1\ -\frac{7}{3} < x < -\frac{5}{3}$

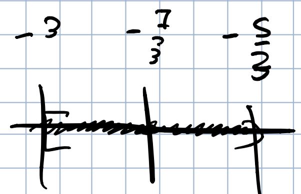
$\cup$

$$\begin{cases} x < -\frac{7}{3} \\ -3x < 9 \to \frac{\cancel{3}x}{\cancel{3}} > -\frac{\cancel{9}^{3}}{\cancel{3}} \end{cases}$$



$S_2 = -3 < x < -\frac{7}{3}$

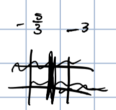
$S_{TOT} =$



$S_{TOT} = \{x \in \mathbb{R} : -3 < x < -\frac{5}{3}\}$

$(-3, -\frac{5}{3}) \cup\ ]-3, -\frac{5}{3}[$

$$-2 < \underbrace{-3x+7}_{} < +2 = \begin{cases} 3x+7 < 2 \to 3x < -5 \\ 3x+7 > -2 \to 3x > 9 \to \end{cases} \begin{cases} x < -\frac{5}{3} \\ x >_- 3 \end{cases}$$



$S = -\frac{5}{3} < x < -3$

$|x^2-4|>5$ $\qquad$ $x^2-4<-5 \ \vee \ x^2-4>5$

$x^2<-1 \quad \vee \quad x^2>9$

$-5 \quad +5$

$S=\emptyset \qquad x=\pm 3$

$S_{tot}=\emptyset+S_2=S_2$ $\qquad$ $S=x<-3 \vee x>3$

$x\in\mathbb{R}=\ x<-3 \cup x>3$

$\left|\frac{5x}{x-2}\right|>-\frac{1}{2}$ $\qquad$ $S=\forall x\in\mathbb{R}/\{2\}$

$\left|\frac{x+1}{x-1}\right|<4$ $\qquad$ $-4<\frac{x+1}{x-1}<4$ $\qquad$ $\begin{cases}\frac{x+1}{x+4}<4\\ \frac{x+1}{x-1}>-4\end{cases}$

$\frac{x+1-4(x-1)}{x-1}<0 \quad \vee \quad \frac{x+1+4(x-1)}{x-1}>0$

$\frac{x+1-4x+4}{x-1}<0 \qquad \frac{x+1+4x-4}{x-1}>0$

$\frac{5-3x}{x-1}<0 \qquad \frac{5x-3}{x-1}>0$

$5-3x>0 \quad 3x<5 \quad x<\frac{5}{3}$

$x-1>0 \quad x>1$

$1 \quad 5/3$

$- \quad + \quad -$

$+$

$S_2 \ x<+1 \cup x>\frac{5}{3}$

$5x-3>0 \quad 5x>3 \quad x>\frac{3}{5}$

$x-1>0 \quad x>1$

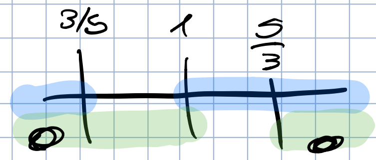
$\frac{3}{5} \quad 1$

S_1

$+ \quad - \quad +$

$x<\frac{3}{5} \vee x>1$

S_{TOT}



$S_{TOT} = x \in \mathbb{R} : \left(-\infty, \frac{3}{5}\right)\left(\frac{5}{3}, +\infty\right)$

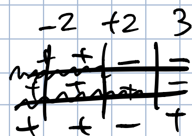
$x < \frac{3}{5} \cup x > \frac{5}{3}$

Diseq. contenenti uno o più moduli

$|4-x^2| - |3-x| > x$

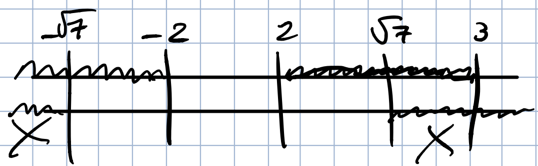
$r: \ 4-x^2 \geq 0 \quad x^2-4 \leq 0 \quad x \leq \pm 2$

$r_2: \ 3-x > 0 \quad x < +3$

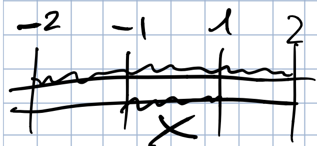


$$\begin{cases} x < -2 \cup 2 \leq x < 3 \\ x^2-4-3+x > x \end{cases} \cup \begin{cases} -2 \leq x < 2 \\ 4-x^2-3+x > x \end{cases} \cup \begin{cases} x > 3 \\ x^2-4+3-x > x \end{cases}$$

$x^2-7 = x = \pm\sqrt{7}$

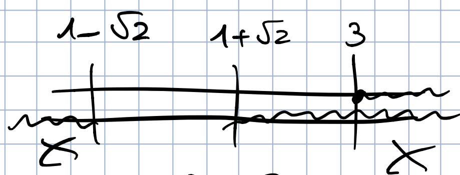


S_1 $\quad x < -\sqrt{7} \cup \quad \sqrt{7} < x < 3$



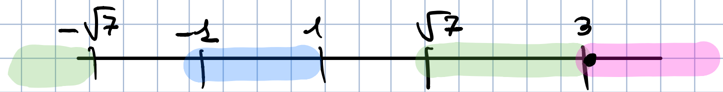
S_2 $\quad -1 < x < 1$

S_3 $\quad \Delta = 8$



$x \geq 3$

S_{TOT}



$= \ x \in \mathbb{R} : x < -\sqrt{7}$

$-1 < x < 1$

$x \geq \sqrt{7}$

Disequazioni irrazionali

$\sqrt{f(x)} < g(x)$

$$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) \leq [g(x)]^2 \end{cases} \quad CE$$

$\sqrt{f(x)} > g(x)$

$$\begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \vee \begin{cases} f(x) > [g(x)]^2 \\ g(x) \geq 0 \end{cases}$$

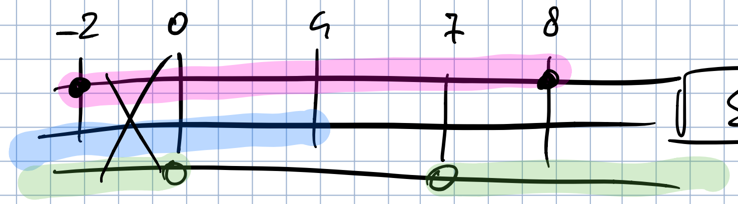
es:

$$\sqrt{\underbrace{6x - x^2 + 16}_{f(x)}} < \underbrace{4 - x}_{g(x)}$$

$(x-8)(x+2) = 0 \quad x = 8, \quad x = 2$

$x(x-7) = 0 \quad x = 0, \quad x = 7$

$$\begin{cases} 6x - x^2 + 16 \geq 0 \\ 4 - x > 0 \\ 6x - x^2 + 16 < 16 - 8x + x^2 \end{cases} \quad \begin{cases} x^2 - 6x - 16 \leq 0 \\ 2x^2 - 14x > 0 \end{cases} \quad \begin{cases} -2 \leq x \leq 8 \\ x < 4 \end{cases}$$



$\boxed{S = -2 \leq x < 0}$

$$\sqrt{x^2 - 8x + 15} + 2 \geq x \rightarrow \sqrt{x^2 - 8x + 15} \geq x - 2$$

$$\begin{cases} x^2 - 8x + 15 \geq 0 \\ x - 2 < 0 \end{cases} \quad \begin{cases} x - 2 \geq 0 \\ x^2 - 8x + 15 \geq x^2 - 4x + 4 \end{cases}$$

$\Delta = 4 \quad x_{1,2} = \frac{8 \pm 2}{2} \begin{matrix} 5 \\ 3 \end{matrix} \quad x \leq 3 \cup x \geq 5$

$\begin{cases} x \leq 3 \cup x \geq 5 \\ x < 2 \end{cases}$

2 3 5

$x < 2 \quad \cup \quad \begin{cases} x \geq +2 \\ x \leq \frac{11}{4} \end{cases}$

2 11/4

$2 \leq x < \frac{11}{4}$

Stot 2 $\frac{11}{4}$

$x \in \mathbb{R}: x \leq \frac{11}{4}$

$|x+2| + |3x-2| > 5x$

$\begin{cases} x+2 \geq 0 \\ x+2 < 0 \end{cases} \cup \begin{cases} x+2 < 0 \\ -x+2 < 0 \end{cases} \cup \begin{cases} 3x+2 \geq 0 \\ 3x+2 < 0 \end{cases} \cup \begin{cases} 3x+2 < 0 \\ -3x+2 < 0 \end{cases}$

$\begin{cases} x \geq -2 \\ x < 2 \end{cases}$

$-2 \quad 2$

$-2 < x < 2$

$\begin{cases} 3x \geq -2 \\ 3x < -2 \end{cases}$

$\begin{cases} x \geq -\frac{2}{3} \\ x < -\frac{2}{3} \end{cases}$

$-\frac{2}{3} \quad \frac{2}{3}$

$-\frac{2}{3} < x \leq \frac{2}{3}$

$-2 \quad -\frac{2}{3} \quad \frac{2}{3} \quad 2$

S: 1° $x+2 \geq 0 \to x \geq -2$

2° $3x-2 \geq 0 \to x \geq \frac{2}{3}$

$-2 \quad \frac{2}{3}$

$\begin{cases} x > \frac{2}{3} \\ x+2+3x-2 > 5x \end{cases}$

$S_1 = x < -2$

$S_2 = -2 \leq x < \frac{4}{7}$

$S_3 = \emptyset$

Forse mi conviene riguardarmele

$S_{TOT} = S_1 + S_2$

$-2 \quad \frac{9}{7} \quad \frac{2}{3}$

$-2 \quad \frac{4}{7}$

$S_{TOT} = x < \frac{4}{7}$

$\sqrt[n]{f(x)} \gtreqless g(x)$

$n \to$ dispari

$\sqrt[3]{x^3+1} < x+1$

$x^3+1 < (x+1)^3$

$x^3+1 < x^3+3x^2+3x+1$

$3x^2+3x > 0$

$3x(x+1) > 0 \quad x=0 \quad x=-1$

$S:$

$\{x \in R : x<-1 \vee x>0\}$

$\sqrt{1-x} > |1+3x|$

$1-x > (1+3x)^2$

$1-x > 1+9x^2+6x$

$-x-9x^2-6x > 1-1$

$-7x-9x^2 > 0$

$7x+9x^2 < 0$

$x(7+9x) < 0$

$x<0$

$7+9x<0$

$9x<-7$

$x<-\frac{7}{9}$

$-7/9 \quad 0$

$+ \quad - \quad +$

$x<-7/9<0$

LEZIONE TEORIA

Tornando alle equaz. di secondo grado

$ax^2+bx+c=0 \quad a,b,c \in \mathbb{R} \quad a \neq 0$

$(2ax+b)^2 = b^2-4ac$ (dalla dim. del delta)

$y^2 = \Delta$

$y = 2ax+b$

3 Casi per le soluzioni

① $\Delta > 0$	② $\Delta < 0$	③ $\Delta = 0$
$y = \sqrt{\Delta} \vee y = -\sqrt{\Delta}$	$i \notin \mathbb{R}$ unità immaginaria	$y=0$ cioè
$x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$	$i^2 = -1$	2 radici reali coincidenti
2 radici reali distinte	$y = i\sqrt{\lvert\Delta\rvert} \vee y = -i\sqrt{\lvert\Delta\rvert}$	
	$y^2 = -9 \quad y = 3i \vee y = -3i$	

$x = \dfrac{y-b}{2a} \quad (a \neq 0)$

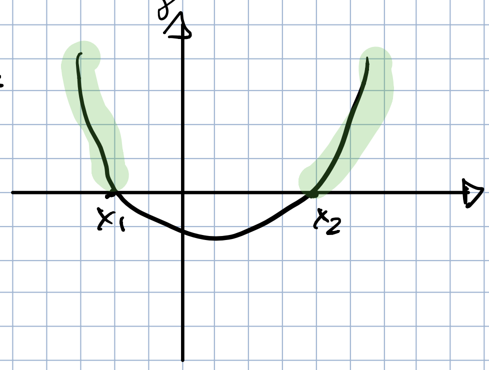
$(3i)^2 = 3^2 \cdot i^2 = (9)(-1)$
$= -9$

e anche $i^2 = -1$
$i^4 = 1$

Quindi sempre riguardo alle diseq di 2° grado:

$\{x \in \mathbb{R} : ax^2+bx+c \geq 0\} =$

$(a \neq 0) \qquad y = ax^2+bx+c$



1a) $\Delta > 0, a > 0$

$x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$(-\infty, x_1] \cup [x_2, +\infty) = S$

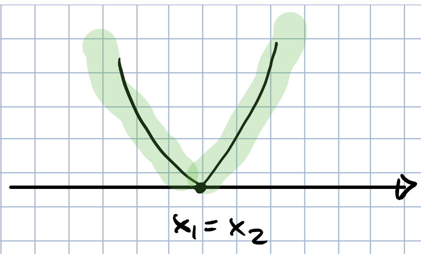
1b

$a>0$

$\Delta=0$

$x_1=x_2=-\frac{b}{2a}$

$S=\mathbb{R}$



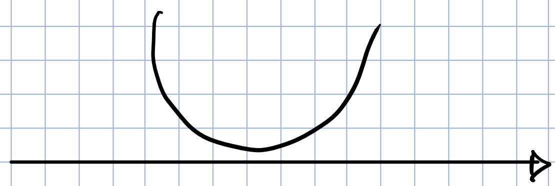
1c

$a>0$

$\Delta<0$

$S=\mathbb{R}$

$x^2+1\geq 0 \quad \forall x\in\mathbb{R}$

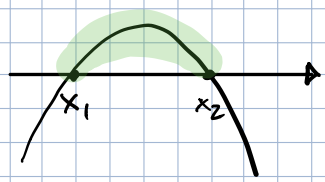


2a

$a<0$

$\Delta>0$

$S=[x_1, x_2]$

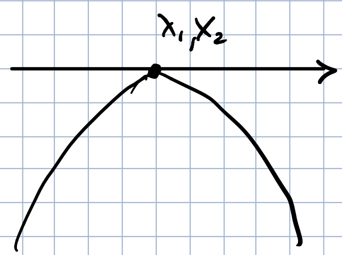


2b

$a<0$

$\Delta=0$

$S=\left\{-\frac{b}{2a}\right\}$



2c

$a<0$

$\Delta<0$

$S=\emptyset$

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Numeri Complessi

$\to \mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

$i^2 = (-i)^2 = -1$

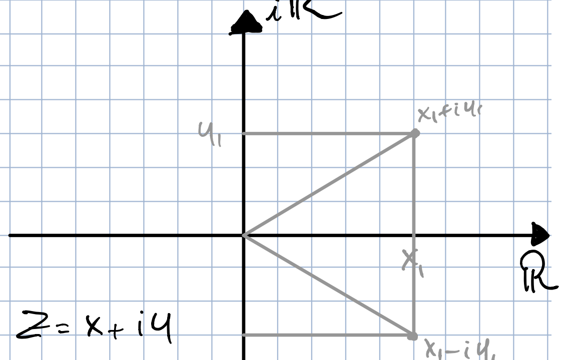
$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2 =$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$$\frac{x_1 + iy_1}{\underset{\neq 0}{x_2 + iy_2}} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 - \cancel{ix_2y_2} + \cancel{ix_2y_2} + y_2^2}$$

$$= \frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$



$\mathbb{R}^2$ è il piano cartesiano $\quad x_1 + iy_1 \in \mathbb{C}$

$(x_2 + iy_2)(x_2 - iy_2) = \quad (x_1, y_1) \in \mathbb{R}^2$

$= x_2^2 + y_2^2 \quad x_1 - iy_1 \in \mathbb{C}$

$(x_1, -y_1) \in \mathbb{R}^2$

$z = x + iy$

$\bar{z} = x - iy$

Questi due punti si dicono complessi coniugati

$z\bar{z} = |z|^2$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

$\sqrt{2} \in \mathbb{R} - \mathbb{Q}$

$x^2 - 2 = 0$ IRRAZIONALE ALGEBRICO

π, i, e NUMERI TRASCENDENTI

$\pi = 3{,}141592\ldots$

$e = 2{,}718281\ldots$

• Operazioni

Sia X un insieme scelto fra $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ o $\mathbb{C}$

$\forall a, b \in X$ sono definiti $a+b \in X$ somma

$a \cdot b \in X$ prodotto

X è chiuso rispetto a queste operazioni

Proprietà commutativa

$\forall a, b \in X$ si ha $a+b = b+a$

$a \cdot b = b \cdot a$

Proprietà associativa

$\forall a, b, c \in X$ si ha $(a+b)+c = a+(b+c)$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Proprietà distributiva

$\forall a, b, c \in X$ si ha $a \cdot (b+c) = a \cdot b + a \cdot c$

$a \cdot b + a \cdot c = a(b+c)$

• Elementi Neutri

$\exists x \in X$ t.c. $\forall a \in X$ $a+x = x+a = a$ x è elemento neutro per l'addizione $x=0$

$\exists y \in X$ t.c. $\forall a \in X$ $a \cdot y = y \cdot a = a$ y è l'elem. neutro per la moltiplicaz $y=1$

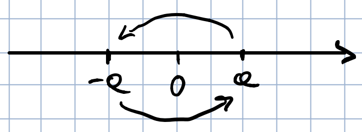
0 ed 1 sono unici

OPPOSTO

Sia X un insieme scelto tra $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ o $\mathbb{C}$

$\forall x \in X$ $\exists b \in X$ t.c. $a+b = b+a = 0$

b si dice opposto di a $b = -a$



$-(-a) = a$

$(-1)^2 = 1$

• Inverso e Reciproco

Sia X un insieme scelto fra $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

$\forall a \in X - \{0\} \quad \exists b \in X - \{0\}$ t.c. $\quad ab = ba = 1$

$b = \frac{1}{a} = 1/a = a^{-1}$

NO!

$a = b \qquad a^2 = ab \qquad a^2 - b^2 = ab - b^2 \qquad (a-b)(a+b) = (a-b) \cdot b$

$a + b = b \qquad 2a = a$

$f(x) = \dfrac{x}{x^2 - 1} \qquad \{x \in \mathbb{R} \mid x^2 - 1 \neq 0\} \quad \mathbb{R} - \{x \in \mathbb{R} \mid x^2 - 1 = 0\}$

• Legge di annullamento del prodotto

<u>OSS</u> $\forall x \in \mathbb{C}(\mathbb{R})$ si ha $\quad 0x = x0 = 0$

TEO: $\forall a, b \in \mathbb{C}$ se $a \cdot b = 0$ allora $a = 0 \quad \vee \quad b = 0$

DIM: $a = 0$ FINE

se $a \neq 0$ allora $\exists\, a^{-1} = \frac{1}{a}$

$a \cdot b = 0$

$\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$

$\left(\frac{1}{a} a\right) \cdot b = 0$

$1 \cdot b = 0$

$b = 0$

• Sottrazione e divisione

$a - b = a + (-b)$

$b \neq 0 \qquad \frac{a}{b} = a \frac{1}{b}$

- Relazione tra opposti e reciproci

$\forall a \in \mathbb{C} \quad -(-a) = a$ $\qquad \forall a, b \in \mathbb{C} - \{0\} \quad \frac{1}{ab} = \left(\frac{1}{a}\right)\left(\frac{1}{b}\right)$

$\forall a, b \in \mathbb{C} \quad (-a)b = a(-b)$ $\qquad \forall a, b \in \mathbb{C} - \{0\} \quad \frac{1}{(-a)} = -\frac{1}{a}$

$(-a)(-b) = ab$

$\forall a, b \in \mathbb{C} - \{0\} \quad \frac{1}{\frac{1}{a}} = a$

$a, b, \; a+b \neq 0 \quad \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$ $\qquad a, b > 0 \quad \frac{1}{a+b} < \frac{1}{a} < \frac{1}{a} + \frac{1}{b}$

Induzione

Dimostrare per ind che $\forall n \in \mathbb{N} \quad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$S = \{ n \in \mathbb{N} \mid \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \}$

1. base induttiva

$0 = 0 \rightarrow 0 \in S$

2. Passo induttivo [Dim che se $n \in S \rightarrow n+1 \in S$]

$n \in S$ vuol dire che $\sum_{i=0}^{n} i^2 = 0^2 + 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

? $n+1 \in S$ vuol dire che

$$\sum_{i=0}^{n+1} i^2 = (0^2 + 1^2 + 2^2 + \dots + n^2) + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6} \; ?$$

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

è sufficiente verificare l'uguaglianza fra i polinomi

$$\frac{(n+1)(n+2)(2n+3)}{6} = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$\frac{1}{6}(n+1)(2n+7n+6) = \frac{1}{6}(n+1)\left(2n^2+n+6(n+1)\right)$$

$$2n^2+7n+6$$

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad \sum_{i=0}^{n} i^k = \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k$$

$\forall k \in \mathbb{N}$

• Dimostrazione che $\sum_{m=1}^{n} \frac{1}{m(m+1)} = 1 - \frac{1}{n+1}$

$\forall n \in \mathbb{N} - \{0\}$ $\quad S = \{n \in \mathbb{N} - \{0\} \mid \sum_{m=1}^{n} \frac{1}{m(m+1)} = 1 - \frac{1}{n+1}\}$

$n=1 \quad \frac{1}{1\cdot 2} + \frac{1}{2} \overset{?}{=} 1 - \frac{1}{1+1} = \frac{1}{2}$ bose induttive $n=1 \in S$ (v)

$n=2 \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3} \overset{?}{=} 1 - \frac{1}{1+2} = \frac{2}{3}$

$n=3 \quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4} \overset{?}{=} 1 - \frac{1}{1+3} = \frac{3}{4}$

1) b. ind

2) Passo induttivo $\quad n \in S \overset{?}{\Rightarrow} n+1 \in S$

$$\sum_{m=1}^{n+1} \frac{1}{m(m+1)} = 1 - \frac{1}{n+2}$$

$$\sum_{m=1}^{n+1} \frac{1}{m(m+1)} = \sum_{m=1}^{n} \frac{1}{m(m+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$$

è vero che $\forall n \in \mathbb{N} - \{0\}$ si ha

$$1 - \frac{1}{n+2} \overset{?}{=} 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$\frac{(n+1)}{(n+2)} \stackrel{?}{=} \frac{(n+1)(n+2)-(n+2)+1}{(n+1)(n+2)} = \frac{n^2+3n+\cancel{2}-n-\cancel{2}+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$\frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

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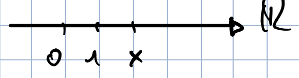
ORDINAMENTO TOTALE IN $\mathbb{R}$



$\forall x, y \in \mathbb{R}$ si ha $x \leq y \vee y \leq x$

$\leq$ ordinamento debole

$<$ ordinamento stretto

- Proprietà riflessiva: $\forall x \in \mathbb{R} \quad x \leq x$
- Proprietà antisimmetrica: $\forall x, y \in \mathbb{R} \quad x \leq y \cup y \leq x \Rightarrow x = y$
- Proprietà transitiva: $\forall x, y, z \in \mathbb{R} \quad x \leq y \vee y \leq z \Rightarrow x \leq z$
- Totalità: $\forall x, y \in \mathbb{R} \quad x \leq y \vee y \leq x$

es: $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{1,2\}, \{2\}\}$

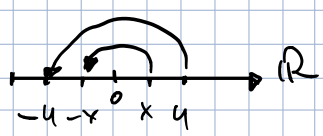
$A, B \in \mathcal{P}(\{1,2\})$; $A \leq B \Longleftrightarrow A \subseteq B$

$$\begin{array}{ccccc} & & \{1\} & & \\ \emptyset & \leq & & \leq & \{1,2\} \\ & \leq & \{2\} & \leq & \end{array}$$

- ## Compatibilità con le operazioni

•) $\forall x, y, z \in \mathbb{R} \quad x \leq y \Rightarrow x + z \leq y + z$

•) $\forall x, y, z \in \mathbb{R} \quad x \leq y \wedge z > 0 \Rightarrow xz < yz$!!!



- ## NOTAZIONE

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} \qquad \mathbb{R}^- = \{x \in \mathbb{R} \mid x < 0\}$

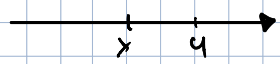
$\mathbb{R}^{0+} = \{x \in \mathbb{R} \mid x \geq 0\} \qquad \mathbb{R}^{0-} = \{x \in \mathbb{R} \mid x \leq 0\}$

$\mathbb{R}^* = \{ x \in \mathbb{R} \mid x \neq 0 \} = \mathbb{R} - \{0\}$

$\mathbb{Q}^* = \{ x \in \mathbb{Q} \mid x \neq 0 \} = \mathbb{Q} - \{0\}$

• Calcolo con le disuguaglianze

$\forall x, y \in \mathbb{R} \qquad x \leq y \leftrightarrow y - x \geq 0$



$x < y \leftrightarrow y - x > 0$ in particolare $x > 0 \leftrightarrow -x < 0$

$x > 0 \leftrightarrow 0 > -x \leftrightarrow -x < 0$

$\forall x, y, z \in \mathbb{R} \quad x \leq y \wedge z < 0 \rightarrow xz \geq yz$

$\forall x \in \mathbb{R} \quad x^2 \geq 0$

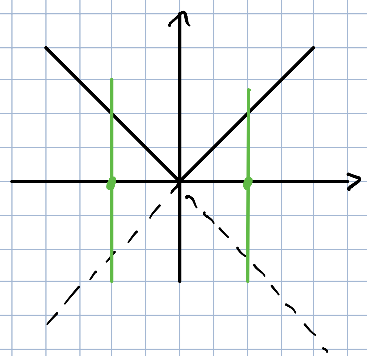
$\forall x \in \mathbb{R}^+ \quad x^2 > 0$

$\forall x \in \mathbb{R}^* \quad \left(x > 0 \rightarrow \frac{1}{x} > 0\right) \wedge \left(x < 0 \rightarrow \frac{1}{x} < 0\right)$

$$\forall x, y \in \mathbb{R}^* \begin{cases} 0 < x < y & \frac{1}{x} > \frac{1}{y} \\ x < 0 < y & \frac{1}{x} < 0 < \frac{1}{y} \\ x < y < 0 & \frac{1}{y} < \frac{1}{x} < 0 \end{cases}$$

• Valore assoluto

Per $x \in \mathbb{R} \qquad |x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$



oss: $|x| \geq 0 \quad \forall x \in \mathbb{R}$

$|x| = 0 \leftrightarrow x = 0$

$|x| = \max(x, -x)$

$|x| = |-x|$ (f. pari)

Prop.

$|5| = 5$

$|-5| = 5$

$[\sqrt{x^2} = |x|]$

$\forall x \in \mathbb{R} \quad \sqrt{x^2} = x \qquad |-5| = \sqrt{[-5]^2} = \sqrt{25} = \sqrt{5^2} = 5$

- Valore assoluto e moltiplicazione

$\forall x, y \in \mathbb{R} \quad |xy| = |x| \cdot |y|$

$\forall x, y \in \mathbb{R} \quad y \neq 0 \rightarrow \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

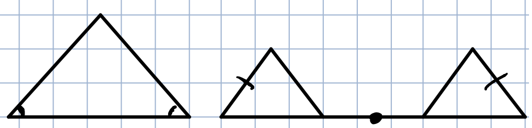
$\forall x \in \mathbb{R} \quad |x^2| = |x|^2 = x^2$

- Disuguaglianza triangolare

$\forall x, y \in \mathbb{R} \quad |x+y| \leq |x| + |y| \qquad x=5 \quad y=-3$

$|5-3| \quad |5| + |-3|$

$2 < 5+3 = 8$



<u>Notazione</u>

<u>Intervalli</u>

Dati $a, b \in \mathbb{R} \quad a \leq b$

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ intervallo chiuso

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ intervallo aperto

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

$(a,b) =]a,b[$

<u>Semirette</u> dato $a \in \mathbb{R}$

$[a, +\infty) = \{x \in \mathbb{R} \mid x \geq a\}$ Semiretta chiusa

$(a, +\infty) = \{x \in \mathbb{R} \mid x > a\}$ Semiretta aperta

$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$

$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$

$\mathbb{R}^+ = (0, +\infty)$

- Massimi e minimi di sottoinsiemi di $\mathbb{R}$

Sia $S \subseteq \mathbb{R}$ e $S \neq \emptyset$

diciamo che $m \in \mathbb{R}$ è il minimo di S se:

•) $m \in S$

•) $\forall x \in S \quad m \leq x$

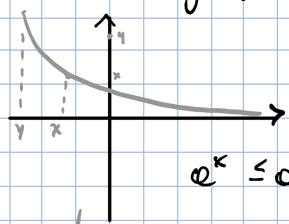
$\mathbb{Q}^+$

m

Equazioni esponenziali e logaritmiche - Grafici e regole

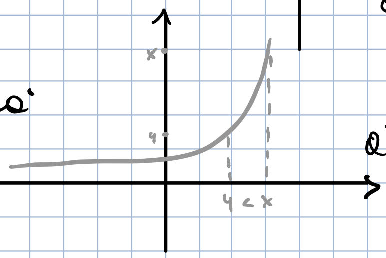
- Funzione esponenziale - $y = a^x$

Se $0<a<1$ il grafico sarà



$a^x \le a^y \Rightarrow x \ge y$

se $a>1$ il grafico sarà



$a^x \le a^y \Rightarrow x \le y$

Possono essere del tipo:

- $\dfrac{8-2^x}{2^{x+1}} \le 0$ $\quad N\ge 0: 8-2^x \ge 0 \quad 2^3 > 2^x \to 3 \ge x \quad x \le 3$

$D>0: 2^{x+1} > 0 \quad 2^{x+1} > 0 \quad \forall x \in R$ ~~sistema~~ 3

La funzione esponenziale è sempre positiva

$S = \{x \in R : x \ge 3\}$ $[+3, +\infty)$

$e = 2,71 \quad e > 1$

- $\dfrac{e^{x-1} - e^x}{e^{5x} - 1} \le 0$ $\quad N \ge 0: e^{x+1} - e^x \ge 0 \to e^{x+1} \ge e^x \quad x+1 > x$

$\forall x \in R$

$D>0: e^{5x} - 1 > 0 \to e^{5x} > 1 \to e^{5x} > e^0 \Rightarrow \frac{5x}{5} > \frac{0}{5} \quad x > 0$

$S = \{x \in R : x < 0\}$ $(-\infty, 0[$ $\quad - - - - 0 + + + +$

- $\dfrac{9^x + 3^x}{3^{2x} - 1} < 0$ $\quad N>0: 9^x + 3^x > 0$; pongo $3^x = t \quad 3^{2x} + 3^x = t^2 + t > 0$

$t(t+1) = \begin{cases} t=0 \\ t=-1 \end{cases}$

$t < -1 \cup t > 0$

$3^x < -1 \cup 3^x > 0 \quad \forall x \in R$

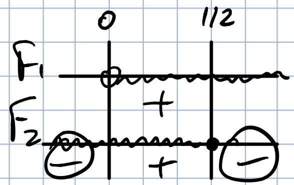
$D>0: 3^{2x} - 1 > 0 \quad 3^{2x} - 1 > 0; \quad 3^{2x} > 1 \quad - 0 + \quad x > 0 \quad \forall x \in R$

$(-\infty, 0)$ $\quad S = \{x \in R : -\infty < x < 0\}$

$$\underbrace{(5^{3x}-5^{2x})}_{f_1}\underbrace{(e^{\frac{1}{x}}-e^{2})}_{f_2}\le 0$$

$F_1 \ge 0$; $5^{3x}-5^{2x}\ge 0 \rightarrow 5^{3x} > 5^{2x} \rightarrow 3x > 2x \rightarrow x > 0$

$F_2 \ge 0$; $e^{\frac{1}{x}}-e^{2}\ge 0 \rightarrow e^{\frac{1}{x}} \ge e^{2} \rightarrow \frac{1}{x} \ge 2 \rightarrow x \le \frac{1}{2}$



$S: \{x\in\mathbb{R} : x \le 0 \cup x \ge 1/2\}$

$(0, \frac{1}{2})$

$5^{2x}-2\cdot 5^{x}-3\ge 0$ Pongo $5^x = t \rightarrow t^2-2\cdot t-3\ge 0$

$(t-3)(t+1)=0$ $\begin{cases} t=+3 \\ t=-1 \ \emptyset \end{cases}$ $t\le 1 \cup t \ge 3$

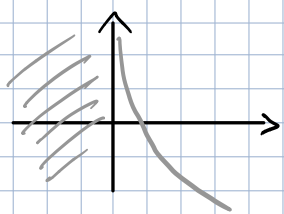
$5^x \le -1 \cup 5^x \ge 3$

$\cancel{\frac{x\log 5}{\log 5}} \ge \frac{\log 3}{\log 5}$

$S: x \ge \frac{\log 3}{\log 5}$

Funzione logaritmica

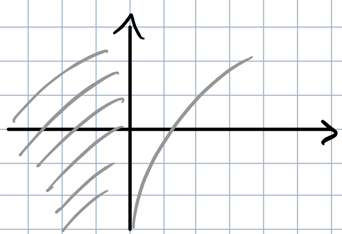
se $0 < a < 1$



$\log_a x \le \log_a 4$

$x \ge 4$

se $a > 1$



$\log_a x \le \log_a 4$

$x \le 4$

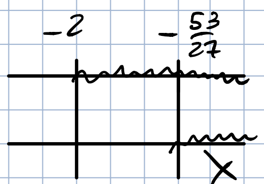
$\log_3(x+2) > -3$

$$\begin{cases} x+2>0 \\ \log_3(x+2) > \log_3 \frac{1}{27} \end{cases}$$

$$\begin{cases} x>-2 \\ x+2 > \frac{1}{27} \end{cases} \rightarrow x > \frac{1}{27} - 2 = \frac{1-54}{27} \qquad x > -\frac{53}{27}$$

$-\frac{2}{1} < -\frac{53}{27}$

$-54 < -53$



$S: x > -\frac{53}{27}$ $\left(-\frac{53}{27}; +\infty\right)$

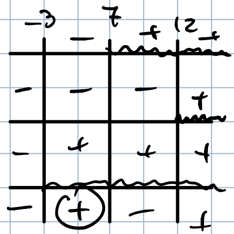
$\log(7-x) + \log(12-x) > 2\log(x+3)$

$7-x>0$

$12-x>0$

$x+3>0$

$$\begin{cases} 7>x \\ 12>x \\ x>-3 \end{cases}$$



$ce: -3<x<7$

$\log(7-x)(12-x) > \log(x+3)^2$

$(7-x)(12-x) > (x+3)^2$

$84 - 7x - 12x + \cancel{x^2} > \cancel{x^2} + 6x + 9$

$-19x - 6x > 9 - 84$ $\quad -25x > -75$ $\quad x < \frac{\cancel{75}}{\cancel{25}}\ 3$

$\log^2 x - 3\log x - 4 \leq 0$ $\qquad \log x = t$

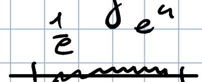
$t^2 - 3t - 4 \leq 0$ $\qquad t=4$ $\quad \log x = 4 \vee \log x = -1$

$(t-4)(t+1) = 0 \begin{cases} \\ t=-1 \end{cases}$ $\quad e^4 = x \vee e^{-1} = x$ $\quad x = \frac{1}{e}$

$-1 < t < 4$ $\qquad \log x > -1$ $\qquad \log_{e^4} x > \log \frac{1}{4}$

$-1 < \log x < 4$

$$\begin{cases} x>0 \\ x<e^4 \\ x>\frac{1}{e} \end{cases}$$



$\left[\frac{1}{e}, e^4\right]$

• Rappresentazione di funzioni

• Trasformazione di grafici

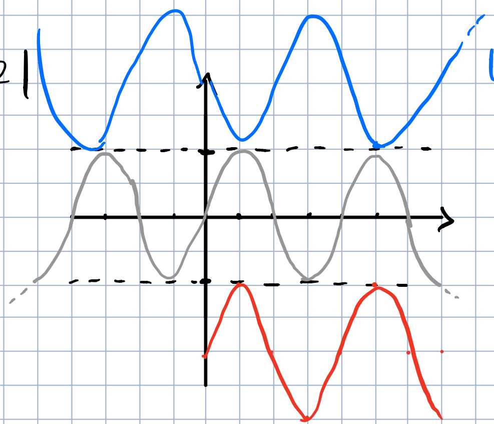
$y = f(x)$ (γ) grafico $(x, f(x))$

- $-f(x)$ è il grafico simmetrico rispetto all'asse x di γ
- $|f(x)|$ " " boh? [...]
- $f(-x)$ è il simmetrico di γ rispetto all'asse y
- $-f(-x)$ " " " " " O
- $f(\pm\alpha)$ traslo in orizzontale γ
- $f(x) \pm \beta$ traslo in verticale γ
- $f(x \pm \alpha) \pm \beta$ traslo in verticale e/o orizzontale γ
- $\alpha \cdot f(x)$ moltiplico per le coordinate dei p.ti
- $|\alpha| > 1$ si ha una dilatazione nella direzione y
- $0 < |\alpha| < 1$ " contrazione " "

$y = |\operatorname{sen} x - 2|$

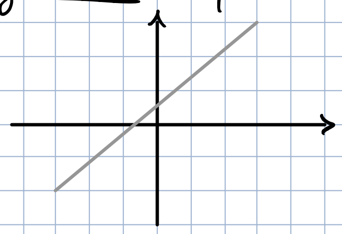
γ: $y = \operatorname{sen} x$

$\operatorname{sen} x - 2$

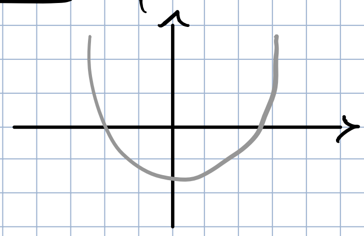


Partendo dal grafico delle funzioni polinomiali

grado 1 : $y = ax + b$

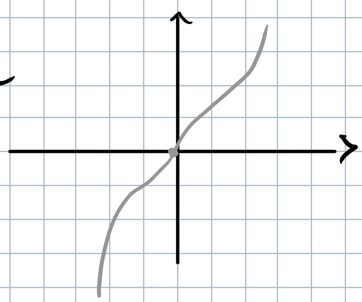


grado 2 : $y = ax^2 + bx + c$



grado 3 : $y = ax^3 + bx^2 + cx + d$

in particolare $y = ax^3$



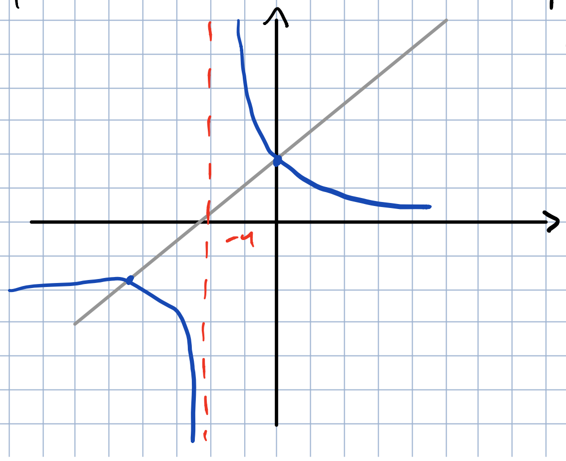
• Grafico del reciproco della funzione $y = f(x)$

- Se $f(x)$ ha p.ti che intersecano l'asse x : $f(x_0) = 0$

$g(x) = \frac{1}{f(x)}$ $\quad f(x_0) = 0$ da fz reciproca $\frac{1}{f(x_0)}$ assume valori molto grandi → la retta $x = x_0$

- se $f(x) \to \pm\infty$ il suo reciproco si avvicina a 0 $\frac{1}{f(x)} \to \infty$

- $f(a) = \pm 1$ anche la sua reciproca passa per [???]



$y = x + 1$

$g(x) = \frac{1}{f(x)}$

$g(x) = \frac{1}{x+1}$

Grafico di $g(x) = f^2(x)$

Se $|f(x)| = 1 \longrightarrow f^2(x) = 1$

Se $f(x) = 0 \longrightarrow f^2(x) = 0$

Se $|f(x)| < 1 \longrightarrow f^2(x) < |f(x)|$

Se $|f(x)| > 1 \longrightarrow f^2(x) > |f(x)|$

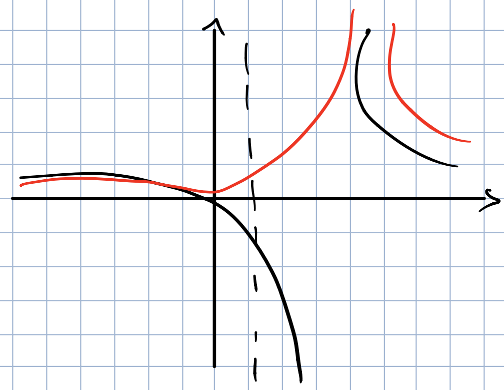
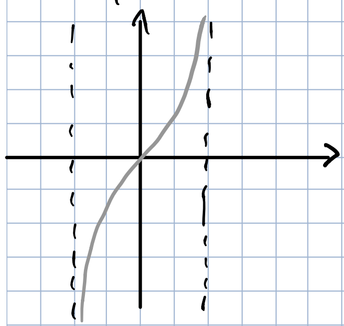


Grafico di $g(x) = \sqrt{f(x)}$

$y = \tan x$ $\qquad$ $y = \sqrt{\tan x}$



- Se $f(x) < 0$ la $\sqrt{f(x)}$ non esiste
- Se $f(x) = 0 \to \sqrt{f(x)} = 0$
- Se $f(x) = 1 \to \sqrt{f(x)} = 1$
- Se $0 < f(x) < 1 \to f(x) < \sqrt{f(x)} < 1$
- Se $1 < f(x) \to \sqrt{f(x)} < f(x)$

17/10/18

MASSIMI E MINIMI

$S \subseteq \mathbb{R} \quad S \neq \emptyset$

DEF $m \in \mathbb{R}$ si dice **minimo** di S se:

- $m \in S$
- $\forall x \in S$ si ha $m \leq x$

esempi: $S = [a, b] \qquad a = \min(S), \; b = \max(S)$

$S = (a, b] \qquad \nexists \min(S), \; b = \max(S)$

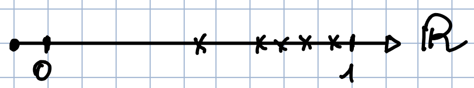
$0 = \min(\mathbb{N})$

$\nexists \max(\mathbb{N})$

$S = [1, +\infty] \quad \min(S) = 1, \; \nexists \max(S)$

$$S = \left\{ \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots \right\}$$

$\min(S) = 0, \; \nexists \max(S)$



$\forall n \in \mathbb{N} \quad \frac{n}{n+1} < 1$

ed $\frac{n}{n+1} = 1 - \frac{1}{n+1}$

$$\exists n \in \mathbb{N} \text{ t.c. } \left| \frac{n}{n+1} - 1 \right| < \frac{1}{1000} \; ?$$

$$\left| \cancel{1} - \frac{1}{n+1} - \cancel{1} \right| < \frac{1}{1000}$$

$$\frac{1}{n+1} < \frac{1}{1000} \qquad n = 1000$$

Dato $\varepsilon > 0 \quad \exists n \in \mathbb{N}$ t.c. $\left| \frac{n}{n+1} - 1 \right| < \varepsilon$

$\frac{1}{n+1} < \varepsilon \iff n+1 > \frac{1}{\varepsilon}$ $(n \in \mathbb{N} \ \ \varepsilon > 0)$

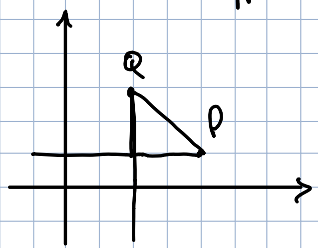
$\boxed{n > \frac{1}{\varepsilon} - 1}$

Buon ordinamento di $\mathbb{N}$ (osservazione)

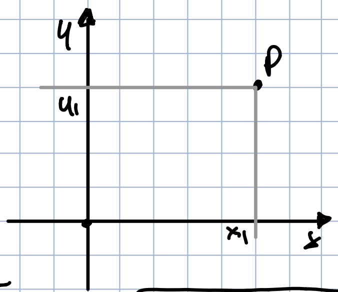
Se $S \subseteq \mathbb{N}$ ed $S \neq 0$ allora si ha <u>minimo</u>

COORDINATE CARTESIANE NEL PIANO

$P = (x, y)$ coppia ordinata



$Q(2,3)$
$P(4,1)$



$$d(P,Q) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

$C = (\alpha, \beta) \in \mathbb{R}^2$ $\quad r > 0 \quad$ centro raggio

$$\{(x,y) \in \mathbb{R}^2 \mid d((x,y),(\alpha,\beta)) = r\} =$$
$$= \{(x,y) \in \mathbb{R}^2 \mid (x-\alpha)^2 + (y-\beta)^2 = r^2\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0$$

$C = (0,0)$
$r = 1$

$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ Circonferenza trigonometrica

Equazione della retta

$\{(x,y) \in \mathbb{R}^2 \mid ax + by + c = 0\}$ dove $a, b, c \in \mathbb{R}$ con A or $B \neq 0$

① $b=0 \rightarrow a\neq 0$ $\{(x,y)\in\mathbb{R}^2 \mid ax+c=0\}$

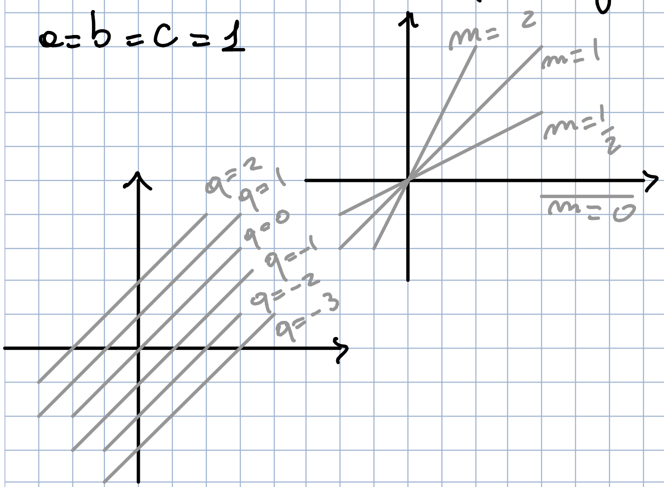
$\{(x,y)\in\mathbb{R}^2 \mid x=-\frac{c}{a}\}$ retta verticale

② $a=0 \rightarrow b\neq 0$ $\{(x,y)\in\mathbb{R}^2 \mid ax+by+c=0\}$

$=\{(x,y)\in\mathbb{R}^2 \mid y=-\frac{a}{b}a-\frac{c}{b}\}$ $\quad y=mx+q$

$a=b=c=1$ $\quad m=$ coeff. angolare

$a=b=c=1$



$y=mx$

$m\in\mathbb{R}$

$m=$ coeff. ang.
$q=$ termine noto

COPPIE ORDINATE - PRODOTTO CARTESIANO

X, Y insiemi

$X\times Y=\{(x,y) \mid x\in X,\ y\in Y\}$

$X=\{a,b\}$ $\quad Y=\{1,2,3\}$ $\quad X\times Y=\{(a,1)(a,2)(a,3)(b,1)(b,2)(b,3)\}$

$Y\times X=\{(1,a)(1,b)(2,a)(2,b)(3,a)(3,b)\}$

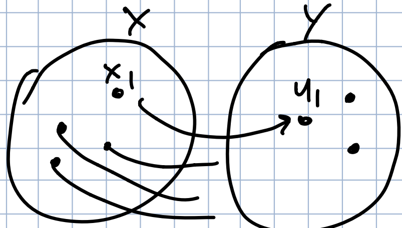
Il prodotto cartesiano non è commutativo

Funzioni

X, Y insiemi (non vuoti)

f funzione con dominio X e codominio Y

$f: X \to Y$



La funzione è definita, devo poterla calcolare per ogni valore di $x \in$ al dom.

Def:

$f: X \to Y$

◦) $f \subseteq X \times Y$

◦) f è definita dunque: $\forall x \in X \; \exists y \in Y$ t.c. $(x, y) \in f$

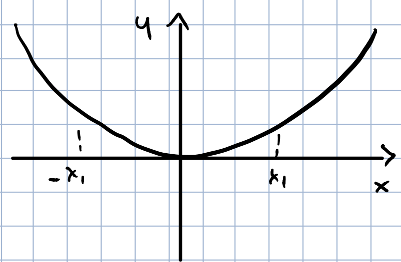
◦) f è ben definita: $\forall x \in X$ se $\exists y_1, y_2 \in Y$ t.c. $(x, y_1) \in f$ $\wedge (x, y_2) \in f \Rightarrow y_1 = y_2$ short: (x, y_2) $y_2 = f(x)$

↑

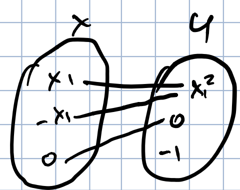
ho usato una notazione diversa che non useremo. useremo la classica $y = f(x)$

esempio:

$$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 \end{cases}$$



$$\begin{cases} y = x^2 \\ y = -1 \end{cases}$$



Funzioni

$f: X \to Y$

X Dominio

Y codominio

$f(x)$ immagine di $x \in X$ tramite f

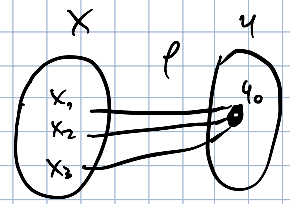
Ci interessa studiare quelle funzioni dove sia x che $y \in \mathbb{R}$

$G(f) \to$ grafico = insieme di coppie

$\{(x,y) \in X \times Y \mid (x,y) \in f\}$

$\{(x,y) \in X \times Y \mid y = f(x)\}$

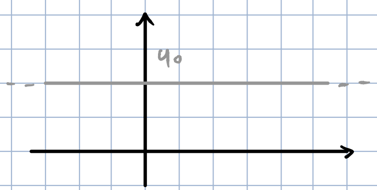
$\{(x, f(x)) \in X \times Y\}$



Funzioni Costanti

$y_0 \in Y$

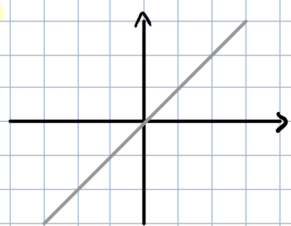
$$\begin{cases} f: X \to Y \\ f(x) = y_0 \end{cases}$$

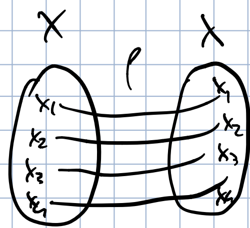


PS: non ci vedo un cazzo e Zaccagnini scrive piccolo

Funzione Identica

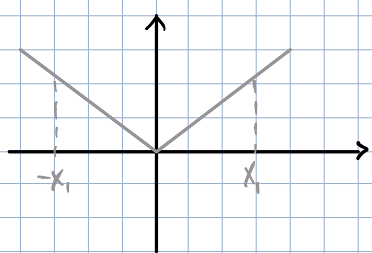
$$\begin{cases} I_X : X \to X \\ I_X(x) = x \end{cases}$$





Funzione Valore Assoluto

$$\begin{cases} |\ | : \mathbb{R} \to \mathbb{R} \end{cases}$$



Funzioni Pari

$f: \mathbb{R} \to \mathbb{R}$ è pari se

$\forall x \in \mathbb{R}$ si ha $f(x) = f(-x)$

es: $f(x) = x^2, x^4, |x|, \cos(2x)$

Funzioni dispari

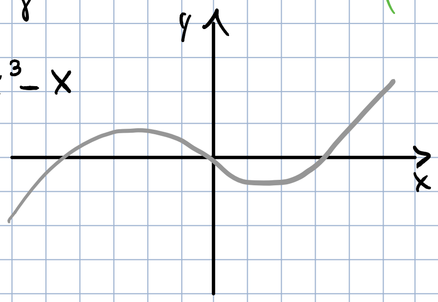
$f: \mathbb{R} \to \mathbb{R}$ è dispari se

$\forall x \in \mathbb{R}$ si ha $f(x) = -f(-x)$

si controlla facendo così: $\begin{cases} f(-x) \begin{cases} f(x) = \text{pari} \\ -f(x) = \text{dispari} \end{cases} \end{cases}$

es grafico

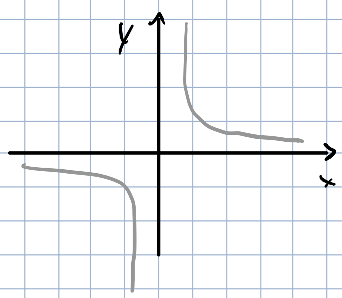
$y = x^3 - x$



Funzione Reciproca

$f: \mathbb{R} \to \mathbb{R}$

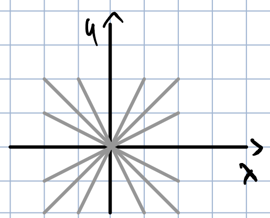
$f(x) = \frac{1}{2}$



Funzione lineare

$\begin{cases} f_m \ \mathbb{R} \to \mathbb{R} \\ f_m(x) = mx \end{cases}$

$m \in \mathbb{R}$



Funzioni Affini

$\begin{cases} f_{mq} : \mathbb{R} \to \mathbb{R} \\ f_{mq}(x) = mx + q \end{cases}$

$m, q \in \mathbb{R}$

Immagine Diretta

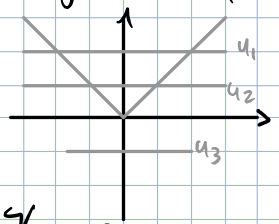
$f: X \to Y$ $\qquad f(S) = \{ f(x) \mid x \in S \}$

$S \subseteq X$, $S \neq \emptyset$ Immagine di S tramite f

$\left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} = f(\mathbb{N})$ $\begin{cases} F: \mathbb{R}^{0+} \to \mathbb{R} \\ f(x) = \frac{x}{x+1} \end{cases}$ $S = \mathbb{N}$

Def: f si dice suriettiva se $Y = f(x)$

in generale $f(x) \subseteq Y$; lo è se $\forall x \in Y \ \exists x \in X$ t.c. $y = f(x)$



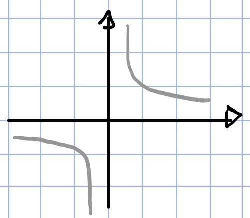
$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = |x| \end{cases}$ $\begin{cases} g: \mathbb{R} \to \mathbb{R}^{0+} \\ g(x) = |x| \end{cases}$

$\nexists\, x \in \mathbb{R}$ $|x| = -1$

f non è suriettiva

g è suriettiva

$\begin{cases} f: \mathbb{R}^{+} \to \mathbb{R} \\ f(x) = \frac{1}{x} \end{cases}$



Non è suriettiva (f)

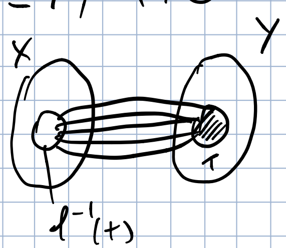
$\begin{cases} g: \mathbb{R}^{*} \to \mathbb{R}^{*} \\ g(x) = \frac{1}{x} \end{cases}$ questa è suriettiva

f. iniettive
f. suriettive NB

Immagine Inversa

$f: X \to Y$ $f^{-1}(T) = \{ x \in X \mid f(x) \in T \}$

$T \subseteq Y$, $T \neq \emptyset$



OSS: f è suriettiva $\iff f^{-1}(t) \neq \emptyset \ \forall t \neq \emptyset$

DEF: f. si dice iniettiva se $\forall x_1, x_2 \in X$
se $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

OSS: f è iniettiva $\iff$ l'eq. $y_1 = f(x)$ ha al max <u>una</u> soluzione $x \in X$ $\forall y_1 \in Y$

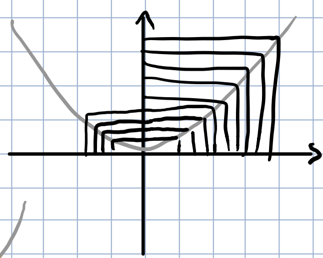
Esercizi

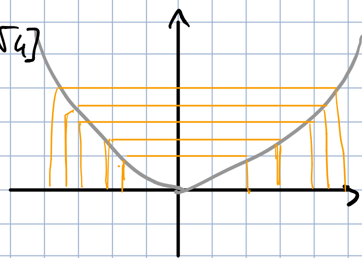
S	$f(S)$
$[0,1]$	$[0,1]$
$[-1,1]$	$[0,1]$
$[-1,2]$	$[0,4]$
$[3,4]$	$[9,16]$

T	$f(T)$
$[0,1]$	$[-1,1]$
$[-1,1]$	$[-1,1]$
$[-1,2]$	$[-\sqrt{2},\sqrt{2}]$
$[3,4]$	$[\sqrt{3},\sqrt{4}]$

$y = f(x)$

$$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) \doteq x^2 \end{cases}$$





PRATICA:

$f(x)$ $\quad f(x) = y = 2x-3 \quad g(x) = \frac{1}{f(x)} = \frac{1}{2x-3}$

Presi un po' a cozzo di fretta

$\frac{1}{f(x)} \quad f(x) = 2x-3=1 \quad \frac{2}{2}x = \frac{4}{2}$

$x \to \frac{3}{2}$ la funzione $g(x) \to \infty$

$\sqrt{f(x)} \quad f(B) = 2x-3=1$

$\to \frac{2x}{3} = \frac{2}{3}$

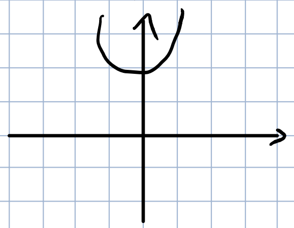
$x = \frac{3}{2}$ è un asintoto verticale

$\sqrt{f(x)} = x^2+2$

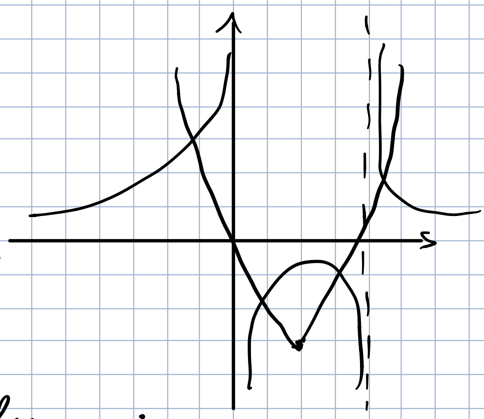
$x \to \infty$

$y =$ asint orizz $\quad g(x)$

$f(x) = \frac{1}{f(x)} = \frac{1}{x^2+2}$



$x^2 = -1, +3$



$f(x) = x^2-4x \quad x_V = -\frac{b}{2a} = \frac{+4}{2} = 2$

$y_x = -4$

$f(B) = x^2 - 4x = 1$

$x^2 - 4x + 1 = 0$

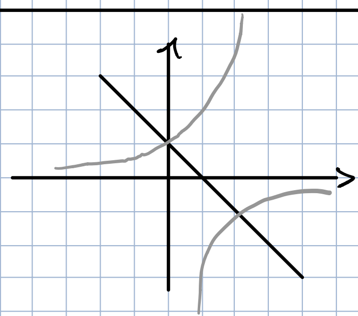
$x_{1,2} = \sqrt{4-1}$

$f(\alpha) = x^2-4x = 1$

$x^2-4x-1=0$

$x_{1,1} = 2 \pm \sqrt{4+1} = 2 \pm \sqrt{5}$

$y = \frac{1}{\sqrt{1-x}} \qquad f(x) = 1-x$



$f(x) = 1-x \qquad x=0$

$f(1) = 0 \qquad x=2$

$\frac{1}{f(x)} = g(x) = \frac{1}{1-x}$

[. . .] Qui mancano appunti

$h(x) = \dfrac{1}{\sqrt{1-x^2}}$

gli esercizi li riporto sotto

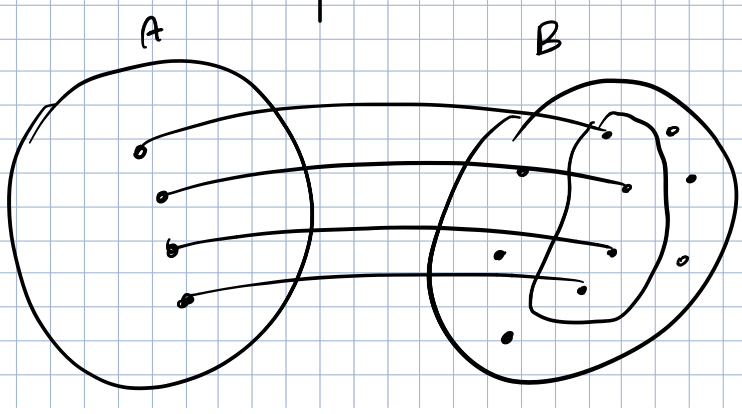
- $y = 2x - 3$
- $y = \dfrac{1}{\sqrt{1-x}}$

$(6a^4 - 17a^3 + 12a^2 - 11a + 2) : (3a^2 - a + 2)$

$$\begin{array}{l|l} 6a^4 - 17a^3 + 12a^2 - 11a & 3a^2 - a + 2 \\ \hline -6a^4 + 2a^3 + 4a^2 & 2a^2 - 5 \\ \quad -15a^3 + 8a^2 - 11a + 2 & \\ \quad +15a^3 \mp 8a^2 + 10a & \\ \qquad [\ldots] & \end{array}$$

$(2x^3 + x + 1) : (2x + 1)$

$$\begin{array}{l|l} 2x^3 + 0 + x + 1 & 2x - 1 \\ \hline -2x^3 & x^2 - \frac{1}{2}x \\ \hline x^2 + x + 1 & \\ -x^2 + \frac{1}{2}x & \\ \hline \frac{3}{2}x + 1 & \end{array}$$



$A = B \Longleftrightarrow A \subseteq B$
$B \subset A$

$f(x) = -x^2+2 \qquad f([-1,0]) \subset [1,2]$

$1 \leq x \leq 0$

$0 \leq x^2 \leq 1$

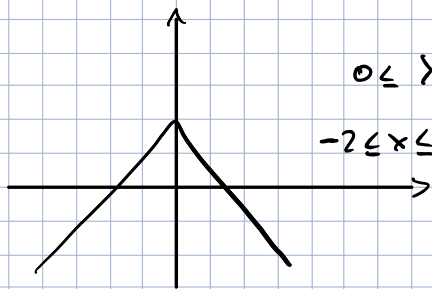
$-1 \leq -x^2 \leq 0$

$1 \leq x^2+2 \leq 2$

$$f(x) = \begin{cases} x+2 \text{ per } & x \leq 0 \quad (D_1) \\ -x^2+2 & x > 0 \quad (D_2) \end{cases} \qquad f^{-1}([0,1])$$

prendo $b \in [0,1]$ $\quad \exists e \in D_1 : f(e) = b$?

$0 \leq x+2 \leq 1$



$0 \leq x+2 \leq 1$

$-2 \leq x \leq -1 \cap D_1 \; [-2,-1]$

$\exists e \in D_2 : f(e) = b$?

$$\begin{cases} x^2 \leq 2 \\ x^2 \geq 1 \end{cases} \qquad \begin{cases} -\sqrt{2} \leq x \leq \sqrt{2} \\ x \leq -1 \vee x \geq 1 \end{cases}$$

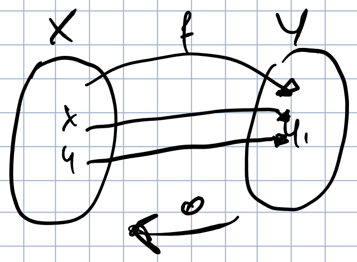
$f(x)$ è iniettiva $\qquad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$f(]1,2[)$ $\qquad f: \mathbb{R} \to \mathbb{R} \qquad f(x_1) = f(x_2)$

$f^{-1}(]0,3[)$ $\qquad x \to f(x) = x^2 - 1$

Funzioni Iniettive, suriettive, biettive. TEORIA

$f: X \to Y$

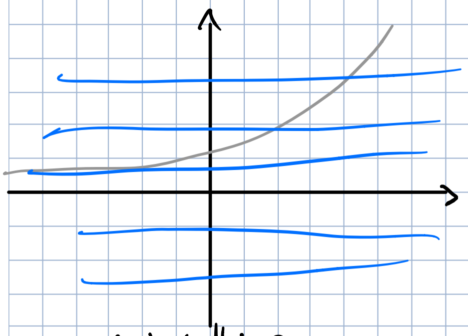


Def: $f: X \to Y$ è iniettiva se $\forall x_1, x_2 \in X$ se $f(x_1) = f(x_2)$ allora $x_1 = x_2$

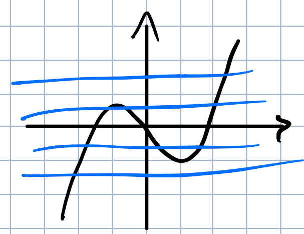
In altre parole $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Def: $f: X \to Y$ è suriettiva se $\forall y \in Y$ $\exists x \in X$ t.c. $f(x) = y$

$f(x) = y$ in altre parole $f(X) = Y$



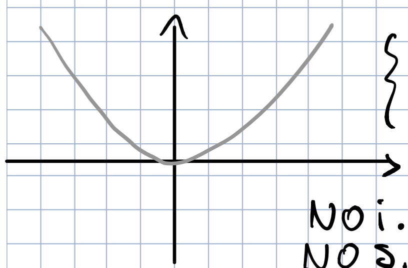
qualunque retta orizz. interseca il grafico al massimo una volta



NON iniettiva
f. suriettiva

f. iniettiva
non suriettiva

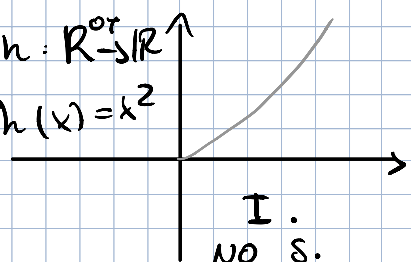
Def: $f: X \to Y$ si dice biettiva se è suriettiva e iniettiva. In altre parole $\forall y \in Y$ $\exists!\, x \in X$ t.c. $f(x) = y$



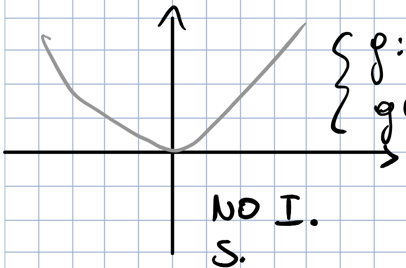
$$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 \end{cases}$$

NO i.
NO S.

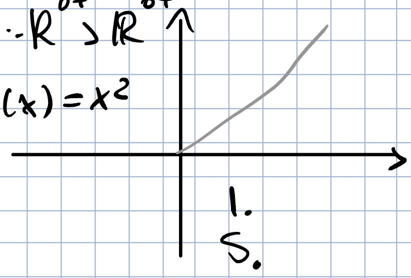
$$\begin{cases} h: \mathbb{R}^{0+} \to \mathbb{R} \\ h(x) = x^2 \end{cases}$$



I.
NO S.

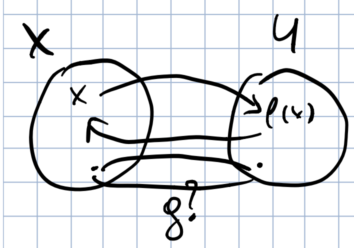


$$\begin{cases} g: \mathbb{R} \to \mathbb{R}^{0+} \\ g(x) = x^2 \end{cases} \qquad \begin{cases} k: \mathbb{R}^{0+} \to \mathbb{R}^{0+} \\ h(x) = x^2 \end{cases}$$



$f: X \to Y$ $\exists g$ come sopra

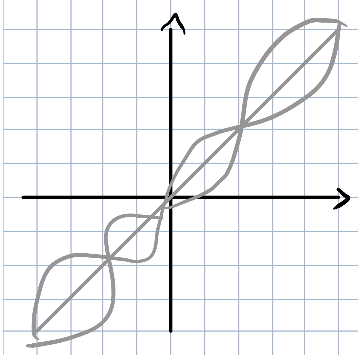
$$\begin{cases} \forall x \in X \\ g(f(x)) = x \\ \forall y \in Y \\ f(g(y)) = y \end{cases} \longrightarrow *$$



<u>Def</u>: se $f: X \to Y$ è biettiva la funzione $g: Y \to X$ con le proprietà $*$ si dice inversa di f e si indica con f^{-1}

$\sqrt{\ }: \mathbb{R}^{0+} \to \mathbb{R}^{0+}$ $\quad \sqrt{x^2} = x \ \ \forall x \in \mathbb{R}^{0+} = X$ $\quad \sqrt{x^2} = |x| \ \forall x \in \mathbb{R}$

$(\sqrt{y})^2 = y \ \ \forall y \in \mathbb{R}^{0+} = Y$



Esercizio: dire per quali $m, q \in \mathbb{R}$ la funzione
$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = mx + q \end{cases}$ è invertibile e determinare l'inversa

$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = mx + q \end{cases}$ ① Quando è iniettiva? Può succedere che $f(x_1) = f(x_2)$ anche se $x_1 \neq x_2$?

$mx_1 + \cancel{q} = mx_2 + \cancel{q}$?

$mx_1 = mx_2$?

$m \neq 0 \to x_1 = x_2$

② Quando è suriettiva?
$\forall y \in Y \ \exists x \in X$ t.c. $f(x) = y$ Dato $y \in \mathbb{R}$ riusciamo a trovare $x \in \mathbb{R}$ t.c. $mx + q = y$?

$mx = y - q$

se $m \neq 0$ $\quad x = \frac{y - q}{m}$ $\quad f^{-1}(x) = \frac{x - q}{m}$

Dato $y \in \mathbb{R}$ riusciamo a trovare $x \in \mathbb{R}$ t.c. $mx + q = y$?

$mx = y - q$ se $m \neq 0$ $\quad x = \frac{y - q}{m}$

$f^{-1}(x) = \frac{x - q}{m}$ $\quad f^{-1}(f(x)) = f^{-1}(mx + q) = \frac{(mx + q) - q}{m} = \frac{mx}{m}$

$= x$

- $y = 2x - 3$
- $y = \frac{1}{\sqrt{1 - x}}$

Studio di funzione

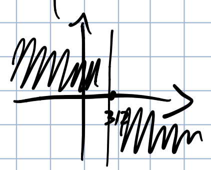
① $\boxed{y = 2x - 3}$ Dom $f (-\infty, +\infty)$

② $f(-x) = -2x-3 \rightarrow \neq -f(x)$ dispari

③ Intersez X $f(x)=0 \quad 2x-3=0 \rightarrow 2x=3 \quad x=\frac{3}{2}$

$f(0)=2\cdot 0-3=-3$ Interseca X in -3, Y in $\frac{3}{2}$

④ Segno: $f(x)>0 \quad 2x-3>0 \quad 2x>3$



$x>\frac{3}{2}$ 3/2 − | + la f è pos da $\frac{3}{2}$ in poi

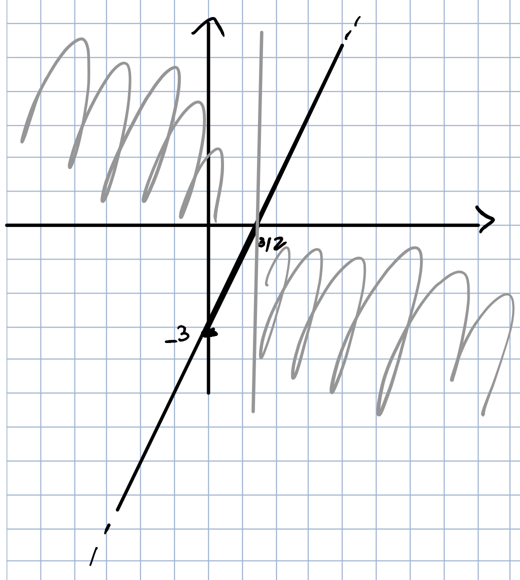
⑤ limiti

$\lim_{x\to+\infty} 2x-3=+\infty \qquad \lim_{x\to-\infty} 2x-3=-\infty$

no asintoti

⑥ Derivata

$f(x)=2x-3 \qquad f'(x)=2>0 \ \forall x\in\mathbb{R}$ quindi è sempre crescente



$\frac{1}{\sqrt{1-x}}$

ce $\sqrt{1-x} \neq 0 \quad x \neq 1$

$1-x \geq 0 \quad -x \geq -1 \quad x \leq 1$

① Dom f $(-\infty, 1) / \neq 1$

② P/D $\quad f(-x) = \frac{1}{\sqrt{1+x}}$ non P non D

③ Intersez

(y) $f(0) = \frac{1}{\sqrt{1-0}} = \frac{1}{1} = 1$ Non interseca x, interseca y in 1

(x) $f(x) = 0 \quad \frac{1}{\sqrt{1-x}} = 0 \rightarrow x \in \emptyset$

④ segno

$1>0$

$\frac{1}{\sqrt{1-x}} > 0 \quad \forall x \in \mathbb{R}$ quindi sempre positiva

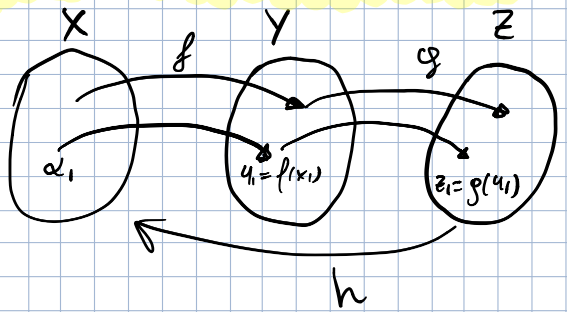
⑤ limiti

$\lim_{x \to \infty} \frac{1}{\sqrt{1-x}} = \frac{1}{+\infty} = 0$ asintoto verticale a $\pm\infty$

$\lim_{x \to -\infty} \frac{1}{\sqrt{1-x}} = \frac{1}{-\infty} = 0$

24/10/18

Composizione di funzioni AKA f. composta



$f: X \to Y$

$g: Y \to Z$

$\begin{cases} h: X \to Z \\ h: g \circ f \end{cases}$

$h(z) = g(f(x))$

$f(x) = \frac{x}{x^2-1} \xrightarrow{\text{posso vederlo come}} x \cdot \frac{1}{x^2-1} \to x(x^2-1)^{-1}$

- Dom $= x \in \mathbb{R} \setminus \{\pm 1\}$

Esempi:

$\begin{cases} f: \mathbb{R} \to \mathbb{R} & f(x) = 2x \\ g: \mathbb{R} \to \mathbb{R} & g(x) = x+1 \end{cases}$

$(g \circ f)(x) = g(f(x)) = g(2x) = 2x+1$

$(f \circ g)(x) = f(g(x)) = f(x+1) = 2x+2$

La composizione di funzioni non è commutativa

$\begin{cases} f(x) = x^2 \\ g(x) = x+1 \end{cases}$

$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2+1$

$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2+2x+1$

Se $\exists\, h^{-1}$, cioè se f e g sono entrambe biiettive allora

$\begin{cases} h^{-1}: Z \to X \\ h^{-1}: f^{-1} \circ g^{-1} \end{cases}$

NB: $(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x))))$

così sto tornando indietro

$f^{-1}(f(x)) = x$

CARDINALITÀ

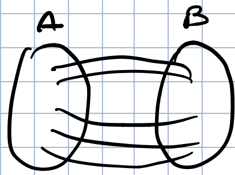
Diciamo che A e B hanno la stessa cardinalità se

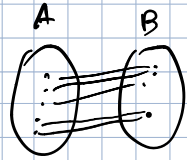
$\exists\, f: A \to B$

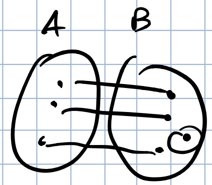
$$\begin{cases} f: \mathbb{N} \to \mathbb{N} \\ f(n) = n+1 \end{cases}$$

$0 \to 1$
$1 \to 2$
$2 \to 3$

Biiettiva







$$\begin{cases} f: \mathbb{N}^2 \to \mathbb{N} \\ f(x,y) = \dfrac{(x+y)^2 + (x+y)}{2} + y \end{cases}$$

$$\begin{cases} g: \mathbb{N}^2 \to \mathbb{N} \\ g(x,y) = 2^x(2y+1) - 1 \end{cases}$$

x \ y	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

$\exists (x,y) \in \mathbb{N}^2$ t.c. $g(x,y) = 15$?
$g(x,y) = 16$?
$g(x,y) = 17$?

$2^x(2y+1) - 1 = 15$
$2^x(2y+1) = 16$

Potenze

obiettivo: definire x^y nel più grande sottoinsieme possibile di $\mathbb{R}^2$

$7^4,\ 7^{\sqrt{2}},\ \cancel{(-1)^{\pi}},\ \dots,\ 7^{-1}$ $\qquad 7^{\sqrt{2}} \simeq 15.663\dots$

Def: Dato $x \in \mathbb{R}$, $m \in \mathbb{N}^*$ definiamo x^m mediante la relazione di ricorrenza $\begin{cases} x^1 = x \\ x^{m+1} = x \cdot x^m \end{cases}$

$$7^4 = \underset{m=3}{7^{3+1}} = 7\cdot 7^3 = \underset{m=2}{7\cdot 7^{2+1}} = 7\cdot 7\cdot 7^2 = \underset{m=1}{7\cdot 7\cdot 7^{1+1}} = 7\cdot 7\cdot 7\cdot 7^1 = 7\cdot 7\cdot 7\cdot 7$$

$7^{100} = (7^{50})^2 = ((7^{25})^2)^2 \qquad 7^{25} = 7\cdot 7^{24} = 7(((7^3)^2)^2)^2$

$7^3 = 7\cdot 7^2$

Proprietà delle potenze

$\forall x \in \mathbb{R}, \quad \forall n,m \in \mathbb{N}^*$ si ha $\quad x^{n+m} = x^n \cdot x^m$

Vorrei definire (quando possibile) il valore di x^0 ?

$x^m = x^0 \cdot x^m$?

Def: se $x \in \mathbb{R}^*$ allora poniamo $x^0 = 1$ quindi $\forall x \in \mathbb{R}^*$
$\forall n,m \in \mathbb{N}$ si ha $\quad x^{n+m} = x^n \cdot x^m$

? $1 = x^{-m+m} = x^{-m} \cdot x^m$? $(x \neq 0)$

Vorrei definire il valore $x^{-m} \, (m \in \mathbb{N})$

Def: se $x \in \mathbb{R}^*$ ed $m \in \mathbb{N}$ definiamo $\quad x^{-m} = \frac{1}{x^m}$

quindi $\forall x \in \mathbb{R}^* \quad \forall n,m \in \mathbb{Z}$ si ha $\quad e^{n+m} = x^n \cdot x^m$

Vorrei definire (quando possibile) $(x^n)^m = x^{nm} \quad x \in \mathbb{R}^*, \; n,m \in \mathbb{Z}$

$n = \frac{1}{2}, \; m = 2$

? $(x^{1/2})^2 = x^1 = x$?

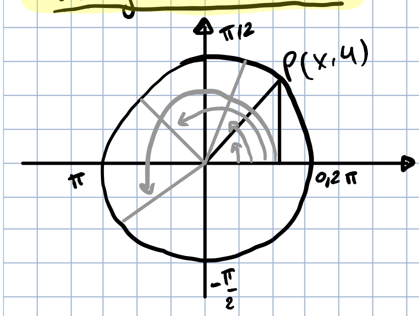
Def: per $x \in \mathbb{R}^{0+}$
$x^{1/2} = \sqrt{x}$

$\sqrt{7}$
$7^{0.5}$

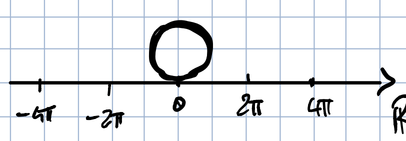
In generale, per $x \in \mathbb{R}^{0+}$, $m \in \mathbb{N}^*$

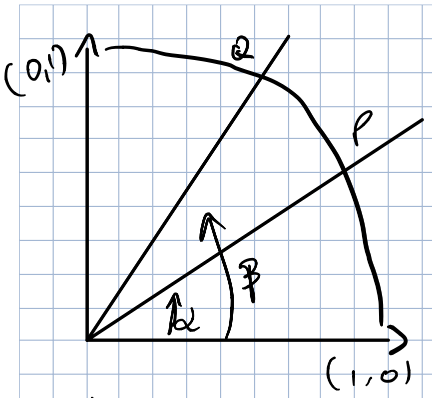
$x^{\frac{1}{m}} = \sqrt[m]{x} \quad$ inoltre $\forall n,m \in \mathbb{N}^* \quad x^{\frac{n}{m}} = \sqrt[m]{x^n} = (\sqrt[m]{x})^n$

Trigonometria



$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$





$P(x,y)$

$$\begin{cases} x = \cos(\alpha) \\ y = \operatorname{sen}(\alpha) \end{cases}$$

α	$\operatorname{sen}\alpha$	$\cos\alpha$
0	0	1
$\pi/2$	1	0
π	0	-1
$3/2\,\pi$	-1	0
2π	0	1

$\forall k \in \mathbb{Z}$ $\quad \operatorname{sen}(x+2k\pi) = \operatorname{sen}(x)$

$\cos(x+2k\pi) = \cos(x)$

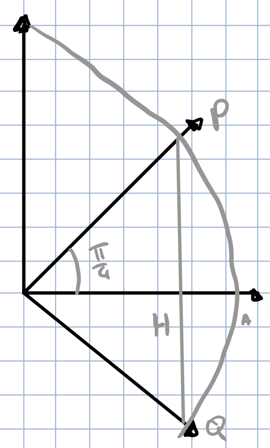
Periodicità $\quad \forall \alpha \in \mathbb{R} \quad \operatorname{sen}^2(\alpha)+\cos^2(\alpha) = 1$

$$\begin{cases} x^2+y^2=1 \\ y=x \end{cases} \qquad 2x^2=1 \qquad x^2=\frac{1}{2} \qquad x=\pm\sqrt{\frac{2}{2}}$$

Trigonometria

$\cos\left(\frac{\pi}{4}\right) = \text{sen}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\text{sen}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

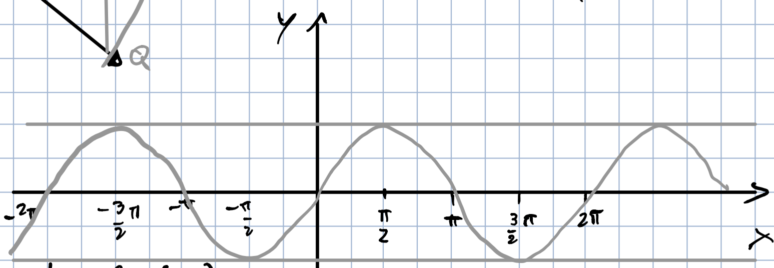


$\cos\left(\frac{\pi}{6}\right) = ?$

$= \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$

$\text{sen}\left(\frac{\pi}{5}\right),\ \cos\left(\frac{\pi}{6}\right)$

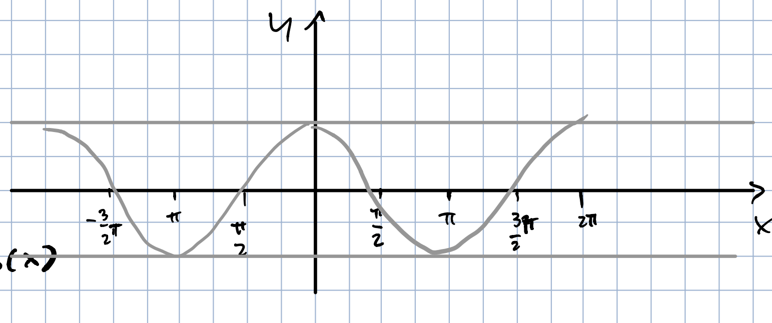
Grafici delle funzioni sen e cos



$y = \text{sen}(x)$

$\text{sen}(\mathbb{R}) = [-1, 1]$

$\text{sen}(x) = y_0 = \frac{1}{2}$



$y = \cos(x)$

Qualche formula

$\text{sen}(\alpha + 2k\pi) = \text{sen}\,\alpha$ $\qquad$ $\text{sen}(-\alpha) = -\text{sen}(\alpha)$ dispari

$\cos(\alpha + 2k\pi) = \cos\alpha$ $\qquad$ $\cos(-\alpha) = \cos(\alpha)$ pari

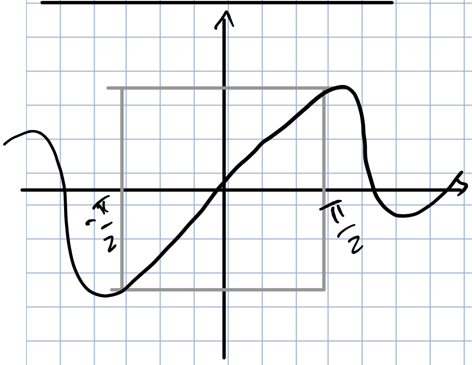
Formule di addizione e sottrazione

$\forall \alpha, \beta \in \mathbb{R}$ $\quad sen(\alpha \pm \beta) = sen(\alpha)\cos(\beta) \pm \cos(\alpha)\, sen(\beta)$

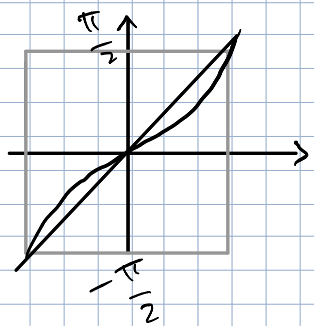
$\cos(\alpha \pm \beta) = \cos^2(\alpha) - sen^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\, sen^2(\alpha)$

$\cos^2(\alpha) = \dfrac{1 + \cos(2\alpha)}{2}$

Funzioni inverse

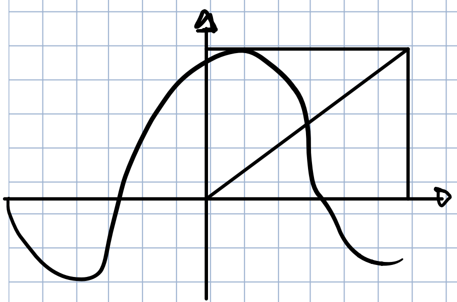


$sen\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

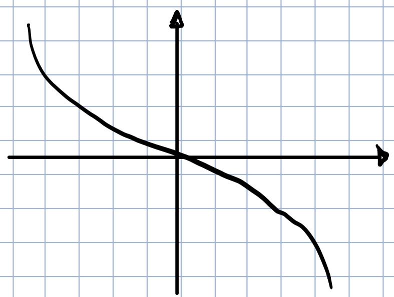


$arcsen(z)$

$sen^{-1}(z)$

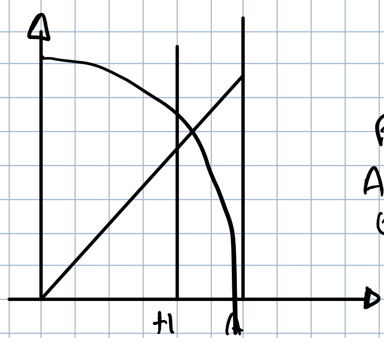


$\cos : [0, \pi] \to [-1, 1]$



$arccos: [-1, 1] \to [0, \pi]$

Funzione tangente



$P = \cos(x)\, sen(x)$

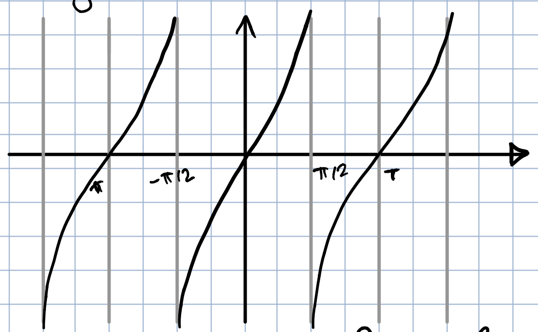
$A = (1; 0)$

$Q = (1, ?)$

$$\frac{x(A)}{x(H)} = \frac{y(Q)}{y(P)} \qquad y(Q) = \frac{y(P)}{x(H)} = \frac{sen(x)}{cos(x)} = tg(x)$$

$$tg : \mathbb{R} - \{x \mid cos(x) = 0\} \to \mathbb{R}$$

$$y : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$



$m \neq 0 \qquad f : \mathbb{R} \to \mathbb{R} \qquad f(x) = mx + q \qquad y = mx - q$

$$x = \frac{y - q}{m}$$

scambio x con y $\qquad g = \dfrac{x - y}{m} \qquad f^{-1}(f(x)) = f^{-1}(mx + q) =$

$$f^{-1}(x) = \frac{x - q}{m} \qquad = \frac{(mx + q) - q}{m} = \frac{mx + y - y}{m} = \frac{mx}{m} = x$$

$f(x) = (x^2-1)$ $\quad A \subset B$ e $B \subset A \Rightarrow A = B$ $\quad$ 30/10/18

Esercitazione

$f(\overbrace{]1,2[}^{A}) = f(\{a \in \mathbb{R} : 1 < a < 2\} \subset (0,3)$

viceversa $b \in (0,3) \Rightarrow b \in f(]1,2[)$

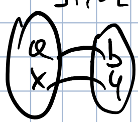
$1 < a < 2$

$1 < a^2 \ 4$

$1 < a^2 - 1 < 3$

$a < b < 3$

$4 \ ]1,2[$

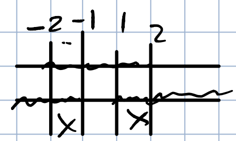


$a < b < 3$

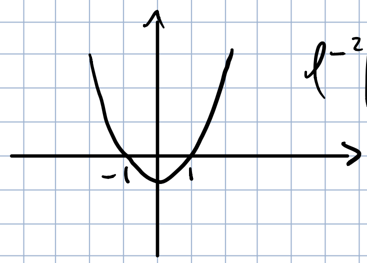
$a < a^2 - 1 < 3$

$1 < a^2 < 4$

$\begin{cases} a^2 < 4 & a = \pm 2 & -2 < a < 2 \\ a^2 > 1 & a = \pm 1 & a < -1 \cup a > 1 \end{cases}$



$1 < a < 2 \ \vee$
$-2 < a < -1$

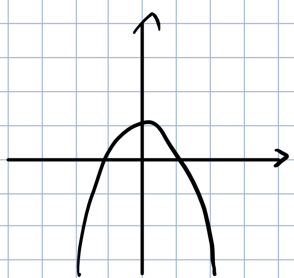


$f^{-2}(]0,3[)$

x compito $f([1,2]) = [2,5]$ $\quad f(x) = x^2+1$

$\begin{cases} x^2 \leq 4 & x = \pm 2 \\ x^2 \geq 1 \end{cases}$

trovo $f^{-1}([2,-5])$



$\begin{cases} -2 \leq x \leq -1 \\ \quad \cup \\ 1 \leq x \leq 2 \end{cases}$

[...]

[...] qui manca delle robe, stupida prof di esercitazione

Passiamo al concetto di f suriettive, biiettive e iniettive

$\forall a_1, a_2 \in A \quad f(a_1) \cdot f(a_2) \Rightarrow a_1 = a_2$

non è iniettiva $\exists a_1, a_2 \in A : [a_1 \neq a_2$ e $f(a_1) = f(a_2))$

analiticamente 1) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

2) $f(x)=k$

$f(x)=\dfrac{x-2}{x-3}$ $D: x\neq 3$

$f(x_1)=f(x_2)$ $x_2=x_1$ $\dfrac{x_1-x_2}{x_1-3}=\dfrac{x_2-2}{x_2-3} \rightarrow (x_1-2)(x_2-3)=$

$(x_2-2)(x_1-3)$ $x_1x_2-3x-2x+6=x_1x_2-3x-2x+6$

esercizio ~~(non lo copio, lo rifaccio a casa)~~ mentivo

$f:\mathbb{R}\setminus\{0\}\rightarrow\mathbb{R}$

$f(x)=x+\dfrac{1}{x}$

Vedere se la funzione è iniettiva

$a_1=\frac{1}{2}$ $f(a_1)=\frac{1}{2}+\dfrac{1}{\frac{1}{2}}=\frac{1}{2}+2=\frac{5}{2}$

non è iniettiva

$1\rightarrow\frac{5}{2}$

$2\nearrow$ $a_2=2$ $f(a_2)=2+\frac{1}{2}=\frac{5}{2}$

$f:\mathbb{R}^+\rightarrow\mathbb{R}$ $f(x)=x\cdot\frac{1}{x}$ $f(x)=f(y)$

$x-\frac{1}{x}=y-\frac{1}{y}$

$(x-y)+\dfrac{(x-y)}{xy}=0$ $x-y=\frac{1}{x}-\frac{1}{y}$

$(x-y)\left[1+\frac{1}{xy}\right]=0$ $\begin{cases} x=y \\ \frac{1}{xy}=x-1 \end{cases}$ $x-y=\dfrac{y-x}{xy}$

$xy=-1$ impossibile in $\mathbb{R}^+$

Suriettive

$\forall_{\text{rett}}$ $\exists$ $b\in B$ e $f(a)=b$

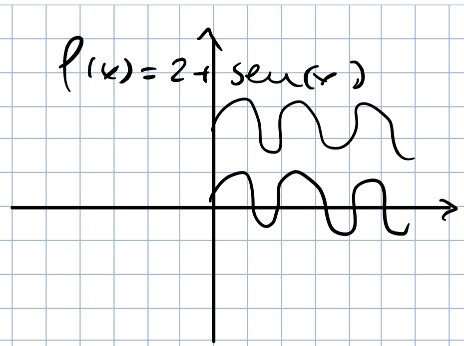
$f(x)=k$ ammette almeno una soluzione

vediamo

$\operatorname{sen} x + 2 = 5 \Rightarrow \operatorname{sen} x = 5-2$

$\operatorname{sen} x = 3$ **No!**

$(-1, 1)$

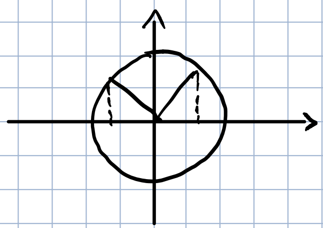


Risoluzione di eq. e diseq. elementari con sen cos tan

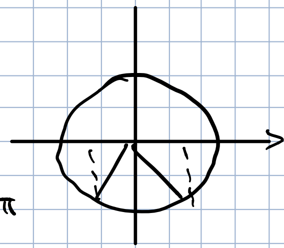
$\operatorname{sen} x = \frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4} + 2k\pi$

$x = \frac{3}{4} + 2k\pi$



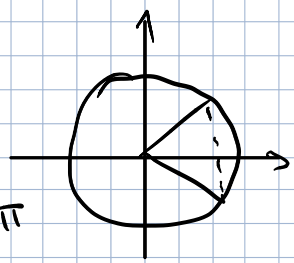
$\operatorname{sen} x = \frac{1}{2}$



$x = \pi/6 + 2k\pi$

$x = \frac{7}{6}\pi + 2k\pi$

$\cos x = \frac{\sqrt{3}}{2}$

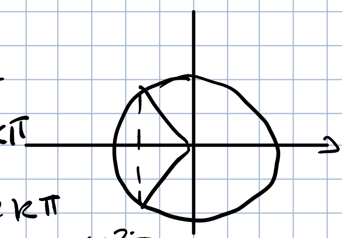


$x = \pm \frac{\pi}{6} + 2k\pi$

$\cos x = -\frac{1}{2}$

$x = \frac{2}{3}\pi + 2k\pi$

$x = \frac{4}{3}\pi + 2k\pi$



$-\frac{2}{3}\pi$

$\operatorname{sen} x = k$

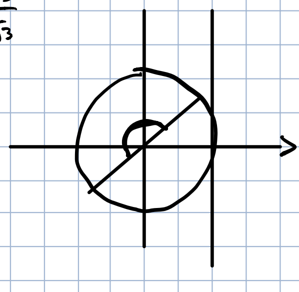
$\cos x = k$

$\operatorname{tg} x = k$

$\frac{\sqrt{3}}{3}$ $\operatorname{tg} x = \frac{1}{\sqrt{3}}$ $\frac{\sqrt{3}}{\sqrt{3}}$

$\operatorname{tg} x = \frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6} + k\pi$



$\operatorname{sen}^2 x = 1$

$\operatorname{sen} x = \pm 1$

$\operatorname{sen} x = 1 \quad x = \frac{\pi}{2} + 2k\pi$

$\operatorname{sen} x = -1 \quad x = \frac{3}{2}\pi + 2k\pi$ opp

$2\cos^2 x - 1 = 0$

$\cos^2 x = \frac{1}{2}$

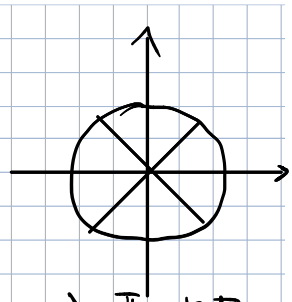
$\cos x = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$

$x = \pm\frac{\pi}{4} + 2k\pi$

$\cos x = \frac{\sqrt{2}}{2}$

$x = \pm\frac{3}{4}\pi + 2k\pi$

$x = -\frac{\pi}{2} + 2k\pi$ $\cos x = \frac{\sqrt{2}}{2}$

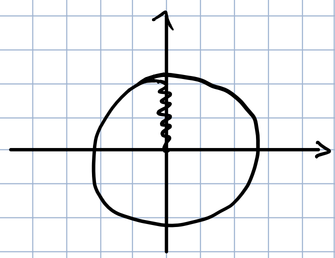


$x = \frac{\pi}{4} + k\frac{\pi}{2}$

sen $x > 0$

$0 + 2k\pi < x < \pi + 2k\pi$

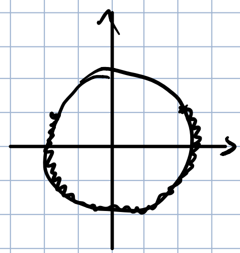
$2k\pi < x < \pi + 2k\pi$

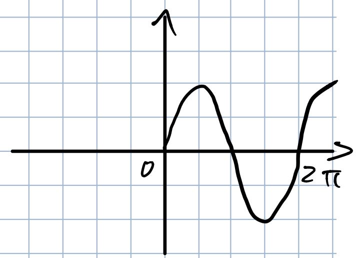


$2 \operatorname{sen} x - 1 < 0$

sen $x < \frac{1}{2}$ π

$\frac{5}{6}\pi + 2k\pi < x < \frac{13}{6}\pi + 2k\pi$





$\cos x > -1$ $\forall x \neq \pi + 2k\pi$

sen $\left(x + \frac{\pi}{4}\right) > 0$

$2k\pi < x + \frac{\pi}{4} < \pi + 2k\pi$

$-\frac{\pi}{4} + 2k\pi < x < \frac{3}{4}\pi + 2k\pi$

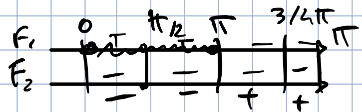
$\pi - \frac{\pi}{4}$

sen$^2 x$ − sen $x \leq 0$

sen x(sen $x - 1) \leq 0$

$F_1 \geq 0$: sen $x \geq 0$

$F_2 \geq 0$: sen $x - 1 \geq 0$



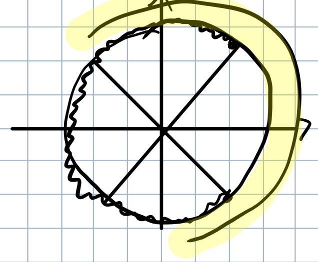
sen $x \geq 1$ $x = \frac{\pi}{2} + 2k\pi$

$2\cos^2 x - 1 < 0$

$\cos^2 x < \frac{1}{2}$

$\cos x = \pm\frac{\sqrt{2}}{2}$

$-\frac{\sqrt{2}}{2} < \cos x < \frac{\sqrt{2}}{2}$



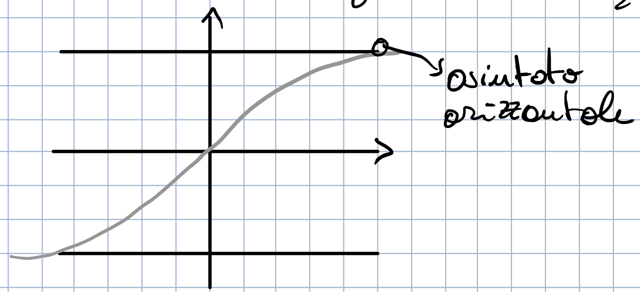
TEORIA

Funzione tangente

$$D = \{ x \in \mathbb{R} \mid \cos(x) \neq 0 \} = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi \mid \forall k \in \mathbb{Z} \right\}$$

$\text{arctg} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\text{tg} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

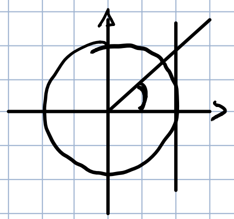


$$\begin{cases} \text{tg} : D \to \mathbb{R} \\ \text{tg}(x) = \dfrac{\text{sen}(x)}{\cos(x)} \end{cases}$$

$$\text{tg}(\alpha \pm \beta) = \frac{\text{tg}(\alpha) \pm \text{tg}(\beta)}{1 \mp \text{tg}(\alpha)\,\text{tg}(\beta)} \qquad \alpha, \beta, \alpha \pm \beta \in D$$

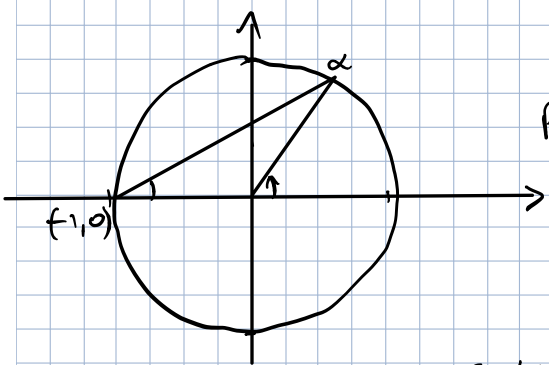
$\pi \cong 3{,}14159265\ldots$

$\text{tg}\left(\frac{\pi}{4}\right) = 1$



$4 \cdot \text{arctg}(1) = \pi$

$\text{arctg}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$



$t = \text{tg}\left(\frac{\alpha}{2}\right)$

$P = (\cos(x), \text{sen}(x))$

$$\begin{cases} x^2 + y^2 = 1 \\ y = tx + t \end{cases}$$

$y = t(x+1)$

$(1+t^2)x^2 + 2t^2 x + t^2 - 1 = 0$

$(1+t^2) - 2t^2 + t^2 - 1 = 0$

$y = mx + q$

$$\begin{cases} 0 = -m + q \\ t = 0 + q \end{cases}$$

$$\begin{cases} m = q = t \\ q = t \end{cases}$$

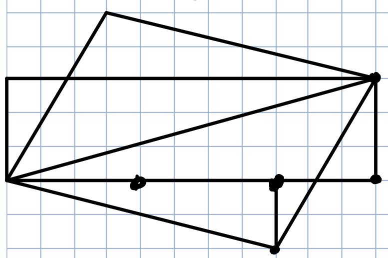
$$(1+t^2)\ x^2 + \ 2t^2 x + t^2 - 1 \ \Big|\ x+1$$

$$-(1+t^2)x^2 - (1+t^2)x \qquad (1+t^2)\,x + (t^2-1)$$

$$(t^2-1)\,x + t^2 - 1$$

$$-(t^2-1)x - (t^2-1)$$

Zaccagnini wtf ? 0



$$(1+t^2)\left((1+t^2)x + (t^2-1)\right) = 0$$

$$x = \frac{1-t^2}{1+t^2}$$

$$\begin{cases} x = \cos(\alpha) = \dfrac{1-t^2}{1+t^2} \\ y = \operatorname{sen}(\alpha) = \dfrac{2t}{1+t^2} \end{cases}$$

$$t = \operatorname{tg}\left(\frac{\alpha}{2}\right)$$

$\alpha = \operatorname{arctg}\left(\frac{1}{3}\right) \qquad \operatorname{tg}(\alpha) = \frac{1}{3}$

$\beta = \operatorname{arctg}\left(\frac{1}{2}\right) \qquad \operatorname{tg}(\beta) = \frac{1}{2}$

$\alpha \simeq 0.32175\ldots$

$\beta \simeq 0.46365$

$0.78830\ldots$

$$|x| \leq 1 \qquad \operatorname{arctg}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^3 - \frac{1}{7}x^3 + \frac{1}{9}x^3 - \frac{1}{11}x^3 \ldots$$

$$\operatorname{arctg}\left(\frac{1}{3}\right) \cong \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 = \frac{1}{3} - \frac{1}{81} = \frac{26}{81}$$

$$\operatorname{arctg}\left(\frac{1}{2}\right) \simeq \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$$

Potenze

obiettivo: definire x^y nel più grande dominio possibile

① $x \in \mathbb{R}$, $y \in \mathbb{N}^*$

② $x \in \mathbb{R}^*$, $y \in \mathbb{Z}$

DEF: $x^0 = 1$

③ $x \in \mathbb{R}^+$ $\quad y = \frac{1}{n}$ $\quad n \in \mathbb{N}^*$

$x^{1/n} = \sqrt[n]{x}$

Proprietà:

$\forall x, y \in \mathbb{R}^* \quad \forall n, m \in \mathbb{Z}$

$x^n y^n = (xy)^n$

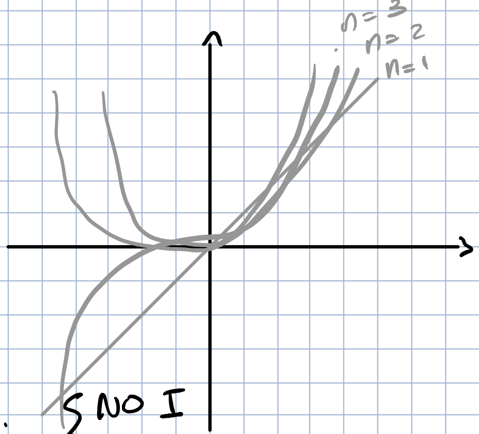
$(x^n)^m = x^{nm}$

$\forall x \in \mathbb{R}^+, \forall n, m \in \mathbb{N}^* \quad x^{m/n} = (\sqrt{2}) = \sqrt[n]{x^m}$

Funzioni Potenza

$n \in \mathbb{N}^*$

$\begin{cases} p_n : \mathbb{R} \to \mathbb{R} \\ p_n(x) = x^n \end{cases}$



$p_1(2) = 2$

$p_2(2) = 4$

$p_3(2) = 8$

$p_4(2) = 16$

$p_1(\frac{1}{2}) = \frac{1}{2}$

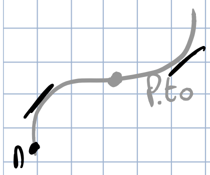
$p_2(\frac{1}{2}) = \frac{1}{4}$

$p_3(\frac{1}{2}) = \frac{1}{8}$

$p_4(\frac{1}{2}) = \frac{1}{16}$

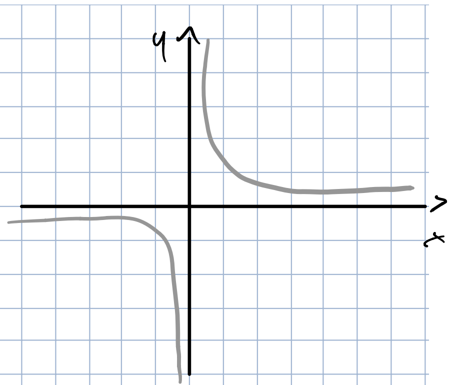
p_n è pari se n è pari $\begin{cases} \text{NO I} \\ \text{NO S} \end{cases}$

p_n è dispari se n è dispari $\begin{cases} \text{I} \\ \text{S} \end{cases}$



$n \in \mathbb{N}^*$

$$\begin{cases} P_{-n} : \mathbb{R}^* \to \mathbb{R} \\ P_{-n}(x) = x^{-n} = \frac{1}{x^n} \end{cases}$$



$x = 10$

$P_{-1}(10) = \frac{1}{10}$

$P_{-2}\left(\frac{1}{10}\right) = \frac{1}{100}$

$P_{-3}\left(\frac{1}{100}\right) = \frac{1}{1000}$

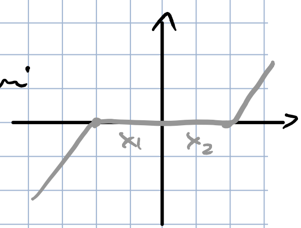
MONOTONIA

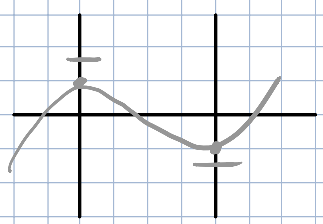
Siano $X, Y \subseteq \mathbb{R}$
$f : X \to Y$

def: diciamo che f è monotona e debolmente crescente se $\forall x_1, x_2 \in X$ se $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

diciamo che f è strettamente crescente se $\forall x_1, x_2 \in X$ se $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Analogamente si possono definire le funzioni debolmente e strettamente decrescenti.





TEO: Sia $f : X \to Y$ biiettiva e monotona strett. crescente con $X, Y \subseteq \mathbb{R}$. Allora, $f^{-1} : Y \to X$ è monotona strett. crescente.

Come definire $7^{\sqrt{2}}$

$x^q \quad x \in \mathbb{R}^+ \quad q \in \mathbb{Q}$

$\sqrt{2} = 1,41\ldots$

$1 < \sqrt{2} < 2$
$1.4 < \sqrt{2} < 1.5$
$1.41 < \sqrt{2} < 1.42 \to \frac{142}{100}$
$1.414 < \sqrt{2} < 1.415$
$1.4142 < \sqrt{2} < 1.4143$

$1.4 = \frac{14}{10} = \frac{7}{5} \qquad 1.5 = \frac{15}{10} = \frac{3}{2}$

$\left(\frac{7}{5}\right)^2 = \frac{49}{25} < 2 \qquad \left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2$

$x_0 < x_1 < x_2 < x_3 < \ldots$

$y_0 > y_1 > y_2 > y_3 > \ldots$

$\forall n \in \mathbb{N} \quad x_n^2 < 2 < y_n^2$

$y_n - x_n = \frac{1}{10^n}$

$7^1 = 7 \qquad < 7^{\sqrt{2}} < 7^2 = 49$

$7^{14/10} = 15.25 \qquad < 7^{\sqrt{2}} < 7^{15/10} = 18.52$

$7^{140/100} = 15.54 \qquad < 7^{\sqrt{2}} < 7^{14?/100} = 15.83$

$7^{1414/1000} = 15.56 \qquad < 7^{\sqrt{2}} < 7^{1415/1000} = 15.69$

$7^{\sqrt{2}} = 15.6728\ldots$

Funzione Esponenziale

TEO: Sia $a \in \mathbb{R}^+$ $\exists$ un' unica funz. monotòna $\exp_a: \mathbb{R} \to \mathbb{R}$ tale che
- $\forall x_1, x_2 \in \mathbb{R} \quad \exp_a(x_1 + x_2) = \exp_a(x_1)\exp_a(x_2)$
- $\exp_a(1) = a$

Scriveremo $\exp_a(x) = a^x$

$$\begin{cases} a^{x_1+x_2} = a^{x_1} \cdot a^{x_2} \\ a^1 = a \end{cases}$$

$x_1 = x_2$

$\exp_a(2x_1) = (\exp_a(x_1))^2$

$a^{2x} = (a^x)^2$

$x_2 = 0$

$\exp_a(x_1 + 0) = \exp_a(x_1)\exp_a(0)$

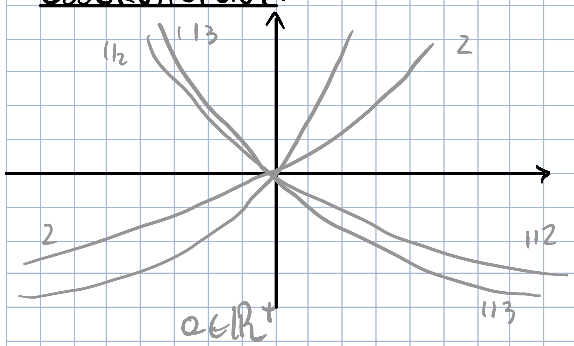
$\exp_a(x_1) = \exp_a(x_1)\exp_a(0)$

$exp(0) = 1 \qquad x_2 = -x_1$

$$1 = exp_a(x_1 - x_2) = exp_a(x_1)\, exp_a(-x_1)$$

$$a^{x_1} \cdot a^{-x_1} = 1$$

<u>OSSERVAZIONI</u>:



exp_a è str. cresc. se $a > 1$

exp_a è str. decresc. se $a \in (0,1)$

$a = e \cong 2.71\ldots$ $exp_a(\mathbb{R}) = \mathbb{R}^+$ se $a \in (0, +\infty) - \{1\}$

Fisso $a \in (0, +\infty) - \{1\}$

$exp_a : \mathbb{R} \to \mathbb{R}^+$ è una f. biettiva dunque invertibile

la funzione inversa si chiama logaritmo in base a

$log_a : \mathbb{R}^+ \to \mathbb{R}$

$\forall x \in \mathbb{R}$ $\quad \forall a \in (0, +\infty) - \{1\}$ si ha $log_a(exp_a(x)) = x$

$\forall y \in \mathbb{R}$ $\quad exp_a(log_a(y)) = y$

$$\begin{cases} log_a(a^x) = x \\ a^{log_x(y)} = y \end{cases}$$

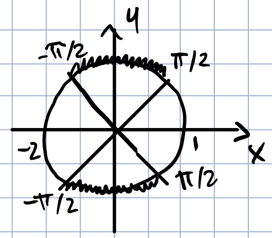
Esercitazione

Diseq. goniometriche

- $|\text{sen } x| > 0$ $\quad$ $\text{sen} \neq 0$ $\quad$ $\forall x \in \mathbb{R} \setminus \{k\pi\}$

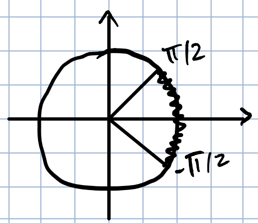
  $x = 0 + k\pi$

- $|\cos x| < \frac{\sqrt{2}}{2}$ $\quad$ $-\frac{\sqrt{2}}{2} < \cos x < \frac{\sqrt{2}}{2}$



$$\frac{\pi}{4} + k\pi \leq x \leq \frac{3}{4}\pi + k\pi$$

- $\cos\left(\frac{x}{3}\right) > \frac{\sqrt{3}}{2}$



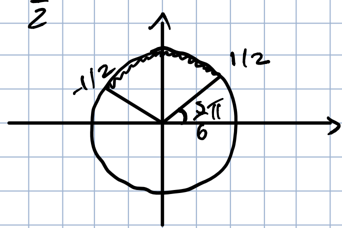
$$-\frac{\pi}{2} + 2k\pi < \frac{x}{3} < \frac{\pi}{6} + 2k\pi$$

$$-\frac{\pi}{2} + 6k\pi < x < \frac{\pi}{2} + 6k\pi$$

- $2\,\text{sen}^2 x - 3\,\text{sen } x + 1 < 0$

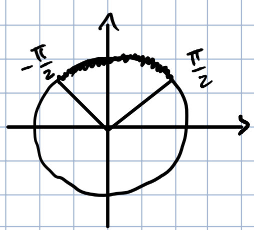
$$\text{sen } x_{1,2} = \frac{3 \pm \sqrt{9-8}}{4} = \begin{cases} \frac{\cancel{2}}{\cancel{4}} & \frac{1}{2} \\ \frac{\cancel{4}}{\cancel{4}} & 1 \end{cases}$$

$$\frac{1}{2} < \text{sen } x < 1$$



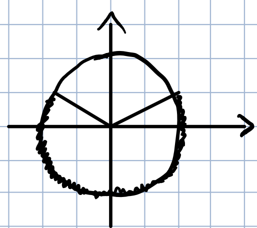
$$\frac{\pi}{6} - 2k\pi < x < \frac{5}{6}\pi + 2k\pi \qquad x \neq \frac{\pi}{2} + 2k\pi$$

- $\text{sen } x > \frac{\sqrt{2}}{2}$



$$\frac{\pi}{4} + 2k\pi < x < \frac{3}{4}\pi + 2k\pi$$

- $\text{sen } x < \frac{1}{2}$



$$-\frac{7}{6}\pi + 2k\pi < x < \frac{\pi}{6} + 2k\pi$$

- $\cos^2 x - 2\cos x - 3 < 0$

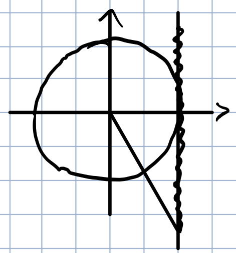
$\cos x_{1,2} = +1 \pm \sqrt{1+3} = \begin{cases} -1 \\ 3 \end{cases}$ $\quad -1 < \cos x < 3 \ \forall x \in \mathbb{R} \setminus \{\pi + 2k\pi\}$

- $3 - 4\cos^3 x > 0$

$-4\cos^2 x > -3 \Rightarrow \cos x_{1,2} = \pm\sqrt{\frac{3}{4}} \begin{cases} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{cases}$

$\frac{4\cos^2 x}{4} < \frac{3}{4}$ $\quad -\frac{\sqrt{3}}{2} < \cos x < \frac{\sqrt{3}}{2}$

- $\sqrt{3} + tg\, x > 0$ $\quad tg\, x > -\sqrt{3}$



$-\frac{\pi}{3} + k\pi < x < \frac{\pi}{2} + k\pi$

- $\sqrt{3} + tg\, x > 0$ $\quad tg\, x > -\sqrt{3}$ $\quad -\frac{1}{3} + k\pi < x < \frac{\pi}{2} + k\pi$

$1 - \sqrt{3}\, tg\, x > 0$ $\quad tg\, x < \frac{\sqrt{3}}{3}$ $\quad -\sqrt{3}\, tg\, x > -1$

$-\frac{\pi}{2} + k\pi < x < \frac{\pi}{6} + k\pi$ $\quad \frac{\cancel{\sqrt{3}}\, tg\, x}{\cancel{\sqrt{3}}} < \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

- Algoritmo di divisione fra polinomi:

ordinare decrescente e lasciare un buco dove serve

- $(4x^3 - 2x^2 + 1) : (2x - 3) = 2x^2 + 2x + 3$ $\quad R = 10$

$4x^3 - 2x^2 \quad + 1 \mid 2x + 3$

$$
\begin{array}{l|l}
-4x^3+6x^2 & \\
\hline
// \quad 4x^2 \qquad\qquad +1 & 2x^2+2x+3 \\
-4x^2+6x & \\
\hline
// \quad 6x+1 & \\
-6x+9 & \\
\hline
10 &
\end{array}
$$

- $\left(a^3-\frac{2}{5}a^2-\frac{5}{2}a+1\right):\left(\frac{a}{2}-\frac{1}{5}\right)=2a^2-5$

$$
\begin{array}{l|l}
a^3-\frac{2}{5}a^2-\frac{5}{2}a+1 & \frac{a}{2}-\frac{1}{5} \\
\hline
-a^3+\frac{2}{5}a^2 & 2a^2-5 \\
\hline
// \quad // \quad -\frac{5}{2}a^2+1 & \\
+\frac{5}{2}a^2-1 & \\
\hline
// \quad // &
\end{array}
$$

- $\left(-\frac{1}{3}a^5+2a^4-5a\right):(a+2)$

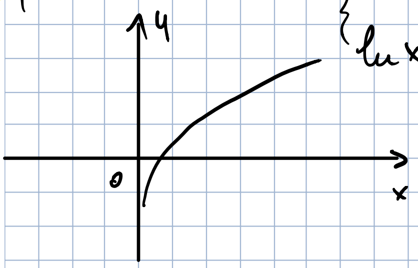
$$
\begin{array}{l|l}
-\frac{1}{3}a^5+2a^4 \qquad -5a & a+2 \\
\hline
+\frac{1}{3}a^5+\frac{2}{3}a^4 & -\frac{1}{3}a^5+\frac{8}{3}a^3-\frac{16}{3}a^2-\frac{79}{3} \\
\hline
\frac{8}{3}a^4 \quad -5a & \\
-\frac{8}{3}a^4-\frac{16}{3}a^3 & \\
\hline
-\frac{16}{3}a^3-5a & \\
+\frac{16}{3}a^3+\frac{32}{3}a^2 & \\
+\frac{32}{3}a^2-5a & \\
-\frac{32}{3}a^2-5a & \\
\hline
// -\frac{79}{3}a & \frac{158}{3}=R
\end{array}
$$

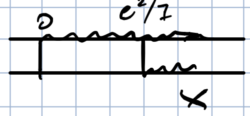
In direzione del parziale, studio di f. fino ai limiti, dom, codom, segno, P/D, lim.

- $y = \dfrac{1}{2 - \ln x}$ $\quad 2 - \ln x \neq 0 \quad \begin{cases} \ln x \neq 2 \\ x > 0 \end{cases} \quad \ln_e^2 \neq \ln x \quad x \neq e^2$

Dom f: $\forall x \in \mathbb{R}_{>0} - \{e^2\} \,//\, (0, +\infty)/\{e^2\} \,//\, 0 < x < e^2 \vee x > e^2$

- $y = \sqrt{\ln 7x - 2}$ $\quad \begin{cases} 7x > 0 \\ \ln x7 - 2 \geq 0 \end{cases} \quad \begin{cases} x > 0 \\ \ln 7x \geq 2 \end{cases} \rightarrow \ln 7x > \ln e^2 \rightarrow \dfrac{7x}{7} > \dfrac{e^2}{7}$





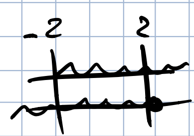
- $y = 3^{2x-1} \longrightarrow D = \mathbb{R}$

- $y = 3^{\sqrt{1-x}}$ $\quad 1 - x \geq 0 \quad x \leq 1 \quad D: \{x \in \mathbb{R} : x \leq 1\}$

- $y = \log_2 \left[\log_{\frac{1}{2}} (2-x) + 2 \right]$

$\begin{cases} \log_{\frac{1}{2}}(2-x) + 2 \geq 0 \\ 2 - x \geq 0 \end{cases} \quad \begin{cases} \log_{\frac{1}{2}}(2-x) > -2 \\ x \leq 2 \end{cases}$

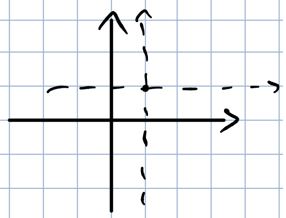
$\begin{cases} \log_{\frac{1}{2}}(2-x) > \log_{\frac{1}{2}} 4 \\ x \leq 2 \end{cases} \quad \begin{cases} 2 - x < 4 \\ x \leq 2 \end{cases} \quad \begin{cases} x > -2 \\ x \leq 2 \end{cases}$



Dom f: $(-2, 2)$ $\qquad -2 < x < 2$

- $y = \frac{x+1}{x-2}$ $y\frac{(x-2)}{x-2} = \frac{x+1}{x+1}$ $D: x \neq 2$

[...]



- $y = 3 - \ln x$ $D: x > 0$

$\ln x = 3 - y$ $e^{\ln x} =$ $x = e^{3-y}$ $C = \mathbb{R}$

[...]

TEORIA

Potenze a esponente reale

x^y ?

se $x \in \mathbb{R}^+$ $a \in \mathbb{R}^+ - \{1\}$

$x^y = \exp_a(y \log_a(x))$

$x^y = \exp_a(\log_a(x^y))$

- **Costruzione di $\mathbb{R}$**

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$

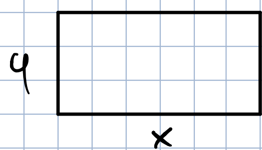
$2(x+y) \in [6, 6.04]$

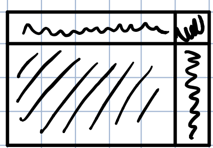
$x, y \in [2, 2.0301]$

- **Calcolo approssimato**

$x \in [2, 2.01]$

$y \in [1, 1.01]$





$x \in [a, a+h]$

$y \in [b, b+k]$

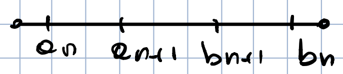
$$x+y \in [a+b,\ a+b+(h+k)]$$
$$x\cdot y \in [ab,\ ab+(ak+bh+hk)]$$

$\mathbb{Q}$ "scatola cinese":

Abbiamo una successione di intervalli $[a_n, b_n]$, con queste proprietà:

- $\forall n \in \mathbb{N} \quad a_n \le b_n$
- $\forall n \in \mathbb{N} \quad a_n, b_n \in \mathbb{Q}$
- $\forall n \in \mathbb{N} \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

$$a_n < a_{n+1} < b_{n+1} < b_n$$



- $\forall \varepsilon \in \mathbb{Q}^+ \quad \exists n = n(\varepsilon) \in \mathbb{N} \quad$ t.c. $b_n - a_n \le \varepsilon$

$0.\bar{9} = 1 \qquad b_n = 1 \ \forall n \in \mathbb{N} \qquad a_n = 1 - \frac{1}{10^n}$

$0.\bar{3} = \frac{1}{3} \qquad a_0 = 0$

$0.\bar{9} = 1 \qquad a_1 = \frac{9}{10} = 0.9$

$$a_2 = \frac{9}{10} + \frac{9}{100} = 0.99$$

$$a_3 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} = 0.999$$

$$1 + q + q^2 + q^3 + \dots + q^n = \frac{1-q^{m+1}}{1-q} \quad \forall q \in \mathbb{R} - \{1\} \quad \forall m \in \mathbb{N}$$

$$a_n = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots + \frac{9}{10^n}$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \dots + \frac{1}{10^n}\right) = \frac{9}{10} \cdot \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} = 1 - \frac{1}{10}$$

$\forall \varepsilon \in \mathbb{Q}^+ \quad \exists n = n(\varepsilon) \in \mathbb{N}$

t.c. $1 - \left(1 - \frac{1}{10^n}\right) < \varepsilon$? $\Leftrightarrow \frac{1}{10^n} < \varepsilon \Leftrightarrow 10^n > \frac{1}{\varepsilon}$

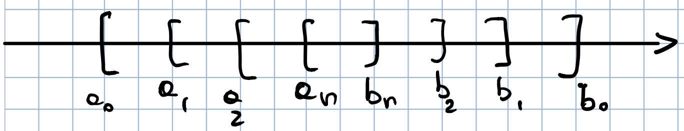
Costruzione di $\mathbb{R}$

"Scatole cinese" $\{[a_n, b_n] \mid n \in \mathbb{N}\}$

- $\forall n \in \mathbb{N}$
- $\forall n \in \mathbb{N}$
- $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
- $\forall \varepsilon \in \mathbb{Q}^+ \quad \exists n = n(\varepsilon) \in \mathbb{N} \quad$ t.c. $b_n - a_n \leq \varepsilon$

Numero reale

$\bigcap_{n \in \mathbb{N}} [a_n, b_n]$



$[a_0, b_0] \cap [a_1, b_1] = [a_1, b_1]$

$[a_0, b_0] \cap [a_1, b_1] \cap [a_2, b_2] = [a_n, b_n]$

$$\bigcap_{n=0}^{N} [a_n, b_n] = [a_N, b_N]$$

$1 < \sqrt{2} < 2$

$1.4 < \sqrt{2} < 1.5 \qquad \varepsilon = \frac{1}{10}$

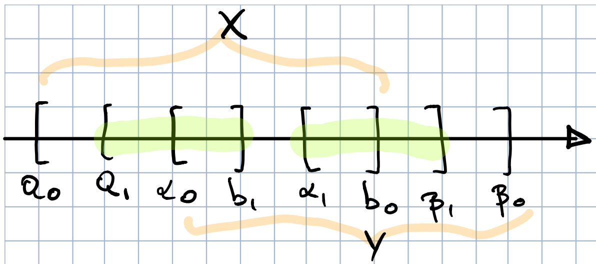
$1.41 < \sqrt{2} < 1.42$

$\{[a_n, b_n] \mid n \in \mathbb{N}\}$ corrisponde ad x

$\{[\alpha_n, \beta_n] \mid n \in \mathbb{N}\}$ corrisponde ad y

$x+y$ corrisponde ad A

$\{[a_n + \alpha_n, b_n + \beta_n] \mid n \in \mathbb{N}\}$

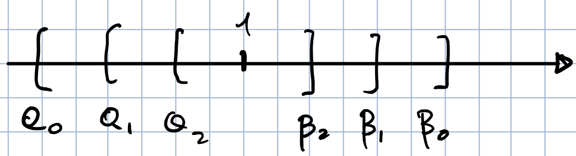


$x<y$ se $\exists n\in\mathbb{N}$ t.c. $b_n<\alpha_n$

$y<x$ se $\exists n\in\mathbb{N}$ t.c. $\beta_n<a_n$

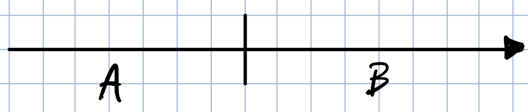
$x=y$ se $\forall n$ $[a_n,b_n]\cap[\alpha_n,\beta_n]\neq 0$

$$\begin{cases} a_n = 1-\frac{1}{10^n} \\ b_n = 1 \\ \alpha_n = 1 \\ \beta_n = 1+\frac{1}{10^n} \end{cases}$$



$[a_n,b_n]\cap[\alpha_n,\beta_n]=\{1\}$ $\forall n\in\mathbb{N}$

- Eudosso di Cnido
- Dedekind sezioni
- Cauchy

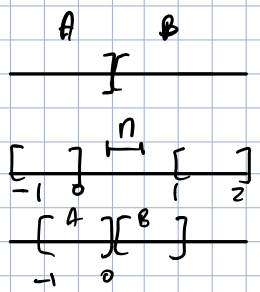


La completezza di $\mathbb{R}$

DEF: dati $A,B\subseteq\mathbb{R}$, $A \text{ e } B\neq\emptyset \to (A,B)$ si dice coppia di classi separate se $\forall a\in A$ $\forall b\in B$ si ha $a\le b$

Una coppia di classi separate (A,B) si dice **coppia di classi contigue** se $\forall \varepsilon \in \mathbb{R}^+ \ \exists a \in A, b \in B$ t.c. $b - a \leq \varepsilon$

A	B	
$\mathbb{R}^{0-}$	$\mathbb{R}^{0+}$	CS
$[-1,1]$	$[0,2]$	CS
$[-1,0]$	$[1,2]$	NO
$[-1,0]$	$(0,1]$	CS



CC

NCC

$a = -\frac{1}{3}\varepsilon \in A$

$b = \frac{1}{3}\varepsilon \in B$

Def: dato una coppia di classi separate (A,B) si chiama elemento separatore di (A,B) qualunque $x \in \mathbb{R}$ t.c. $\forall a \in A, \forall b \in B \quad a \leq x \leq b$

Teorema di completezza:

Ogni coppia di classi contigue ha un unico elemento separatore in $\mathbb{R}$.

$A = \{x \in \mathbb{Q} \mid x \leq 0 \vee x^2 \leq 2\}$

$B = \{x \in \mathbb{Q} \mid x > 0 \vee x^2 \geq 2\}$

$A = \mathbb{Q} \cap (-\infty, \sqrt{2}] \qquad B = \mathbb{Q} \cap [\sqrt{2}, +\infty)$

(A,B) è una coppia di classi contigue, l'elemento separatore è $\sqrt{2} \notin \mathbb{Q}$

- **Estremo superiore**

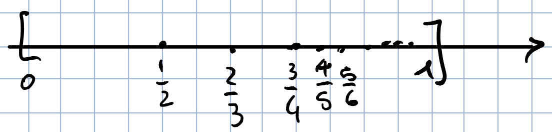
$A = [0,1] \qquad \max(A) = 1 \qquad \min(A) = ?$

$B = (0,1) \qquad \max(B) = \qquad \min(B) =$

Def: dato $A \subseteq \mathbb{R}$ $x \in A$ si dice massimo di A se $\forall a \in A$ si ha $a \leq x$

$$A = \left\{ \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \right\}$$

$\min(A) = 0$ $\quad \max(A) = \nexists$ $\quad \boxed{\frac{n}{n+1} = 1 - \frac{1}{n+1}}$



$\forall \varepsilon \in \mathbb{R}^+ \; \exists x \in A$ t.c. $1 - x < \varepsilon$

$x = \frac{n}{n+1} = 1 - \frac{1}{n+1} \qquad 1 - x = 1 - \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$

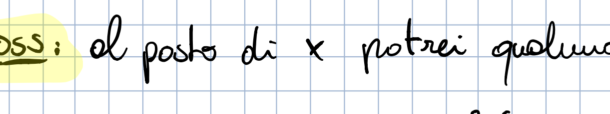
Problema: dato $\varepsilon > 0$, trovare $n \in \mathbb{N}$ t.c. $\frac{1}{n+1} \leq \varepsilon$

$\frac{1}{n+1} \leq \varepsilon \quad (\varepsilon > 0, n \geq 0) \longleftrightarrow n+1 \geq \frac{1}{\varepsilon} \longleftrightarrow n \geq \frac{1}{\varepsilon} - 1$

Risposta: basta prendere l'intero successivo a $\frac{1}{\varepsilon} - 1$

Def: Dato $a \subseteq \mathbb{R}$, $A \neq 0$ diciamo che A è limitato superiormente se $\exists x \in \mathbb{R}$ t.c. $\forall a \in A$ si ha $a \leq x$.

OSS: al posto di x potrei qualunque numero più grande di x



Def: Se $A \subseteq \mathbb{R}$, $A \neq \emptyset$, A è limitato superiormente, si chiama estremo superiore di A la più piccola limitazione superiore per A, e si indica con $\sup(A)$

OSS: Se $\exists \max(A)$, allora $\sup(A) = \max(A)$

TEO: Se $A \subseteq \mathbb{R}$, $A \neq 0$, A è limitato superiormente, allora A ha un estremo superiore in $\mathbb{R}$

Dim: è una conseguenza del teorema di completezza

Se A non è limitato superiormente scriviamo $\sup(A) = +\infty$

OSS: $A \subseteq \mathbb{R}$, $A \neq \emptyset$, A limitato superiormente. Sia $L = \sup(A) \in \mathbb{R}$ allora
- $\forall a \in A$, $a \leq L$
- $\forall \varepsilon > 0$ $\exists a \in A$ t.c. $a \in (L - \varepsilon, L)$ $\varepsilon < a \leq L$

Successioni

$a : \mathbb{N} \to \mathbb{R}$ $(\mathbb{C})$

$a(n) = \frac{n}{n+1}$

(a_n)

Def. per ricorrenza

$$\begin{cases} a_0 = x \\ a_{n+1} = a_n + h \end{cases}$$

$a_0 = x$

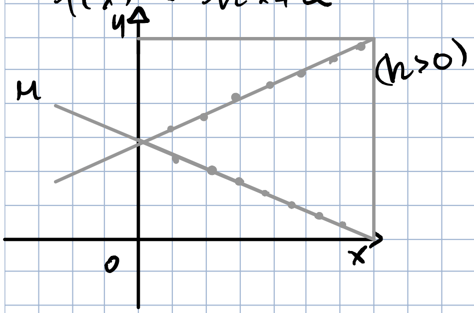
$a_1 = a_0 + h = h + a$

$a_2 = a_1 + h = 2h + a$

$a_3 = a_2 + h = 3h + a$

$\boxed{a_n = nh + a}$

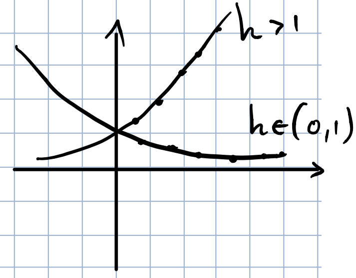
$f(x) = hx + \alpha$



$$\begin{cases} a_0 = \alpha \\ a_{n+1} = a_n \cdot h \end{cases}$$

$a_0 = h$

$a_1 = h a_0 = \alpha h$
$a_2 = h a_1 = \alpha h^2$
$a_3 = h a_2 = \alpha h^3$
$a_n = \alpha h^n$



$$\begin{cases} a_0 = \alpha \\ a_{n+1} = q a_n + h \end{cases}$$

Fattoriale:

$$\begin{cases} 0! \\ (n+1)! = n!\,(n+1) \end{cases}$$

$0! = 1$ $5! = 120$
$1! = 1$ $6! = 720$
$2! = 2$ $7! = 5040$
$3! = 6$ $8! = 40320$
$4! = 24$ $9! = 362880$

$10! = 3.628.800$

Fibonacci:

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_{n+2} = f_n + f_{n+1} \end{cases}$$

$f_0 = 0$ $f_6 = 8$
$f_1 = 1$ $f_7 = 13$
$f_2 = 1$ $f_8 = 21$
$f_3 = 2$ $f_9 = 34$
$f_4 = 3$ $f_{10} = 55$
$f_5 = 5$ $f_{11} = 89$

$f_n =$

Lemma $\forall n \in \mathbb{N} \quad \forall \delta \in \mathbb{R}^{0+}$ si ha $(1+\delta)^n \geq 1+\delta n$

più precisamente $(1+\delta)^n \geq 1+\delta n = 1+(h-1)n$

$\boxed{a_n = h^n}$

Def: diciamo che la successione (a_n) tende a $+\infty$ e scriviamo $\lim_{n \to +\infty} a_n = +\infty$

Se $\forall M \in \mathbb{R} \quad \exists \bar{n} = \bar{n}(M) \in \mathbb{N}$ t.c. $a_n > M \quad \forall n \geq \bar{n}$

Def: diciamo che la successione $(a_n)_{n\in\mathbb{N}}$ ha limite $L \in \mathbb{R}$ e scriviamo $\lim_{n \to +\infty} a_n = L$

Se $\forall \varepsilon > 0 \quad \exists \bar{n} = \bar{n}(\varepsilon) \in \mathbb{N}$ t.c. $|a_n - L| \leq \varepsilon \quad \forall n \geq \bar{n}$

Successioni:

$a : \mathbb{N} \to \mathbb{R}$

$a_n = f(n)$

$$\begin{cases} a_0 = \alpha \\ a_{n+1} = f(a_n, n) \end{cases}$$

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_{n+2} = f_{n+1} + f_n \end{cases}$$

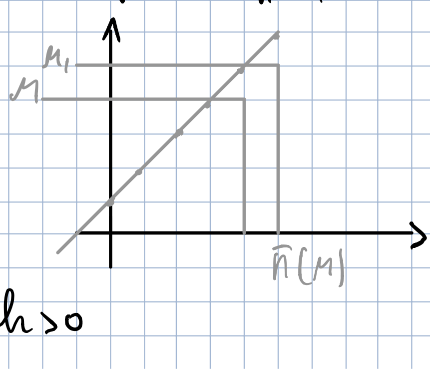
$$\begin{cases} a_0 = \alpha \\ a_{n+1} = h + a_n \end{cases}$$

$a_n = hn + \alpha$

$h > 0$

Fissato $M \in \mathbb{R}$, $\exists \bar{n} = \bar{n}(M) \in \mathbb{N}$

T.C. $a_n > M$



$hn + \alpha > M \qquad h > 0$

$n > \frac{M - \alpha}{h}$ $\quad$ $\bar{n}(M)$ è l'intero successivo

$\begin{cases} a_0 = 1 \\ a_{n+1} = h a_n \end{cases}$ $\quad$ $a_n = h^n$ $\quad$ $h > 1$

Dato $M \exists \bar{n} = \bar{n}(M) \in \mathbb{N}$ t.c. $a_n > M$ $\quad$ $M > 0$

$h^n > M$

$n \log(h) > \log M$

$n > \frac{\log(M)}{\log(h)}$ $\quad$ $\bar{n}$ è l'intero successivo

$a_n = \alpha^n$ $\quad$ $\alpha, \beta > 1$

$b_n = \beta^n$

$$\lim_{k \to +\infty} \frac{\alpha^n}{\beta^n} = \left(\frac{\alpha}{\beta}\right)^n = \begin{cases} +\infty & \alpha > \beta \\ 1 & \alpha = \beta \\ 0 & \alpha < \beta \end{cases}$$

$h = \frac{\alpha}{\beta}$

Prendiamo $k \in \mathbb{N}^*$ $(k \in \mathbb{R}^*)$ $\quad$ $\lim_{n \to +\infty} n^{\frac{1}{k}} = +\infty$

$n^k > M \iff n > M^{1/k} = \sqrt[k]{M}$ $\quad$ $\bar{n}(M)$ è l'intero successivo

$k, J \in \mathbb{N}$ $(\mathbb{R}^{0+})$

$a_n = n^k$

$b_n = n^J$

$$\lim_{n \to +\infty} \frac{a_n}{b_n} = \lim_{n \to +\infty} n^{k-J} = \begin{cases} 1 & \text{se } J = k \\ + & \text{se } k > J \\ 0 & \text{se } k < J \end{cases}$$

Polinomi:

$p(n) = a_k n^k + a_{k+1} n^{k+1} + \ldots + a_1 n + a_0$

$q(n) = b_J n^J + b_{J-1} n^{J-1} + \dots + b_1 n + b_0$

Dove $a_0 \dots a_1, b_0 \dots b_1, b_J \in \mathbb{R}$ $a_k, b_1 \neq 0$

Problema calcolare:

$$\lim_{n \to +\infty} \frac{p(n)}{q(n)} = ?$$

$$\frac{p(n)}{q(n)} = \frac{a_k n^k + \dots}{n^k} \cdot \frac{n^J}{b_J n^J + \dots} \cdot \frac{n^k}{n^J}$$

Problema (+ facile) calcolare

$$\lim_{n \to +\infty} \frac{a_k n^k + a_{k_1} n^{k-1} + \dots}{n^k}$$

$$= \lim_{n \to +\infty} \left(a_k + \frac{a_{k-1}}{n} + \frac{a_{k-2}}{n} + \dots + \frac{a_1}{n^k} + \frac{a_0}{n^k} \right) = 0$$

$$\frac{p(n)}{q(n)} = \frac{a_k n^k + a_{k-1} n^{k-1} + \dots}{n^k} \cdot \frac{n^J}{b_J n^J + \dots} \quad \frac{n^k}{n^J}$$

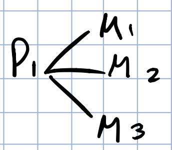
Dunque (manca un teorema)

$$\lim_{n \to +\infty} \frac{p(n)}{q(n)} = \lim_{n \to +\infty} \frac{a_k}{b_J} \frac{n^k}{n^J} = \begin{cases} \{ & \text{se } k > J \\ \frac{a_k}{b_n} & \text{se } k = J \\ 0 & \text{se } k < J \end{cases}$$

ESERCITAZIONE

Calcolo combinatorio (terne) $0 < k \leq n$

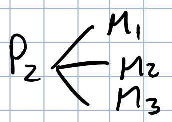
raggruppamenti

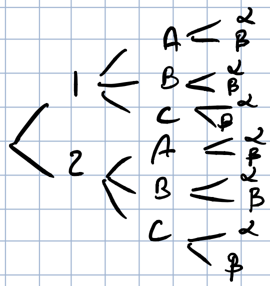


Possibili abbinamenti

Disposizione

→ raggruppamenti





$2 \cdot 3 \cdot 2 = 12$

- diversità di almeno un el t
- ordine

es) 15 squadre 3 posti

$D_{n,k} = n(n-1)(n-2)\ldots(n-k+1)$

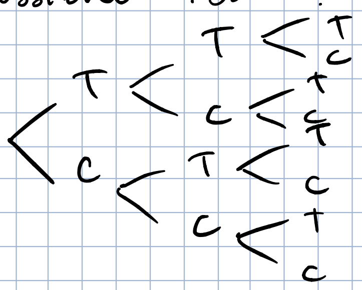
es)
15 squadre
5 posti $D_{n,k}$

$D_{15,5} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$

- **Disposizioni con ripetizione**

esempio: lancio della moneta

possibili terne:



* Argomenti non d'esame

[... altri esempi stupidi sul calcolo combinatorio ...]

[... mancano una marea di esercizi di esempio ...]

[...]

Successioni $\mathbb{N} \to \mathbb{R}$

$1 \to a_1$

$2 \to a_2$

$3 \to a_3$

$\dots$

$n \to a_n$

$\boxed{a_n = \frac{1}{n}}$ $1, \frac{1}{2}, \frac{1}{3}$ $\frac{1}{n}$ $\forall \varepsilon > 0$

$a_n = \frac{n-1}{n}$ $0, \frac{1}{2}, \frac{2}{3}$ $\frac{n+1}{n}$

$a_n = \frac{(-1)^n}{n}$ $(-1), \frac{1}{2}, -\frac{1}{3}, +\frac{1}{4}$

$a_n = (-1)^n$ $(-1),$

$\lim\limits_{n \to +\infty} a_n = a$ $\forall \varepsilon > 0$ $\exists \nu$ $a - \varepsilon < a_n < a + \varepsilon$

$|a_n - a| < \varepsilon$ $\forall n > \nu$

Verifico che $\lim\limits_{n \to +\infty} \frac{1}{n} = 0$

$\lim\limits_{n \to +\infty} \frac{n-1}{n} = 1$

$\left| \frac{n-1}{n} - 1 \right| < \varepsilon$ $\forall n > \nu$

$\left| \frac{\cancel{n} - 1 - \cancel{n}}{n} \right| < \varepsilon$

$\left| \frac{-1}{n} \right| < \varepsilon$ $\frac{1}{n} < \varepsilon$ $n > \frac{1}{\varepsilon}$

$\lim\limits_{n \to +\infty} \frac{n}{n+5} = \lim\limits_{n \to +\infty} \frac{\cancel{n}/\cancel{n}\ 1}{\frac{n}{n}\ 1 + \frac{5}{n}\ 0} = 1$

tutto fratto n

Teoria

Limiti di successione

Lemma: $\forall n \in \mathbb{N}$ $\forall \delta \in \mathbb{R}^{0+}$ si ha $(1+\delta)^n \geq 1 + n\delta$

$$\boxed{\lim_{n\to+\infty} \frac{a^n}{n^k} = +\infty}$$

Dim: induzione su n

$$\lim_{n\to+\infty} \frac{2^n}{n^{100}} = ?$$

$a>1$
$k\in\mathbb{R}^+$

o) Caso base $n=0$ $(1+\delta)^0 = 1 \geq 1+\delta\cdot 0 = 1$

Supponiamo che per un certo $n\geq 0$ si abbia
$(1+\delta)^n \geq 1+n\delta$

Vogliamo dim. che:

o) Passo induttivo:

? $(1+\delta)^{n+1} \geq 1+(n+1)\delta$?

$$(1+\delta)^{n+1} = (1+\delta)(1+\delta)^n \geq (1+\delta)(1+n\delta) =$$

$$= 1+n\delta+\delta+n\delta^2 = 1+(n+1)\delta+n\delta^2 \geq 0 \geq 1+(n+1)\delta$$

$n\delta^2\geq 0 \to \forall A\in\mathbb{R} \quad A+n\delta^2\geq A$

LEMMA: $\forall n\in\mathbb{N}$ $\forall\delta\in\mathbb{R}^{0+}$ si ha $(1+\delta)^n \geq 1+n\delta+\frac{n(n-1)}{2}\delta^2$

$\delta=1$ $\forall n\in\mathbb{N}$ $2^n\geq 1+n+\frac{n(n-1)}{2}$

$$\lim_{n\to+\infty} \frac{2^n}{n} \geq \lim_{n\to\infty} \frac{1+n+n(n-1)}{n} = +\infty$$

$$\begin{cases} x_0 = 0.1 \\ x_{n+1} = x_n(2-7x_n) \end{cases}$$

$x_0 = 0,1$
$x_1 = 0.13$
$x_2 = 0.1417$
$x_3 = 0.14284777$
$x_4 = 0.14285714 2242$

$$\begin{cases} x_0 = 0.1 \\ x_{n+1} = x_{n+1}(2-3x_n) \end{cases}$$

$x_0 = 0.1$
$x_1 = 0.17$
$x_2 = 0.17(2-0.51) = 0.2533$
$x_3 = 0.3141$

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_2 = f_{n+1}+f_n \end{cases}$$

$f_n \leq 2^n \ \forall n\in\mathbb{N}$

$f_n \geq \left(\frac{3}{2}\right)^n \forall n\in\mathbb{N}$

0 1 1 2 3 5
8 13 21 34 55 89

$\alpha^2 = \alpha + 1$

$f_n \approx \alpha^n$ $\alpha^{n+2} \approx \alpha^{n+1} + \alpha^n$

$f_{n+1} \approx \alpha^{n+1}$ $\alpha^2 \approx \alpha + 1$

$f_{n+2} \approx \alpha^{n+2}$

$\alpha = \dfrac{1 \pm \sqrt{5}}{2}$

$\alpha_1 = \dfrac{\sqrt{5}+1}{2} = 1{,}61\ldots$

Formula di Binet

$\forall n \in \mathbb{N} \quad f_n = \dfrac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}$

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_{n+2} = a_{n+1} - a_n \end{cases}$$

$a_0 = 0$ $a_6 = 0$

$a_1 = 1$ $a_7 = 1$

$a_2 = 1$ $a_8 = 1$

$a_3 = 0$

$a_4 = -1$

$a_5 = -1$

Limiti

$$\boxed{\forall a>1,\ \forall k\in\mathbb{R}^{o+} \quad \lim_{n\to\infty}\frac{a^n}{n^k}=+\infty}$$

$\forall \delta\in\mathbb{R}^{+}$

$$(1+\delta)^n \geq 1+n\delta+\frac{n(n-1)}{2}\delta^2$$

$$\delta=a-1>0$$

$$a^n=(1+\delta)^n\geq 1+n\delta+\frac{n(n+1)}{2}\delta^2=$$

$$=1+n(a-1)+\frac{n(n+1)}{2}(a-1)^2$$

$$\lim_{n\to\infty}\frac{a^n}{n}\geq\lim_{n\to\infty}\frac{1+n(a-1)+\frac{n(n-1)}{2}(a-1)^2}{n}$$

$$b=a^{1/k}>1$$

$$\lim_{n\to\infty}\frac{a^n}{n}=+\infty$$

$$\left(\lim_{n\to\infty}\frac{b^n}{n}\right)^k=+\infty \Longrightarrow \lim_{n\to\infty}\frac{b^{nk}}{n^k}=+\infty$$

$$\lim_{n\to\infty}\frac{(\sqrt{2})^2}{n}=+\infty$$

$$\lim_{n\to\infty}\frac{a^n}{n^k}=+\infty$$

N.B.: Si tratta fondamentalmente della velocità degli infiniti.

$$\lim_{n\to\infty}\frac{n!}{a^n}=+\infty$$

$$\frac{a\in\mathbb{R},\ a>1}{a=10}$$

$$\frac{20!}{10^{20}}=\left(\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\ldots\cdot10}{10\cdot10\cdot10\cdot10\cdot10\ldots}\right)\left(\frac{11\cdot12\cdot\ldots\cdot20}{10\cdot10\cdot10\cdot10\cdot10}\right)$$

Formula di Stirling

$$\log(n!)=n\log(n)-n+\frac{1}{2}\log(2\pi n)+\cdots\approx\frac{1}{12\pi}$$

$$\boxed{n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

$\forall k\in\mathbb{R}^{o+}$

$$\lim_{n\to\infty}\frac{(\log(n))^k}{n}=0$$

la funzione logaritmo è la più lenta di tutte a tendere ad infinito

$(a_n)_{n\in\mathbb{N}}$ SUCCESSIONE

$\lim_{n\to\infty} a_n = +\infty \quad (-\infty)$ $\qquad \lim_{n\to+\infty} a_n = L \in \mathbb{R}$

$a_n = (-1)^n$
$a_n = (-2)^n$

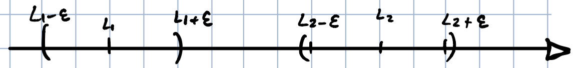
$\lim_{n\to+\infty} \frac{(-2)^n}{3^n} = \lim_{n\to\infty} \left(-\frac{2}{3}\right)^n = 0$

$h^n, \ h \in \mathbb{R}$

$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \qquad \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon) \text{ t.c. } -\varepsilon \leq \frac{(-1)^n}{n} \leq \varepsilon$

Teorema: Unicità del limite

Se $(a_n)_{n\in\mathbb{N}}$ ha il limite $L \in \mathbb{R}$ oppure $+\infty$ o $-\infty$, allora questo limite è unico.



Dim:

$\lim_{n\to+\infty} a_n = L_1 \quad \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon) \in \mathbb{N} \text{ t.c. } \forall n \geq \bar{n} \quad L_1 - \varepsilon \leq a_n \leq L_1 + \varepsilon$

$\lim_{n\to\infty} a_n = L_2 \quad \forall \varepsilon > 0 \ \exists \bar{m} = \bar{m}(\varepsilon) \in \mathbb{N} \text{ t.c. } \forall m > \bar{m} \quad L_2 - \varepsilon \leq a_n \leq L_2 + \varepsilon$

Supponiamo $L_1 < L_2$

Scelgo $\varepsilon = \frac{L_2 - L_1}{4} > 0 \qquad \forall n \geq \max(\bar{n}(\varepsilon), \bar{m}(\varepsilon))$ valgono le 4 disuguaglianze: in particolare:

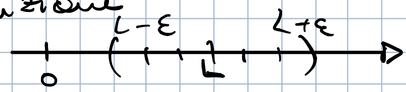
$$\begin{cases} a_n \leq L_1 + \varepsilon \\ a_n \geq L_2 + \varepsilon \end{cases} \Rightarrow \begin{array}{l} L_2 - \varepsilon \leq L_1 + \varepsilon \\ L_2 - L_1 \leq 2\varepsilon = 2 \cdot \frac{L_2 - L_1}{4} = \frac{L_2 - L_1}{2} \end{array}$$

ASSURDO!

TEO: (Permanenza del segno)

Se $(a_n)_{n\in\mathbb{N}}$ ha limite $L \in \mathbb{R}^+$ allora $\exists \bar{n} \in \mathbb{N}$ t.c. $a_n > 0 \quad \forall n \geq \bar{n}$

Dim: Prendere $\varepsilon = \frac{1}{2} L$ nella definizione



Nomenclatura: $(a_n)_{n\in\mathbb{N}}$ successione

$(a_n)_{n\in\mathbb{N}}$ è monotòna (deb) crescente se $\forall n \in \mathbb{N}$ $a_n \leq a_{n+1}$

$(a_n)_{n\in\mathbb{N}}$ è monotòna (deb) decrescente se $\forall n \in \mathbb{N}$ $a_n \geq a_{n+1}$

$(a_n)_{n\in\mathbb{N}}$ è infinitesima se $\lim_{n\to\infty} a_n = 0$

$(a_n)_{n\in\mathbb{N}}$ è convergente se $\lim_{n\to\infty} a_n = L \in \mathbb{R}$

$(a_n)_{n\in\mathbb{N}}$ è divergente se $\lim_{n\to\infty} a_n = +\infty$ o $-\infty$

$(a_n)_{n\in\mathbb{N}}$ è oscillante se non ha limite

$$\lim_{n\to\infty} a_n = L \in \mathbb{R} \longleftrightarrow \lim_{n\to\infty} (a_n - L) = 0$$

$\alpha_n = a_n - L$ è infinitesima

TEO: Sia $(a_n)_{n\in\mathbb{N}}$ una successione monotòna crescente e superiormente limitata: allora $\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}$

$$a_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

$$a_0 \leq a_1 \leq a_2 \leq \dots$$

Dim:

Sia $L = \sup\{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}$

- $\forall n \in \mathbb{N}$ $a_n \leq L$
- $\forall \varepsilon > 0$ $\exists \bar{n} = \bar{n}(\varepsilon)$ t.c. $a_{\bar{n}} \geq L - \varepsilon$

Voglio dim. che: $\forall \varepsilon > 0$ $\exists \bar{m} = \bar{m}(\varepsilon) \in \mathbb{N}$ t.c. $\forall n > \bar{m}$ si ha $L - \varepsilon \leq a_n \leq L + \varepsilon$.

Dunque basta prendere $\bar{m}(\varepsilon) = \bar{n}(\varepsilon)$ perché

$$a_{\bar{n}} \leq a_{\bar{n}+1} \leq a_{\bar{n}+2} \leq \dots \leq L$$

Operazioni con i limiti:

TEO: se $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} b_n = L_2$ e $L_1, L_2 \in \mathbb{R}$

allora
- $\lim_{n\to\infty} (a_n + b_n) = L_1 + L_2$
- $\lim_{n\to\infty} (a_n b_n) = L_1 \cdot L_2$

Se $L_2 \neq 0$ allora $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$

Lemma: se (a_n) (b_n) sono infinitesime, allora $(a_n + b_n)$ è infinitesima

Dim: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

$\to \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon) \in \mathbb{N}$ t.c. $-\frac{\varepsilon}{2} \leq a_n \leq \frac{\varepsilon}{2} \quad \forall n \in \bar{n}$

$\to \forall \varepsilon > 0 \ \exists \bar{m} = \bar{m}(\varepsilon) \in \mathbb{N}$ t.c. $-\frac{\varepsilon}{2} \leq b_n \leq \frac{\varepsilon}{2} \quad \forall m \in \bar{m}$

Dunque $\forall n \geq \max(\bar{n}(\varepsilon), \bar{m}(\varepsilon))$ abbiamo $-\varepsilon \leq a_n + b_n \leq \varepsilon$

$\lim_{n\to\infty} a_n = L_1 \qquad \alpha_n = a_n - L_1$ è infinitesima

$\lim_{n\to\infty} b_n = L_2 \qquad \beta_n = b_n - L_2$ " "

$\Downarrow$

$\alpha_n + \beta_n = (a_n + b_n - L_1 - L_2) =$ è infinitesima

$\lim_{n\to\infty} (a_n + b_n) = L_1 + L_2$

$a_n b_n = (L_1 + \alpha_n)(L_2 + \beta_n)$

$L_1 L_2 + \underset{\downarrow 0}{L_1 \beta_n} + \underset{\downarrow 0}{L_2 \alpha_n} + \underset{\downarrow 0}{\alpha_n \beta_n}$

$$\begin{cases} b_0 = 0 \\ b_1 = 1 \\ b_{n+2} = b_{n+1} - b_n \end{cases}$$

$f_n = a^n$?

$\alpha_2 = \frac{1 \pm \sqrt{5}}{2}$

$\boxed{\alpha^2 = \alpha + 1}$

$f_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}$

$b_n \approx \beta^n$

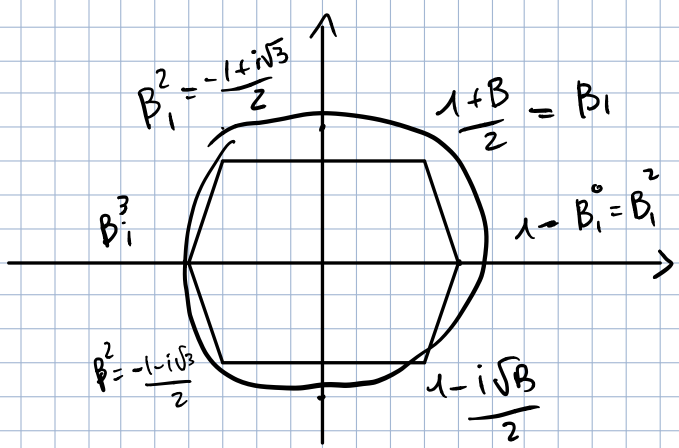
$\beta^{n+2} \approx \beta^{n+1} - \beta^n$

$\beta^2 \approx \beta - 1$

$\beta = \frac{1 \pm \sqrt{3}}{2}$

$\beta^2 = \beta - 1 \longrightarrow \beta^6 = 1$

ok(?)



Algoritmo di Erone-Newton

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \frac{a_n^2 + 2}{2a_n} \end{cases}$$

$a_0 = 1$

$a_1 = \frac{3}{2} = 1.5$

$a_2 = \frac{17}{12} = 1.416$

$$a_3 = \frac{\left(\frac{17}{12}\right)^2 + 2}{2 \cdot \frac{17}{12}} = \frac{\frac{289}{144} + 2}{\frac{17}{6}} = \frac{289 + 288}{17 \cdot 24} = \frac{577}{408}$$

$$\frac{\left(\frac{3}{2}\right)^2 + 2}{2 \cdot \frac{3}{2}} = \frac{\frac{9}{4} + 2}{3} = \frac{9+8}{12}$$

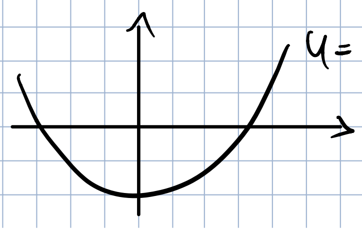
$a_4 = 1.414429$

$a_n > \sqrt{2}$

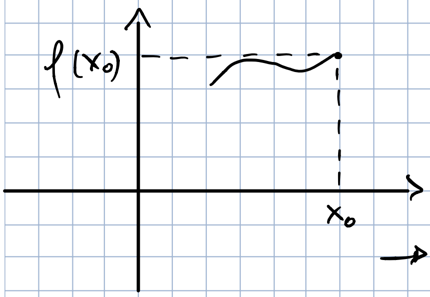
$\frac{2}{a_n} < \sqrt{2}$

$$\frac{a_n + \frac{2}{a_n}}{2} = \frac{a_n^2 + 2}{2a_n}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$



Funzioni Continue



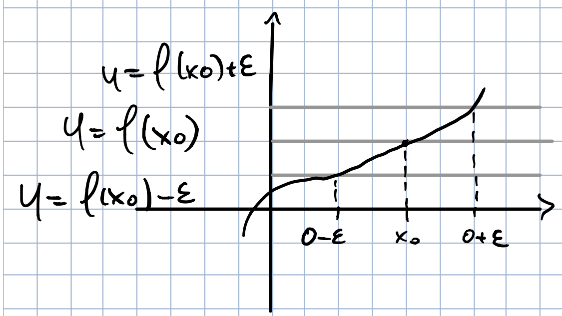
Sia $f(x)$ $f: X \to \mathbb{R}$ $(X \subseteq \mathbb{R})$

Si dice che f è continua in $x_0 \in X$

se $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon, x_0) /$ $\to$

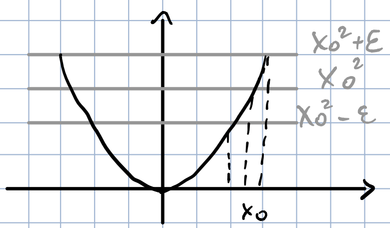
$\to$ $\forall x \in (x_0 - \delta, x_0 + \delta) \cap X$ | $\to$

$\to$ $|f(x) - f(x_0)| \leq \varepsilon$



Esercizio:

Dato $x_0 \in \mathbb{R}^+$, dimostrare che $f(x) = x^2$ è continua in x_0



Per quali x si ha $f(x) \in (x_0^2 - \varepsilon, x_0^2 + \varepsilon)$?

$f(x_1) = f(x_0) + \varepsilon$ | $f(x_2) = f(x_0) - \varepsilon$

$\begin{cases} x_1 > 0 \\ x_1^2 = x_0^2 + \varepsilon \end{cases}$ | $\begin{cases} x_2 > 0 \\ x_2^2 = x_0^2 - \varepsilon \end{cases}$

$x_1 = \sqrt{x_0^2 + \varepsilon}$ | $x_2 = \sqrt{x_0 - \varepsilon}$

$0 < \varepsilon < \ldots$

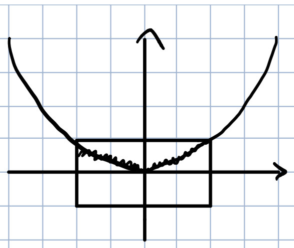
Abbiamo dimostrato che

se $x \in (x_2, x_1) = (\sqrt{x_0^2 - \varepsilon}, \sqrt{x_0^2 + \varepsilon})$ allora $x^2 \in (x_0^2 - \varepsilon, x_0^2 + \varepsilon)$

Prendo $\delta = \min(x_0 - \sqrt{x_0^2 - \varepsilon}, \sqrt{x_0^2 + \varepsilon} + x_0) > 0$

$$\sqrt{x_0^2 + \varepsilon} - x_0 = (\sqrt{x_0^2 + \varepsilon} - x_0) \frac{\sqrt{x_0^2 + \varepsilon} + x_0}{\sqrt{x_0^2 + \varepsilon} + x_0} = \frac{\cancel{x_0^2} + \varepsilon - \cancel{x_0^2}}{\sqrt{x_0^2 + \varepsilon} + x_0} \cdot \frac{\varepsilon}{\sqrt{x_0^2 + \varepsilon} + x_0}$$

$\approx \frac{\varepsilon}{2x_0}$



Def: $f: X \to \mathbb{R}$ $\quad \emptyset \neq X \subseteq \mathbb{R}$

Diciamo che f è continua se è continua in ogni punto del suo dominio

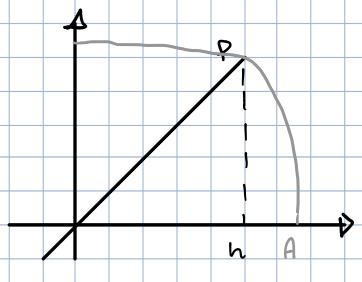
esempi:

$\begin{cases} f: \mathbb{R}^* \to \mathbb{R} \\ f(x) = \frac{1}{x} \end{cases}$ è continua $\qquad \begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x^n \end{cases}$ è continua

$\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = \operatorname{sen}(x) \end{cases}$ è continua

$|\operatorname{sen}(x_0+h) - \operatorname{sen}(x_0)| = 2|\cos(x_0+\frac{h}{2})\operatorname{sen}(\frac{h}{2})|$

$\leq 2|\operatorname{sen}(\frac{h}{2})| \leq 2 \cdot \frac{|h|}{2} = |h|$



AREA$(O\hat{A}P)$ $\frac{1}{2}\operatorname{sen}(\alpha)$

AREA$(O\hat{A}P) = ?$

α	AREA
2π	π
π	$\pi/2$
$\pi/2$	$\pi/4$

Se $\alpha \in (0, \pi/2)$ allora $0 < \operatorname{sen}(\alpha) < \alpha$

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x_0) \in \mathbb{R}^+ / |\operatorname{sen}(x) - \operatorname{sen}(x_0)| \leq \varepsilon$?

$\forall x \in (x_0 - \delta, x_0 + \varepsilon)$

$|\operatorname{sen}(x) - \operatorname{sen}(x_0)| =$

Dato ε, basta prendere $\delta = \varepsilon$ $\qquad |\operatorname{sen}(x_0+h) - \operatorname{sen}(x_0)| \leq |h| \leq \delta = \varepsilon$

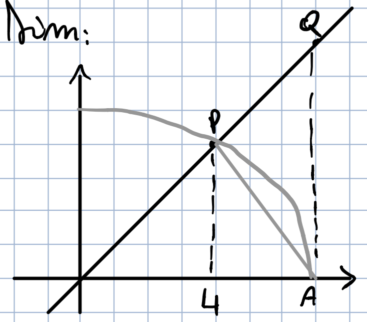
$x \in (x_0 - \delta, x_0 + \delta)$

$x = x_0 + h$ dove $h = x - x_0 \in (-\delta ; \delta)$

Teorema

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$ limite notevole

Dim:



$AREA(O\hat{A}P) = \frac{1}{2} \operatorname{sen}(x)$

$AREA(O\hat{A}P) = \frac{1}{2} x$

$AREA(O\hat{A}Q) = \frac{1}{2} tg(x)$

$\frac{1}{2} \operatorname{sen}(x) \leq x \leq \tan(x) \rightarrow \operatorname{sen}(x) \leq x \leq \frac{\operatorname{sen}(x)}{\cos(x)}$

$1 \leq \frac{x}{\operatorname{sen} x} \leq \frac{1}{\cos x} \rightarrow \underset{\downarrow \, 0}{\cos(x)} \leq \underset{\downarrow \, 1}{\frac{\operatorname{sen}(x)}{x}} \leq \underset{\downarrow \, 1}{1}$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \left(\frac{1 - \cos(x)}{x} \cdot \frac{1 + \cos(x)}{1 + \cos(x)} \right) =$$

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \left(\frac{\operatorname{sen}(x)}{x} \cdot \operatorname{sen} x \cdot \frac{1}{1 + \cos x} \right)$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x^2(1 + \cos x)} = \lim_{x \to 0} \left(\frac{\operatorname{sen}(x)}{x} \right)^2 \cdot \frac{1}{1 + \cos x} = \frac{1}{2}$$

$\frac{\operatorname{sen} x}{2} \simeq 1$ (x piccolo)

$\operatorname{sen} x = x$

$1 - \cos(x) \simeq \frac{1}{2}$ (x piccolo)

$1 - \cos(x) \simeq \frac{1}{2} x^2$

$\cos(x) \simeq 1 - \frac{1}{2} x^2$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right) = \lim_{x\to 0}\frac{\cancel{1}+x-\cancel{1}}{x(\sqrt{1+x}+1)} =$$

$$\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\left(1+\frac{1}{2}x\right)}{x^2} = -\frac{3}{8}$$

Esercitazione

Stessa identica cosa che ho fatto a casa

Teoria

$$\lim_{n\to\infty} \frac{p(n)}{q(n)}$$

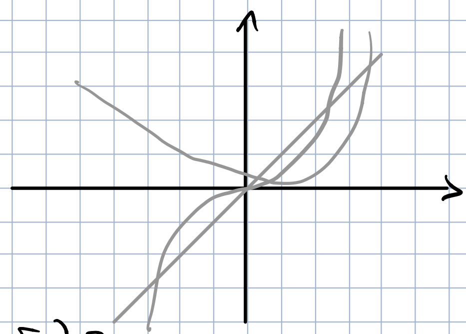
p, q polinomi

p di grado k

q di grado j

$$\lim_{n\to+\infty} \frac{p(x)}{q(x)}$$

$$\lim_{n\to-\infty} \frac{p(x)}{q(x)}$$



$$\lim_{n\to\infty} (\sqrt{n+1}-\sqrt{n})\sqrt{n}$$

$$\lim_{x\to\infty} (\sqrt{2x+1}-\sqrt{x})\sqrt{x}$$

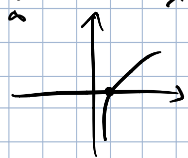
$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\boxed{e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^2}$$

$$e = \lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x$$

$$1 = \log\left(\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x\right)$$

$$1 = \lim_{x\to\infty} \log\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \underset{\infty}{x} \log\left(1+\frac{1}{x}\right)$$



$$y = \frac{1}{x} \to 0+$$

$$1 = \lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right) = \lim_{y\to 0+} \frac{1}{y}\log(1+y)$$

$$x = \frac{1}{y} \qquad = \lim_{y\to 0^+} \frac{\log(1+y)}{y} = \lim_{y\to 0-} \frac{\log(1+y)}{y}$$

$$\lim_{x\to 0\pm} \frac{\log(1+y)}{y} = 1 \qquad t = \log(1+y)$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \qquad e^{t} = 1+y$$

$$y = e^{t} - 1$$

$$\lim_{x \to 0} \frac{\log(1+3x)}{x} = \lim_{y \to 0} \frac{\log(1+y)}{y/3} = \lim_{y \to 0} \frac{3\log(1+y)}{y} = 3$$

$y = 3x$ $\qquad = \lim_{x \to 0} \log(1+3x)$

Calcolo limiti più complessi

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to+\infty} \exp(\log(n^{2/n})) =$$

$$= \log \lim_{n\to\infty} (\log(n^{1/n}) = \exp\left(\lim_{n\to\infty} 1/n(\log n)\right)$$

$$y = \log n \quad \exp \lim_{y\to\infty} \frac{y}{e^y} = 0$$

$y \to \infty$, $n \to \infty$, $n = e^y$

$$\exp(0) = 1$$

- esponenziale > polinomiale

ricorda il lemma: $(1+\delta)^n \geq 1 + n\delta$

prendo: $\delta = \frac{\sqrt{n}-1}{n} \leq \frac{1}{\sqrt{n}}$

Forme Indeterminate

- del tipo $\frac{0}{0}$ $\quad \lim_{x\to\infty} \frac{f(x)}{g(x)}$ dove $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$

- del tipo $\frac{\infty}{\infty}$ $\quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \pm\infty$

- del tipo $0 \cdot \infty$ $\quad \lim_{x\to\infty} f(x)g(x)$ $\quad$ • $\lim g(x) = 0$

$$\lim_{x\to\infty} \frac{f(x)}{1/g(x)} \to \left[\frac{0}{0}\right] \qquad \lim_{x\to\infty} g(x) = \pm\infty$$

$$\lim_{x\to\infty} \frac{g(x)}{1/f(x)} \to \left[\frac{\infty}{\infty}\right]$$

• del tipo 1^{∞} $\qquad \lim_{x\to\infty} f(x)^{g(x)}$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = \pm\infty$$

• del tipo $\infty^{0^{\pm}}$ $\qquad \lim_{x\to+\infty}$ "

$$\lim_{x\to+\infty} f(x) = \infty \quad \lim_{x\to\infty} g(x) = 0$$

• $\lim_{x\to\infty} f(x)^{g(x)} = \lim_{x\to\infty} \exp\log\left(f(x)^{g(x)}\right) =$

con $f(x)>0$

$$= \exp\left(\lim_{x\to\infty} g(x)\log(f(x))\right)$$

forme ora esaminate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{n\to\infty} n^{1/n} = 1$$

Calcolo: $\lim_{x\to+\infty}\left(1+\frac{1}{3x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{1}{3x}\right)^{3x}\right]^{\frac{1}{3}} = \sqrt[3]{e}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " $\quad e^{1/3}$ "

Uso $y=3x$

per $x\to+\infty$ $\qquad \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y/3} =$ " $\left[\left(1+\frac{1}{y}\right)^y\right]^{\frac{1}{3}} = \sqrt[3]{e}$

$y\to+\infty$ $\qquad\qquad\qquad\qquad$ $y=x$

es: • $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2} =$ " $\left[\left(1+\frac{1}{x}\right)^x\right]^x = e^x = +\infty$

• $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{1}{x^2}\right)^{x^2}\right]^{1/x} = e^{1/x} = e^0 = 1$

$$\bullet \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^2 = \exp\left(\lim_{x\to\infty} x \log\left(1+\frac{1}{x^2}\right)\right)$$

lim notevole	$\lim_{y\to 0} \frac{\log(1+y)}{y}$

con $x=\sqrt{1/y}$, $y=1/x^2$, $x\to\infty$, $y\to 0$

come?

$$= \exp\left(\lim_{y\to 0} \sqrt{1/y}\cdot\log(1+y)\right)$$

$$= \exp\left(\lim_{y\to 0} \frac{\log(1+y)}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}}\right) = \exp(1\cdot\sqrt{0}) = 1$$

$$\bullet \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2} = \exp\left(\lim_{x\to\infty} x^2\log\left(x+\frac{1}{x}\right)\right)$$

$\boxed{y=1/x \quad x\to\infty \quad y\to 0 \quad x=1/y}$

$$= \exp\left(\lim_{y\to 0} \frac{\log(1+y)}{y^2}\cdot\frac{y^2}{y^2}\right)$$

$$= \exp\left(\frac{1}{0}\cdot\frac{1}{0}\right) = +\infty$$

es: $\lim_{x\to\infty} \frac{1}{x^n}\cdot x^m = \lim_{x\to\infty}\frac{x^m}{x^n} = \text{"} x^{m-n} \text{"} = \begin{cases}\infty & m>n\\ 1 & m=n\\ 0 & m<0\end{cases}$

$m,n\in\mathbb{N}^*$ — dipende da $\longrightarrow$

① Grafico $f: \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \dfrac{e^{2x}+8}{e^x+1}$

① Dom f $\quad e^x+1 \neq 0 \quad e^x \neq -1 \quad$ Dom f: $(-\infty, +\infty)$

$\mathbb{R}\ \forall x \in \mathbb{R} / \{\underline{+}3\}$

② P/D $\quad f(-x) = \dfrac{e^{-2x}+8}{e^{-x}+1}$ $\quad$ la f non è né P né D

non è $-f(x)$

non è $f(x)$

$f(-x) \begin{cases} -f(x)\ D \\ f(x)\ P \end{cases}$

③ Intersez. ossi

y $\quad f(0) = \dfrac{e^{2\cdot 0}+8}{e^0+1} = \dfrac{9}{2}$ $\quad$ (asse y) $\quad \left(\frac{9}{2}\right)$

int. $\left(0, \frac{9}{2}\right)$

x $\quad f(x)=0 \quad \dfrac{e^{2x}+8}{e^x+1} = 0$ $\quad$ (asse x) $\quad 0$

$e^x+1=0 \quad e^x=-1$ mai vero

essendo il 1° membro sempre >0

allora la eq non ha sol

④ Segno $\quad f(x)>0$

$\dfrac{e^{2x}+8}{e^x+1} > 0$ $\quad$ l'eq. è $>0\ \forall x \in \mathbb{R}$

la f è sempre positiva

⑤ limiti

$$\lim_{x \to \infty} \frac{e^{2x}+8}{e^x+1} = \lim_{x \to \infty} \frac{e^{\infty}+8}{e^{\infty}+1} = \frac{\infty}{\infty}$$

$$\lim_{x\to+\infty} \frac{\cancel{e^{2x}}\left(1+\frac{8}{e^{2x}}\right)}{\cancel{e^{x}}\left(1+\frac{1}{e^{x}}\right)} = e^x = +\infty$$

$$\lim_{x\to-\infty} \frac{e^{2x}+8}{e^x+1} = \lim_{x\to-\infty} \frac{0+8}{0+1} = 8$$ asintoto orizzontale a 8

6 $$f = \frac{e^{2x}+8}{e^x+1} \qquad f' = \left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2} =$$

$$\frac{2e^{2x}\cdot(e^x+1)-(e^{2x}+8)\cdot e^x}{(e^x+1)^2} = \frac{e^{2x}\cdot 2(e^x+1)-(e^{2x}+8)\cdot e^x}{(e^x+1)^2} =$$

$$\frac{e^{2x}\cdot(2e^x+2)-e^{3x}+8e^x}{(e^x+1)^2} = \frac{2e^{3x}+2e^{2x}-e^{3x}-8e^x}{(e^x+1)^2}$$

$$\frac{e^{3x}+2e^{2x}-8e^x}{(e^x+1)^2} =$$

$$\frac{e^{3x}+2e^{2x}-8e^x}{(e^x+1)^2} \geq 0 \qquad (e^x+1)^2 \geq 0 \quad \forall x \in R$$

$$\frac{e^x(e^{2x}+2e^x-8)}{(e^x+1)^2} \geq 0 \qquad \frac{e^x(e^{2x}+(4e^x-2e^x)-8)}{\text{"}} \geq 0$$

$$\frac{e^x(e^x(e^x+4)-2e^x-8)}{(e^x+1)^2} \geq 0$$

$$\frac{e^x(e^x(e^x+4)-2(e^x-4))}{//} \geq 0$$

$$\frac{e^x\,(e^x+4)(e^x-2)}{(e^x+1)^2} \geq 0$$

$$\frac{a}{b} \geq 0 \quad \begin{cases} a \geq 0 & x \in \mathbb{R} \\ b > 0 & (\ln 2, +\infty) \end{cases} \qquad \begin{cases} a \leq 0 & x \in \emptyset \\ b < 0 & x \in \emptyset \end{cases}$$

$$\begin{cases} e^x(e^x+4)(e^x-2) \geq 0 \\ (e^x+1)^2 > 0 \quad \forall x \in \mathbb{R} \end{cases}$$

decresce da $-\infty, \ln(2)$

min $\ln(2)$

$$\begin{cases} e^x \geq 0 \\ (e^x+4)(e^x-2) > 0 \end{cases} \qquad \begin{cases} x \in \mathbb{R} \\ x > \ln(2) \end{cases}$$

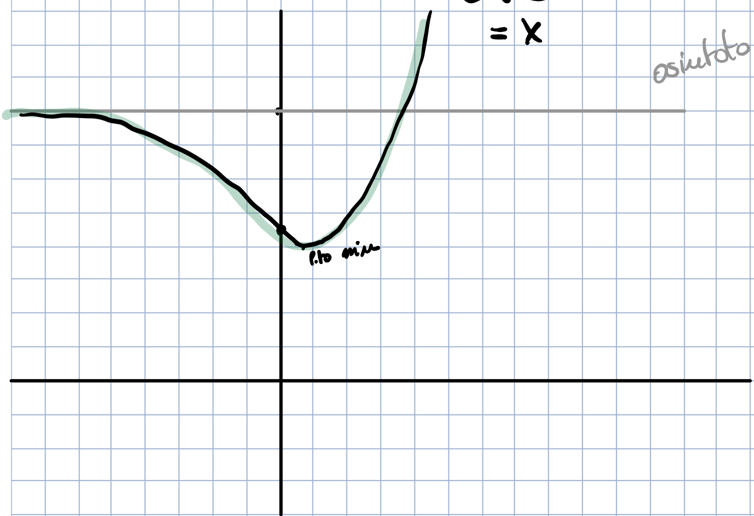
$0 \quad \ln 2$

$]\ln 2, +\infty)$

$e^x \geq -4$ l'exp è sempre >0, $x \in \emptyset$

$e^x \geq 2$ $\quad \underbrace{\ln e^x}_{=x} > \ln 2 \rightarrow x > \ln 2$

asintoto $x=8$



intersez $(0, \frac{9}{2})$

Per passare da exp a log → def di log → se $a, b \in \mathbb{R}^+$ con $a \neq 1$, il log in base a di b è quel numero c per cui a elevato alla c è $=$ a b

$\log_a(b) = c \longleftrightarrow a^c = b$, con $a, b > 0$, $a \neq 1$

Si ha che:

$e^{\ln(x)} = x$

$\ln(e^x) = x$ *

P.to minimo

$$\ln 2 \to \frac{e^{2\ln 2}+8}{e^{\ln 2}+1} = \frac{4+8}{2+1} = \frac{\cancel{12}}{\cancel{3}} = 4$$

$(\ln(2), 4)$
P.to min

$$\lim_{x\to 0} \frac{\operatorname{sen}(4x)(1-\cos(2x))}{3x - \operatorname{sen}(3x)}$$

$$\lim_{x\to 0} \frac{\overset{0}{\operatorname{sen}(0)}(1-\overset{1}{\cos(0)})}{0-\underset{0}{\operatorname{sen}(0)}} = \frac{1}{0} = 1 \text{ NO!}$$

$\lim_{x\to 0} \operatorname{sen}(4x)($ WTF bro?

27 Nov 2014 Parziale

① Grafico della funzione $f(x) = \frac{x(x^2+4)}{x^2-36}$
No Derivate, si immagina.

1) Dom: $x^2-36 \neq 0$ $x^2 \neq 36$ $x \neq \pm\sqrt{36}$ $x \neq \pm 6$

Dom $f = x \in \mathbb{R} / \{\pm 6\}$

$(-\infty, -6)(-6, +6)(+6, +\infty)$

2) P/D

$$f(-x) = \frac{-x(-x^2+4)}{-x^2-36} = \text{dispari}$$

3) Intersez assi

$f(0) \stackrel{(4)}{=} \frac{0(0^2+4)}{0-36} = \frac{0+0}{36} = \frac{0}{36} = 0$

$f(x) \stackrel{(x)}{=} 0 \to \frac{x(x^2+4)}{x^2-36} = 0 \qquad \frac{\cancel{x^3}+4x}{\cancel{x^2}-36} = \frac{5x}{-36} = 0$ falso

$x \notin R \longrightarrow 0$

Intersez assi $(0,0)$

$x^2 > 4 \quad x > \pm 4$

$x > 0$

④ Segno funzione

$\frac{x(x^2-4)}{x^2-36} > 0 \qquad \frac{x^3-4x}{x^2-36} > 0$

$x(x^2-4) > 0$

$\begin{cases} x^3-4x > 0 \ \forall x \in R \\ x^2-36 > 0 \end{cases}$

$\downarrow$

$|x| > 6$

$x < -6 \cup x > 6$

$\begin{cases} x > 0 \\ x^2 > +4 \end{cases} \quad \begin{cases} x > 0 \\ x \in R \end{cases} \to (0, +\infty)$

$-6 \quad 6$

$+ \quad - \quad +$

$x < -6 \quad x > 6$

$(-\infty, -6) \cup (6, +\infty)$

La f è pos da $-\infty$ a -6

neg da -6 a 6

pos da 6 a $+\infty$

⑤ limiti

$\lim_{x \to +\infty} \frac{x(x^2-4)}{x^2-36} > 0 \qquad \frac{\cancel{x^3}-4x}{\cancel{x^2}-36} = +\infty$

no asint. orizzontali

$\lim_{x \to -\infty} \frac{x(x^2-4)}{x^2-36} > 0 \qquad \frac{\cancel{x^3}-4x}{\cancel{x^2}-36} = -\infty$

$\lim_{x \to 6^-} \frac{x^3-4x}{x^2-36} = \lim_{x \to 6^-} x^2(x-4) = \overset{36}{6^2}(6-4) = 216-144 = 72$

$\lim_{x \to 6^-} \frac{1}{x^2-36} = \frac{1}{6^2-36} = \frac{1}{0}$ e $x^2-36 < 0$

$\to 0^-$

$\forall x \in R\ 6^-$

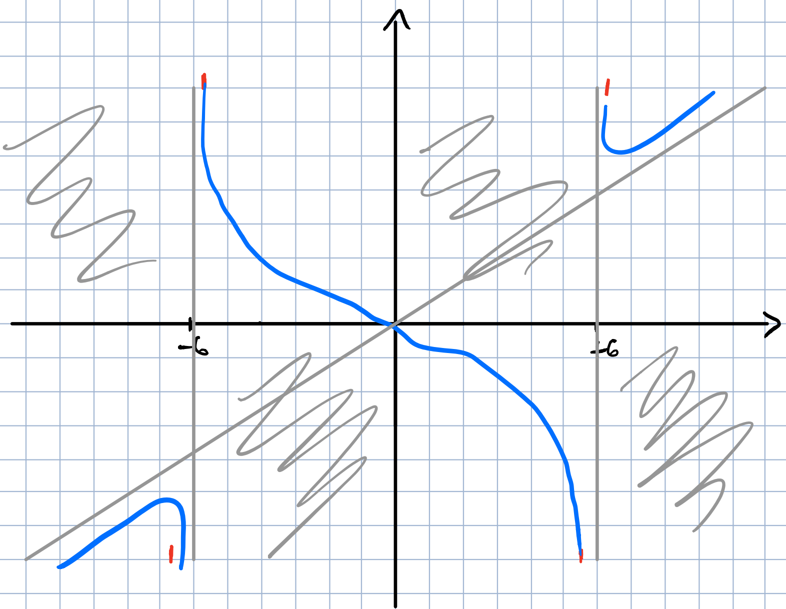
$= -\infty$

$= -\infty$

$\lim_{x \to 6^+} = \frac{72}{0^+} = +\infty$ asintoti verticali a 6^+ e 6^-

$\lim_{x \to -6^+} \frac{x^2(x-4)}{x^2-36} = +\infty$ perché dispari

-6^- $= -\infty$ asintoti a $-6^\pm$



pos in $-6<x<0$
pos in $x>6$
neg in $(-\infty,-6)$
neg in $(0,6)$

Positività della f. da controllare

Facoltativo → immagine di f

Analizzando il grafico l'immagine della funzione DOVREBBE essere $im(f) = (-\infty, +\infty)$ quindi $\mathbb{R}$

$im(f) = \mathbb{R}$

Controllo segno della f / Segno

$f(x) = \frac{x(x^2+4)}{x^2-36}$ $\qquad \frac{x(x^2+4)}{x^2-36} > 0$

$$\begin{cases} x(x^2+4) > 0 \\ x^2-36 > 0 \end{cases}$$

$x(x^2+4) > 0$: $x>0$; $x^2>-4 \ \forall x\in\mathbb{R}$

$\downarrow$

$0 \ (0,+\infty)$

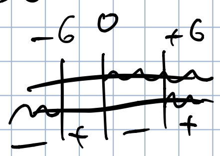
$x>0$

$x^2-36>0$

$\downarrow$

$x > \pm 6$

$x<-6 \cup x>6$



$-6<x<0$

$x>6$

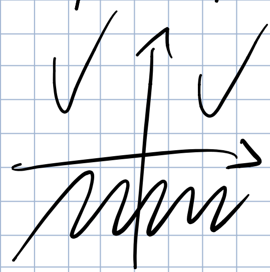
pos in $-6<x<0$

pos in $x>6$

neg in $(-\infty,-6)$

neg in $(0,6)$

La radice del grafico è la parte pos



26 Novembre 2015 – Parziale

SDF $\rightarrow$ $f(x)=\dfrac{(x-4)^2}{x^2-36}$

1) Dom $f \rightarrow$ $x\neq\pm6$ $\quad(-\infty,-6)(-6,6)(6,+\infty)$

② P/D $f(-x)=\frac{(-x-4)^2}{-x^2-36}$ f. Dispari

$\qquad\qquad \parallel$

$\qquad\quad -f(x)$

3) Assi

$y = f(0)=\frac{(0-4)^2}{0^2-36}=-\frac{\cancel{16}^{\,4}}{\cancel{36}_{\,9}} \qquad -\frac{4}{9}$

$x = f(x)=0 \qquad \frac{(x-4)^2}{x^2-36}=0 \qquad \emptyset$

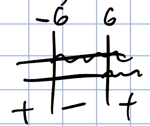
Interseca in $(0,-\frac{4}{9})$

④ Segno

$\frac{(x-4)^2}{x^2-36}\geq 0$

$(x-4)^2>0 \quad \forall x\in R\setminus\{x=4\}$

$x^2-36>0 \qquad x>\pm 6$



$x<-6 \cup x>6$

Pos in $(-\infty,-6)$

Pos in $(6,+\infty)$ neg $(-6,6)$

Radice $(4,0)$

⑤ Limiti

$$\lim_{x\to+\infty}\frac{(x-4)^2}{x^2-36}=\lim_{x\to\infty}\frac{x^2-8x+16}{x^2-36}=\frac{\cancel{x^2}(1-\frac{8}{x}+\frac{16}{x^2})}{\cancel{x^2}(1-\frac{36}{x})}=1$$

$$\lim_{x\to-\infty}\frac{(x-4)^2}{x^2-36}=1$$ perché dispari

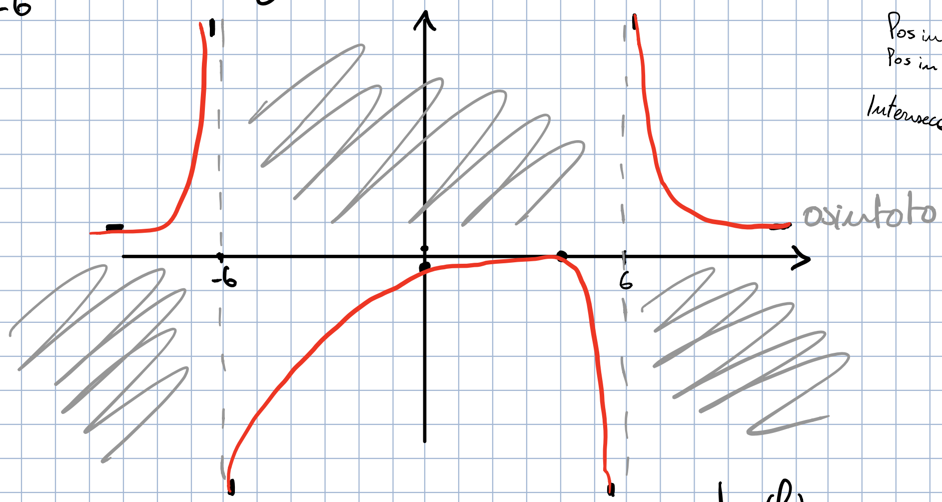
Asintoto orizzontale a 1 per $\pm\infty$

$$\lim_{x\to+6^+}\frac{(x-4)^2}{x^2-36}=\frac{4}{0^+}=+\infty$$

$$\lim_{x\to+6^-}=\frac{4}{0^-}=-\infty$$

$$\lim_{x \to -6^-} = \frac{4}{0^-} = -\infty$$

$$\lim_{x \to -6^+} = \frac{4}{0^+} = +\infty$$



Pos in $(-\infty, -6)$
Pos in $(6, +\infty)$ neg $(-6, 6)$
Interseca in $(0, \frac{4}{9})$

$Im(f) = (-\infty, 0)\ (1, +\infty)$

Limite

$$\lim_{x \to +\infty} \left(\frac{4x^2+6}{4x^2+5}\right)^{x^2} = \frac{(4x^2+6)^{x^2}}{(4x^2+5)^{x^2}} = \frac{\cancel{4x^2}(1+\frac{6}{4x^2})^{x^2}}{\cancel{4x^2}(1+\frac{5}{4x^2})^{x^2}} = \frac{(1)^{\cancel{x^2}}}{(1)^{\cancel{x^2}}} = 1$$

$\sqrt{f(x)}$
- $f(x) < 0 \to$ non esiste
- $f(x) = 1 \to \sqrt{f(x)} = 1$
- $f(x) > 1$?! lo scopriremo

$y = mx + q$

Asob per 1° $= m = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3+4x}{x^3-36x} = \frac{\infty}{\infty}$

$$\frac{\cancel{x^3}(1+\frac{4}{x^2})}{\cancel{x^3}(1-\frac{36}{x^2})} = 1$$

$$q = \lim_{x\to\infty} [f(x) - mx] = \lim_{x\to\infty} \left[\frac{x^3+4x}{x^2-36} - x\right] =$$

$$\lim_{x\to\infty} \frac{\cancel{x^3}+4x-\cancel{x^3}+36x}{x^2-36} = \lim_{x\to\infty} \frac{40\cancel{x}}{\cancel{x^2}\left(1-\frac{36}{x^2}\right)} = \frac{40}{\infty} = 0$$

$$y = \frac{x^2-5x+6}{x-4}$$

$$\lim_{n\to\infty} \frac{2^n+4n^2}{6^n+5} = \frac{\cancel{2^n}\left(1+\frac{4n^2}{2^n}\right)}{\cancel{2^n}\left(3^n+\frac{5}{6^n}\right)} = \frac{1}{3}$$

$$\left(\frac{(4n+5)+1}{4n+5}\right) \to \left(\frac{4n+5}{4n+5}+\frac{1}{4n+5}\right) \to \frac{n}{4n+5} = \frac{x}{n\left(4+\frac{5}{n}\right)}$$

$$\left[\left(1+\frac{1}{4n+5}\right)^{4n+5}\right]^{\frac{n}{4n+5}} = e^{\frac{\infty}{\infty}} = e^{\frac{1}{4}} = \sqrt[4]{e}$$

$$\lim_{n\to\infty} a^n \begin{cases} +\infty & \text{se } a>1 \\ 1 & \text{se } a=1 \\ 0 & \text{se } -1<a<1 \\ \nexists & \text{non esiste} \end{cases}$$

Es sui limiti

$$\lim_{n\to\infty} \frac{2^n+4^n}{3^n} = \underbrace{\lim_{n\to\infty} \left(\frac{2}{3}\right)^n}_{=0} + \underbrace{\lim_{n\to\infty} \left(\frac{4}{3}\right)^n}_{=\infty} = +\infty$$

$$\lim_{n\to\infty} \frac{2^{n+1}-4^{n-1}}{3^n} \to -\infty$$

↓ dividi N e D per 3^n

Ris forma $[+\infty - \infty]$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n-1} = [+\infty - \infty]$$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n-1} \cdot \frac{\sqrt{n+1} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}} = \frac{n+1-(n-1)}{\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{+\infty}$$

$$\lim_{n\to+\infty} \left[\sqrt{2n^2+n+1} - n+1\right] = [+\infty - \infty] =$$

$$\lim_{n\to\infty} \left[\sqrt{2n^2+n+1} - (n-1)\right] \cdot \frac{\sqrt{2n^2+n+1} + (n-1)}{\sqrt{2n^2+n+1} + (n-1)} =$$

$$= \lim_{n\to\infty} \frac{2n^2+n+\cancel{1}-n^2+2n-\cancel{1}}{\sqrt{2n^2+n+1}+(n-1)} = \frac{n^2+3n}{\sqrt{n^2\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}+n-1}$$

$$\lim_{n\to\infty} \frac{n^{\cancel{2}}\left(1+\frac{3}{n}\right)}{\cancel{n}\left[\sqrt{2+\frac{1}{n}+\frac{1}{n}}+1-\frac{1}{n}\right]} = \lim_{n\to\infty} \frac{n}{\sqrt{2}+1} = \infty$$

$$\lim_{n\to+\infty} \left[\sqrt{n^2+n} - \sqrt{n^3+1}\right]$$

24 Novembre 2016

$$f(x) = \frac{x^2 - 5x + 6}{x-5}$$

$x - 5 \neq 0 \quad x \neq 5$

1) $\text{Dom } f = (-\infty, 5[\cup]5, +\infty)$

$\in \mathbb{R} \setminus \{5\}$

2) P/D

$$f(-x) = \frac{-x^2 + 5x + 6}{-x-5}$$ f dispari

3) Intersezione assi:

$$f(0) = \frac{0^2 + 5 \cdot 0 - 6}{0-5} = +\frac{6}{5}$$

$f(x) = 0 \quad \frac{x^2+5x-6}{x-5} = 0 \quad \rightarrow x \neq 5$

$x^2 + 5x + 6 = 0$

$\Delta = b^2 - 4ac \rightarrow 25 - 4 \cdot (-6)$

$\Delta = 49$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm 7}{2} \begin{cases} = \frac{\cancel{2}}{\cancel{2}} = 1 \\ = -\frac{\cancel{12}}{\cancel{2}} = -6 \end{cases} \quad (0)$$

Intersez $\left(0, \frac{6}{6}\right)$ pos in $-6, 1$

pos in $5, +\infty$

④ Segno $\frac{x^2+5x-6}{x-5} > 0$

$x \neq 5$

$\begin{cases} x^2+5x-6 > 0 \\ x-5 > 0 \end{cases} \quad \begin{cases} x < -6 \cup x > 1 \\ x > 5 \end{cases}$

$x^2 + 5x - 6 > 0$

$x > 1$

$x > -6$

$-6 \quad 1$

$+ \quad - \quad +$

$x < -6 \cup x > 1$

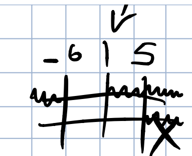
$\begin{cases} x^2+5x-6 < 0 \\ x-5 < 0 \end{cases} \quad \begin{cases} -6 < x < 1 \\ x < 5 \end{cases}$

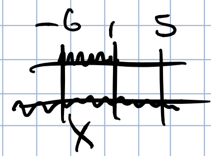
$x^2+5x-6<0$ $\qquad -6<x<1$

$x \in (-6,1) \cup (5,+\infty)$

pos in $-6, 1$

pos in $5, +\infty$





$\begin{cases} x>5 \\ -6<x<1 \end{cases}$

⑤ limiti

$$\lim_{x\to\infty} \frac{x^2+5x-6}{x-5} = \frac{\cancel{x^2}\left(1+\frac{5}{x}-\frac{6}{x^2}\right)}{x\left(1-\frac{5}{x}\right)} = +\infty$$

$\lim_{x\to-\infty} = -\infty$ perché dispari $\qquad \frac{1}{0^+}=\infty \quad \frac{1}{0^-}=-\infty$

$\lim_{x\to5^+} x^2+5x-6 = 44 \qquad \lim_{x\to5^+} x-5 = 0^+ \qquad 44\cdot0^+=+\infty$

$44\cdot0^-=-\infty$

Asintoti verticali a 5^+ e 5^-

$y=mx+q \qquad m=\frac{f(x)}{x} \qquad q=f(x)-mx$

$$m=\lim_{x\to\infty}\frac{\frac{x^2+5x-6}{x-5}}{x} = \lim_{x\to\infty}\frac{x^2+5x-6}{x-5}\cdot\frac{1}{x} = \lim_{x\to\infty}\frac{x^2+5x-6}{x^2-5x}$$

$$\frac{\cancel{x^2}\left(1+\frac{5}{x}{}^{0}-\frac{6}{x^2}{}^{0}\right)}{\cancel{x^2}\left(1-\frac{5}{x}{}^{0}\right)} = \frac{1}{1} = 1$$

$$q=\lim_{x\to\infty}\frac{x^2+5x-6}{x-5}-x = \frac{x^2+5x-6-x(x-5)}{x-5} = \frac{\cancel{x^2}+5x-6-\cancel{x^2}+5x}{x-5}$$

$$\frac{10x-6}{x-5}$$

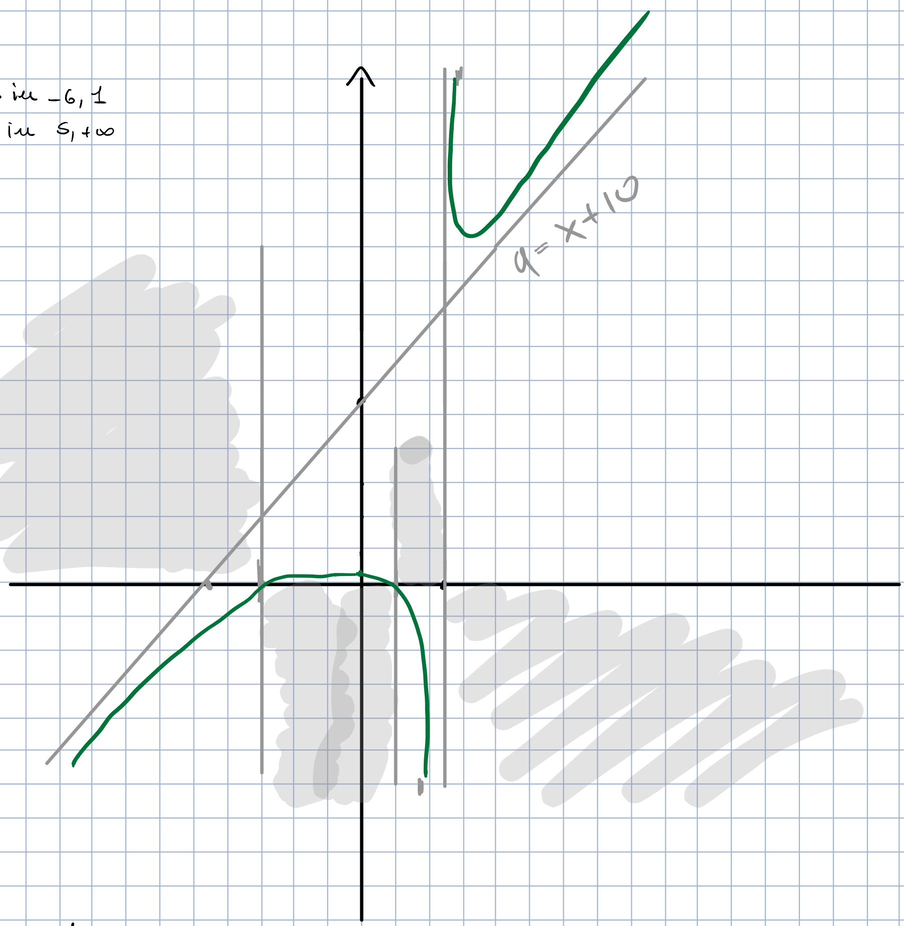
$$\frac{x\left(10-\frac{6}{x}\right)}{x\left(1-\frac{5}{x}\right)} = 10$$

$y = \overset{1}{m}x + \overset{10}{q}$ $\quad y = 11$

$y = 11$

x	y
1	11
2	22

Intersez $\left(0, \frac{6}{6}\right)$ pos in $-6, 1$
pos in $5, +\infty$



a b c d
2, 3, 4, 5

$$\lim_{n\to+\infty} \frac{2^n+4n^2}{6^n+5} = \lim_{n\to\infty} \frac{2^n\left(1+\frac{4n^2}{2^n}\right)}{6^n\left(1+\frac{5}{6^n}\right)} = \frac{2^n}{6^n} = \frac{1}{3^n} = 0$$

$\frac{4n^2}{2^n} \to 0$ $\quad \frac{5}{6^n} \to 0$ $\quad 3^{-n} = 0$

SDF: $\dfrac{3x^2-3x+1}{3x-1}$ ① Dom $f: \in \mathbb{R} \setminus \{\frac{1}{3}\}$

$3x-1 \neq 0$

$3x \neq 1$

$x \neq \frac{1}{3}$

② P/D

$f(-x) = \dfrac{-3x^2+3x+1}{-3x-1}$ Dispari

③ Intersez

$f(0) = \dfrac{0-0+1}{0-1} = -1$

$f(x)=0 \quad \dfrac{3x^2-3x+1}{3x-1} = 0$

$\Delta = 9 - 4 \cdot 3 \cdot 1 = -3$

$x_{1,2} = \dfrac{+3 \pm \sqrt{-3}}{2 \cdot 3} = x \notin \mathbb{R}$

Interseca $(0,-1)$

$x_{1,2} = \dfrac{+3 \pm \sqrt{-3}}{6} = x \in \mathbb{R}$

④ Segno

$\dfrac{3x^2-3x+1}{3x-1} \geq 0$

$\begin{cases} 3x^2-3x+1 \geq 0 \\ 3x-1 > 0 \end{cases} \quad \begin{cases} x \in \mathbb{R} \\ x > \frac{1}{3} \end{cases}$

1/3

$\begin{cases} 3x^2-3x+1 < 0 \\ 3x-1 < 0 \end{cases} \quad \begin{cases} x \in \emptyset \\ x < \frac{1}{3} \end{cases}$

$x \in \left(\frac{1}{3}, +\infty\right)$

f pos in $\frac{1}{3}, +\infty$

⑤ limiti

$\lim_{x \to \infty} \dfrac{3x^2-3x+1}{3x-1} = \dfrac{\cancel{x}(3x-3+\frac{1}{\cancel{x}})}{\cancel{x}(3-\frac{1}{\cancel{x}})} = \dfrac{3x-3}{3} = +\infty$

$\lim_{x \to -\infty}$ NO AS OR $= -\infty$ perchè dispari

$\lim_{x \to \frac{1}{3}^+} \dfrac{3x^2-3x+1}{3x-1} = \dfrac{3\left(\frac{1}{3}\right)^2 - 3 \cdot \frac{1}{3} + 1}{\cancel{3} \cdot \frac{1}{\cancel{3}}^{+} - 1 \quad 0^+} = +\infty$ As verticali a $\frac{1}{3}^{\pm}$

$$\lim_{x \to \frac{1}{3}^-} \qquad 0^- = -\infty \qquad e \pm\infty$$

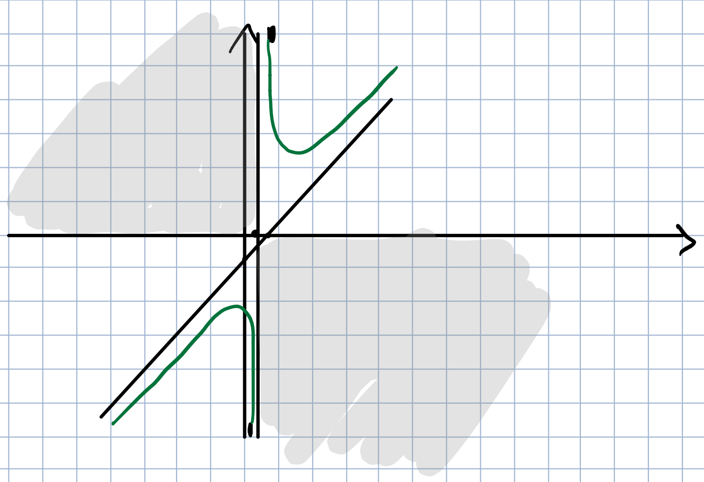
$$y = mx + q \qquad m = \lim_{x \to \infty} \frac{f(x)}{x} \qquad q = \lim_{x \to \infty} f(x) - mx$$

$$m = \lim_{x \to \infty} \frac{3x^2 - 3x + 1}{3x - 1} \cdot \frac{1}{x} = \frac{3x^2 - 3x + 1}{3x^2 - 1x} = \frac{3x^2\left(1 - \frac{1}{x} + \frac{1}{3x^2}\right)}{3x^2\left(1 - \frac{1}{3x}\right)} = 1$$

$$q = \lim_{x \to \infty} \frac{3x^2 - 3x + 1}{3x - 1} - x = \frac{3x^2 - 3x + 1 - x(3x - 1)}{3x - 1} =$$

$$\frac{\cancel{3x^2} - 3x + 1 - \cancel{3x^2} + x}{3x - 1} = \frac{-3x + 1 + x}{3x - 1} = \frac{-2x + 1}{3x + 1} = \frac{x\left(-2 + \frac{1}{x}\right)}{x\left(3 + \frac{1}{x}\right)} = -\frac{2}{3}$$

$$y = x - \frac{2}{3}$$



$$\lim_{x \to \infty} \left(\frac{4x + 6}{4x + 5}\right)^x$$

Limite stronzo

$$\lim_{x \to \infty} \left(\frac{4x + 6}{4x + 5}\right)^x = \text{ricorda} \quad \left(1 + \frac{1}{x}\right)^x = e \quad \text{notevole}$$

$$\lim_{x \to} \left(\frac{4x + 5 + 1}{4x + 5}\right)^x = \text{“}\left(\frac{\cancel{4x + 5}}{\cancel{4x + 5}} + \frac{1}{4x + 5}\right)^x = \text{“}\left(1 + \frac{1}{4x + 5}\right)^x =$$

$$\sqrt[4]{\left(1+\frac{1}{4x+5}\right)^{4x}} = \sqrt[4]{\left(1+\frac{1}{4x+5}\right)^{4x} \cdot \frac{\left(1+\frac{1}{4x+5}\right)^{5}}{\left(1+\frac{1}{4x+5}\right)^{5}}} =$$

$$\sqrt[4]{\frac{\left(1+\frac{1}{4x+5}\right)^{4x+5}}{\left(1+\underbrace{\frac{1}{4x+5}}_{\to 0}\right)^{5}}} = \sqrt{\frac{e}{1^5}} = \sqrt[4]{e}$$

Esercizi vari sui limiti:

- $\lim_{n\to\infty} \left(1+\frac{1}{3n}\right)^{2n}$ ricorda $\left(1+\frac{1}{x}\right)^{x} = e$ devo liberarmi dell'esponente

$$2 \cdot \frac{1}{3} = \frac{2}{3}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{3n}\right)^{3n} \to \left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{2}{3}} = e^{2/3}$$

Altrimenti $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = e^{a} \to \lim_{n\to\infty}\left[\left(1+\frac{1}{3n}\right)^{n}\right]^{2} =$

$(e^{1/3})^2 = e^{2/3}$

- $\lim_{n\to\infty} \frac{2^n-3^n}{1+3^n} = \lim_{n\to\infty} \frac{\cancel{3^n}\left(-1+\left(\frac{2}{3}\right)^{n}\right)}{\cancel{3^n}\left(1+\underbrace{\frac{1}{3^n}}_{\to 0}\right)} = \frac{-1}{+1} = -1$, con $\left(\frac{2}{3}\right)^{n} \to 0$

- $\lim_{n\to\infty} \frac{\sqrt{n}-n+n^2}{2n^2-n^{3/2}+1} = \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n}-\frac{1}{n^{3/2}}\right)}{n^2\left(2-\frac{1}{\sqrt{n}}+\frac{1}{n^2}\right)} = \frac{1}{2}$, con $\sqrt{n} = n^{\frac{1}{2}}$

- $\lim_{n\to\infty} \frac{2^n+n^2}{3^n+n^3} =$ $\boxed{\log n, n, n^2, x^n, e^n, n!, n^n}$

$$= \frac{2^n\left(1+\frac{n^2}{2^n}\right)}{3^n\left(1+\frac{n^3}{3^n}\right)} = \frac{2^n}{3^n} \to \left(\frac{2}{3}\right)^n = 0$$

$$\underline{(4\cdot 6\cdot 8)} = 2(2\cdot 3\cdot 4)$$

- $\lim_{n\to\infty} \frac{n\log n}{(n+1)(n+2)} = \frac{n\log n}{n^2+3n+2} = \frac{\cancel{n}\log n}{n^{\cancel{2}}\left(1+\frac{3n}{n^2}+\frac{2}{n^2}\right)} = \frac{\log n}{n} = 0$

- $\lim_{n\to\infty} \frac{1+\log n}{\sqrt{n}-\log n} = \frac{\log n\left(1+\frac{1}{\log n}\right)}{\sqrt{n}\left(1-\frac{\log n}{\sqrt{n}}\right)} = \frac{\log n}{\sqrt{n}} = 0$

- $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n^2+1} = 0$ perché la succ $(-1)^n$ è limitata mentre $\frac{n}{n^2+1}$ infinitesima

- $\lim_{n\to\infty} (-1)^n \cdot \frac{n^2+1}{n+1} =$ Non esiste perché la succ dei termini pari $\to +\infty$ mentre quella dei termini di indice dispari $\to -\infty$

- $\lim_{n\to\infty} \sqrt[n]{2^n+3^n} = \left(2^n+3^n\right)^{\frac{1}{n}} = 3\left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} = 3$

- $\lim_{n\to\infty} \sqrt[n]{\frac{2n}{3n^2+1}} = \sqrt[n]{\frac{2\cancel{n}}{n^{\cancel{2}}\left(3+\frac{1}{3n^2}\right)}} = \sqrt[n]{\frac{2}{3n}} = \sqrt[\infty]{0} = 1$

$$\bullet \lim_{n\to\infty} \frac{n^2(3^n-3^{-n})}{4^n+n^2} = \frac{n^2\left(3^n-\left(\frac{1}{3}\right)^n\right)}{4^n+n^2} = \frac{\cancel{n^2}\left(3^n-\left(\frac{1}{3}\right)^{n}\right)}{\cancel{n^2}\left(\frac{4^n}{n^2}+1\right)} = \frac{3^n}{\left[\frac{4^n}{n^2}\right]} = 0$$

$\frac{1}{+} \to +$

$$\left(\frac{3}{4}\right)^n \cdot n^2 = 0\cdot\infty = 0$$

$$\text{N.B.} = \lim_{n\to\infty}\left(\frac{1}{3}\right)^n = 0$$

$$\bullet \lim_{n\to\infty} \frac{n^6+\log n+3^n}{2^n+n^4+\log^5 n} = \frac{3^n\left(\frac{n^6}{3^n}+\frac{\log n}{3^n}+1\right)}{2^n\left(1+\frac{n^4}{2^n}+\frac{\log^5 n}{2^n}\right)} = \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n = +\infty$$

$$\bullet \lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^n = \left(\frac{\cancel{n+1}}{\cancel{n+1}} + \frac{2}{n+1}\right)^n = \left(1+\frac{2}{n+1}\right)^n =$$

$$\left(1+\frac{1}{n}\right)^n$$

$$\left[\left(1+\frac{1}{\frac{n+1}{2}}\right)^{\frac{n+1}{2}}\right]^{\frac{2n}{n+1}} = e^2 \qquad \frac{\cancel{n+1}}{\cancel{2}}\cdot\frac{\cancel{2}n}{\cancel{n+1}} = n$$

$$\left(1-\frac{1}{n}\right)^n = e^{-1}$$

$$\bullet \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n^2} = \left[\left(\frac{\cancel{n}}{\cancel{n}}-\frac{1}{n}\right)^n\right]^n = (e^{-1})^{+\infty} = 0$$

$$\bullet \lim_{n\to\infty}\frac{\log(n+1)}{\log n} = \text{si ha } \frac{\log\left[n\left(1+\frac{1}{n}\right)\right]}{\log n} = \frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n}$$

$$\text{dunque} \quad \lim_{n\to\infty}\left(1+\frac{1}{\log n}\log\left(1+\frac{1}{n}\right)\right) = 1$$

- $\lim_{n\to\infty} (\sqrt[n]{3}-1)^n = (0)^{+\infty} = 0$

- $\lim_{n\to\infty} \sqrt[n]{n\log n}$ = È facile verificare che per ogni $n \geq 3$ vale $1 \leq \log n \leq n$. Moltiplicando per n abbiamo che $n \leq n\log n \leq n^2$, e dunque $\sqrt[n]{n} \leq \sqrt[n]{n\log n} \leq (\sqrt[n]{n})^2$ $\forall n \geq 3$. Ricordando che $\sqrt[n]{n} \to 1$ e applicando il teorema del doppio confronto otteniamo:

$$\lim_{n\to\infty} \sqrt[n]{n\log n} = 1$$

- $\lim_{n\to\infty} n^2 \cdot 2^{-\sqrt{n}} = e^{2\log n - \sqrt{n}\log 2} = e^{\sqrt{n}\left(-\log 2 + 2\frac{\log n}{\sqrt{n}}\right)} = e^{-\infty} = 0$

- $\lim_{n\to\infty} n^{\sqrt{n}} - 2^n = e^{\sqrt{n}\log n} - e^{n\log 2} = -e^{n\log 2}\left(1 - e^{\sqrt{n}\log n - n\log 2}\right)$

Ora $e^{\sqrt{n}\log n - n\log 2} = e^{n\left(-\log 2 + \frac{\log n}{\sqrt{n}}\right)} \to e^{-\infty} = 0$

E quindi $\lim_{n\to\infty} \left(n^{\sqrt{n}} - 2^n\right) = -\lim_{n\to-\infty} e^{n\log 2} = -\infty$

$$f(x) = \frac{4-5x^2}{x^2+x-2}$$

1) Dom $f = x^2+x-2 \neq 0$ $\quad \Delta = 1 - 4 \cdot 1 \cdot (-2) = 9$

$$x_{1,2} = \frac{-1 \pm 3}{2} = \begin{matrix} 1 \\ -\cancel{\frac{4}{2}} = -2 \end{matrix}$$

Dom $f \in \mathbb{R} / \{1, -2\}$

$(-\infty, -2) \cup (1, +\infty)$

2) P/D

$$f(-x) = \frac{4-5x^2}{x^2+x-2} = \frac{4+5x^2}{-x^2-x-2} = \frac{4+5x^2}{x^2-x-2}$$

$-5x^2$

$-x^2-x$

3) intersez

$$f(0) = \frac{4+0}{0-0-2} = \frac{\cancel{4}}{-\cancel{2}} = -2 \quad \text{intersez} \ (0,-2)$$

$$f(x) = 0 \rightarrow \frac{4-5x^2}{x^2+x-2} = 0 \qquad 4-5x^2=0 \quad x \notin \mathbb{R}$$

4) Segno

$$x < \pm\frac{2\sqrt{5}}{5} \qquad \left(-\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

$$\frac{4-5x^2}{x^2+x-2} > 0$$

① $\begin{cases} 4-5x^2 > 0 \\ x^2+x-2 > 0 \end{cases}$ $\quad -5x^2 > -4 \rightarrow 5x^2 < 4 \quad x^2 < \frac{4}{5}$

$\begin{cases} \left(-\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right) \\ (-\infty, -2) \cup (1, +\infty) \end{cases}$ $\quad x < \pm\sqrt{\frac{4}{5}}$

$$\frac{2}{\sqrt{5}} \rightarrow \frac{2\sqrt{5}}{5}$$

② $\begin{cases} 4-5x^2 < 0 \\ x^2+x-2 < 0 \end{cases}$

$-2 \quad + \quad - \quad 1$

$\begin{cases} \left(-\infty, -\frac{2\sqrt{5}}{5}\right) \cup \left(\frac{2\sqrt{5}}{5}, +\infty\right) \\ (-2, 1) \end{cases}$

$-2 \quad - \quad + \quad 1$

$$\left(-2, -\frac{2\sqrt{5}}{2}\right) \cup \left(\frac{2\sqrt{5}}{2}, 1\right)$$

5) limiti

$$\lim_{x\to+\infty} \frac{4-5x^2}{x^2+x-2} = " \quad \frac{x^2\left(\frac{4}{x^2}-5\right)}{x^2\left(1+\frac{1}{x}-\frac{2}{x^2}\right)} = -5$$

$$\lim_{x\to-\infty} \frac{4-5x^2}{x^2+x-2} = " \quad \frac{x^2\left(\frac{4}{x^2}-5\right)}{x^2\left(1+\frac{1}{x}-\frac{2}{x^2}\right)} = -5$$

$$\lim_{x\to1^+} \frac{4-5x^2}{x^2+x-2} = \frac{4-5\cdot1^2}{1+1-2} \qquad \lim_{x\to1^+} 4-5 = -1$$

$$-1\lim_{x\to1^+} \frac{1}{x^2+x-2} = -1\cdot\infty = -\infty$$

$$1^- \quad = - \cdot -\infty = +\infty$$

$$\lim_{x\to-2^+} \frac{4-5x^2}{x^2+x-2} = \lim_{x\to-2^+} 4-5\cdot(-2)^2 = 4-5\cdot4 = -16$$

$$-16\lim_{x\to-2^+} \frac{1}{-2^2-2-2 = 0^+} = -16\cdot\infty = +\infty$$

$$4-2-2=0$$

$$-2^- = -\infty$$