Digital Signal Processing

Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

Recap: Complex numbers

- real and imaginary part
- **modulus and phase**
- graphical interpretation
- Euler formula
- modulus and phase of e^{jx}

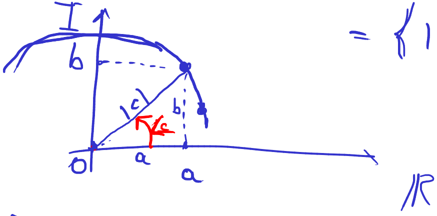
$$c = a + j \cdot b$$

$$|c| = \sqrt{a^2 + b^2}$$

$$= d\big((a,b),(0,0)\big) = \sqrt{(a-0)^2 + (b-0)^2}$$

$$\measuredangle c \quad \angle c = \operatorname{atan}\left(\frac{b}{a}\right)$$

$$\tan = tg(\angle c) = \frac{b}{a}$$

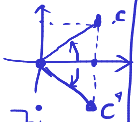


$$c = (a, b) = \{|c|, \angle c\}$$

$$\boxed{e^{j \cdot x} = \underbrace{\cos x}_{a} + j \cdot \underbrace{\sin x}_{b}}$$

$$|e^{jx}| = \sqrt{\cos^2 x + \sin^2 x} = 1$$

$$c = 2 + 7j$$



$$\bar{c} = c^* = 2 - 7j$$

complex conjugation

$$c = |c| \cdot e^{j \angle c} = a + jb$$

$$c^* = |c| \cdot e^{-j \angle c} = a - jb$$

Definition of Z transform

- The Z Transform of a signal $x[n]$, called $\underline{X(z)}$, is defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n} \quad = \cdots + x[-2]\cdot z^{2} + x[-1]\cdot z^{1} + x[0] + x[1]\cdot z^{-1} + x[2]\cdot z^{-2} + \cdots$$

- Notation:

$$\mathcal{Z}\left(x[n]\right) = X(Z) \qquad \mathcal{Z}\{x[n]\} = X(z)$$

$$x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(Z) \qquad x[n] \overset{Z}{\longleftrightarrow} X(z)$$

Definition of Z transform

- Similar to the Laplace transform for continuous signals
- The Z transform associates **a polynomial** to a signal (think Information Theory class)
- Why?
  - Easy representation of convolution
  - Convolution of two signals = multiplication of polynomials
  - Efficient descriptions of complicated systems with poles and zeros

Region of convergence

$X(z) \quad , \; z \in \mathbb{C}$

- $X(Z)$ is a sum dependent on some variable z (complex number)
- The **Region Of Convergence (ROC)** $=$ the values of z for which the sum is convergent (does not go to $\pm\infty$)

Examples

$$X(z) = \sum_n x[n] \cdot z^{-n}$$

Exercises:

- Compute Z transform for the following signals:

  $x[n] = 1, 2, 5, 7, 0,$ (with time origin in 1 or in 5)

  $\delta[n]$, $\delta[n-k]$, $\delta[n+k]$

  $x[n] = \left(\frac{1}{2}\right)^n u[n]$

a) a1) $x[n] = \{0 \ldots \underset{x[0]}{1}, \underset{x[1]}{2}, \underset{x[2]}{5}, \underset{x[3]}{7}, 0, \ldots\}$

$$X(z) = 1 + 2z^{-1} + 5\cdot z^{-2} + 7\cdot z^{-3}$$

a2) $x[n] = \{0 \ldots \underset{x[-2]}{1}, \underset{x[-1]}{2}, \underset{x[0]}{5}, \underset{x[1]}{7}, 0, \ldots\}$

$$X(z) = 1\cdot z^{2} + 2\cdot z + 5 + 7\cdot z^{-1}$$

$\cdot z$

b). $0\,0\,0\,1\,0\,0 \ldots$

$\delta[n] \longleftrightarrow 1$

$\delta[n-k] \longleftrightarrow z^{-k}$ $\quad \left(0 + 0\cdot z^{-1} + 0\cdot z^{-2} + \ldots + 1\cdot z^{-k} + 0 \ldots\right)$

$0\,0\,0 \ldots \underset{k}{1}\,0\,0 \ldots$

c) $x[n] = \frac{1}{2}^n \cdot u[n]$

$= \{0 \ldots \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots \left(\frac{1}{2}\right)^n \ldots\}$

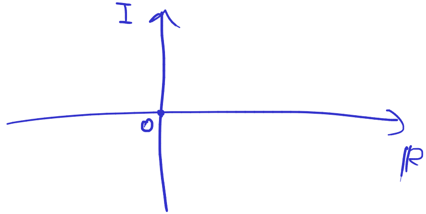
$$\frac{z}{z - 1/2}$$

Region of convergence

$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3}$, ROC: $Z - \{0\}$

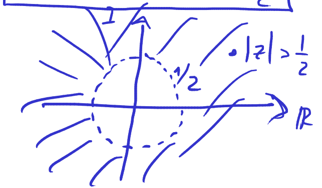
$z = 0$

$z^{-1} = \frac{1}{0} = \infty$



- z is a complex number
- Region of convergence (ROC) is displayed as an area in the complex plane (also known as the Z plane)

$X(z) = \frac{z}{z - \frac{1}{2}}$, $\boxed{ROC: |z| > \frac{1}{2}}$

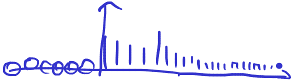


Region of convergence

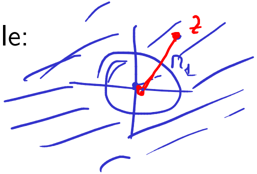
- For **finite-support** signals, the ROC is the **whole** Z plane, possibly except 0 or ∞

$\{\dots 0 \dots 1, 2, 5, 7, \dots 0 \dots\}$

- For **causal** signals, the ROC is the **outside** of a circle:



$\boxed{|z| > r_1}$

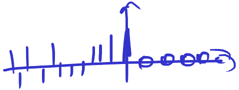


$x[n] = u[n] = \{\dots 0, 1, 1, 1, \dots 1 \dots\}$

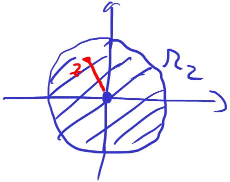
$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots z^{-n} + \dots$

  - e.g. if $|z|$ is big enough, the sum is convergent

- For **anti-causal** signals, the ROC is the **inside** of a circle:



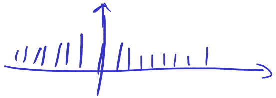
$\boxed{|z| < r_2}$



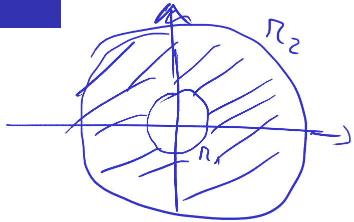
  - e.g. if $|z|$ is small enough, the sum is convergent

- Why circles? Because only modulus of z matters
  - complex numbers on a circle have the same modulus

Region of convergence



- For **bilateral** signals, the ROC is the area **between** two circles:



$$\boxed{r_1 < |z| < r_2}$$ most general

  - bilateral signals have a causal part and an anti-causal part
- For finite-support signals, the two "circles" are of "radius" 0 and ∞
- Two different signals can have the same expression of $X(z)$, but with different ROC!
  - ROC is an essential part in specifying a Z transform
  - it should never be omitted

The Inverse Z Transform

$X(z) \longleftrightarrow x[n] = ?$

- From a purely mathematical point of view, $X(z)$ is a complex function

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Proper definition of inverse transform is based on the theory of complex functions
- Multiply with z^{n-1} and integrate along a contour C inside the ROC:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

The Inverse Z Transform

- The Cauchy integral theorem says that:

$$\frac{1}{2\pi j}\oint_C z^{n-k-1}dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- And therefore:

$$\boxed{x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz}$$

- We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

$$\mathcal{Z}\left\{a \cdot x_1[n] + b \cdot x_2[n]\right\} = a \cdot \mathcal{Z}\{x_1[n]\} + b \cdot \mathcal{Z}\{x_2[n]\}$$

1. Linearity

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with ROC1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with ROC2, then:

$$a x_1[n] + b x_2[n] \overset{Z}{\longleftrightarrow} a X_1(z) + b X_2(z)$$

and ROC is at least the intersection of ROC1 and ROC2.

Proof: use definition

$$\mathcal{Z}\left\{a \cdot x_1[n] + b \cdot x_2[n]\right\} = \sum_{n} \left(a x_1[n] + b \cdot x_2[n]\right) \cdot z^{-n} = a \cdot \underbrace{\sum_{n} x_1[n] z^{-n}}_{\mathcal{Z}\{x_1[n]\}} + b \cdot \underbrace{\sum_{n} x_2[n] \cdot z^{-n}}_{\mathcal{Z}\{x_2[n]\}}$$

Properties of Z transform

$$x[n] \longleftrightarrow X(z)$$

$$x[n-k] \longleftrightarrow z^{-k} \cdot X(z)$$

$$x[n+k] \longleftrightarrow z^{k} \cdot X(z)$$

2. Shifting in time

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$x[n-k] \overset{Z}{\longleftrightarrow} z^{-k} X(z)$$

with same ROC, possibly except 0 and ∞.

Proof: by definition

- valid for all k, also for $k < 0$
- delay of 1 sample $= \boxed{z^{-1}}$

$$\mathcal{Z}\{x[n-k]\} = \sum_{n} x[n-k] \cdot z^{-n} \cdot \underbrace{z^{-k} \cdot z^{+k}}_{1} = z^{-k} \underbrace{\sum_{n=-\infty}^{\infty} x[n-k] \cdot z^{-(n-k)}}_{X(z)} = z^{-k} \cdot X(z)$$

Properties of Z transform

3. Modulation in time

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$e^{j\omega_0 n}x[n] \overset{Z}{\longleftrightarrow} X\left(e^{-j\omega_0}z\right)$$

with same ROC
Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC $r_1 < |z| < r_2$, then:

$$x[-n] \overset{Z}{\longleftrightarrow} X\left(z^{-1}\right)$$

with ROC $\frac{1}{r_2} < |z| \frac{1}{r_1}$
Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with ROC, then:

$$nx[n] \xleftrightarrow{\mathrm{Z}} -z\frac{dX(z)}{dz}$$

with same ROC
Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \stackrel{\mathrm{Z}}{\longleftrightarrow} X(z)$ with ROC, then:

$$x[n] - x[n-1] \stackrel{\mathrm{Z}}{\longleftrightarrow} (1 - z^{-1})X(z)$$

with same ROC except $z = 0$.

Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with ROC, then:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\mathrm{Z}} \frac{X(z)}{(1 - z^{-1})}$$

with same ROC except $z = 1$.
Proof: $x[n] = y[n] - y[n-1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with ROC, and $x[n]$ is a complex signal, then:

$$x^*[n] \overset{\mathrm{Z}}{\longleftrightarrow} X^*(z^*)$$

with same ROC except $z = 0$.
Proof: apply definition

Consequence

If $x[n]$ is a real signal, the poles / zeroes are either real or in complex pairs

$x[n] \in \mathbb{R}$

$x[n] \in \mathbb{C}$

Properties of Z transform

9. Convolution in time

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with ROC1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with ROC2, then:

$$x[n] = x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X(z) = X_1(z) \cdot X_2(z)$$

and ROC the intersection of ROC1 and ROC2.

Proof: use definition

- **Very important property!**
- Can compute the convolution of two signals via the Z transform

$$\mathcal{Z}\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} \left(\sum_k x_1[k] \cdot x_2[n-k] \right) \cdot z^{-n} \cdot \underbrace{z^{-k} \cdot z^{k}}_{1} = \underbrace{\sum_k x_1[k] \cdot z^{-k}}_{X_1(z)} \cdot \underbrace{\sum_n x_2[n-k] \cdot z^{-(n-k)}}_{\substack{n-k=m \\ X_2(z)}}$$

Properties of Z transform

10. Correlation in time

If $x_1[n] \xleftrightarrow{\mathrm{Z}} X_1(z)$ with ROC1, and $x_2[n] \xleftrightarrow{\mathrm{Z}} X_2(z)$ with ROC2, then:

$$r_{x_1x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n-l] \xleftrightarrow{\mathrm{Z}} R_{x_1x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and ROC the intersection of ROC1 and with the ROC of $X_2(z^{-1})$ (see reflection property)
Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

Properties of Z transform

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

When $z \to \infty$, all terms z^{-k} vanish.

Common Z transform pairs

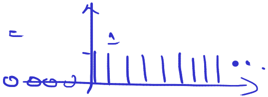
- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

causal

anti-causal

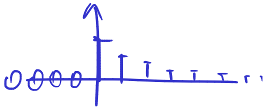
$u[n] =$



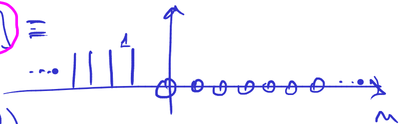
$a^n \cdot u[n]$ = causal

$= \{0, \ldots 0, 1, a, a^2, a^3, a^4, \ldots\}$

$n = 0$



$u[-n-1] =$



(anti-causal)

III.2. Z transforms which are Rational Functions

Rational functions

- Many Z transforms are in the form of a **rational function**, i.e. a **fraction** where
  - numerator = **polynomial** in z^{-1} or z
  - denominator = **polynomial** in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z{-}2 + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-}2 + \ldots + a_N z^{-N}} \quad , \ \text{ROC}$$

- Example:

$$X(z) = \frac{\overset{b_0}{3} + \overset{b_1}{2z^{-1}} + \overset{b_2}{4z^{-2}}}{\underset{a_0}{1} \underset{a_1 = 0}{\uparrow} \underbrace{-\,5z^{-2}}_{a_2} \underset{a_3 = 0}{\uparrow} \underbrace{+\,7z^{-4}}_{a_4}} = \frac{B(z)}{A(z)} \quad , \ \text{ROC}$$

Poles and zeros

$\boxed{P(x) = 0}$

$P(x) = x^3 + 3x^2 + 79x^1 + 80 = G \cdot (x - x_1)(x - x_2)(x - x_3)$

- Any polynomial is completely determined by its **roots** and a **scaling** factor

$$Any\ polynomial(X) = G \cdot (X - x_1) \ldots (X - x_n)$$

- The **zeros** of $X(z)$ are the **roots of the numerator** $B(z)$
- The **poles** of $X(z)$ are the **roots of the denominator** $A(z)$
- The zeros are usually named $z_1, z_2, ...z_M$, and the poles $p_1, p_2, ...p_N$.

Poles and zeros

$$= \frac{(z-2)(z-1)}{(z-0.5)(z+0.3)} \cdot \frac{z^{-1} \cdot z^{-1}}{z^{-1} \cdot z^{-1}} \; ? = \frac{(1-2z^{-1})(1-z^{-1})}{(1-0.5z^{-1})(1+0.3z^{-1})}$$

- The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot \underbrace{z^{N-M}} \cdot \underbrace{\frac{(z-z_1)\dots(z-z_M)}{(z-p_1)\dots(z-z_N)}} = \frac{b_0}{a_0} \cdot \frac{(1-z_1z^{-1})\dots(1-z_Mz^{-1})}{(1-p_1z^{-1})\dots(1-z_Nz^{-1})}$$

- It has:
  - M zeros with finite values
  - N poles with finite values
  - and either N-M zeros in 0, if N $>$ M, or N-M poles in 0, if N $<$ M (trivial poles/zeros)

Poles and zeros

- Example:

$$
\begin{aligned}
X(z) =& \frac{2z^2 + 0.4z - 1}{3z^3 + 2.4z^2 - 3z - 2.4} \\
=& \frac{2}{3} \cdot \frac{(z-0.3)(z+0.5)}{(z-1)(z+1)(z+0.8)} \\
=& z^{-1} \cdot \frac{(2 + 0.4z^{-1} - 1z^{-2})}{3 + 2.4z^{-1} - 3z^{-2} - 2.4z^{-3}} \\
=& z^{-1} \cdot \frac{2}{3} \cdot \frac{(1-0.3z^{-1})(1+0.5z^{-1})}{(1-z^{-1})(1+z^{-1})(1+0.8z^{-1})}
\end{aligned}
$$

$$\cdot \frac{z^{-3}}{z^{-3}}$$

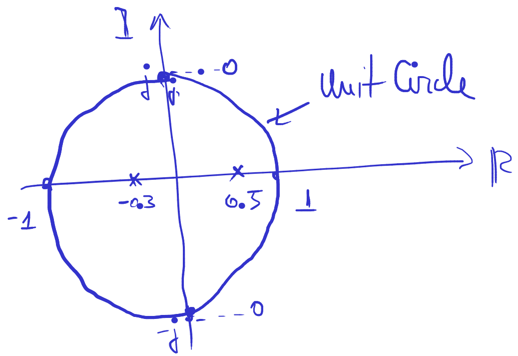
- Multiple ways of writing same expression

Graphical representation

$p_1 = 0.5$ $\quad z_1 = 0.5 + j$

$p_2 = -0.3$ $\quad z_2 = 0.5 - j$

- The graphical representation of poles and zeros in the complex place is called **the pole-zero plot**
- Graphical: poles = "x", zeros = "0"
- ROC cannot contain poles
- Example: at whiteboard



III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

Inverse Z Transform:

- We have X(z) and the ROC, what is the signal $x[n] =?$

Methods:

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition** (the one we'll actually use)

Partial fraction decomposition

Any rational function

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z{-}2 + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-}2 + \ldots + a_N z^{-N}} \longrightarrow (z-p_1)\cdot(z-p_2)\cdot(z-p_3)$$

can be decomposed in **partial fractions**:

simple fractions

$$X(z) = c_0 + c_1 z^{-1} + \ldots c_{N-M} z^{-(M-N)} + \frac{A_1}{z-p_1} + \ldots \frac{A_N}{z-p_N}$$

$x[n] = c_0\delta[n] + c_1\delta[n-1]$

$(p_1)^n u[n] \ldots$

- Each pole p_i has a corresponding partial fraction $\frac{A_i}{z-p_i}$
- First terms appear if $M \leq N$
- Based on linearity, we invert each term individually (simple)

Procedure for Inverse Z Transform

~~See E~~

See Exercise

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, **divide numerator to denominator** to obtain the first terms.
   - The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller than denominator
2. In the remaining fraction, **eliminate the negative powers** of z by multiplying with z^N. We want all powers like z^N, not z^{-N}
3. **Divide by** z:

$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$

Procedure for Inverse Z Transform

See Exercise

4. Compute the poles of $\frac{X_1(z)}{z}$ and **decompose in partial fractions**:

$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + \ldots$$

5. **Multiply back with** z:

$$X_1(z) = A_1 \frac{z}{z - p_1} + \ldots$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- If all poles are distinct:

$$A_k = (z - p_k)\frac{X(z)}{z}|_{z=p_k}$$

See Exercise

- If poles are in complex conjugate pairs
  - group the two fractions into a single fraction of degree 2
- If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \ldots + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z - p_k)^m \cdot \frac{X(z)}{z}\right]|_{z=p_k}$$

  - example for m = 2

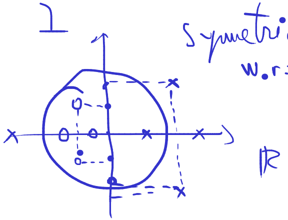
Real signals and complex poles/zeros

$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \text{ceva} \pm j \cdot \text{altceva}$

$P_1 = 0.5 + 0.3j$

$P_2 = 0.5 - 0.3j$

- Consequence of the complex-conjugate property of Z transform:
- A signal $x[n]$ with **real values** can have only:
  - **real-valued** poles or zeroes
  - complex poles and zeroes in **conjugate pairs**, which can be grouped into a single term of degree 2, with real coefficients
- If a Z transform has a complex pole or zero **without** its conjugate pair, then the corresponding signal $x[n]$ is **complex**



symmetric: w.r.t. ℝ axis

Position of poles and signal behavior

//TODO: Graph. examples (more)

- A rational Z transform $X(z)$ = sum of partial fractions, as we just saw
    - and some initial terms z^k in front
- Each **partial fraction** (pole) generates an **exponential signal**:

- $a^n u[n]$, or $-a^n u[-n-1]$

  - $-a^n u[-n-1]$

- For a single partial fraction (one pole only), we will analyze the relation between the position of the pole and the signal in time

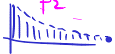
$$X(z) = \frac{1}{z^2 - 0.8z + 0.15}$$

$$= \frac{1}{\underbrace{(z-0.3)}_{P_1}\underbrace{(z-0.5)}_{P_2}}$$

$$= \frac{A \cdot z}{z - \underbrace{0.3}_{P_1}} + \frac{B \cdot z}{z - \underbrace{0.5}_{P_2}}$$

$$= x[n] = A \cdot 0.3^n u[n] + B (0.5)^n u[n]$$

P1 P2



Position of poles and signal behaviour - 1 pole

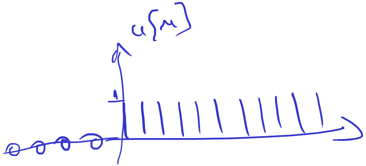
- Consider a single partial fraction with **1 pole** $p_1 = a$:

$$X(z) = C \cdot \frac{z}{z - a}, \quad ROC : |z| > |a|$$

  - Consider only real signals $x[n] \in \mathbb{R}$ —> a is real
  - Consider only causal signals $x[n]$ —> ROC is $|z| > |a|$

- Let's analyze how the corresponding signal looks like
  - use the formulas:

$$x[n] = a^n u[n]$$

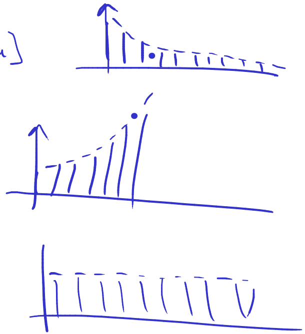


Position of poles and signal behavior - 1 pole

$\boxed{a^n \cdot u[n]}$

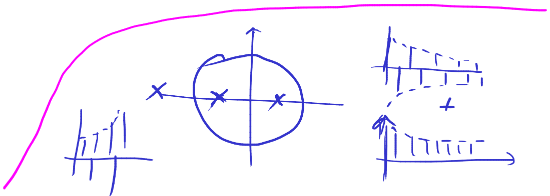
How does the signal look like, depending on the pole value $p_1 = a$:

$a^n u[n]$



- Pole **inside** the unit circle ($|a| < 1$) = **exponentially decreasing** signal
- Pole **outside** the unit circle ($|a| > 1$) = **exponentially increasing** signal
- Pole **exactly on** unit circle ($|a| = 1$)= not increasing, not decreasing, but **constant** signal
- **Negative** pole ($a < 0$) —> **alternating** signal

- **Positive** value ($a > 0$) –> **non-alternating** signal



Position of poles and signal behavior - 1 pole

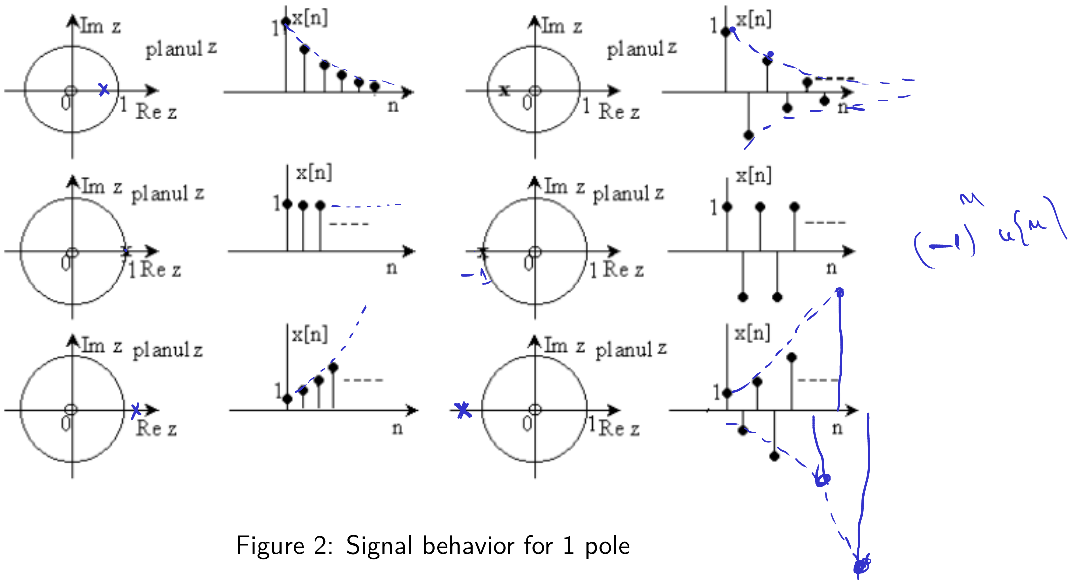


Figure 2: Signal behavior for 1 pole

Position of poles and signal behavior - 1 double pole

$$X(z) = \frac{1}{z^2 - 2z + 1} = \frac{1}{(z-1)(z-1)}$$

- Consider a **double pole** ($p_1 = a$, $p_2 = a$):

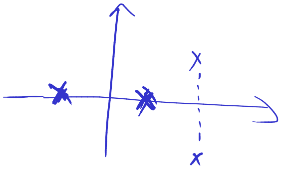
$$X(z) = C\frac{az}{(z-a)^2} = C\frac{az^{-1}}{(1-az^{-1})^2}, ROC : |z| > |a|$$

- The corresponding signal is:

$$\boxed{x[n] = na^n u[n]}$$

Effect of double pole in $p_1 = p_2 = a$:

$a \in \mathbb{R}$



- Pole inside the unit circle ($|a| < 1$) = decreasing signal
- Pole outside the unit circle ($|a| > 1$) = increasing signal
- Pole **exactly on unit** circle ($|a| = 1$) = **increasing signal**
- Negative pole ($a < 0$) = alternating signal
- Positive value ($a > 0$) = non-alternating signal

Position of poles and signal behavior - 1 double pole

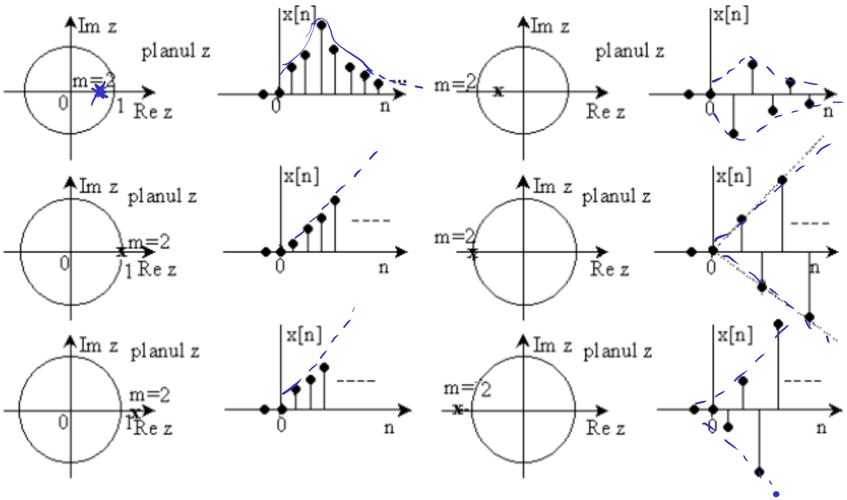


Figure 3: Signal behavior for 1 double pole

Position of poles and signal behavior - conjugate poles

$a = 2 + 0.5j$

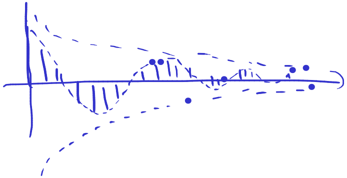
$a^* = 2 + 0.5j$

- Consider a **pair of complex conjugate** poles ($p_1 = a$, $p_2 = a^*$):

$$X(z) = \frac{1 - a\cos\omega_0 z^{-1}}{1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}}, ROC : |z| > |a|$$

- The correponding signal is:

$$x[n] = a^n \cos(\omega_0 n) u[n]$$



- Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**

Position of poles and signal behavior - conjugate poles

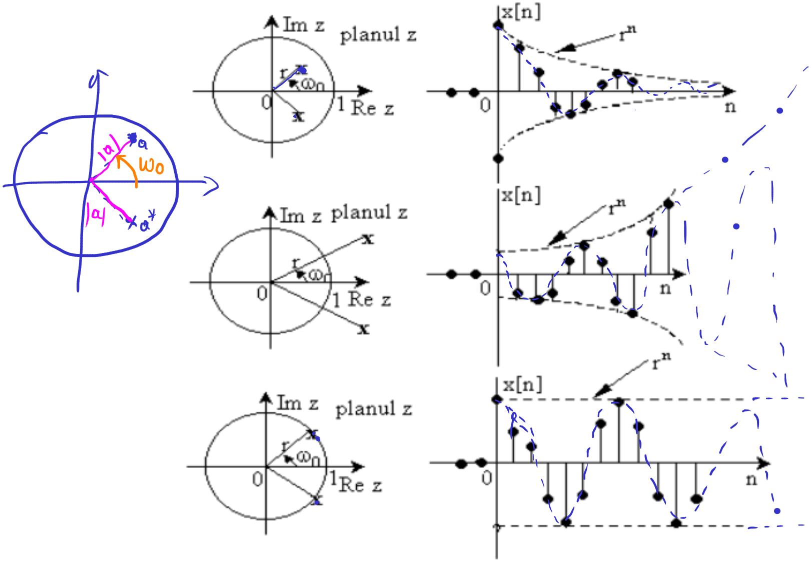
Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**

- **phase** of poles gives the **frequency** of the sinusoidal
- **modulus** of poles gives the **exponential envelope**
  - poles **inside** unit circle = **decreasing** signal
  - poles **outside** unit circle –> **increasing** signal
  - poles **on** unit circle –> **oscillating signal**, constant amplitude, neither increasing nor decreasing

What if the poles are double?

- poles **on** unit circle –> **increasing** signal
- otherwise, similar to above

Position of poles and signal behavior - conjugate poles



$a^n \cdot \cos(\omega_0 n)$, $|a| < 1$

$a^n \cdot \cos(\omega_0 n)$, $|a| > 1$

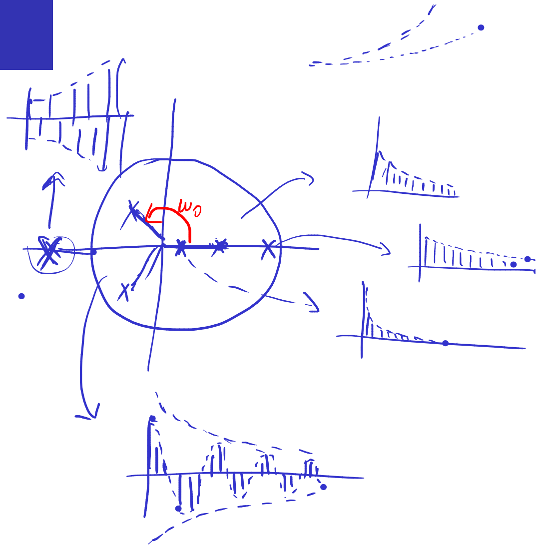
$a^n \cdot \cos(\omega_0 n)$, $|a| = 1$

Figure 4: Signal behavior for 1 double pole

Position of poles and signal behavior

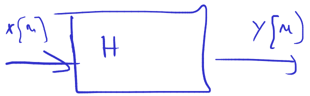
Summary: position of poles and behavior of signal

- A Z transform can be decomposed into **partial fractions**, i.e. separate poles
- Each pole means a separate fraction, means a separate component within the signal
- Conclusions (for real signals, causal):
  - **all poles inside** unit circle = **bounded** signal
    - because all components are exponentially decreasing
  - **simple** poles **on** unit circle = **bounded** signal
    - not increasing to infinity, but also not decreasing
  - **otherwise** = **unbounded** signal
  - poles **closer to 0** = **faster decreasing** signal
  - poles **farther from 0** = **slower decrease** of signal



III.4 LTI systems and the Z Transform

System function of a LTI system



- Consider a LTI system with impulse response $h[n]$
- If we aplpy an input signal $x[n]$, the output is (convolution):

$$y[n] = x[n] * h[n]$$

- In Z transform, **convolution = product** of transforms

$Z\{\ \}$

$$\boxed{Y(z) = X(z) \cdot H(z)}$$

- **The system function H(z)** of a LTI system = the **Z transform of the impulse response** $h[n]$
- The system function of a LTI system is(you know this from SCS):

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

System function and the system equation

- Reminder: any LTI system has an equation:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$
$$= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] +$$
$$+ b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

- which can be rewritten as:

$$y[n] + \sum_{k=1}^{n} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \ldots + a_N y[n-N] =$$
$$= b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

$x[n] \longleftrightarrow X(z)$

$x[n-5] \longleftrightarrow z^{-5} \cdot X(z)$

"Difference Equation"

$$1 \cdot y[n] = 2 \cdot y[n-1] + 3 x[n] - 1 \cdot x[n-1]$$

$(\Leftrightarrow)$

$$y[n] - 2 \cdot y[n-1] = 3x[n] - x[n-1]$$

$\Downarrow Z$

$$Y(z) - 2 \cdot z^{-1} \cdot Y(z) = 3X(z) - z^{-1} \cdot X(z)$$

$$Y(z)(1 - 2z^{-1}) = X(z)(3 - z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - 1 z^{-1}}{1 - 2z^{-1}}$$

System function and the system equation

- The system function $H(z)$ can be written **directly from the equation**
- We apply the Z transform to the whole equation
  - every $y[n-k]$ becomes $z^{-k}Y(z)$
  - every $x[n-k]$ becomes $z^{-k}X(z)$
  - $Y(z)$, $X(z)$ are pulled in front as common factors
- We obtain:

$$Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$
$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$y[n] = \underbrace{(-2)}_{-a_1} y[n-1] + \underbrace{(4)}_{-a_3} y[n-3] + \underbrace{1}_{b_2} \cdot x[n-2] + \underbrace{0.7}_{b_3} x[n-3] - \underbrace{0.33}_{b_5} x[n-5]$$

$$H(z) = \frac{1\cdot z^{-2} + 0.7 z^{-3} - 0.33\cdot z^{-5}}{1 + 2z^{-1} - 4z^{-3}}$$

System function and the system equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- coefficients b_k of $x[n]$, $x[n-1]$ ... appear at **numerator**
- coefficients a_k of $y[n-1]$, $y[n-2]$ ... appear at **denominator**
  - beware of the sign change of a_k
  - the coefficient of $y[n]$ itself is always $a_0 = 1$

System function of FIR systems

Finite Impulse Response (FIR)

Eq: $y[n] = x[n] \quad x[n-1] \quad x[n-2] \quad \ldots\ldots \quad x[n-M]$

Particular cases:

- **FIR systems**: when all $a_k = 0$ (except $a_0 = 1$)
  - only zeros, no poles ("all-zero system"), no denominator in $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

  - the coefficients b_k are really the impulse response $h[n]$

EXAMPLE:

$$y[n] = 2\,x[n] + 3\,x[n-1] - 7\,x[n-2] + 2.5\,x[n-3]$$

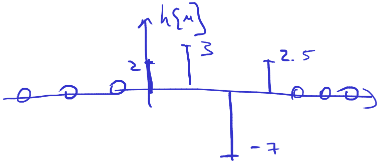
$$= 2 + 3z^{-1} - 7z^{-2} + 2.5z^{-3}$$

$$\updownarrow \; Z^{-1}$$

$$h[n] = 2\cdot\delta[n] + 3\cdot\delta[n-1] - 7\,\delta[n-2] + 2.5\,\delta[n-3]$$

$\delta[n] \longleftrightarrow 1$

$\delta[n-1] \longleftrightarrow 1\cdot z^{-1}$



$h[n] = \{0, 2, 3, -7, 2.5, 0 \ldots\}$

System function of IIR systems

infinite impulse Response

Particular cases:

- If some $a_k \neq 0$ we have an **IIR system**
  - $H(z)$ has some polynomial at the denominator
  - If denominator is just b_0: **all-pole system**
    - has only poles, no zeros

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  - Or it can be a general IIR system with both poles and zeros (i.e. polynomials both at numerator or denominator)

Stability of a system and H(z)

Reminders from chapter 2:

- **Stable** system = a **bounded input** implies a **bounded output** (BIBO)
- A system is stable if:

$$\sum_{n} |h[n]| < \infty (\quad \textit{is convergent})$$

Stability of a system and H(z)

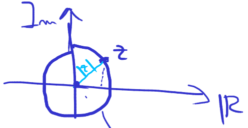
$|a+b| \le |a|+|b|$

$|a\cdot b| = |a|\cdot|b|$

$$\left|H(z)\right| = \left|\sum_n h[n]\, z^{-n}\right| \le \sum_n \left|h[n]\cdot z^{-n}\right| = \sum_n |h[n]|\cdot|z^{-n}|$$

- For a system with system function $H(z)$ we have:

$$|H(z)| = |\sum h[n]z^{-n}| \le \sum |h[n]| \cdot \underbrace{|z^{-n}|}_{1}$$



z on unit circle

- Now let's consider z **on the unit circle**, i.e. $|z| = |z^{-n}| = 1$:

$$|H(z)|\Big|_{|z|=1} = |H(z)|\Big|_{|z|=1} \le \sum |h[n]|$$

$|z|$ on unit circle $\Longleftrightarrow$ $|z| = 1$

$|z^{-n}| = 1$

- If the system is **stable**, $\sum |h[n]| < \infty$ (**convergent**), so $|H(z)|\big|_{|z|=1} < \infty$
  - i.e. the $\boxed{\text{unit circle } |z| = 1 \textbf{ is in the ROC}}$

$|H(z)|$ on unit circle should not be ∞!

Stability of a system and H(z)

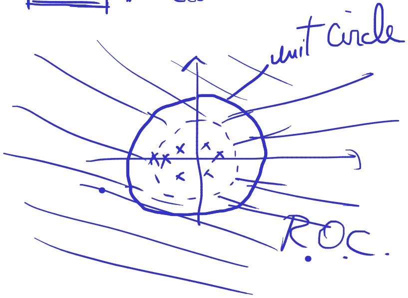
- A LTI system is stable if ⟺ the **unit circle in inside the Region of Convergence** of $H(z)$
  - one can also prove the reciprocal, so there is equivalence

FIR : stable

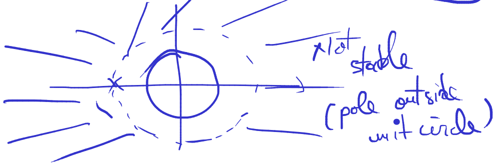
IIR : causal :

- When the system is also **causal**: $y[n] = x[n], [n-1]$

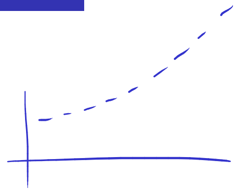
  $h[n]$ = causal
  - ROC of causal system = exterior of a circle given by the largest pole
  - stable = unit circle inside the ROC
  - therefore stable = all poles **inside** unit circle



- A **causal** LTI system is stable if **all the poles are inside the unit circle**



Stability of a system and H(z)

$h[n] =$ 

- Alternative explanation:
  - If one pole is **outside** unit circle, the signal component for that partial fraction will be exponentially **increasing** –> whole signal is **unbounded**

Natural and forced response

- Consider a causal LTI system with initial conditions $= 0$
  - I.C. are relevant for recursive implementations (IIR)
- Consider an input signal:

$$x[n] \stackrel{Z}{\longleftrightarrow} X(z) = \frac{N(z)}{Q(z)}$$

- Consider an impulse response (system function):

$$h[n] \stackrel{Z}{\longleftrightarrow} H(z) = \frac{B(z)}{A(z)}$$

- Then the output signal is:

$$y[n] = x[n] * h[n] \stackrel{Z}{\longleftrightarrow} Y(z) = X(z)H(z) = \frac{N(z)B(z)}{Q(z)A(z)} = Y(z)$$

- (Some poles and zeros might simplify, if exactly identical)

$y[n] = 0.7\, y[n-1] + x[n]$

$\boxed{\text{I.C.: } y[-1] = 3}$

$Q(z) = (z-0.1)(z-0.2)$

$A(z) = (z-0.3)(z-0.4)$

Natural and forced response

- Denote the poles of $X(z)$ as q_i and the poles of $H(z)$ as p_i
  - Assume all poles are *simple* (i.e. no multiplicity)
  - Assume all poles $\neq$ all zeros, so no simplification
- The output signal has components dependent on the **input signal** and also of the **system itself**

$$Y(z) = \underbrace{\sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}}_{\text{from } h[n]} + \underbrace{\sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}}_{\text{from input } x[n]}$$

- and $y[n]$ is

$$y[n] = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u[n]}_{\textit{natural response}} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u[n]}_{\textit{forced response}}$$

from $h[n]$ from input $x[n]$

Natural and forced response

Any output $y[n]$ is the **sum of two signals**:

- **Natural response** $y_{nr}[n]$ = the part given by the poles of **the system** itself
- **Forced response** $y_{fr}[n]$ = given by the poles of **the input signal**
- Together they form the **zero-state response** of the system = the output signal when initial conditions are 0

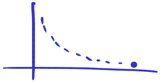
Zero-input response

If there are **non-zero** initial conditions, there is a **third component** as well:

- **Zero-input response** $y_{zi}[n]$ = given by the initial conditions of the system
  - It behaves similarly to the natural response, i.e. depends on the system's poles

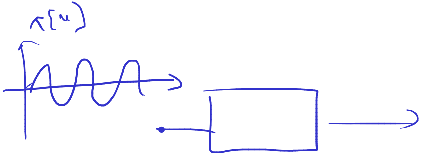
Transient and permanent response

poles of $H(z)$

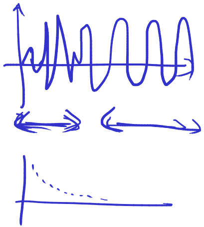


Note to self: Consider removing this part next year

- For a **stable** system, all system poles $|p_k| < 1$
  - therefore, both natural response and zero-input repsonse are made of decreasing exponentials



- For a stable system, the natural response and the zero-input response **die out exponentially**
- Thus, the natural response and the zero-input repsonse are called **transient** response
  - they fade~~a~~ away, usually quickly



- Input signals last indefinitely => the forced response is a **permanent response**

Transient and permanent regime

Operating regimes:

- When the input signal is first applied, and the transient response is present, the system is in **transient regime**
- When the transient response has died out, the system remains in **permanent regime**, where only the input signal determines the output

Example: apply a infinitely long sinusoidal, starting from $n = 0$

- the output has some irregularities at the beginning, due to the natural responses
- afterwards, it becomes perfectly regular