

Cryptographic Secret Sharing

Girls Talk Math

Introduction

In this problem set, you will learn how to use math to split secrets into pieces. This is the cryptographic technique known as secret sharing and is the basis for many useful tools being deployed by companies today.

One last note about reading mathematical texts: it is very normal when reading math to read a passage or even a single sentence several times before understanding it properly. Also, never trust the author! Check every claim and calculation (time permitting). Take your time and never give up. Let's talk math!

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Optional sections are marked with an asterisk (*).

1 Probability and Randomness

In mathematics, the word “random” has a more precise meaning than in everyday speech. Something is random if its outcome is governed by some *probability distribution*. For example, the outcome c of a coin toss is not fixed, but takes on one of the values in the set $\{H, T\}$, where H means heads and T means tails. (This is called the *sample space* of c .) We write that $c \in \{H, T\}$, where the symbol \in is read aloud as “is an element of” or simply “is in”. The value of c is described by the distribution $\{\frac{1}{2}, \frac{1}{2}\}$, which says that each outcome happens with probability one-half.

A variable like c that represents an outcome or event is called a *random variable*, since it can take on one of several values depending based on an underlying probability distribution. A random variable is *uniformly random* if every possible value is equally likely. In other words, its probability distribution is the *uniform distribution*. The variable c is uniformly random (or simply *uniform*).

Exercise 1.1 Write “uniform” or “not uniform” for each of the following random variables:

- (a) Drawing a card from a shuffled deck.
- (b) Drawing a card from an ordered deck.
- (c) The weather on a given day.
- (d) The outcome of a dice roll.
- (e) The birthday of the first person you encounter on the street.

When a random variable is uniform, we say it is chosen *uniformly at random* (sometimes shortened to chosen *uniformly*) and denote this with the symbol $\leftarrow_{\$}$. For example, the outcome c of a coin toss can be written as $c \leftarrow_{\$} \{H, T\}$ to mean that c is sampled uniformly from the set $\{H, T\}$.

Exercise 1.2 How would you represent a dice roll using the $\leftarrow_{\$}$ notation?

We use Pr to denote the probability of some event. For instance, the probability that c is heads is one-half. In mathematical notation this is written as

$$\text{Pr}[c = H] = \frac{1}{2}$$

Similarly, c is tails with probability one-half:

$$\Pr[c = T] = \frac{1}{2}$$

To find the probability of one event or another happening, we add probabilities. For example, the probability that c is heads or tails is

$$\Pr[c = H] + \Pr[c = T] = \frac{1}{2} + \frac{1}{2} = 1$$

Notice that these probabilities sum to 1. This is a common requirement of all probability distributions: the sum of their probabilities must be 1. This is the same as saying that *one of* the possible events in the distribution happens with 100% probability. If the sum was less than 1, it would mean that there's some other event that could happen instead, so the distribution is incomplete.

When the value of one random variable is not related to that of another, we say the variables are *independent*. For example, a good coin toss should be independent of the coin used for the toss or the person doing the toss. When two random variables are independent, we can multiply the probabilities of two events together to get the probability that both events happen simultaneously.

Example 1.1 Suppose I roll a die and flip a coin. What's the probability that I roll a 1 *and* get heads?

The dice and coin outcomes are independent, so we can multiply the probabilities of the two events together to get the probability of both events occurring. Let d be the random variable representing the outcome of the die roll and c be the random variable representing the outcome of the coin toss. Then

$$\begin{aligned} & \Pr[d = 1 \text{ and } c = H] \\ &= \Pr[d = 1] \cdot \Pr[c = H] \\ &= \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12} \end{aligned}$$

Exercise 1.3 Suppose I flip two coins. What's the probability of both coins landing on heads?

Let's put everything we've learned so far together and work through some examples. Remember, for *or*, we add; for *and*, we multiply.

Example 1.2 Say I flip a fair coin twice. What's the probability of getting the same result both times?

There are two ways of getting the same result on both flips: we either get heads twice (HH) or tails twice (TT):

$$\Pr[HH] + \Pr[TT]$$

Because subsequent coin flips are independent of each other, this is equal to

$$\Pr[H] \cdot \Pr[H] + \Pr[T] \cdot \Pr[T]$$

Since the coin is fair, each outcome (heads or tails) happens with equal probability, so this equals

$$\begin{aligned} & \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

Conditional probability is the probability that something occurs given that another event occurred. This is written with the symbol $|$, which is read aloud as "given". So, $\Pr[A | B]$ is read as "the probability that A given B ".

For example, it's more likely to be raining if the sky is cloudy than if it's clear. We can write this as

$$\Pr[\text{raining} | \text{cloudy sky}] > \Pr[\text{raining} | \text{clear sky}]$$

When two events A and B are independent, the following equality holds:

$$\Pr[A \text{ and } B] = \Pr[A | B] \cdot \Pr[B] \tag{1}$$

Example 1.3 Suppose I only flip the coin again if my first coin toss came out heads. What's the probability of getting one heads outcome?

There are two ways of getting one heads outcome: (1) the first toss is heads (H), and we don't do another coin toss, or (2) the first toss comes up tails, and the second one comes up heads (TH).

Now we need to use conditional probability, since our second coin toss depends on the first:

$$\begin{aligned} & \Pr[H] + \Pr[TH] \\ &= \Pr[H] + \Pr[TH \mid \text{first toss is } T] \cdot \Pr[\text{first toss is } T] \end{aligned}$$

Now we plug in the numbers:

$$\begin{aligned} &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Exercise 1.4 The following questions deal with rolling a 6-sided die.

- (a) What's the probability of rolling an even number?
- (b) What's the probability of rolling a 1 followed immediately by a 2?
- (c) If I roll twice, what's the probability that my second roll will be higher than my first?
- (d) Suppose that when I roll a 1, I get to reroll and use the second number instead. What's the probability of rolling a 5 or higher?

When two random variables have the same underlying probability distribution, we say they are *indistinguishable*. For example, if $c \in \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ is the random variable representing the outcome of three coin tosses, the probability distribution of c is

$$\left\{ \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right\}$$

where each probability is the probability that c takes on the corresponding value in the sample space.

Compare this to the probability distribution of the random variable $d \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ for rolling an 8-sided die:

$$\left\{ \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right\}$$

They're the same distribution! Therefore c and d are indistinguishable.

Exercise 1.5 Write “indistinguishable” or “not indistinguishable” for each pair of probability distributions:

- (a) Rolling and even or an odd number on a 6-sided die, getting heads or tails when you flip a coin
- (b) Drawing a random card from a deck, rolling a 6-sided die and flipping 3 coins
- (c) Whether or not you get at least one heads when flipping two coins, whether or not you draw a club from a shuffled deck.

1.1 Randomness in Cryptography

As we'll see, randomness plays an extremely important role in modern cryptographic schemes. A cryptographic *scheme* is a well-defined procedure for accomplishing some goal (called a *cryptographic primitive*). For example, the Caesar cipher is an *encryption* scheme. In this packet, you'll learn about several *secret sharing* schemes.

When a scheme's security rests only on the randomness used in it, and not on solving a difficult or very long problem, we say that scheme is *information-theoretically secure*. This is different from, for example, the schemes in the RSA cryptography packet, whose security depends on the assumed difficulty factoring. (When a scheme is protected by how hard it is to compute something, it is said to be *computationally secure*).

The exact level of security of a scheme is determined by a *security parameter*, which is a number that's often denoted by the symbol λ (the Greek letter lambda). The people using a scheme will generally decide on the security parameter ahead of time based on how secure they want to be. There's usually a tradeoff between security and efficiency, so the parties will settle on a λ that

makes them feel safe enough without making their computations prohibitively slow (hours, days, or weeks). You'll learn more about this in Section 2.3.

Lambda is usually set to a large power of two, like 128 or 256; to keep numbers reasonable in this packet, we'll let $\lambda = 10$.

2 Secret Sharing

Secret sharing is a way to “split” a secret value (call it s for “secret”) into pieces, called *shares*. Two useful properties a secret sharing scheme might have are *correctness* and *privacy*. Informally, correctness means that if we put shares back together, we get back the original secret; privacy says that each share by itself reveals nothing about the secret s .

These properties are important for practical uses of secret sharing. For example, using secret sharing, we can distribute shares among a large set of people so that no one knows the secret but some subset of them can recover the secret if they pool their information.

We’ll be a little more rigorous about these definitions soon, but first, let’s see an example.

2.1 A simple secret sharing

Here is a simple scheme for sharing integers using nothing but addition and subtraction:

Share(s)	Rec(s_1, s_2)
$s_1 \leftarrow_{\$} \{1, \dots, 2^\lambda\}$ return ($s_1, s - s_1$)	return $s_1 + s_2$

Figure 1: Additive secret sharing scheme

Let’s go through the notation together. First, on the left, we see that the **Share** algorithm is being defined. (An *algorithm* is simply a procedure.) The parentheses after the name tell us that it takes an input s , which in this context is the secret to be shared. The first line tells us to sample¹ an element from the set $\{1, \dots, 2^\lambda\}$ and call it s_1 . The dots in the set are shorthand for all the numbers in between. For instance, when $\lambda = 10$, we sample s_1 uniformly from the set $\{1, 2, 3, 4, \dots, 1022, 1023, 1024\}$. Now we’re almost done! The next line says to return (output) a pair of numbers: the random number s_1 and the difference $s - s_1$.

To summarize, **Share** converts a secret s into a two shares. Okay, so how can two people, each with one of the shares, get back the original secret?

¹We introduced the symbol $\leftarrow_{\$}$ in Section 1.

That's what the right side of Figure 1 tells us. The reconstruction algorithm **Rec** takes two integers (call them s_1 and s_2) and returns their sum. That's it!

If **Rec** receives two shares that were produced by the **Share** algorithm, it will return $s_1 + (s - s_1) = s$. This means the scheme has correctness! What about privacy?

It turns out the scheme is private as well: someone who sees only one of the two shares learns nothing about the secret. This is pretty straightforward if the one share you see is s_1 : we picked this value randomly (remember this means we picked it independently of s), so it has nothing to do with s . This is the case with the share $s - s_1$ as well. If we're given a number s_2 calculated as $s - s_1$, we don't know the other share s_1 , so s_2 could be anything: $s - 0$, $s - 1$, $s - 2$, and so on. Another way to think about this is that we don't know s_1 , so we can't undo the subtraction and recover s . In this case we say that s_1 "masks" s .

These arguments for correctness and privacy are not very rigorous. We'll take a look at how cryptographers prove these properties in Sections 2.2 and 2.3.

Exercise 2.1 Pick your favorite number and secret share it using the **Share** algorithm defined above, with $\lambda = 10$. (Hint: $2^{10} = 1024$.) (Repeat this exercise until you are comfortable with this secret sharing scheme.)

Exercise 2.2 What is

- (a) $\text{Rec}(2, 6)$?
- (b) $\text{Rec}(4, 1)$?
- (c) $\text{Rec}(10, 2)$?
- (d) $\text{Rec}(115, -103)$?
- (e) $\text{Rec}(559, -544)$?

Exercise 2.3 Can you adapt this additive secret sharing scheme to output 3 shares instead of 2? How does this change the reconstruction algorithm?

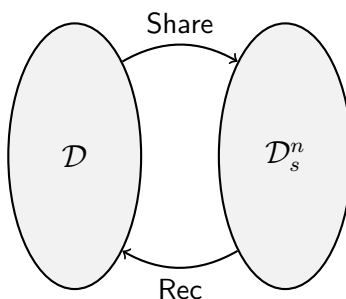
2.2 Formal Definitions*

We'll now formally what it means to be a secret sharing scheme and the properties such a scheme might have, using the standard notation in cryptography. First, we'll define what a secret sharing scheme does without giving implementation details (there could be multiple ways of achieving the same thing, after all).

Definition 1 (Secret sharing scheme). *Let \mathcal{D} be the input domain and \mathcal{D}_s be the share domain. A secret sharing scheme is a pair of efficient algorithms (Share, Rec) and an associated natural number n such that*

- Share takes as input a secret $s \in \mathcal{D}$ and outputs n shares in \mathcal{D}_s .
- Rec takes as input m shares $s_1, \dots, s_m \in \mathcal{D}_s$ and output some value $y \in \mathcal{D}$ or a special symbol \perp indicating failure. (If $m \neq n$, it outputs \perp .)

Here's a visual representation: (NG: Might labeling the right side \mathcal{D}_s^n be confusing?)



2.2.1 Correctness

Definition 2 (Correctness). *A secret sharing scheme is correct if for all $s \in \mathcal{D}$, $\text{Rec}(\text{Share}(s)) = s$.*

Example 2.1 Consider the following secret sharing scheme:

Share(s)	Rec(s_1, s_2)
$s_1 \leftarrow \{1, \dots, 2^\lambda\}$ return ($s_1, s + s_1$)	return $s_1 + s_2$

This scheme meets Definition 1 with $n = 2$. The input and share domains are both the integers, **Share** correctly outputs two integers, and **Rec** takes two integers and outputs another integer (it never returns \perp).

Does this scheme have correctness? Answer in your head before reading on.

If we compare this to the simple scheme in Figure 1, the answer is pretty clear: no, this does not meet the correctness requirement. A little bit of arithmetic confirms this: $\text{Rec}(\text{Share}(s)) = \text{Rec}(s_1, s + s_1) = s_1 + (s + s_1) = s + 2s_1$, which is not equal to s (s_1 always greater than or equal to 1).

Example 2.2 Let's look at another scheme:

<u>Share(s)</u>	<u>Rec(s_1, s_2)</u>
return $(3, \lfloor s/3 \rfloor)$	return $s_1 \cdot s_2$

The $\lfloor \cdot \rfloor$ notation is the “floor” operation, which rounds a decimal down to the integer below it. For example, $\lfloor 3.725 \rfloor = 3$.

Again, this is a secret sharing scheme according to Definition 1. But is it correct? Answer to yourself again before reading on.

This example is a little trickier because it works in some cases – but not all! When $s = 15$,

$$\text{Rec}(\text{Share}(15)) = \text{Rec}(3, 5) = 3 \cdot 5 = 15,$$

which is correct. But in the case of $s = 14$,

$$\text{Rec}(\text{Share}(14)) = \text{Rec}(3, 4) = 3 \cdot 4 = 12,$$

which is not equal to 14. So this scheme isn't always correct, and since the definition of correctness is all or nothing², the scheme doesn't have correctness.

²we said $\text{Rec}(\text{Share}(s)) = s$ must hold for *all* s in the set \mathcal{D} , which in this example is the integers. But 14 was an integer for which the equality didn't hold.

Example 2.3 So setting s_1 to always be 3 works when sharing some numbers but not others. Let's go back to picking s_1 randomly:

$\text{Share}(s)$	$\text{Rec}(s_1, s_2)$
$s_1 \leftarrow_{\$} \{1, \dots, 2^\lambda\}$	return $s_1 \cdot s_2$
return $(s_1, \lfloor s/s_1 \rfloor)$	

This still meets Definition 1. Is the adjusted scheme correct now?

What happens if we share the number 14? First, we pick s_1 randomly. Say we happen to choose 7. Then

$$\text{Rec}(\text{Share}(14)) = \text{Rec}(7, 2) = 14,$$

which is correct. But what if we had randomly chosen s_1 to be 5? Then

$$\text{Rec}(\text{Share}(14)) = \text{Rec}(5, 2) = 10,$$

which is not 14! So even though this scheme *can* work for all $s \in \mathcal{D}$, correctness could fail randomly depending on the choice of s_1 . So it is still not correct.

Even though all the examples above didn't meet correctness, remember that correct secret sharing schemes do exist. For example, we already argued informally that the simple additive scheme from Figure 1 meets correctness by observing that $\text{Rec}(\text{Share}(s)) = \text{Rec}(s_1, s - s_1) = s_1 + (s - s_1) = s + (s_1 - s_1) = s$.

Now that we understand correctness, let's look at privacy.

2.2.2 Privacy

In cryptography, security properties like privacy are defined using what are called “games”. A game is a challenge in which an attacker (called the *adversary* and usually denoted by the curly letter \mathcal{A}) is given some information and tries to break the security property of the scheme. \mathcal{A} “wins” the game if it can give an answer that proves it broke the security property of the scheme.

For example, in the case of privacy for a secret sharing scheme, we give the attacker a share and ask it to give us some information about the secret it came from. This should be almost impossible if the scheme is private.

Here is the *privacy game* for any secret sharing scheme $\mathcal{S} = (\text{Share}, \text{Rec})$:

1. The adversary \mathcal{A} picks two values x_0, x_1 and a number i between 1 and n .

2. The game flips a coin to randomly choose one of those two values. This is usually written as picking a random value b from the set $\{0, 1\}$.
3. Now the game runs the **Share** algorithm on this randomly chosen value x_b to get shares s_1, \dots, s_m . It gives the i th share to \mathcal{A} .
4. \mathcal{A} tries to guess which of the two values x_0, x_1 were shared. More specifically, it outputs a guess $b' \in \{0, 1\}$.
5. If $b = b'$, we say the result of the game is 1 (to signify **true** or **success**), in which case we say that \mathcal{A} “wins” the game; otherwise, it’s 0.

In cryptography, these games are generally written much more compactly by using symbols. The privacy game above would be written as follows (without the comments, which are there to explain the notation):

<div style="border-bottom: 1px solid black; margin-bottom: 5px;"> $\text{SS-priv}_{\mathcal{A}, \mathcal{S}}$ </div> <div> $(x_0, x_1, i) \leftarrow \mathcal{A} \quad // \text{ get input values from } \mathcal{A}$ $b \leftarrow_{\\$} \{0, 1\} \quad // \text{ decide randomly which value to share}$ $s_1, \dots, s_m \leftarrow \text{Share}(x_b) \quad // \text{ share the chosen value}$ $b' \leftarrow \mathcal{A}(s_i) \quad // \mathcal{A} \text{ uses the } i\text{th share to guess } b'$ return $b = b' \quad // \text{ return 1 if } b = b' \text{ and 0 otherwise}$ </div>

Figure 2: The secret sharing privacy game.

Where did this definition of the game come from, you might ask? The simple answer is that usually the person who invents a new cryptographic primitive (what we call a general idea like secret sharing or encryption, as opposed to a specific algorithm for actually achieving what it describes) also gives definitions for its potential properties. This includes defining security games. (Sometimes, others come along later and describe new properties for an existing primitive; in this case, they might describe a game for the new property.)

If you look at many security games, though, you’ll see that they are usually pretty similar to each other. This is because it’s useful for a new game to be easy to work with, since it makes other people more likely to build on that work. If a new game is similar to an already existing game, people who are familiar with the previous game can understand the new game quickly and prove things about a new scheme more easily.

***Bonus Exercise* 2.4** With a partner, play through the privacy game a couple times using the additive secret sharing scheme. One of you should take on the role of the game while the other acts as the adversary. If you’re playing the part of the game, make sure you use something truly random, like a coin flip or the Python command `random.randint(0,1)`³, to pick your bit b . After you’ve done this a couple of times, switch roles and repeat.

How successful was the adversary? If you were the adversary, what was difficult about your role? What was the key part of the scheme that ensured privacy?

Now that we have an idea of why the game is difficult to win consistently, let’s rigorously define privacy by specifying how often the adversary should be able to win:

Definition 3 (secret sharing privacy). *A secret sharing scheme $\mathcal{S} = (\text{Share}, \text{Rec})$ is private if, for all adversaries \mathcal{A} ,*

$$\Pr[\text{SS-priv}_{\mathcal{A},\mathcal{S}}(t, n) = 1] - \frac{1}{2}$$

*is small*⁴. *This quantity is called \mathcal{A} ’s advantage.*

What does this actually mean? If \mathcal{A} ’s advantage is small, it means that $\Pr[\text{SS-priv}_{\mathcal{A},\mathcal{S}}(t, n) = 1]$ is very close to one-half. In other words, no matter what adversary \mathcal{A} we are dealing with, the probability that it can win privacy game (i.e., the game outputs 1) should be very close to one-half, which is what it should be if \mathcal{A} were to randomly guess which value was shared.

This might sound very different from the informal definition from Section 2: each share by itself reveals nothing about the secret. The game instead says that \mathcal{A} can’t tell the difference between two different secrets. But if you think about this a little more, you can see that they are related: if \mathcal{A} can tell the difference between two secrets based on a single share, it means the share gave away some information about the secret it came from.

Let’s work through some examples. For the remainder of the packet, you can assume that all schemes are secret sharing schemes that meet Definition 1.

³Be sure to `import random` first.

⁴In cryptography, this usually means bounded by a *negligible* function. For the purposes of this packet, “small” means very close to 0, for example $\frac{1}{2^\lambda}$, which for $\lambda = 10$ is $\frac{1}{1024}$. Since the advantage ranges from 0 to $\frac{1}{2}$, a value like $\frac{1}{4}$ or $\frac{1}{8}$ is not small.

Example 2.4 Here’s a secret sharing scheme for sharing integers 1 to 99 (i.e., $\mathcal{D} = \{1, \dots, 99\}$) among two parties:

Share (s)	Rec (s_1, s_2)
s_1 = the tens digit of s	return $s_1 s_2$
s_2 = the ones digit of s	
return (s_1, s_2)	

where $||$ means concatenation (for example, $1||2 = 12$). Do you think this scheme is private?

Let’s put ourselves in the adversary’s shoes. We want to win the privacy game. First, we get to pick two numbers $x_0, x_1 \in \mathcal{D}$ and an index i . Let’s say $x_0 = 18$ and $x_1 = 24$. We’ll let $i = 2$ (it doesn’t actually matter what i is in this case). Now the game picks a random b unknown to us and shares x_b . It’ll give us the second share, s_2 .

What are the possibilities for s_2 ? If $b = 0$, the game runs **Share**(18), which outputs (1, 8). Then $s_2 = 8$. If $b = 1$, on the other hand, the game will run **Share**(24) to get (2, 4) and $s_2 = 4$. Say we get 8. Then we’ll guess $b' = 0$. If we get 4 from the game, we’ll guess $b' = 1$. Because of how the secret sharing scheme works (“by definition”), we’ll always guess correctly, and $b = b'$ with probability 1! This means that our advantage is $\frac{1}{2}$, which is not small! Therefore, this scheme is not private.

You might say, well, duh! This was obvious from the informal definition of privacy: if we’re given a digit of the secret, we’re clearly learning something about the secret! Looking at the adversary’s advantage in the game, however, makes more sense mathematically. In the case of a much more complicated scheme, it might be unclear what “learning something about the secret” really means.

Example 2.5 We already saw that the scheme in Example 2 is not correct. But is it private?

You might think so at first, since the number 3 (the first share) is unrelated to the secret, and the second share isn’t giving away the secret if the adversary doesn’t know that it’s just the secret

minus 3. But in reality, we can't assume that the adversary doesn't know the inner workings of the **Share** algorithm.

This means that \mathcal{A} can consistently win the game if it sets $i = 2$. It first picks two values x_0, x_1 that are multiples of 3. When it gets s_2 from the game, it multiplies it by 3 and compares that value to x_0 and x_1 . If it equals x_0 , it outputs $b' = 0$; otherwise, it outputs $b' = 1$. By the definition of the scheme, \mathcal{A} wins with probability 1, so its advantage is $\frac{1}{2}$, so the scheme is not private.

***Bonus Exercise* 2.5** Is the following scheme private?

Share (s)	Rec (s_1, s_2)
if s is even then $s_1 \leftarrow \{1, \dots, 2^\lambda/2\}$ else $s_1 \leftarrow \{1, \dots, 2^\lambda\}$ return $(s_1, s - s_2)$	return $s_1 + s_2$

2.3 Proving Security*

So far, we've only proven that secret sharing schemes are *not* private. Let's work through a proof of privacy using the additive secret sharing scheme as an example.

Theorem 1. *The additive secret sharing scheme defined in Figure 1 is private.*

Proof. Let x_0 and x_1 be any two integers. Define $s_{0,i}$ as the i th share output by **Share**(x_0) and $s_{1,i}$ as the i th share output by **Share**(x_1).

If $i = 1$, then $s_{0,1}$ and $s_{1,1}$ are distributed uniformly at random by the definition of **Share** (the first share is a uniformly random value). Thus, they are indistinguishable, and any adversary \mathcal{A} can only randomly guess the value of b . So

$$\Pr[b = b' \mid i = 1] = \frac{1}{2}.$$

If $i = 2$, then $s_{0,2} = s - s_{0,1}$ and $s_{1,2} = s - s_{1,1}$. But as we said before, $s_{0,1}$ and $s_{1,1}$ are uniform, which implies that $s_{0,2}$ and $s_{1,2}$ are also uniform. Therefore they are indistinguishable, and

$$\Pr[b = b' \mid i = 2] = \frac{1}{2}.$$

Putting the two cases together, we see that for all possible values of x_0, x_1, i (and thus for all adversaries possible adversaries \mathcal{A}),

$$\begin{aligned} \Pr[b = b'] &= \Pr[b = b' \mid i = 1] \cdot \Pr[i = 1] + \\ &\quad \Pr[b = b' \mid i = 2] \cdot \Pr[i = 2] \\ &= \frac{1}{2} (\Pr[i = 1] + \Pr[i = 2]) \\ &= \frac{1}{2} (1) \\ &= \frac{1}{2} \end{aligned}$$

From the definition of the game,

$$\Pr[\text{SS-priv}_{\mathcal{A},\mathcal{S}} = 1] = \Pr[b = b'],$$

so the advantage for any adversary \mathcal{A} is

$$\Pr[\text{SS-priv}_{\mathcal{A},\mathcal{S}} = 1] - \frac{1}{2} = 0$$

which is clearly close to 0! So the additive secret sharing scheme is private. \square

3 Shamir's Secret Sharing

Until now, we've only seen n -out-of- n secret sharing schemes. The “ n -out-of- n ” part means that the reconstruction algorithm needs at least n out of the n total shares to recover the secret: that is, it needs all of the shares to recover the secret. For example, if the **Share** algorithm outputs 2 shares, we need both shares to reconstruct.

In general, though, $(t + 1)$ -out-of- n secret sharing schemes exist for any integers t and n . (t stands for “threshold”, since it determines the minimum number of parties necessary for reconstruction.) In this section, we'll see how to construct such a secret sharing scheme using the properties of polynomials.

3.1 Polynomials

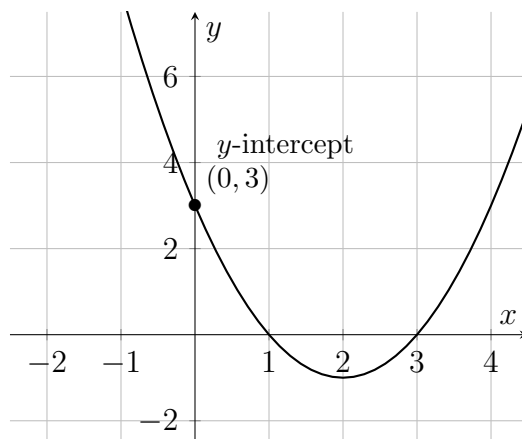
A *polynomial* is an expression consisting of powers of a variable (or several variables, but we'll stick with polynomials in one variable in this packet) multiplied by numbers called coefficients. Here's an example:

$$x^2 - 4x + 3$$

The standard form for a polynomial in one variable, called a univariate polynomial, is

$$a_n x^n + a_{n-2} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \quad (2)$$

where the a_n, \dots, a_1 are constant (fixed) values, usually integers, and n is a positive integer called the *degree* of the polynomial. The example polynomial above is a degree-2 polynomial.



You may have plotted polynomials before to show how the value of the polynomial changes with different values of x . A plot of our example polynomial is shown above. In that case, we are plotting the equation

$$f(x) = x^2 - 4x + 3$$

where $f(x)$ is read as “ f of x ” and indicates that the expression to the right of the equal sign is a function of the variable x . The *y-intercept* of a function is the place it crosses the y -axis, i.e. its value when $x = 0$. In our example, the y -intercept is 3. Notice that we could have calculated it without plotting the equation:

$$f(0) = (0)^2 - 4(0) + 3 = 3$$

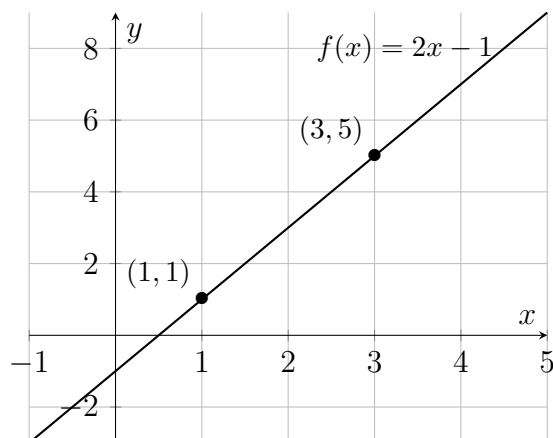
When a polynomial is written in standard form as in (2), the y -intercept is a_0 .

Exercise 3.1 Write down the degree and y -intercept of each of the following polynomials:

- (a) $f(x) = x^2 + 3x - 1$
- (b) $f(x) = 5x^2 + 11$
- (c) $f(x) = -2x^3 - x^2 + 9x$
- (d) $f(x) = 3x^5 - 2x^3 - 15$
- (e) $f(x) = (x^2 - 1)(x + 3)$
- (f) $f(x) = 2(x - 6)(x + 2)(x - 5)$
- (g) $f(x) = 2x^3(x + 16)$

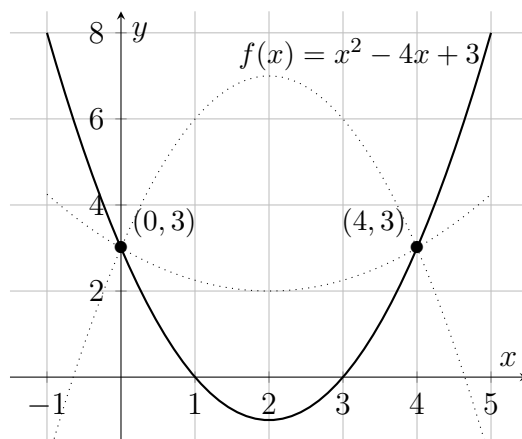
3.1.1 Uniqueness

An important property of polynomials is that they can be uniquely defined by a set of points of the right size. For example, two points uniquely define a line:

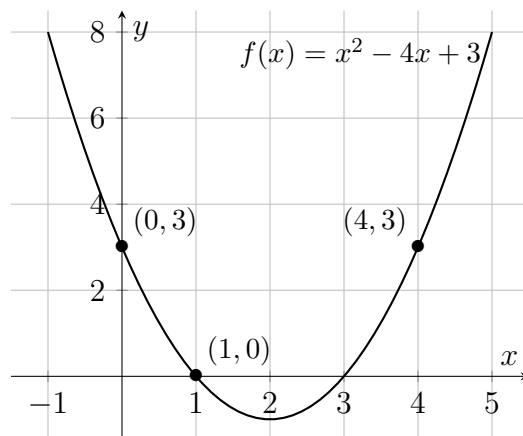


Remember that a line can be viewed as a degree-1 polynomial! So, instead of giving someone the equation for this line, you could give them 2 points (for example, (1, 1) and (3, 5)) and they'd still know exactly what line you're talking about.

To specify a particular degree-2 polynomial, two points aren't enough. For example, $x^2 - 4x + 3$ goes through the points (0, 3) and (4, 3), but so do many other degree-2 polynomials (represented here with dotted lines):



Instead, 3 points are needed to uniquely define a degree-2 polynomial:



How many points do you think are needed to uniquely define a degree- n polynomial? Once you think you have an answer, flip to the next page.

The process of recovering a polynomial passing through a set of points is called *interpolation*. Accordingly, this general rule about the uniqueness of a polynomial described by a set of points is called the interpolation theorem:

INTERPOLATION THEOREM.

Given a set of $t+1$ points, there exists a unique degree- t polynomial passing through those points.

Exercise 3.2 How many points are needed to uniquely define each polynomial in the previous exercise?

If getting from a set of points to an equation sounds difficult, don't worry because there are well-known techniques that always work. Most programming languages actually have these built-in, so you don't need to know the details; you just plug points in and out comes a polynomial! In case you want to know more, though, the next section explains how one of these techniques works.

3.1.2 Lagrange Interpolation*

In this section we'll learn about one method of doing polynomial interpolation called *Lagrange interpolation*. It's described by a single equation:

$$P(x) = \sum_{i=0}^{n-1} y_i \ell_i(x), \text{ where } \ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n-1} \frac{x - x_j}{x_i - x_j} \quad (3)$$

Let's break it down bit by bit. First, in case you aren't familiar with the symbols, $\sum_{i=0}^{n-1}$ means to evaluate the expression after \sum for each value of i and then add all those terms together. For example:

$$\sum_{i=1}^n i = 1 + 2 + \dots + n$$

\prod is similar, but we multiply the expressions instead of adding them:

$$\prod_{i=1}^n i = 1 \cdot 2 \cdot \dots \cdot n$$

So, $\sum_{i=1}^4 i = 1 + 2 + 3 + 4 = 10$ and $\prod_{i=1}^4 i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

***Bonus Exercise* 3.3** Evaluate the following expressions:

- (a) $\sum_{i=1}^5 2i$
- (b) $\sum_{i=1}^5 1$
- (c) $\sum_{i=0}^4 x_i$ where x_i means the i th elements of the set $\{1, 0, 5, -3, 8\}$ and the numbering starts at 0 (that is, $x_0 = 1$).
- (d) $\sum_{i=0}^2 x_i$ for the same set.

***Bonus Exercise* 3.4** Repeat the previous exercise, substituting \prod for \sum .

Now, back to Lagrange interpolation. We start with a set of points $(x_0, y_0), \dots, (x_n, y_n)$. For each point (x_i, y_i) , we compute the expression $\ell_i(x)$. For instance, for $i = 2$, it will be of the form

$$\frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1} \frac{x - x_3}{x_2 - x_3} \dots \frac{x - x_n}{x_2 - x_n}$$

Then we plug that expression $\ell_i(x)$, along with the y -value y_i for that point, into the summation, and we'll end up with a polynomial. If we did everything right, that will be exactly the polynomial defined by those points.

Let's work through an example using the degree-1 polynomial from Section 3.1.1: we'll use the points $(1, 1), (3, 5)$ to recover the equation of the line. Remember that we start numbering the elements of a set at 0, so $(1, 1)$ is the 0th point and $(3, 5)$ is the 1st.

Example 3.1 Use Lagrange interpolation to get a degree-2 polynomial passing through the points $(1, 1), (3, 5)$.

Since we have 2 points, $n = 2$:

$$\begin{aligned} P(x) &= \sum_{i=0}^1 y_i \ell_i(x) \\ &= y_0 \ell_0(x) + y_1 \ell_1(x) \end{aligned}$$

Next, let's substitute in the y -coordinates of our points:

$$= 1 \cdot \ell_0(x) + 5 \cdot \ell_1(x)$$

Now let's evaluate the polynomials $\ell_i(x)$:

$$\begin{aligned}
 \ell_0(x) &= \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{x - x_j}{x_0 - x_j} \\
 &= \frac{x - x_1}{x_0 - x_1} \\
 &= \frac{x - 3}{1 - 3} = \frac{x - 3}{-2} \\
 \ell_1(x) &= \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{x - x_j}{x_1 - x_j} \\
 &= \frac{x - x_0}{x_1 - x_0} \\
 &= \frac{x - 1}{3 - 1} = \frac{x - 1}{2}
 \end{aligned}$$

Plugging that back into the sum, we get

$$\begin{aligned}
 P(x) &= 1 \cdot \ell_0(x) + 5 \cdot \ell_1(x) \\
 &= 1 \left(\frac{x - 3}{-2} \right) + 5 \left(\frac{x - 1}{2} \right) \\
 &= \frac{-(x - 3)}{2} + \frac{5(x - 1)}{2} \\
 &= \frac{5x - 5 - (x - 3)}{2} \\
 &= \frac{4x - 2}{2} = 2x - 1
 \end{aligned}$$

That's the same equation as the one we graphed!

Example 3.2 Let's do the degree-3 example from Section 3.1.1

next. Our set of points is $\{(0, 3), (1, 0), (4, 3)\}$ and $n = 3$.

$$\begin{aligned}
 P(x) &= \sum_{i=0}^2 y_i \ell_i(x) \\
 &= y_0 \ell_0(x) + y_1 \ell_1(x) + y_2 \ell_2(x) \\
 &= 3 \cdot \ell_0(x) + 0 \cdot \ell_1(x) + 3 \cdot \ell_2(x)
 \end{aligned}$$

The polynomials $\ell_i(x)$ are:

$$\begin{aligned}
 \ell_0(x) &= \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{x - x_j}{x_0 - x_j} \\
 &= \frac{x - x_1}{x_0 - x_1} \frac{x - x_2}{x_0 - x_2} \\
 &= \frac{x - 1}{0 - 1} \frac{x - 4}{0 - 4} \\
 &= \frac{x - 1}{-1} \frac{x - 4}{-4} = \frac{(x - 1)(x - 4)}{4} \\
 \ell_1(x) &= \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{x - x_j}{x_1 - x_j} \\
 &= \frac{x - x_0}{x_1 - x_0} \frac{x - x_2}{x_1 - x_2} \\
 &= \frac{x - 0}{1 - 0} \frac{x - 4}{1 - 4} \\
 &= \frac{x}{1} \frac{x - 4}{-3} = \frac{x(x - 4)}{-3} \\
 \ell_2(x) &= \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{x - x_j}{x_2 - x_j} \\
 &= \frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1} \\
 &= \frac{x - 0}{4 - 0} \frac{x - 1}{4 - 1} \\
 &= \frac{x}{4} \frac{x - 1}{3} = \frac{x(x - 1)}{12}
 \end{aligned}$$

Now we can simplify $P(x)$ to get:

$$\begin{aligned}
P(x) &= 3 \cdot \ell_0(x) + 0 \cdot \ell_1(x) + 3 \cdot \ell_2(x) \\
&= 3 \cdot \frac{(x-1)(x-4)}{4} + 0 \cdot \frac{x(x-4)}{-3} + 3 \cdot \frac{x(x-1)}{12} \\
&= \frac{3(x-1)(x-4)}{4} + \frac{3x(x-1)}{12} \\
&= \frac{9(x-1)(x-4) + 3x(x-1)}{12} \\
&= \frac{9(x^2 - 5x + 4) + (3x^2 - 3x)}{12} \\
&= \frac{9x^2 - 45x + 36 + 3x^2 - 3x}{12} \\
&= \frac{12x^2 - 48x + 36}{12} \\
&= x^2 - 4x + 3
\end{aligned}$$

and again we've arrived at the same polynomial we graphed.

***Bonus Exercise* 3.5** Use Lagrange interpolation to find the unique degree-2 polynomial through the points $\{(-1, -16), (1, -2), (2, 28)\}$.

3.2 Sharing Secrets Using Polynomials

Shamir secret sharing is a $(t+1)$ -out-of- n secret sharing scheme, for some numbers t and n . This means that we split the secret s into n values and distribute them to n people. Then, at least $t+1$ of those people must work together to recover s .

Let's update our definition of secret sharing from Section 2.2 to include $(t+1)$ -out-of- n secret sharing schemes where $t+1 \neq n$.

Definition 4 (Secret sharing scheme (updated)). *Let \mathcal{D} be the input domain and \mathcal{D}_s be the share domain. A secret sharing scheme is a pair of efficient algorithms $(\text{Share}, \text{Rec})$ and two associated natural numbers t, n such that*

- *Share takes as input a secret $s \in \mathcal{D}$ and outputs n shares in \mathcal{D}_s .*

- **Rec** takes as input m shares $s_1, \dots, s_m \in D_s$ and output some value $y \in \mathcal{D}$ or a special symbol \perp indicating failure. (If $m < t + 1$, it outputs \perp .)

Now we're ready to put everything we've learned together and define Shamir's secret sharing scheme⁵.

<u>Share(s)</u>	<u>Rec(s_1, \dots, s_m)</u>
$a_1, \dots, a_t \leftarrow \mathbb{S} \{1, \dots, 2^\lambda\}$	if $m < t + 1$ then
$a_0 = s$	return \perp
$f(x) = a_t x^t + \dots + a_1 x + a_0$	else
return $((1, f(1)), \dots, (n, f(n)))$	$f(x) = \text{interpolate}(s_1, \dots, s_m)$
	return $f(0)$

Figure 3: Shamir's secret sharing scheme

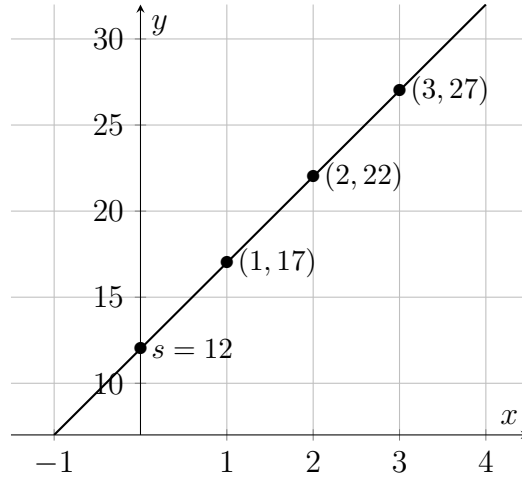
To share a secret s in \mathcal{D} , we choose a random degree- t polynomial f by sampling t random coefficients and setting f 's y -intercept to the secret. Then we pick n points on f (the convention is to evaluate f at 1, 2, and so on, up to n)⁶. These n points are the shares.

To reconstruct, we need at least $t + 1$ points. The reconstruction algorithm takes these points $s_i = (x_i, y_i)$ and tries to recover f using interpolation. (If there are not enough points, interpolation would fail, so we return the symbol \perp to indicate an error.) Once we recover a polynomial, we evaluate it at 0 to find its y -intercept and output that.

Example 3.3 Here's how we would compute **Share**(12) with $t = 1$ and $n = 3$. First, we pick one random integer a_1 , say 5. Then $f(x) = 5x + 12$. Our three shares are (1, 17), (2, 22), (3, 27). Below is a visual representation.

⁵Adi Shamir introduced this scheme in a short 1979 paper entitled "How to share a secret"[1]. The paper is only two pages long, so if you're feeling adventurous you could have a go at it! You can find it online at <http://web.mit.edu/6.857/OldStuff/Fall103/ref/Shamir-HowToShareASecret.pdf>.

⁶To be exact, Shamir's secret sharing evaluates f in a way that ensures the values $f(x_i)$ are elements of something called a *finite field*. We won't go into details here about what that means, since such polynomials can't be graphed in two dimensions, but just know that this is important for the scheme to be truly private.



Example 3.4 Now let's say we receive two of the previous shares to reconstruct: $(1, 17), (3, 27)$. We know they are 2-out-of-3 shares (this is realistic; in real life, t and n would be known to all the parties, or the points would be labeled with the values of t and n used to generate them.) How do we reconstruct?

First of all, we know that *can* reconstruct at all because $t+1 = 2$ and we have at least two shares (exactly two, in fact). Now we need to interpolate to recover the degree-2 polynomial they represent. (If you didn't read Section 3.1.2, you can skip to the top of the next page.)

Let's re-number the input shares starting with 0, so $(x_0, y_0) = (1, 17)$ and $(x_1, y_1) = (3, 27)$.

$$\begin{aligned} f(x) &= \sum_{i=0}^m y_i \ell_i(x) \\ &= y_0 \ell_0(x) + y_1 \ell_1(x) \\ &= 17 \ell_0(x) + 27 \ell_1(x) \end{aligned}$$

At this point, we can take a little shortcut. We know we only care about finding the value of f at 0, which means we only need

to find $\ell_0(0)$ and $\ell_1(0)$ instead of the full expressions. So,

$$\ell_0(0) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{0 - x_j}{x_0 - x_j} = \frac{-3}{1-3} = \frac{3}{2}$$

$$\ell_1(0) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{0 - x_j}{x_1 - x_j} = \frac{-1}{3-1} = -\frac{1}{2}$$

Then

$$\begin{aligned} f(0) &= 17\ell_0(0) + 27\ell_1(0) \\ &= 17 \cdot \frac{3}{2} + 27 \cdot -\frac{1}{2} \\ &= \frac{51 - 27}{2} \\ &= \frac{24}{2} = 12 \end{aligned}$$

Exercise 3.6 Work in a small group. Everyone in the group should pick a secret number to share. Let n be the number of people in your group and pick t so that $t + 1 < n$. Use **Share** to compute shares of your secret and give each person in the group one share. Now a subgroup of $t + 1$ people should work together to reconstruct the secret using **Rec**. Once they succeed, form a different group of $t + 1$ people and run **Rec** again using this new group of points. You should get the same result!

Notice that in the scheme presented in Figure 3, it's possible for someone to lie about their point, thereby causing the interpolation algorithm to return a different polynomial besides f . In that case, $f(0)$ might not equal s and we'd recover the wrong secret!

This doesn't violate correctness, however, because in that case we aren't running **Rec** on outputs of the **Share** algorithm, so the correctness definition doesn't apply. Instead, what's happening is that our scheme fails when the parties don't behave honestly. In cryptography, this means that the scheme is

only secure in the presence of *semi-honest* adversaries (the scheme has *semi-honest security*.) There are ways of fixing this scheme to guarantee security against *malicious* adversaries (*malicious security*), but that's outside the scope of this packet.

Exercise 3.7 Visit <https://bit.ly/ShamirSS> in your browser. This is a Google Colab notebook written in the Python programming language. It already has the functions `Share` and `Rec` from Shamir's secret sharing scheme. Work through the examples in the notebook to share and reconstruct any numbers you want using this scheme, then read on to find out how to share secret messages!

References

- [1] Adi Shamir. How to share a secret. *Communications of the ACM*, 22(11):612–613, 1979.