OpenNodal Theory Manual

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Revision Log

Revision	Date	Affected Pages	Revision Description
0	August 11, 2022	All	Initial Release

Acronyms

Contents

1. Background

OpenNodal is an open source nodal solver for nuclear reactors. OpenNodal uses the polynomial nodal method with additive correction factors for the CMFD form solved.

The polynomial solution is achieved via a nonlinear iteration scheme described in [3] and [1]. The correction factors that couple the CMFD solve with the polynomial form of the nodal equations is described in [2].

2. Additional Theory

While the listed references provide a complete description of the polynomial nodal equations and the nonlinear iteration procedure, some correlations are implied but not explicitly stated. One of these implied values is $\bar{S}$, so a more explicit description is given here.

2.1 Explicit Form of $\bar{S}$

Fundamental equations given or implied in [1], [2], and [3]:

$$l, m, n = \begin{cases} i = 1, 2, \ldots, I & u = x, y, z \\ j = 1, 2, \ldots, J & v \neq u \\ k = 1, 2, \ldots, K & w \neq u, v \end{cases}$$

$$J_{gu}^{mn}(u_l) = -\left[\frac{h_u^l}{2D_g^{lmn}} + \frac{h_u^{l-1}}{2D_g^{l-1,mn}}\right]^{-1} \left(\bar{\phi}_g^{lmn} - \bar{\phi}_g^{l-1,mn}\right) + \tilde{D}_{gu}^{l,l-1,mn} \left(\bar{\phi}_g^{lmn} + \bar{\phi}_g^{l-1,mn}\right)$$

$$J_{gu}^{mn}(u) = \frac{1}{h_w^n h_v^m} \int_{v_m}^{v_{m+1}} \int_{w_n}^{w_{n+1}} J_{gu}(u, v, w)\, dv\, dw$$

$$L_{gv}^{mn}(u) = \frac{1}{h_w^n} \int_{w_n}^{w_{n+1}} \left(J_{gv}(u, v_{m+1}, w) - J_{gv}(u, v_m, w)\right) dw$$

$$S_{gu}^{mn}(u) = \frac{1}{h_v^m} L_{gv}^{mn}(u) + \frac{1}{h_w^n} L_{gw}^{mn}(u)$$

$$\bar{S}_{gu}^{lmn} = \frac{1}{h_u^l} \int_{u_l}^{u_{l+1}} S_{gu}^{mn}(u)\, du$$

To get a clear form for how to compute $\bar{S}$, we first get $\bar{L}$ in terms of J:

$$\bar{L}_{gv}^{lmn} = \frac{1}{h_w^n h_u^l} \int_{u_l}^{u_{l+1}} \int_{w_n}^{w_{n+1}} \left(J_{gv}(u, v_{m+1}, w) - J_{gv}(u, v_m, w)\right) dw\, du = J_{gv}^{ln}(v_{m+1}) - J_{gv}^{ln}(v_m)$$

So then $\bar{S}$ can be expressed as:

$$\bar{S}_{gu}^{lmn} = \frac{1}{h_u^l} \int_{u_l}^{u_{l+1}} \left(\frac{1}{h_v^m} L_{gv}^{mn}(u) + \frac{1}{h_w^n} L_{gw}^{mn}(u)\right) du = \frac{1}{h_v^m} \bar{L}_{gv}^{lmn} + \frac{1}{h_w^n} \bar{L}_{gw}^{lmn}$$

Which gives $\bar{S}$ in a form that can now be explicitly computed using flux, ϕ, for a current, J, expressed as a function of ϕ.

2.2 Reduced Two Node Problem Equations

Fundamental equations given or implied in [1], [2], and [3]:

1. Nodal balance equation for node $l-1$:

$$6a_{gu2}^{l-1,mn} - \frac{2}{5}a_{gu4}^{l-1,mn} = \sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{l-1,mn}\bar{\phi}_{g'}^{l-1,mn} + \frac{(h_u^{l-1})^2}{D_g^{l-1,mn}}\bar{S}_{gu}^{l-1,mn}$$

2. First moment equation for node $l-1$:

$$\frac{1}{2}a_{gu3}^{l-1,mn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{l-1,mn}\left[a_{g'u1}^{l-1,mn} + \frac{1}{10}a_{g'u3}^{l-1,mn}\right] = -\frac{(h_u^{l-1})^2}{D_g^{l-1,mn}}S_{gu1}^{l-1,mn}$$

3. Second moment equation for node $l-1$:

$$\frac{1}{5}a_{gu4}^{l-1,mn} + \frac{1}{20}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{l-1,mn}\left[a_{g'u2}^{l-1,mn} + \frac{1}{35}a_{g'u4}^{l-1,mn}\right] = -\frac{(h_u^{l-1})^2}{D_g^{l-1,mn}}S_{gu2}^{l-1,mn}$$

4. Nodal balance equation for node l:

$$6a_{gu2}^{lmn} - \frac{2}{5}a_{gu4}^{lmn} = \sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\bar{\phi}_{g'}^{lmn} + \frac{(h_u^{l})^2}{D_g^{lmn}}\bar{S}_{gu}^{lmn}$$

5. First moment equation for node l:

$$\frac{1}{2}a_{gu3}^{lmn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u1}^{lmn} + \frac{1}{10}a_{g'u3}^{lmn}\right] = -\frac{(h_u^{l})^2}{D_g^{lmn}}S_{gu1}^{lmn}$$

6. Second moment equation for node l:

$$\frac{1}{5}a_{gu4}^{lmn} + \frac{1}{20}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u2}^{lmn} + \frac{1}{35}a_{g'u4}^{lmn}\right] = -\frac{(h_u^{l})^2}{D_g^{lmn}}S_{gu2}^{lmn}$$

7. Discontinuity at the interface:

$$\left(\bar{\phi}_g^{l-1,mn} + \frac{1}{2}a_{gu1}^{l-1,mn} + \frac{1}{2}a_{gu2}^{l-1,mn}\right)f_{gu+}^{l-1,mn} = \left(\bar{\phi}_g^{lmn} - \frac{1}{2}a_{gu1}^{lmn} + \frac{1}{2}a_{gu2}^{lmn}\right)f_{gu-}^{lmn}$$

8. Continuity of the net current at the interface:

$$-\frac{D_g^{l-1,mn}}{h_u^{l-1}}\left(a_{gu1}^{l-1,mn} + 3a_{gu2}^{l-1,mn} - \frac{1}{2}a_{gu3}^{l-1,mn} - \frac{1}{5}a_{gu4}^{l-1,mn}\right) = -\frac{D_g^{lmn}}{h_u^{l}}\left(a_{gu1}^{lmn} - 3a_{gu2}^{lmn} - \frac{1}{2}a_{gu3}^{lmn} + \frac{1}{5}a_{gu4}^{lmn}\right)$$

These equations can be solved as is for an $8G \times 8G$ system of equations, however they can also be reduced to improve efficiency. First solving the nodal balance equation for a coefficient:

$$a_{gu2}^{lmn} = \frac{1}{15}a_{gu4}^{lmn} + \frac{1}{6}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\bar{\phi}_{g'}^{lmn} + \frac{(h_u^{l})^2}{6D_g^{lmn}}\bar{S}_{gu}^{lmn}$$

Which can be substituted into the second moment equation:

$$a_{gu4}^{lmn} + \frac{1}{24}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[\sum_{g''=1}^{G}(\kappa_u^2)_{g'g''}^{lmn}\bar{\phi}_{g''}^{lmn} + \frac{(h_u^l)^2}{D_g'^{lmn}}\bar{S}_{g'u}^{lmn} + \frac{4}{7}a_{g'u4}^{lmn}\right] = -5\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu2}^{lmn}$$

This is now a $G \times G$ system of equations for each a_{gu4}^{lmn} from $g = 1, \ldots, G$ whose solution can then be used to perform a direct solve for a_{gu2}^{lmn} from $g = 1, \ldots, G$ using the previously solved balance equation. This is performed similarly for the $l-1$ node, and indeed the resulting equations are identical except for the index of l vs $l-1$.

The next reduction involves solving the discontinuity and equations to get:

$$a_{gu1}^{l-1,mn} = \alpha_1 + \beta_1 a_{gu3}^{l-1,mn} + \gamma_1 a_{gu3}^{lmn}$$

and:

$$a_{gu1}^{lmn} = \alpha_2 + \beta_2 a_{gu3}^{l-1,mn} + \gamma_2 a_{gu3}^{lmn}$$

where

$$\begin{aligned}\alpha_1 = \Big(&5a_{gu2}^{l-1,mn}(D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} - 3D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l) + a_{gu4}^{l-1,mn}D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l \\ &+ D_g^{lmn}h_u^{l-1}(f_{gu-}^{lmn}(-20a_{gu2}^{lmn} + a_{gu4}^{lmn} - 10\bar{\phi}_g^{lmn}) + 10f_{gu+}^{l-1,mn}\bar{\phi}_g^{l-1,mn})\Big) \\ &/\Big(5D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l - 5D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1}\Big)\end{aligned}$$

$$\beta_1 = \frac{D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l}{2D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l - 2D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1}}$$

$$\gamma_1 = \frac{D_g^{lmn}f_{gu-}^{lmn}h_u^{l-1}}{2D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} - 2D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l}$$

$$\begin{aligned}\alpha_2 = \Big(&10a_{gu2}^{l-1,mn}D_g^{l-1,mn}f_{gu+}^{l-1,mn}h_u^l - a_{gu4}^{l-1,mn}D_g^{l-1,mn}f_{gu+}^{l-1,mn}h_u^l + 15a_{gu2}^{lmn}D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} \\ &+ 5a_{gu2}^{lmn}D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l - a_{gu4}^{lmn}D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} \\ &- 10D_g^{l-1,mn}f_{gu+}^{l-1,mn}h_u^l\bar{\phi}_g^{l-1,mn} + 10D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l\bar{\phi}_g^{lmn}\Big) \\ &/\Big(5D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} - 5D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l\Big)\end{aligned}$$

$$\beta_2 = \frac{D_g^{l-1,mn}f_{gu+}^{l-1,mn}h_u^l}{2D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l - 2D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1}}$$

$$\gamma_2 = \frac{D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1}}{2D_g^{lmn}f_{gu+}^{l-1,mn}h_u^{l-1} - 2D_g^{l-1,mn}f_{gu-}^{lmn}h_u^l}$$

So we can now plug this in to the first moment equations to get:

$$\frac{1}{2}a_{gu3}^{l-1,mn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{l-1,mn}\left[\alpha_1 + \beta_1 a_{g'u3}^{l-1,mn} + \gamma_1 a_{g'u3}^{lmn} + \frac{1}{10}a_{g'u3}^{l-1,mn}\right] = -\frac{(h_u^{l-1})^2}{D_g^{l-1,mn}}S_{gu1}^{l-1,mn}$$

and

$$\frac{1}{2}a_{gu3}^{lmn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[\alpha_2 + \beta_2 a_{gu3}^{l-1,mn} + \gamma_2 a_{gu3}^{lmn} + \frac{1}{10}a_{g'u3}^{lmn}\right] = -\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu1}^{lmn}$$

This is now a $2G \times 2G$ equation for each $\frac{1}{2}a_{gu3}^{l-1,mn}$ and $\frac{1}{2}a_{gu3}^{lmn}$ from $g = 1, \ldots, G$ whose solution can then be used to perform a a direct solve for $a_{gu1}^{l-1,mn}$ and a_{gu1}^{lmn} from $g = 1, \ldots, G$.

These two reductions have now reduced the previous $8G \times 8G$ system of equations down to two $G \times G$ and one $2G \times 2G$ systems of equations.

2.3 Reduced One Node Problem Equations

Fundamental equations given or implied in [1], [2], and [3]:

For left (minus direction) BC:

1. Nodal balance equation:

$$6a_{gu2}^{lmn} - \frac{2}{5}a_{gu4}^{lmn} = \sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\bar{\phi}_{g'}^{lmn} + \frac{(h_u^l)^2}{D_g^{lmn}}\bar{S}_{gu}^{lmn}$$

2. First moment equation:

$$\frac{1}{2}a_{gu3}^{lmn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u1}^{lmn} + \frac{1}{10}a_{g'u3}^{lmn}\right] = -\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu1}^{lmn}$$

3. Second moment equation:

$$\frac{1}{5}a_{gu4}^{lmn} + \frac{1}{20}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u2}^{lmn} + \frac{1}{35}a_{g'u4}^{lmn}\right] = -\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu2}^{lmn}$$

4. Left BC face:

$$\left[\bar{\phi}_g^{lmn} - \frac{1}{2}a_{gu1}^{lmn} + \frac{1}{2}a_{gu2}^{lmn}\right]f_{gu-}^{lmn} = -\sum_{g'=1}^{G}\Gamma_{gg',u-}^{mn}\frac{D_{g'}^{lmn}}{h_u^l}\left[a_{g'u1}^{lmn} - 3a_{g'u2}^{lmn} - \frac{1}{2}a_{g'u3}^{lmn} + \frac{1}{5}a_{g'u4}^{lmn}\right]$$

For right (plus direction) BC:

1. Nodal balance equation:

$$6a_{gu2}^{lmn} - \frac{2}{5}a_{gu4}^{lmn} = \sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\bar{\phi}_{g'}^{lmn} + \frac{(h_u^l)^2}{D_g^{lmn}}\bar{S}_{gu}^{lmn}$$

2. First moment equation:

$$\frac{1}{2}a_{gu3}^{lmn} + \frac{1}{12}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u1}^{lmn} + \frac{1}{10}a_{g'u3}^{lmn}\right] = -\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu1}^{lmn}$$

3. Second moment equation:

$$\frac{1}{5}a_{gu4}^{lmn} + \frac{1}{20}\sum_{g'=1}^{G}(\kappa_u^2)_{gg'}^{lmn}\left[a_{g'u2}^{lmn} + \frac{1}{35}a_{g'u4}^{lmn}\right] = -\frac{(h_u^l)^2}{D_g^{lmn}}S_{gu2}^{lmn}$$

4. Right BC face:

$$\left[\bar{\phi}_g^{lmn} + \frac{1}{2}a_{gu1}^{lmn} + \frac{1}{2}a_{gu2}^{lmn}\right]f_{gu+}^{lmn} = -\sum_{g'=1}^{G}\Gamma_{gg',u+}^{mn}\frac{D_{g'}^{lmn}}{h_u^l}\left[a_{g'u1}^{lmn} + 3a_{g'u2}^{lmn} - \frac{1}{2}a_{g'u3}^{lmn} - \frac{1}{5}a_{g'u4}^{lmn}\right]$$

These boundary conditions appear to be $4G \times 4G$ systems of equations. However, for both of these recall the nodal balance reduction to get the $G \times G$ equation for both a_{gu4}^{lmn} and a_{gu2}^{lmn}. This then leaves a single $2G \times 2G$ system of equations.

But further reductions can be made for boundary conditions that lack a cross group term (that is to say, any boundary condition other than out-scattering albedos, i.e. vacuum, reflective, group independent albedo, and in-scattering only albedos). Taking this form for BC faces and solving for the left BC face:

$$a_{gu1}^{lmn} = \alpha_1 + \beta_1 a_{gu3}^{lmn}$$

where

$$\alpha_1 = \frac{-30a_{gu2}^{lmn}\Gamma_{g,u-}^{mn}D_g^{lmn} + 5a_{gu2}^{lmn}f_{gu-}^{lmn}h_u^l + 2a_{gu4}^{lmn}\Gamma_{g,u-}^{mn}D_g^{lmn} + 10f_{gu-}^{lmn}h_u^l\bar{\phi}_g^{lmn}}{5f_{gu-}^{lmn}h_u^l - 10\Gamma_{g,u-}^{mn}D_g^{lmn}}$$

$$\beta_1 = \frac{\Gamma_{g,u-}^{mn}D_g^{lmn}}{2\Gamma_{g,u-}^{mn}D_g^{lmn} - f_{gu-}^{lmn}h_u^l}$$

And for the right BC face:

$$a_{gu1}^{lmn} = \alpha_2 + \beta_2 a_{gu3}^{lmn}$$

where

$$\alpha_2 = -\frac{30a_{gu2}^{lmn}\Gamma_{g,u+}^{mn}D_g^{lmn} + 5a_{gu2}^{lmn}f_{gu+}^{lmn}h_u^l - 2a_{gu4}^{lmn}\Gamma_{g,u+}^{mn}D_g^{lmn} + 10f_{gu+}^{lmn}h_u^l\bar{\phi}_g^{lmn}}{10\Gamma_{g,u+}^{mn}D_g^{lmn} + 5f_{gu+}^{lmn}h_u^l}$$

$$\beta_2 = \frac{\Gamma_{g,u+}^{mn}D_g^{lmn}}{2\Gamma_{g,u+}^{mn}D_g^{lmn} + f_{gu+}^{lmn}h_u^l}$$

2.4 Errata

2.4.1 Notational Errors in Dr. Gehnin's Thesis

Dr. Gehin's thesis is thorough and correct for the understanding of this method. However, almost all works have some errors contained within, and Dr. Gehin's thesis is no exception. The following are notational errors in Dr. Gehin's thesis that we noticed:

1. Equations 2.41, 2.42, 2.44, and 2.45 all mention a value $\bar{S}_p$ where p is the moment indicator. By Dr. Gehin's previous notation, higher order moments don't have bars over them (see sepcifically equations 2.37 and 2.39) so these terms should be listed as S_p

2. All flux and diffusion coefficient terms on the RHS of equations 2.49 and 2.50 should have a g' index not a g index. This is because it is a substitution of the current $J_{g'}$ in equation 2.48 using the expansion form of the current.

Bibliography

[1] Jess C. Gehin. *A Quasi-Static Polynomial Nodal Method for Nuclear Reactor Analysis*. PhD thesis, Massachusetts Institute of Technology, Department of Nuclear Engineering, 1992.

[2] Kord S. Smith. An analytic nodal method for solving the two-group, multidimensional, static and transient neutron diffusion equations, 1979.

[3] Kord S. Smith. Nodal method storage reduction by nonlinear iteration. *Transactions of the American Nuclear Society*, 44:265–266, 1983.