Wave propagation:

Solid Mechanics

Nicolás Guarín Zapata

nguarin@purdue.edu

Slides available at: https://github.com/nicoguaro/CE597-slides

Civil Engineering Department
Purdue University

September 24, 2014

Outline

Introduction

A wave is the propagation of certain medium property perturbation, e.g., density, pressure, electric field or magnetic field, that travel through the space transporting energy. The perturbed medium could be of diverse nature like air, water, a piece of metal or vacuum.

The property that presents the phenomena is expressed as function of position and time $\psi(\vec{r}, t)$. Mathematically it is said that ψ is a wave if it verifies the wave equation:

$$\nabla^2 \psi(\vec{r}, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \psi(\vec{r}, t) \ ,$$

where v is the wave propagation speed. For instance, some perturbations of the pressure, called sound, verify the above equation.

Global Seismic Wave Propagation Simulation

Figure: Global Seismic Wave Propagation Simulation.[1]

[1] Contributors from ICES, The University of Texas at Austin: Carsten Burstedde, Omar Ghattas, James R. Martin, Georg Stadler, Lucas C. Wilcox Visualization at the Texas Advanced Computing Center, the University of Texas at Austin by Greg Abram: `http://vimeo.com/30813579`

Global Seismic Wave Propagation Simulation

Figure: Vertical Plane SH wave in an asymmetric canyon.[2]

[2] Juan Diego Jaramillo et al. 2012

Scattering of a pressure wave around a long cylinder

Figure: Scattering of a pressure wave around a long cylinder

Wave characteristics

There are great variety of waves but all of them could experiment

- **Reflection**:Occurs when a wave find a new medium, that can not cross, change its direction.
- **Refraction**: Occurs when a wave change its direction when enter in a new medium with different propagation speed.
- **Doppler Effect**: Effect caused by the relative motion between the source and the receptor.
- **Interference**: Occurs when two or more waves coexist in the same place and are superimposed.
- **Diffraction**: Occurs when a wave find the border of an obstacle and change its *form* to round it.

Waves in a string

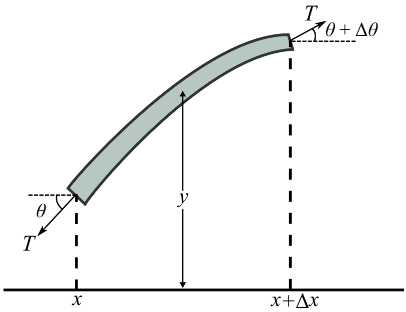

Figure: Forces diagram over an element of the string with length Δx.

For a string with length L, linear mass density μ and a tension T, let's take a small segment with small displacements in y. The force balance over the element showed in the Figure is:

$$F_y = T\sin(\theta + \Delta\theta) - T\sin(\theta)$$
$$F_x = T\cos(\theta + \Delta\theta) - T\cos(\theta) \quad ,$$

Waves in a string

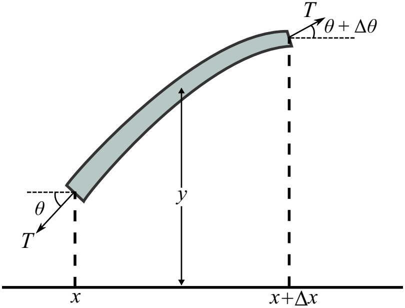

Figure: Forces diagram over an element of the string with length Δx.

Taking small displacements in the string, the angles are also small and then

$$F_y \approx T(\theta + \Delta\theta) - T(\theta) = T\Delta\theta$$
$$F_x \approx 0 \ .$$

Waves in a string I

From the second Newton's law we get

$$T\ \Delta\theta = \underbrace{(\mu\ \Delta x)}_{\text{mass}} a_y\ \ ,$$

if we take the limit $\Delta x \to dx$

$$T\ d\theta = (\mu\ dx) a_y\ \ . \tag{1}$$

And we know that $\tan\theta = \frac{\partial y}{\partial x}$, taking derivative respect x

$$\sec^2\theta \frac{d\theta}{dx} = \frac{\partial^2 y}{\partial x^2}\ \ .$$

Due to small displacements $\sec^2\theta \approx 1$, hence

$$d\theta \approx \frac{\partial^2 y}{\partial x^2} dx \tag{2}$$

Waves in a string II

and replacing (2) in (1)

$$T\frac{\partial^2 y}{\partial x^2}dx = (\mu\ dx)\frac{\partial^2 y}{\partial t^2} \quad ,$$

we finally get the 1D Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} \quad .$$

T/μ have square speed units, and is the square propagation speed

$$v = \sqrt{\frac{T}{\mu}} \quad ,$$

so

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad . \tag{3}$$

Solution for the 1D case I

The wave equation could be solved by separation of variables, i.e., let's suppose it has a solution of the form $y = y(x,t) = X(x)T(t) = X\ T$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{1}{v^2T}\frac{d^2T}{dt^2} = 0 \ ;$$

we can see that the temporal and the spatial terms are isolated, so they should be equal to a constant. For simplicity let's take $-\alpha^2$, with $\alpha \in \mathbb{R}$

$$\frac{d^2X}{dx^2} = -\alpha^2 X, \qquad \frac{d^2T}{dt^2} = -\alpha^2 v^2 T \ ,$$

so the solution is

$$y = [C_1 \sin(\alpha x) + C_2 \cos(\alpha x)]\,[C_3 \sin(\alpha v t) + C_4 \cos(\alpha v t)] \ .$$

Solution for the 1D case II

If we take fixed extremes and assume that the string starts with a deformed shape $f(x)$, i.e., $y(0,t) = y(L,t)$, $y(x,0) = f(x)$ and $y'(x,0) = 0$ can be shown that the complete solution is

$$y(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \right] \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi v t}{L}\right) \quad .$$

Example in 1D

Figure: Wave traveling in a string.

Wavelength

Once the speed of propagation is known, the frequency of the sound produced by the string can be calculated. The speed of propagation of a wave is equal to the wavelength λ divided by the period τ, or multiplied by the frequency f:

$$v \equiv \sqrt{\frac{T}{\mu}} = \frac{\lambda}{\tau} = \lambda f \quad .$$

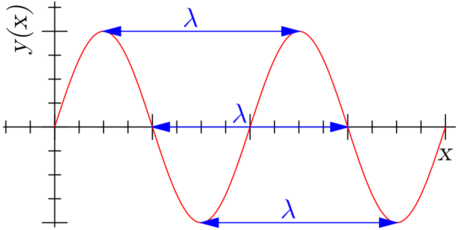

Figure: Wavelength of a sine wave, λ, can be measured between any two points with the same phase, such as between crests, or troughs, or corresponding zero crossings as shown.

Navier-Cauchy Equations: Starting Point

In a three dimensional solid we can describe waves starting from the continuity equation, motion equations and constitutive relationships.

- Continuity:
$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x_i}\left(\rho \frac{\partial}{\partial t} u_i\right)$$
- Equations of Motion:
$$\frac{\partial^2}{\partial t^2}(\rho u_i) = \frac{\partial}{\partial x_j}\sigma_{ij} + f_i$$
- Constitutive relationships:
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \quad ,$$

where u_i is the displacement of a material point in i direction, ρ is the mass density, σ_{ij} the stress tensor, ϵ_{kl} is the strain tensor, f_i are *body forces* and C_{ijkl} depends on the material of the medium.

Continuity equation I

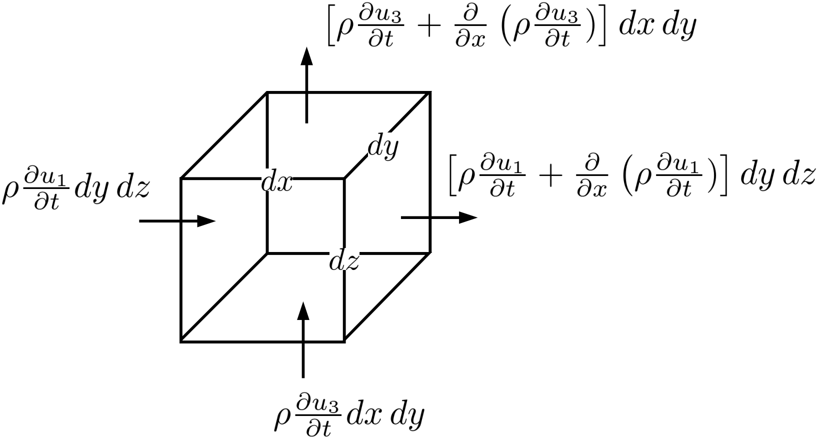

Figure: Fluxes on a parallelepiped element.

Continuity equation II

The total mass of the element is $M_T = \rho dx\, dy\, dz$, and its rate is

$$\frac{\partial M_T}{\partial t} = \frac{\partial \rho}{\partial t} dx\, dy\, dz \quad .$$

We can compute the total flow through the x direction:

$$\left[\rho \frac{\partial u_1}{\partial t} + \frac{\partial}{\partial x}\left(\rho \frac{\partial u_1}{\partial t}\right) dx\right] dy\, dz - \rho \frac{\partial u_1}{\partial t} dy\, dz = \frac{\partial}{\partial x}\left(\rho \frac{\partial u_1}{\partial t}\right) dx\, dy\, dz \quad .$$

Similarly, for the other directions

$$\frac{\partial}{\partial y}\left(\rho \frac{\partial u_2}{\partial t}\right) dx\, dy\, dz, \qquad \frac{\partial}{\partial z}\left(\rho \frac{\partial u_3}{\partial t}\right) dx\, dy\, dz \quad ,$$

and equating to the first expression

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho \frac{\partial u_1}{\partial t}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial u_2}{\partial t}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial u_3}{\partial t}\right) = 0 \quad .$$

Continuity equation III

In index notation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}\left(\rho \frac{\partial u_i}{\partial t}\right) \quad \text{Summation over } i \ ,$$

and, in vector notation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \frac{\partial \vec{u}}{\partial t}\right) \ .$$

Balance of momentum: “Equations of Motion” I

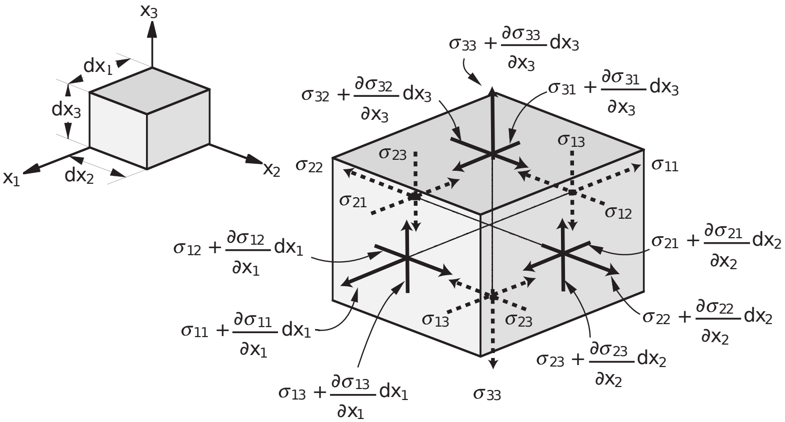

Figure: Stresses on a parallelepiped element.

Balance of momentum: "Equations of Motion" II

We are going to do the balance of forces in the x_1 direction

$$
\begin{aligned}
\rho \frac{\partial^2 u_1}{\partial t^2} dx\, dy\, dz = & \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) dx_2\, dx_3 - \sigma_{11} dx_2\, dx_3 \\
& + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 \right) dx_1\, dx_3 - \sigma_{21} dx_1\, dx_3 \\
& + \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3 \right) dx_1\, dx_2 \\
& - \sigma_{31} dx_1\, dx_2 + f_1 dx_1\, dx_2\, dx_3 \\
= & \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \right) dx_1\, dx_2\, dx_3
\end{aligned}
$$

upon dividing by $dx_1\, dx_2\, dx_3$ we obtain

$$
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho \frac{\partial^2 u_1}{\partial t^2} \quad ,
$$

Balance of momentum: "Equations of Motion" III

similarly in the other two directions

$$\begin{aligned} \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} &= \rho \frac{\partial^2 u_3}{\partial t^2} \quad . \end{aligned}$$

In index notation

$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \text{Summation over } j \;\; ,$$

and in vector notation

$$\nabla \cdot \sigma + \vec{f} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} \quad .$$

Navier-Cauchy Equations: Deduction I

For an isotropic medium the constitutive relation can be expressed as

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \quad ,$$

being λ and μ the Lame parameters.
ϵ_{ij} is usually define as

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad ,$$

so

$$\sigma_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad .$$

If we take the constitutive equation and replace σ_{ij} in the motion equation (assuming that $\frac{\partial \rho}{\partial t} = 0$).

$$\rho \frac{\partial u_i}{\partial t} = \delta_{ij} \lambda \frac{\partial \epsilon_{kk}}{\partial x_j} + 2\mu \frac{\partial \epsilon_{ij}}{\partial x_j} + f_i \quad ,$$

Navier-Cauchy Equations: Deduction II

and rewriting the right hand side in terms of the displacements we have

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \delta_{ij} \lambda \frac{\partial}{\partial x_j} \left(\frac{\partial u_k}{\partial x_k} \right) + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + f_i \ ,$$

operating the Kronecker's delta

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + f_i \ ,$$

and this can be rewritten like

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial^2 u_j}{\partial x_i x_j} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial^2 u_j}{\partial x_j x_i} + f_i \ ,$$

grouping terms

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) + \mu \frac{\partial^2 u_i}{\partial x_j^2} + f_i \ . \tag{4}$$

Navier-Cauchy Equations: Deduction III

This is the Navier Equation that can be expressed also in vector form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \underbrace{\frac{\partial}{\partial x_i}}_{\text{grad}} \nabla \cdot \vec{u} + \mu \nabla^2 u_i + f_i \ ,$$

or

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u} + \vec{f} \ . \tag{5}$$

The body forces will be neglected hereinafter for simplicity. The vector

$$\nabla^2 \vec{a} = \nabla (\nabla \cdot \vec{a}) - \nabla \times (\nabla \times \vec{a}) \ ,$$

identity is used, and we get

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \vec{u}) - \mu (\nabla \times (\nabla \times \vec{u})) \ .$$

Navier-Cauchy Equations: Deduction IV

If we take $\varphi = \nabla \cdot \vec{u}$ and $\vec{\psi} = \nabla \times \vec{u}$ and replace it in the Navier equation we obtain:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu)\nabla\varphi - \mu\nabla \times \vec{\psi} \ . \tag{6}$$

P-waves equation

Taking divergence to (6)

$$\nabla \cdot \left(\frac{\partial^2 \vec{u}}{\partial t^2} \right) = (\lambda + 2\mu) \nabla \cdot \nabla \varphi - \mu \nabla \cdot \nabla \times \vec{\psi} \ ,$$

we finally get

$$\nabla^2 \varphi = \frac{1}{\frac{\lambda+2\mu}{\rho}} \frac{\partial^2 \varphi}{\partial t^2} \ , \tag{7}$$

with speed of propagation

$$v_P := \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

P-wave polarization

Figure: Particle motion in a P wave.

S-waves equation

Taking rotational to (6)

$$\nabla \times \left(\frac{\partial^2 \vec{u}}{\partial t^2} \right) = (\lambda + 2\mu) \nabla \times \nabla \varphi - \mu \nabla \times \nabla \times \vec{\psi} \ ,$$

we finally get

$$\nabla^2 \vec{\psi} = \frac{1}{\frac{\mu}{\rho}} \frac{\partial^2 \vec{\psi}}{\partial t^2} \ , \tag{8}$$

with speed of propagation

$$v_S := \sqrt{\frac{\mu}{\rho}}$$

S-wave polarization

Figure: Particle motion in a SV wave.

References

- Michael A. Slawinski. Waves and rays in Elastic Continua. Second Edition, 2007.
- Wikipedia community. Onda (física) [on line]. Wikipedia, La enciclopedia libre, 2010 ; 22 November 2010.
- George B. Arfken & Hans. J. Weber. Mathematical Methods for Physicists. Elsevier Academic Press, 6th Edition, San Diego, 2005.