

## Chapter IV: The Fourier Transform and its applications

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## IV.1 Vector spaces of signals (crash course)

# Vector spaces

- ▶ **Vector space** = a set  $V\{v_i\}$  with the following two properties:
  - ▶ one element + another element = still an element of the same space
  - ▶ a scalar constant  $\times$  an element = still an element of the same space
- ▶ You **can't escape** a vector space by summing or scaling
- ▶ The elements of a vector space are called **vectors**

# Examples of vector spaces

- ▶ Geometric spaces are great intuitive examples:
  - ▶ a line, or the set  $\mathbb{R}$  (one-dimensional)
  - ▶ a plane, or the set  $\mathbb{C}$  (two-dimensional)
  - ▶ 3D space (three-dimensional)
  - ▶ 4D space (four-dimensional, like the spatio-temporal universe)
  - ▶ arrays with  $N$  numbers ( $N$ -dimensional)
  - ▶ space of continuous signals ( $\infty$ -dimensional)
- ▶ The **dimension** of the space = “how many numbers you need in order to specify one element” (informal)
- ▶ A “vector” like in maths = a sequence of  $N$  numbers = a “vector” like in programming
  - ▶ e.g. a point in a plane has two coordinates = a vector of size  $N = 2$
  - ▶ e.g. a point in a 3D-space has three coordinates = a vector of size  $N = 3$

# Inner product

- ▶ Many vector spaces have a fundamental operation: **the (Euclidean) inner product**

- ▶ for **discrete** signals:

$$\langle \vec{x}, \vec{y} \rangle = \sum_i x_i y_i^*$$

- ▶ for **continuous** signals:

$$\langle \vec{x}, \vec{y} \rangle = \int x(t) y^*(t)$$

- ▶ \* represents **complex conjugate** (has no effect for real signals)
- ▶ The result is one number (real or complex)
- ▶ Also known as **dot product** or **scalar product** (“produs scalar”)

# Inner product

- ▶ Each entry in  $\vec{x}$  times the complex conjugate of the one in  $\vec{y}$ , all summed
- ▶ For discrete signals, it can be understood as a row  $\times$  column multiplication
- ▶ Discrete vs continuous: just change sum/integral depending on signal type

# Inner product properties

► Inner product is **linear** in both terms:

$$\langle \vec{x}_1 + \vec{x}_2, \vec{y} \rangle = \langle \vec{x}_1, \vec{y} \rangle + \langle \vec{x}_2, \vec{y} \rangle$$

$$\langle c \cdot \vec{x}, \vec{y} \rangle = c \cdot \langle \vec{x}, \vec{y} \rangle$$

$$\langle \vec{x}, \vec{y}_1 + \vec{y}_2 \rangle = \langle \vec{x}, \vec{y}_1 \rangle + \langle \vec{x}, \vec{y}_2 \rangle$$

$$\langle \vec{x}, c \cdot \vec{y} \rangle = c^* \cdot \langle \vec{x}, \vec{y} \rangle$$



# The distance between two vectors

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ **The (Euclidean) distance** between two vectors =

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_N - y_N)^2}$$

- ▶ This distance is the **usual geometric distance** you know from geometry
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ if two vectors have small distance, they are close, they are similar
  - ▶ two vectors with large distance are far away, not similar
  - ▶ two identical vectors have zero distance

# The norm of a vector

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ The **norm** of a vector is the distance from  $\vec{x}$  to point  $\vec{0}$ .
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ vector has large norm = has big values, is far from  $\vec{0}$
  - ▶ vector has small norm = has small values, is close to  $\vec{0}$
  - ▶ vector has zero norm = it is the vector  $\vec{0}$
- ▶ Norm of a vector = sqrt(the signal **energy**)

# Norm and distance

- ▶ The norm and distance are related
- ▶ The distance between  $\vec{a}$  and  $\vec{b}$  = norm (length) of their difference

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ Just like in geometry: distance = length of the difference vector

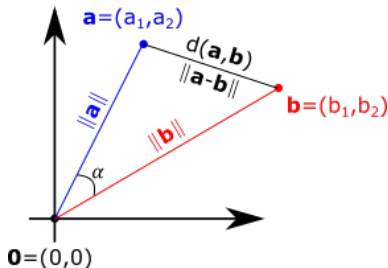


Figure 1: Norm and distance in vector spaces

# Angle between vectors

- ▶ The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

- ▶ is a value between -1 and 1
- ▶ **Otrhogonal vectors** = two vectors with  $\langle x, y \rangle = 0$ 
  - ▶ their angle = 90 deg
  - ▶ in geometric language, the two vectors are **perpendicular**

# Why vector space

- ▶ Why are all these useful?
- ▶ They are a very general **framework** for different kinds of signals
- ▶ We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - ▶ Example in DEDP class: ML decision with 1, 2, N samples

# Vector spaces in DSP class

We deal mainly with the following vector spaces:

- ▶ The vector space of all infinitely-long real signals  $x[n]$
- ▶ The vector space of all infinitely-long periodic signals  $x[n]$  with period  $N$ 
  - ▶ for each  $N$  we have a different vector space
- ▶ The vector space of all finite-length signals  $x[n]$  with only  $N$  samples
  - ▶ for each  $N$  we have a different vector space

- ▶ A **basis** = a set of  $N$  linear independent elements from a vector space

$$B = \{\vec{b}^1, \vec{b}^2 \dots \vec{b}^N\}$$

- ▶ Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \dots + \alpha_N \vec{b}^N$$

- ▶ The vector is defined by these coefficients:

$$\vec{x} = (\alpha_1, \alpha_2, \dots \alpha_N)$$

# Bases and coordinate systems

- ▶ Bases are just like **coordinate systems** in a geometric space
  - ▶ any point is expressed w.r.t. a coordinate system

$$\vec{x} = x_1 \vec{i} + x_2 \vec{j}$$

- ▶ any vector is expressed w.r.t. a basis

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \dots + \alpha_N \vec{b}^N$$

- ▶  $N$  = The number of basis elements = The dimension of the space
- ▶ Example: any color = RGB values (monitor) or  
Cyan-Yellow-Magenta values (printer)



# Bases and coordinate systems

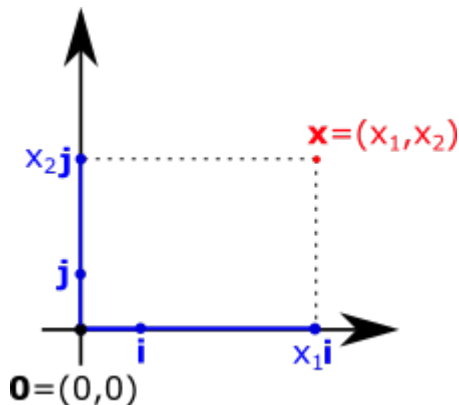


Figure 2: Basis expansion of a vector  $\mathbf{x}$

# Choice of bases

- ▶ There is typically an infinite choice of bases
- ▶ The **canonical basis** = all basis vectors are full of zeros, just with one 1
- ▶ You used it already in an exercise:

- ▶ any signal  $x[n]$  can be expressed of a sum of  $\delta[n - k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n - 1] + 2\delta[n - 2] + \dots$$

- ▶ the canonical basis is  $B = \{\dots, \delta[n], \delta[n - 1], \delta[n - 2], \dots\}$

# Orthonormal bases

- ▶ An **orthonormal basis** a basis where all elements  $\vec{b}^i$  are:
  - ▶ orthogonal to each other:

$$\langle \vec{b}^i, \vec{b}^j \rangle = 0, \forall i \neq j$$

- ▶ **normalized** (their norm = 1):

$$||\vec{b}^i|| = \sqrt{\langle \vec{b}^i, \vec{b}^i \rangle} = 1, \forall i$$

- ▶ Example: the canonical basis  $\{\delta[n - k]\}$  is orthonormal:
  - ▶  $\langle \delta[n - k], \delta[n - l] \rangle = 0, \forall k \neq l$
  - ▶  $\langle \delta[n - k], \delta[n - k] \rangle = 1, \forall k$

# Orthonormal bases

- ▶ Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

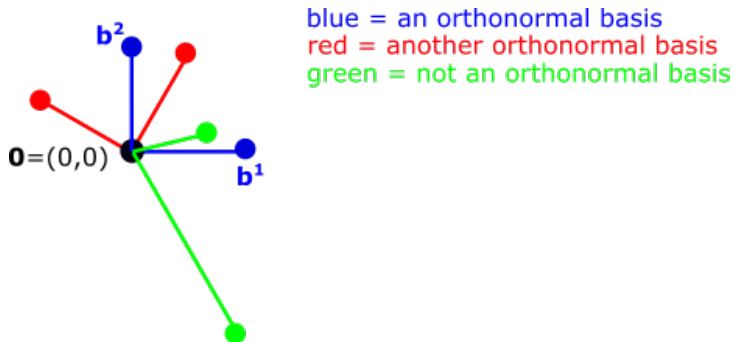


Figure 3: Sample bases in a 2D space

# Basis expansion of a vector

- ▶ Suppose we have an **orthonormal basis**  $B = \{\vec{b}^i\}$
- ▶ Suppose we have a vector  $\vec{x}$
- ▶ We can write (expand)  $\vec{x}$  as:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \cdots + \alpha_N \vec{b}^N$$

- ▶ Question: how to **find** the coefficients  $\alpha_i$ ?

# Basis expansion of a vector

► If the basis is **orthonormal**, we have:

$$\begin{aligned}\langle \vec{x}, \vec{b}^i \rangle &= \langle \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \dots + \alpha_N \vec{b}^N, \vec{b}^i \rangle \\ &= \langle \alpha_1 \vec{b}^1, \vec{b}^i \rangle + \langle \alpha_2 \vec{b}^2, \vec{b}^i \rangle + \dots + \langle \alpha_N \vec{b}^N, \vec{b}^i \rangle \\ &= \alpha_1 \langle \vec{b}^1, \vec{b}^i \rangle + \alpha_2 \langle \vec{b}^2, \vec{b}^i \rangle + \dots + \alpha_N \langle \vec{b}^N, \vec{b}^i \rangle \\ &= \alpha_i\end{aligned}$$

# Basis expansion of a vector

- ▶ Any vector  $\vec{x}$  can be written as:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \dots + \alpha_N \vec{b}^N$$

- ▶ For orthonormal basis: the coefficients  $\alpha_i$  are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle \vec{x}, \vec{b}^i \rangle$$

# Why bases

- ▶ How does all this talk about bases help us?
- ▶ To better understand the Fourier transform
- ▶ The signals  $\{e^{j\omega n}\}$  form an **orthonormal basis**
- ▶ The Fourier Transform of a signal  $x$  = finding the coefficients of  $\vec{x}$  in this basis
- ▶ The Inverse Fourier Transform = expanding  $\vec{x}$  with the elements of this basis
- ▶ Same **generic** thing every time, only the type of signals differ



## IV.2 Introducing the Fourier Transforms

# Reminder

► Reminder:

$$e^{jx} = \cos(x) + j \sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

# Why sinusoidal signals

- ▶ Why are sinusoidal signals  $\sin()$  and  $\cos()$  **so prevalent** in signal processing?
- ▶ Answer: because they are combinations of an  $e^{jx}$  and an  $e^{-jx}$
- ▶ Why are these  $e^{jx}$  so special?
- ▶ Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

# Response of LTI systems to harmonic signals

- ▶ Consider an LTI system with  $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal  $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

- ▶ Output signal = input signal  $\times$  a (complex) constant ( $H(\omega_0)$ )

# Eigen-function

- ▶ **Eigen-function** of a system (“funcție proprie”) = a function  $f$  which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

- ▶ just like **eigen-vectors** of a matrix (remember algebra):  $A\mathbf{v} = \lambda\mathbf{v}$
  - ▶ we call the “functions” to allow for continuous signals as well
- ▶ Complex exponential signals  $e^{j\omega t}$  (or  $e^{j\omega n}$ ) are **eigen-functions** of Linear and Time Invariant (LTI) systems:
  - ▶ output signal = input signal  $\times$  a (complex) constant

# Representation with respect to eigen-functions (-vectors)

- ▶ We can understand the effect of a LTI system very easily if we **decompose all signals  $x[n]$  as a combination of  $\{e^{j\omega n}\}$**
- ▶ Example: RGB color filter
  - ▶ suppose we have some photographic filters (lenses):
    - ▶ one reduces red to 50%
    - ▶ one reduces green to 25%
    - ▶ one reduces blue to 80%
    - ▶ RGB are eigen-functions of the system: input = 200 Blue, output =  $0.8 * 200$  Blue
    - ▶ what is the output color if input is “pink”?
    - ▶ Answer is easy if we represent all colors in RGB

# Representation with respect to eigen-functions (-vectors)

- ▶ We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of  $\{e^{j\omega n}\}$
- ▶ All vector space theory becomes useful now:
  - ▶  $\{e^{j\omega n}\}$  is an **orthonormal basis**
  - ▶ decomposing signals = finding coefficients  $\alpha_i$
  - ▶ we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_{\omega} \cdot e^{j\omega n}$$

$$\alpha_{\omega} = \langle x, e^{j\omega n} \rangle$$

# Discrete-Time Fourier Transform (DTFT)

- ▶ Consider the vector space of **non-periodic infinitely-long signals**
- ▶ This vector space is **infinite-dimensional**
- ▶ The signals  $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$  form an **orthonormal basis**
- ▶ We can expand (almost) any  $\vec{x}$  in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- ▶ The coefficient of every  $e^{j2\pi fn}$  is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$



# Discrete-Time Fourier Transform (DTFT)

## Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n} df$$

- ▶ A signal  $x[n]$  can be written as a linear combination of  $\{e^{j2\pi f n}\}$ ,  $\forall f \in [-\frac{1}{2}, \frac{1}{2}]$ , with some coefficients  $X(f)$

## Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi f n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

- ▶ The coefficient  $X(f)$  of every  $\{e^{j2\pi f n}\}$  is found using the inner product  $\langle \vec{x}, e^{j2\pi f n} \rangle$

# Discrete-Time Fourier Transform (DTFT)

- ▶ Alternative form with  $\omega$
- ▶ We can replace  $2\pi f = \omega$ , and  $df = \frac{1}{2\pi}d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

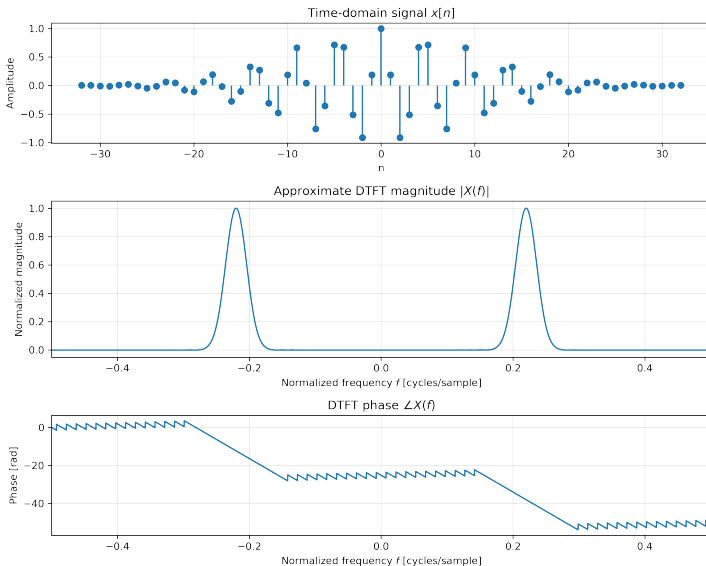
$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_n x[n] e^{-j\omega n}$$

# Discrete-Time Fourier Transform (DTFT)

- ▶ A non-periodic signal  $x[n]$  has a **continuous spectrum**  $X(\omega)$ , with  $f \in [-\frac{1}{2}, \frac{1}{2}]$ 
  - ▶ e.g.  $\omega \in [-\pi, \pi]$

# DTFT example

► A non-periodic, finite energy signal and its DTFT



# Discrete Fourier Transform (DFT)

- ▶ Consider the vector space of **periodic** signals with **period N**
  - ▶ for some fixed  $N = 2, 3$  or ... etc
- ▶ This is a vector space of **dimension N**
  - ▶ we need N numbers to identify a signal (specify its period)
- ▶ We can consider  $x[n]$  only for **one period**, i.e.  $n = 0, \dots, N - 1$
- ▶ The signals  $\{e^{j2\pi f n}\}, \forall f \in \{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}$  form an **orthonormal basis** with N elements
- ▶ It is a **discrete** set of frequencies:  $f = \frac{k}{N}, \forall k \in \{0, 1, \dots, N - 1\}$

# Discrete Fourier Transform (DFT)

## Inverse Discrete Fourier Transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- ▶ A periodic signal  $x[n]$  can be written as a linear combination of  $k$  signals  $\{e^{j2\pi kn/N}\}$ , with some coefficients  $X_k$

## Discrete Fourier Transform

$$X_k = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

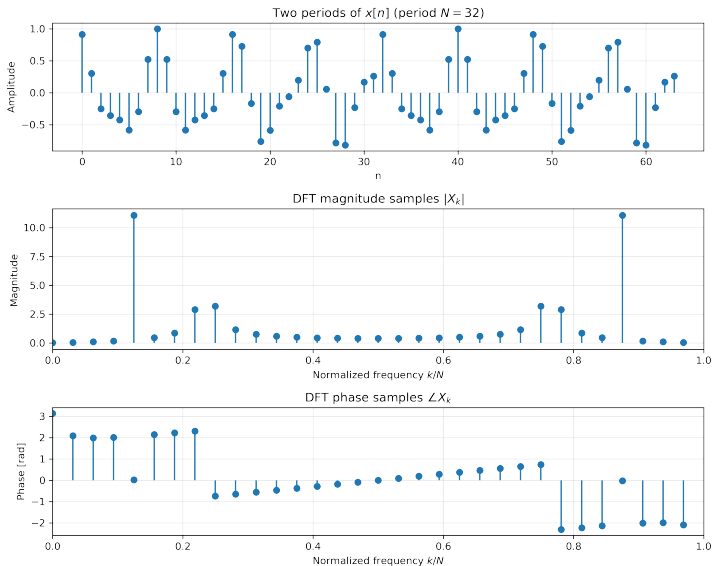
- ▶ The coefficient  $X(f)$  of every  $\{e^{j2\pi fn}\}$  is found using the inner product  $\langle \vec{x}, e^{j2\pi fn} \rangle$

# Discrete Fourier Transform (DFT)

- ▶ A periodic signal  $x[n]$  with period  $N$  has a **discrete spectrum**  $X(\omega)$  composed of only  $N$  frequencies  $\{0, \frac{1}{N} \dots \frac{N-1}{N}\}$
- ▶ Each frequency  $\frac{k}{N}$  has a **coefficient**  $X_k$ 
  - ▶ also written as  $c_k$
  - ▶ The  $N$  coefficients  $X_k$  are the equivalent of  $X(\omega)$
- ▶ It is also known as the “Fourier Series for Discrete Signals”

# DFT example

## ► A periodic signal and its DTFT coefficients





## IV.3 The Discrete-Time Fourier Transform (DTFT)

# Definition

Definitions (again):

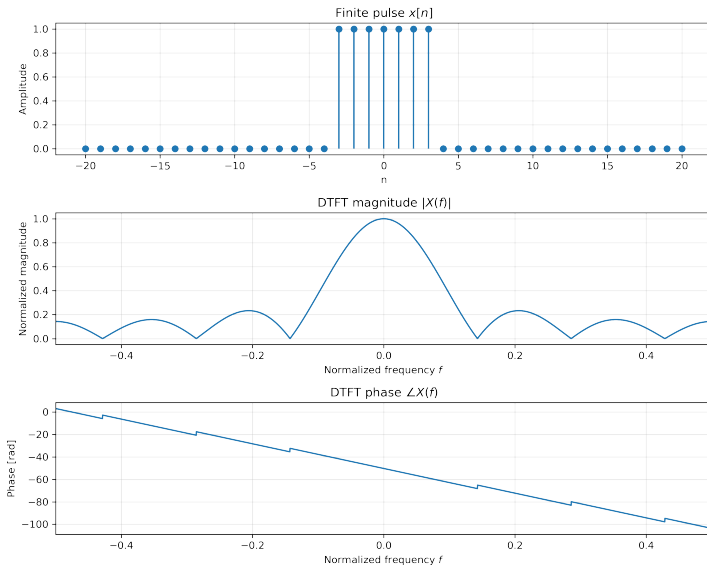
## **Inverse Discrete-Time Fourier Transform (DTFT)**

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

## **Discrete-Time Fourier Transform (DTFT)**

$$X(f) = \langle x[n], e^{j2\pi f n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

# DTFT example



► DTFT of a short rectangular sequence spreads energy over all

# Basic properties of DTFT

- ▶  $X(\omega)$  is **defined** only for  $\omega \in [-\pi, \pi]$ 
  - ▶ or  $f \in [-\frac{1}{2}, \frac{1}{2}]$
- ▶  $X(\omega)$  is **complex** (has  $|X(\omega)|$ ,  $\angle X(\omega)$ )
- ▶ If the signal  $x[n]$  is real,  $X(\omega)$  is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

- ▶ This means:
  - ▶ modulus is even:  $|X(\omega)| = |X(-\omega)|$
  - ▶ phase is odd:  $X(\omega) = -X(-\omega)$

# Expressing as sum of sinusoids

- Grouping terms with  $e^{j\omega n}$  and  $e^{j(-\omega)n}$  we get:

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \int_{-\pi}^0 X(\omega) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} X(\omega) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} (X(\omega) e^{j\omega n} + X(-\omega) e^{j(-\omega)n}) d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} 2|X(\omega)| (e^{j\omega n + \angle X(\omega)} + e^{-j\omega n - \angle X(\omega)}) d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} 2|X(\omega)| \cos(\omega n + \angle X(\omega)) d\omega\end{aligned}$$

- Any signal  $x[n]$  is **a sum of sinusoids with all frequencies**  
 $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$

# Expressing as sum of sinusoids

- ▶ The DTFT shows that any signal  $x[n]$  is a “**sum**” of sinusoids with **all frequencies**  $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$ 
  - ▶ this is the fundamental practical interpretation of the Fourier transform
  - ▶ not really a sum, because we have an integral
- ▶ The **modulus**  $|X(\omega)|$  gives the **amplitude** of the sinusoids ( $\times 2$ )
  - ▶ for  $\omega = 0$ ,  $|X(\omega = 0)|$  = the DC component
- ▶ The **phase**  $\angle X(\omega)$  gives the initial phase

# Properties of DTFT

## 1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

## 2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- ▶ The amplitudes  $|X(\omega)|$  is not affected, shifting in time affects only the phase



# Properties of DTFT

## 3. Modulation in time

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

## 5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- ▶ Not circular convolution, this is the normal convolution

## 6. Product in time

Product in time  $\leftrightarrow$  convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

# Properties of DTFT

## Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega) X_2^*(\omega)$$

## Wiener Khinchin theorem

Autocorrelation of a signal  $\leftrightarrow$  Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

# Parseval theorem

- **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Is true for all orthonormal bases

## IV.4 The Discrete Fourier Transform (DFT)

# Definitions

Definitions (again):

## **Inverse Discrete Fourier Transform (DFT)**

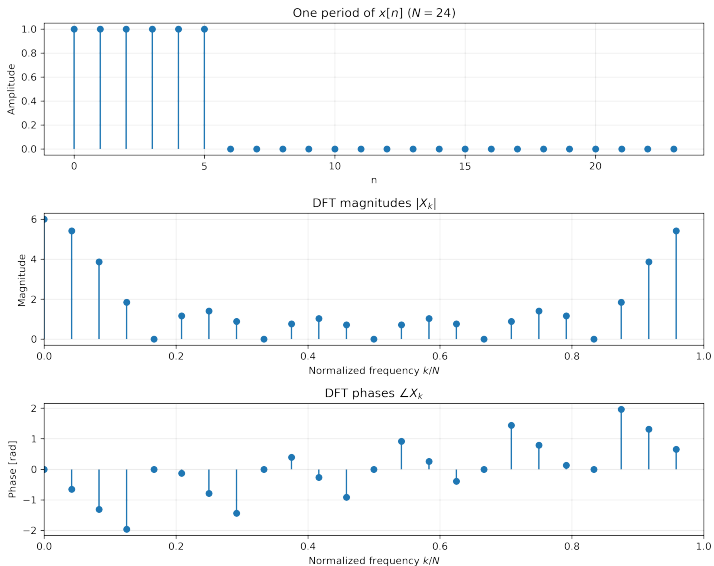
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

## **Discrete Fourier Transform (DFT)**

$$X_k = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- ▶ DFT is defined for periodical signals with period  $N$
- ▶ there are exactly  $N$  terms in each sum

# DFT example





# Definitions

To remember:

- ▶ DFT: takes a vector with  $N$  elements ( $x[n]$ ), produces a vector with  $N$  elements ( $X_k$ )
  - ▶ for this reason, we can compute it e.g. with Matlab
- ▶ DTFT: takes a vector with  $\infty$  elements ( $x[n]$ ), produces a continuous function ( $X(\omega)$ ) between  $[-\pi, \pi]$

# Periodicity and notation

- ▶ DTFT has only  $N$  coefficients  $X_k$ , each  $X_k$  corresponding to a frequency  $f = \frac{k}{N}$
- ▶ In frequency domain,  $f = \frac{N-k}{N} = \frac{-k}{N}$  (aliasing, we can always add/subtract 1 from  $f$ )
- ▶ So we can consider  $X_{N-k}$  as  $X_{-k}$ , due to periodicity
- ▶ Example: a signal with period  $N = 6$  has 6 DFT coefficients
  - ▶ we can call them  $X_0, X_1, X_2, X_3, X_4, X_5$
  - ▶ we have  $X_5 = X_{-1}, X_4 = X_{-2}$
  - ▶ we can also call them  $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$

# Basic Properties of the DFT

- ▶ Has only  $N$  coefficients  $X_k$
- ▶  $X_k$  are **complex** (has  $|X_k|$ ,  $\angle X_k$ )
- ▶ If the signal  $x[n]$  is real, the coefficients are **even**

$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

- ▶ This means:
  - ▶ modulus is even:  $|X_k| = |X_{-k}|$
  - ▶ phase is odd:  $\angle X_{-k} = -\angle X_k$

## Expressing as sum of sinusoids, $N = \text{odd}$

- ▶ Grouping terms with  $k$  and  $-k$ :
- ▶ If  $N$  is odd, we have  $X_0$  and pairs  $(X_k, X_{-k})$ :

$$\begin{aligned}x[n] &= \frac{1}{N}X_0e^{j0n} + \frac{1}{N}\sum_{k=-(N-1)/2}^{-1}X_ke^{j2\pi kn/N} + \frac{1}{N}\sum_{k=1}^{(N-1)/2}X_ke^{j2\pi kn/N} \\&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2}(X_ke^{j2\pi kn/N} + X_{-k}e^{-j2\pi kn/N}) \\&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2}|X_k|(e^{j2\pi kn/N + \angle X(k)} + e^{-j2\pi kn/N - \angle X(\omega)}) \\&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=0}^{(N-1)/2}2|X_k|\cos(2\pi k/Nn + \angle X_k)\end{aligned}$$

- ▶ A sum of sinusoids with frequencies up to  $1/2$

## Expressing as sum of sinusoids, $N = \text{even}$

- ▶ If  $N$  is even, we have  $X_0$  and pairs  $(X_k, X_{-k})$ , with an extra term  $X_{N/2}$  which has no pair
  - ▶ e.g.  $N = 6$ :  $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$
- ▶ The term with no pair,  $X_{N/2}$ , must be a real number, because  $X_{N/2} = X_{-N/2}^* = X_{N/2}^*$
- ▶ The extra term will be  $\frac{1}{N} X_{N/2} e^{j2\pi N/2n/N} = X_{N/2} \cos(n\pi)$
- ▶ Overall:

$$x[n] = \frac{1}{N} X_0 + \frac{1}{N} \sum_{k=0}^{(N-2)/2} 2|X_k| \cos(2\pi k/N n + \angle X_k) + \frac{1}{N} X_{N/2} \cos(n\pi)$$

- ▶ A sum of sinusoids with frequencies up to  $1/2$

# Expressing as sum of sinusoids

- ▶ DFT says that any periodic signal  $x[n]$ , with period  $N$ , is a **sum of  $N$  sinusoids with frequencies:**

$$f = 0, \frac{1}{N}, \frac{2}{N}, \dots \text{ up to } \frac{N-1}{2} \text{ or } \frac{N}{2}$$

(not exceeding  $1/2$ )

- ▶ The **modulus**  $|X_k|$  gives the **amplitude** of the sinusoids (sometimes  $\times 2$ )
  - ▶ for  $\omega = 0$ ,  $|X_0|$  = the DC component
  - ▶ when modulus = 0, that frequency has amplitude 0
- ▶ The **phase**  $\angle X_k$  gives the initial phase

# Example

- ▶ Consider a periodic signal  $x[n]$  with period  $N = 6$  and the DFT coefficients:

$$X_k = [15.0000 + 0.0000i, -2.5000 + 3.4410i, -2.5000 + 0.8123i, -2.5000 - 0.8123i, -2.5000 - 3.4410i]$$

Write  $x[n]$  as a sum of sinusoids.

- ▶ Do the same for a periodic signal  $x[n]$  with period  $N = 5$  and the DFT coefficients:

$$X_k = [21.0000 + 0.0000i, -3.0000 + 5.1962i, -3.0000 + 1.7321i, -3.0000 + 0.0000i, -3.0000 - 1.7321i, -3.0000 - 5.1962i]$$

Write  $x[n]$  as a sum of sinusoids.

- ▶ The DFT (and the inverse IDFT) is equivalent with a matrix multiplication:
  - ▶ on whiteboard
- ▶ In the world of discrete signals, there are many signal transforms possible, and many of them can be expressed as matrix multiplications, just like the DFT.



# Properties of the DFT

## 1. Linearity

If the signal  $x_1[n]$  has the DFT coefficients  $\{X_k^{(1)}\}$ , and  $x_2[n]$  has  $\{X_k^{(2)}\}$ , then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$$

Proof: via definition

# Properties of the DFT

## 2. Shifting in time

If  $x[n] \leftrightarrow \{X_k\}$ , then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi kn_0/N)} X_k\}$$

Proof: via definition

- ▶ The amplitudes  $|X_k|$  are not affected, shifting in time **affects only the phase**

# Properties of the DFT

## 3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow \{X_{-k}^*\}$$

# Properties of the DFT

## 5. Circular convolution

Circular convolution of two signals  $\leftrightarrow$  product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

**Circular convolution** definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- ▶ takes two periodic signals of period  $N$ , result is also periodic with period  $N$
- ▶ Example at the whiteboard: how it is computed

# Example

Example (write on slides)

# Circular convolution

- ▶ We are in the vector space of **periodic signals** with period  $N$
- ▶ Linear (e.g. normal) convolution produces a result which is longer periodic with period  $N$
- ▶ Circular convolution takes two sequences of length  $N$  and produces another sequence of length  $N$ 
  - ▶ each sequence is a period of a periodic signal
  - ▶ circular convolution = like a convolution of periodic signals

# Properties of the DFT

## 6. Product in time

Product in time  $\leftrightarrow$  circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

# Properties of the DFT

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_0^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum |X_k|^2$$

- ▶ Is true for all orthonormal bases



# Plot / sketch DTFT and DFT of various signals

Let's plot / sketch DTFT and DFT of various signals

DTFT of:

- ▶ a constant signal  $x[n] = A$
- ▶ a rectangular signal  $x[n] = A$  between  $-\tau$  and  $\tau$ , 0 elsewhere
- ▶ a cosine of frequency precisely  $f = k/N$
- ▶ a cosine of frequency not  $f = k/N$

DFT, with  $N=20$ , of:

- ▶ a constant signal  $x[n] = A$
- ▶ a rectangular signal  $x[n] = A$  between  $-\tau$  and  $\tau$ , 0 elsewhere
- ▶ a cosine of frequency precisely  $f = k/N$
- ▶ a cosine of frequency not  $f = k/N$

# DFT matrix

- The DFT (and the inverse IDFT) is equivalent with a matrix multiplication:

DFT:

$$\mathbf{X} = \mathbf{W}_N \mathbf{x}$$

IDFT:

$$\mathbf{x} = \mathbf{W}_N^T \mathbf{X}$$

where

$$\mathbf{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix},$$

## DFT matrix (continued)

$$\mathbf{W}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{0 \cdot 0} & W_N^{0 \cdot 1} & \dots & W_N^{0 \cdot (N-1)} \\ W_N^{1 \cdot 0} & W_N^{1 \cdot 1} & \dots & W_N^{1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 0} & W_N^{(N-1) \cdot 1} & \dots & W_N^{(N-1) \cdot (N-1)} \end{bmatrix}$$

with an element  $W_N^{kn} = e^{-j2\pi \frac{k}{N}n}$  ( $k$  = row index,  $n$  = column index)

- ▶ there might be small variations depending on whether we have  $\frac{1}{\sqrt{N}}$  at both DFT and IDFT, or just put  $\frac{1}{N}$  just for the IDFT
- ▶ note that for IDFT we have  $W^{-1} = W^T$  (orthonormal basis)

# DFT matrix multiplication

- ▶ Naive implementation of DFT, IDFT: use matrix multiplication with  $W, W^{-1}$
- ▶ Number of multiplications necessary for a vector of length  $N$  is  $N^2$
- ▶ In the world of algorithms, the **computational complexity** of an algorithm = number of multiplications necessary, depending on some variable  $N$ 
  - ▶ only the dominant term matters, no coefficient, e.g  $O(N^2)$  not  $7.3N^2 + 4N$
- ▶ Naive DFT has computation complexity  $\mathcal{O}(N^2)$ 
  - ▶ prohibitively large

- ▶ The Fast Fourier Transform (FFT) algorithm = a fast algorithm for computing the DFT, exploiting the particular nature (symmetries) in the DFT matrix
- ▶ FFT computational complexity:  $\mathcal{O}(N \log_2(N))$
- ▶ Exercise: for  $N = 1024$ , how much faster is FFT compared to naive DFT multiplication?
- ▶ Invention and adoption of FFT (~'60s, Cooley & Tukey) = “the birth of Digital Signal Processing”

# Other transforms

- ▶ In the world of discrete signals, there are many signal transforms possible, and many of them can be expressed as matrix multiplications, just like the DFT.
- ▶ Transform = expressing a  $N$ -dimensional vector  $x$  as a linear combination of a set of  $N$  basis vectors
- ▶ How:
  1. Put the  $N$  vectors of the basis as columns in a matrix  $A$
  2. Solve the system  $x = AX$  (inverse transform)
  3. Which means  $X = A^{-1}x$  (forward transform)
- ▶ Why:
  - ▶ compression: the discrete cosine transform is the basis for lossy JPEG image compression
  - ▶ ...

# Example

- Consider the exercise from Week 2:

$$x[n] = \{..., 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1, 1, 0, ...\}$$

Write the expression of  $x[n]$  based on the signal  $u[n]$ .

Solve this in Matlab using a matrix approach

## Another example: JPEG

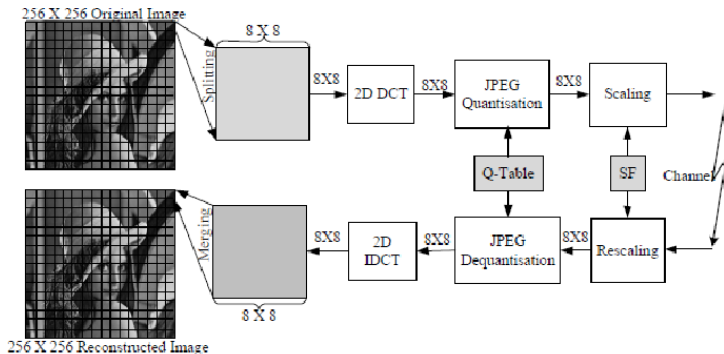


Figure 4: JPEG compression with DCT transform

- ▶ image from: JPEG Picture Compression Using Discrete Cosine Transform, N. K. More, S. Dubey, 2012



## Another example: JPEG (cont'd)

- ▶ Each 8x8 image block is a vector in a 64 dimensional space
- ▶ Each 8x8 image block is decomposed into 64 basis vectors

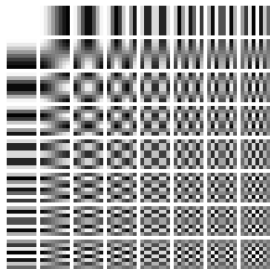


Figure 5: 8x8 DCT basis vectors

- ▶ Result: 64 coefficients, but many are small, negligible, quantizable  
=> compression
- ▶ image from: Wikipedia

# Relationship between DTFT and DFT

- ▶ How are DTFT and DFT related?
- ▶ Discrete Time Fourier Transform (DTFT):
  - ▶ for non-periodical signals
  - ▶ spectrum is continuous
- ▶ Discrete Fourier Transform (DFT):
  - ▶ for periodical signals
  - ▶ spectrum is discrete
- ▶ Duality: periodic in time  $\leftrightarrow$  discrete in frequency
- ▶ The Diracs of the DFT are samples from the continuous DTFT of a single period of the signal

# Relationship between DTFT and DFT

- ▶ Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

- ▶ If we consider a  $x$  surrounded by infinitely long zeros ( $x[n]$  non-periodical), we have a continuous spectrum  $X(\omega)$  (DTFT)

$$x = [\dots 0, 6, 3, -4, 2, 0, 1, 2, 0, \dots] \leftrightarrow X(\omega)$$

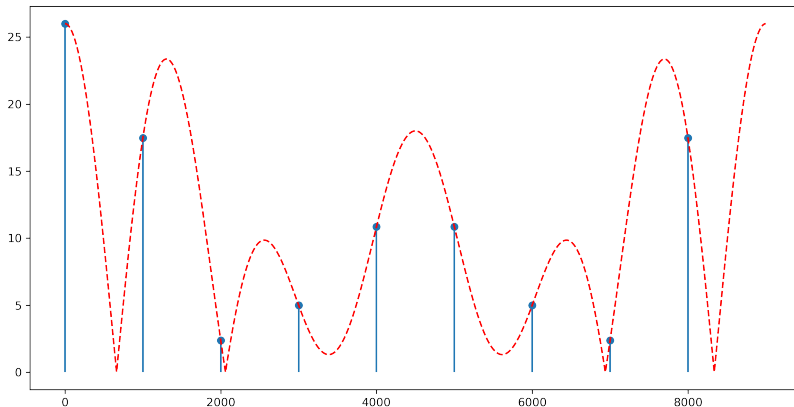
- ▶ If we consider that  $x$  is surrounded by repeating the sequences ( $x[n]$  periodical), we have a discrete spectrum  $X_k$  (DFT)

$$x = [\dots, -4, 2, 0, 1, 2, 0, 6, 3, -4, 2, 0, 1, 2, 0, 6, 3, -4, \dots] \leftrightarrow X_k$$

- ▶ The discrete  $X_k$  are just **samples from**  $X(\omega)$ , at frequencies  $k/N$ :

$$X_k = X(2\pi(k/N)n)$$

# Relationship between DTFT and DFT



$$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$$

- ▶ red line = DTFT of  $x$  (assuming surrounded by zeros)
  - ▶ (actually run as `fft(x, 10000)`,  $x$  is extended with 9991 zeros)
- ▶ blue = DFT of  $x$  (assumes periodic) = `fft(x)`

# Relationship between DTFT and DFT

- ▶ Consider a non-periodic signal  $x[n]$
- ▶ It has a continuous spectrum  $X(\omega)$
- ▶ If we **periodize** it by repeating with period  $N$ :

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- ▶ then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi \sum_k X_k \delta(\omega - k\frac{2\pi}{N})$$

- ▶ The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k/N)$$

- ▶ They are **samples** from the continuous  $X(\omega)$  of the non-periodized signal

# Study case

```
n = 0:99;  
f = 0.015;  
x = cos(2*pi*f*n)  
plot(abs(fft(x)))
```

Discuss:

1. Why is the spectrum not just 2 Diracs, like a normal `cos()`?
2. FFT assumes periodicity. Are there boundary problems?
3. What is the role of the rectangular window?
4. What happens if we run `fft(x, 10000)` instead of `fft(x)`?

# Signal windowing and frequency resolution

- ▶ When you have finite-length cosine vector  $x$ , you have just a part of your signal
- ▶ The true signal  $x$ , infinitely long, is actually multiplied with a rectangular window  $w[n]$

$$x = \cos(2\pi fn) \cdot w[n]$$

- ▶ Multiplication in time = convolution in frequency  
The spectrum of  $x = x[n] \cdot w[n]$  is Diracs  $*$   $W(\omega)$
- ▶ Instead of Diracs, you have  $W(\omega)$ 's:
  - ▶ wide peak
  - ▶ secondary lobes

# Signal windowing and frequency resolution

- ▶ Working with a piece of a signal **always** distorts a signal
  - ▶ every Dirac is “smudged” into a  $W(\omega)$
- ▶ This is unavoidable
- ▶ If we have a segment of a cosine, in the DTFT (continuous) we never see Diracs, but  $W(\omega)$ 
  - ▶ The longer the piece, the better



# Signal windowing and frequency resolution

- ▶ So what's the problem if we see  $W(\omega)$  instead of Diracs?
- ▶ **Frequency resolution** = the ability to distinguish between closely spaced frequency components in a signal
- ▶ Having  $W(\omega)$  instead of a Dirac is bad because it **masks** the surrounding region
  - ▶ a two close Diracs with similar height are be impossible to differentiate, because of the wide central lobe
  - ▶ a Dirac further away but smaller is impossible to differentiate, because of secondary lobes
- ▶ Analyzing a short segment of a signal leads to **low resolution in frequency**
- ▶ Analyzing a longer segment leads to **higher resolution in frequency**
- ▶ Frequency resolution is proportional to the length of the signal

# Signal windowing and frequency resolution

- ▶ We can change the window  $w[n]$ 
  - ▶ Rectangular window
  - ▶ Hamming window
  - ▶ Hann window
  - ▶ ...
- ▶ What they do: trade narrow peak vs small secondary lobes
  - ▶ Rectangular window: widest peak, smallest secondary lobes
  - ▶ Other windows : narrower peak, higher secondary lobes
- ▶ What they do: attenuate endings, to reduce boundary problems

# Signal windowing and frequency resolution

- ▶ Remember: every time we work with a piece of a signal (e.g. we process an audio file in pieces of 1024 samples), we are applying windowing
- ▶ Even the rectangular window is still a window
- ▶ If you need to compute the spectrum, know that is is affected
- ▶ Always consider replacing the rectangular window with another one, if you use `fft()` or other frequency-based operations

# STFT and Spectrogram

- ▶ How to analyze the frequency of a signal whose frequency components change in time (e.g. like a musical song)?
- ▶ Short-Time Fourier Transform (STFT) = a technique for analyzing the frequency content of local sections of a signal as it changes over time.
- ▶ STFT divides a longer time signal into shorter segments of equal length and then computes the Fourier Transform separately on each short segment.
  - ▶ Split the signal into pieces (e.g. 1024-samples long)
  - ▶ Compute the spectrum of every piece (e.g. `fft()`)
  - ▶ Display the resulting sequence of spectra = “spectrogram”

# STFT and Spectrogram

- ▶ The STFT is a **time-frequency representation** of a signal
- ▶ 2-Dimensional: time and frequency
- ▶ Examples: <https://en.wikipedia.org/wiki/Spectrogram>

# STFT: Time and frequency resolution

- ▶ Imagine you look at the spectrogram of a music piece, and you want to pinpoint the moment where the bass guitar starts to play a chord of 100Hz
- ▶ Do a STFT and look for the moment where you see a high spectrum atoun 100Hz
- ▶ If the segments are short:
  - ▶ good time resolution
  - ▶ poor frequency resolution
- ▶ If the segments are long:
  - ▶ poor time resolution
  - ▶ high frequency resolution

# STFT: Time and frequency resolution

## **Time-frequency Uncertainty Principle:**

- ▶ you cannot have very good time resolution and very good frequency resolution simultaneously

# STFT: other issues

- ▶ Other issues with STFT
  - ▶ can change the window type, to alleviate boundary problems / artefacts
  - ▶ allow some overlap between segments (e.g. 10%)



# How to compute the DTFT

- ▶ The DFT is computed with `fft(x)`
- ▶ How to compute the DTFT?
- ▶ You can't. You need to surround  $x$  with infinitely long zeros
- ▶ You can only surround it with many zeros, but still finite
- ▶ Do this with `fft(x, 100000)` (DFT in  $N=100000$  points)
  - ▶  $x$  is surrounded with zeros until total length = 100000
  - ▶ it's still just `fft()`, so DFT, so you have points and not the full continuous function
  - ▶ but it's many many points

# How to compute the DTFT

- ▶ Computing the `fft()` in  $N=100000$  points is unrelated with frequency resolution
- ▶ Frequency resolution is dependent on actual length of  $x$
- ▶ Windowing changes the Diracs into  $W(\omega)$ 's
- ▶ `fft()` in  $N$  points is just taking  $N$  points from the resulting continuous spectrum

# Relation between DTFT and Z transform

- ▶ Z transform:

$$X(z) = \sum_n x[n]z^{-n}$$

- ▶ DTFT:

$$X(\omega) = \sum_n x[n]e^{-j\omega n}$$

- ▶ DTFT can be obtained from Z transform with

$$z = e^{j\omega}$$

- ▶ These  $z = e^{j\omega}$  are **points on the unit circle**

- ▶  $|z| = |e^{j\omega}| = 1$  (*modulus*)

- ▶  $\angle z = \angle e^{j\omega} = \omega$  (*phase*)

# Relation between DTFT and Z transform

- ▶ Fourier transform = Z transform evaluated **on the unit circle**
  - ▶ if the unit circle is in the convergence region of Z transform
  - ▶ otherwise, equivalence does not hold
- ▶ This is true for most usual signals we work with
  - ▶ some details and discussions are skipped

# Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

► Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

► Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

# Geometric interpretation of Fourier transform

- ▶ For complex numbers:
  - ▶ modulus of  $|a - b|$  = the length of the segment between  $a$  and  $b$
  - ▶ phase of  $|a - b|$  = the angle of the segment from  $b$  to  $a$  (direction is important)
- ▶ So, for a point on the unit circle  $z = e^{j\omega}$ 
  - ▶ modulus  $|X(\omega)|$  is **given by the distances to the zeros and to the poles**
  - ▶ phase  $\angle X(\omega)$  is **given by the angles from the zeros and poles to  $z$**

# Geometric interpretation of Fourier transform

- ▶ Consequences:
  - ▶ when a **pole** is very close to unit circle  $\rightarrow$  Fourier transform is **large** at this point
  - ▶ when a **zero** is very close to unit circle  $\rightarrow$  Fourier transform is **small** at this point
- ▶ Examples: ...

# Geometric interpretation of Fourier transform

- ▶ Simple interpretation for modulus  $|X(\omega)|$ :
  - ▶ Z transform  $X(z)$  is like **a landscape**
    - ▶ **poles** = **mountains** of infinite height
    - ▶ **zeros** = **valleys** of zero height
  - ▶ Fourier transform  $X(\omega) =$  “*Walking over this landscape along the unit circle*”
  - ▶ The height profile of the walk gives the amplitude of the Fourier transform
  - ▶ When close to a mountain  $\rightarrow$  road is high  $\rightarrow$  Fourier transform has large amplitude
  - ▶ When close to a valley  $\rightarrow$  road is low  $\rightarrow$  Fourier transform has small amplitude



# Geometric interpretation of Fourier transform

- ▶ Note:  $X(z)$  might also have a constant  $C$  in front!
  - ▶ It does not appear in pole-zero plot
  - ▶ The value of  $|C|$  and  $\angle C$  must be determined separately
- ▶ This “geometric method” can be applied for phase as well

# Time-frequency duality

- ▶ **Duality** properties related to all Fourier transforms
- ▶ Discrete  $\leftrightarrow$  Periodic
  - ▶ **discrete** in time  $\rightarrow$  **periodic** in frequency
  - ▶ **periodic** in time  $\rightarrow$  **discrete** in frequency
- ▶ Continuous  $\leftrightarrow$  Non-periodic
  - ▶ **continuous** in time  $\rightarrow$  **non-periodic** in frequency
  - ▶ **non-periodic** in time  $\rightarrow$  **continuous** in frequency

# Terminology

- ▶ Based on frequency content:
  - ▶ **low-frequency** signals
  - ▶ **mid-frequency** signals (band-pass)
  - ▶ **high-frequency** signals
- ▶ **Band-limited** signals: spectrum is 0 beyond some frequency  $f_{max}$
- ▶ **Bandwidth**  $B$ : frequency interval  $[F_1, F_2]$  which contains 95% of energy
  - ▶  $B = F_2 - F_1$
- ▶ Based on bandwidth  $B$ :
  - ▶ **Narrow-band** signals:  $B \ll$  central frequency  $\frac{F_1 + F_2}{2}$
  - ▶ **Wide-band** signals: not narrow-band