Digital Signal Processing

Chapter IV: The Fourier Transform and its applications

IV.1 Vector spaces of signals (crash course)

Vector spaces

- **Vector space** $=$ a set $V\{v_i\}$ with the following two properties:
  - one element $+$ another element $=$ still an element of the same space
  - a scalar constant $\times$ an element $=$ still an element of the same space
- You **can't escape** a vector space by summing or scaling
- The elements of a vector space are called **vectors**

Examples of vector spaces

- Geometric spaces are great intuitive examples:
  - a line, or the set $\mathbb{R}$ (one-dimensional)
  - a plane, or the set $\mathbb{C}$ (two-dimensional)
  - 3D space (three-dimensional)
  - 4D space (four-dimensional, like the spatio-temporal universe)
  - arrays with N numbers (N-dimensional)
  - space of continuous signals (∞-dimensional)
- The **dimension** of the space = "how many numbers you need in order to specify one element" (informal)
- A "vector" like in maths = a sequence of N numbers = a "vector" like in programming
  - e.g. a point in a plane has two coordinates = a vector of size $N = 2$
  - e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

Inner product

- Many vector spaces have a fundamental operation: **the (Euclidean) inner product**
  - for **discrete** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$
  - for **continuous** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$
- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** ("produs scalar")

Inner product

- Each entry in $\mathbf{x}$ times the complex conjugate of the one in $\mathbf{y}$, all summed
- For discrete signals, it can be understood as a row $\times$ column multiplication
- Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- Inner product is **linear** in both terms:

$$\langle \mathbf{x_1} + \mathbf{x_2}, \mathbf{y} \rangle = \langle \mathbf{x_1}, \mathbf{y} \rangle + \langle \mathbf{x_2}, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y_1} + \mathbf{y_2} \rangle = \langle \mathbf{x}, \mathbf{y_1} \rangle + \langle \mathbf{x}, \mathbf{y_2} \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

The distance between two vectors

- An inner product induces a **norm** and a **distance** function
- **The (Euclidean) distance** between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_N - y_N)^2}$$

- This distance is the **usual geometric distance** you know from geometry
- It has the exact same intuition like in **normal geometry**:
  - if two vectors have small distance, they are close, they are similar
  - two vectors with large distance are far away, not similar
  - two identical vectors have zero distance

The norm of a vector

- An inner product induces a **norm** and a **distance** function
- The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- The **norm** of a vector is the distance from $\mathbf{x}$ to point $\mathbf{0}$.
- It has the exact same intuition like in **normal geometry**:
  - vector has large norm = has big values, is far from $\mathbf{0}$
  - vector has small norm = has small values, is close to $\mathbf{0}$
  - vector has zero norm = it is the vector $\mathbf{0}$
- Norm of a vector = sqrt(the signal **energy**)

Norm and distance

- The norm and distance are related
- The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- Just like in geometry: distance = length of the difference vector

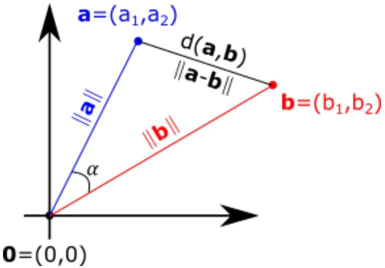


Figure 1: Norm and distance in vector spaces

Angle between vectors

- The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - is a value between -1 and 1
- **Otrhogonal vectors** $=$ two vectors with $\langle x, y \rangle = 0$
  - their angle $= 90\,\deg$
  - in geometric language, the two vectors are **perpendicular**

Why vector space

- Why are all these useful?
- They are a very general **framework** for different kinds of signals
- We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - Example in DEDP class: ML decision with 1, 2, N samples

Vector spaces in DSP class

We deal mainly with the following vector spaces:

- The vector space of all infinitely-long real signals $x[n]$
- The vector space of all infinitely-long periodic signals $x[n]$ with period N
  - for each N we have a different vector space
- The vector space of all finite-length signals $x[n]$ with only N samples
  - for each N we have a different vector space

Bases

- A **basis** = a set of N linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2 ... \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + ... + \alpha_N \mathbf{b}^N$$

- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, ... \alpha_N)$$

Bases and coordinate systems

- Bases are just like **coordinate systems** in a geometric space
  - any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

  - any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- N = The number of basis elements = The dimension of the space
- Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

Bases and coordinate systems

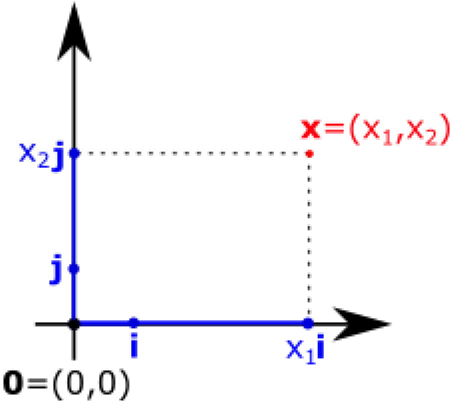


Figure 2: Basis expansion of a vector x

Choice of bases

- There is typically an infinite choice of bases
- The **canonical basis** = all basis vectors are full of zeros, just with one 1
- You used it already in an exercise:
  - any signal $x[n]$ can be expressed of a sum of $\delta[n-k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n-1] + 2\delta[n-2] + \dots$$

  - the canonical basis is $B = \{..., \delta[n], \delta[n-1], \delta[n-2], \dots\}$

Orthonormal bases

- An **orthonormal basis** a basis where all elements $\mathbf{b}^i$ are:
  - orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

  - **normalized** (their norm $= 1$):

$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- Example: the canonical basis $\{\delta[n-k]\}$ is orthonormal:
  - $\langle \delta[n-k], \delta[n-l] \rangle = 0, \forall k \neq l$
  - $\langle \delta[n-k], \delta[n-k] \rangle = 1, \forall k$

Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

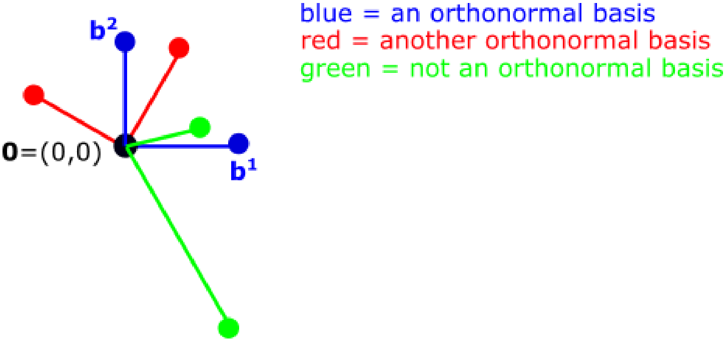


Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- Suppose we have an **orthonormal basis** $B = \{\mathbf{b}^i\}$
- Suppose we have a vector $\mathbf{x}$
- We can write (expand) $\mathbf{x}$ as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- Question: how to **find** the coefficients α_i?

Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{b}^i \rangle =& \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\ &= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\ &= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\ &= \alpha_i\end{aligned}$$

Basis expansion of a vector

- Any vector $\mathbf{x}$ can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle, \mathbf{x}, \mathbf{b}^i \rangle$$

Why bases

- How does all this talk about bases help us?
- To better understand the Fourier transform
- The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- The Fourier Transform of a signal x = finding the coefficients of $\mathbf{x}$ in this basis
- The Inverse Fourier Transform = expanding $\mathbf{x}$ with the elements of this basis
- Same **generic** thing every time, only the type of signals differ

IV.2 Introducing the Fourier Transforms

Reminder

- Reminder:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos(x - \frac{\pi}{2})$$

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

Why sinusoidal signals

- Why are sinusoidal signals sin() and cos() **so prevalent** in signal processing?
- Answer: because they are combinations of an e^{jx} and an e^{-jx}
- Why are these e^{jx} so special?
- Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

- Consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- Output signal = input signal $\times$ a (complex) constant $(H(\omega_0))$

Eigen-function

- **Eigen-function** of a system ("funcție proprie") = a function f which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

  - just like **eigen-vectors** of a matrix (remember algebra): $A\underset{\approx}{x} = \lambda\underset{\approx}{x}$
  - we call the "functions" to allow for continuous signals as well

- Complex exponential signals $e^{j\omega t}$ (or $e^{j\omega n}$) are **eigen-functions** of Linear and Time Invariant (LTI) systems:

  - output signal = input signal $\times$ a (complex) constant

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system very easily if we **decompose all signals $x[n]$ as a combination of $\{e^{j\omega n}\}$**
- Example: RGB color filter
  - suppose we have some photographic filters (lenses):
    - one reduces red to 50%
    - one reduces green to 25%
    - one reduces blue to 80%
    - RGB are eigen-functions of the system: input = 200 Blue, output = 0.8 * 200 Blue
    - what is the output color if input is “pink”?
    - Answer is easy if we represent all colors in RGB

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$
- All vector space theory becomes useful now:
  - $\{e^{j\omega n}\}$ is an **orthonormal basis**
  - decomposing signals = finding coefficients α_i
  - we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

Discrete-Time Fourier Transform (DTFT)

- Consider the vector space of **non-periodic infinitely-long signals**
- This vector space is **infinite-dimensional**
- The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis**
- We can expand (almost) any $\mathbf{x}$ in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

Discrete-Time Fourier Transform (DTFT)

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- A signal $x[n]$ can be written as a linear combination of $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn} \rangle$

Discrete-Time Fourier Transform (DTFT)

- Alternative form with ω
- We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_{n} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform (DTFT)

- A non-periodic signal $x[n]$ has a **continuous spectrum** $X(\omega)$, with $f \in [-\frac{1}{2}, \frac{1}{2}]$
  - e.g. $\omega \in [-\pi, \pi]$

Discrete Fourier Transform (DFT)

- Consider the vector space of **periodic** signals with **period N**
  - for some fixed $N = 2$, 3 or ... etc
- This is a vector space of **dimension N**
  - we need N numbers to identify a signal (specify its period)
- We can consider $x[n]$ only for **one period**, i.e. $n = 0, \dots N-1$
- The signals $\{e^{j2\pi fn}\}, \forall f \in \{0, \frac{1}{N}, \dots \frac{N-1}{N}\}$ form an **orthonormal basis** with N elements
- It is a **discrete** set of frequencies: $f = \frac{k}{N}, \forall k \in \{0, 1, \dots N-1\}$

Discrete Fourier Transform (DFT)

Inverse Discrete Fourier Transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- A periodic signal $x[n]$ can be written as a linear combination of k signals $\{e^{j2\pi kn/N}\}$, with some coefficients X_k

Discrete Fourier Transform

$$X_k = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn} \rangle$

Discrete Fourier Transform (DFT)

- A periodic signal $x[n]$ with period N has a **discrete spectrum** $X(\omega)$ composed of only N frequencies $\{0, \frac{1}{N} \dots \frac{N-1}{N}\}$
- Each frequency $\frac{k}{N}$ has a **coefficient** X_k
  - also written as c_k
  - The N coefficients X_k are the equivalent of $X(\omega)$
- It is also known as the "Fourier Series for Discrete Signals"

IV.3 The Discrete-Time Fourier Transform (DTFT)

Definition

Definitions (again):

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

Basic properties of DTFT

- $X(\omega)$ is **defined** only for $\omega \in [-\pi, \pi]$
  - or $f \in [-\frac{1}{2}, \frac{1}{2}]$
- $X(\omega)$ is **complex** (has $|X(\omega)|$, $\angle X(\omega)$)
- If the signal $x[n]$ is real, $X(\omega)$ is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

  - This means:
    - modulus is even: $|X(\omega)| = |X(-\omega)|$
    - phase is odd: $X(\omega) = -X(-\omega)$

Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n} + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (X(\omega)e^{j\omega n} + X(-\omega)e^{j(-\omega)n})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|(e^{j\omega n+\angle X(\omega)} + e^{-j\omega n-\angle X(\omega)})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|\cos(\omega n+\angle X(\omega))d\omega
\end{aligned}
$$

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

Expressing as sum of sinusoids

- The DTFT shows that any signal $x[n]$ is **a "sum" of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
  - this is the fundamental practical interpretation of the Fourier transform
  - not really a sum, because we have an integral
- The **modulus** $|X(\omega)|$ gives the **amplitude** of the sinusoids ($\times$ 2)
  - for $\omega = 0$, $|X(\omega = 0)|$ = the DC component
- The **phase** $\angle X(\omega)$ gives the initial phase

Properties of DTFT

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

Properties of DTFT

2. Shifting in time

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

Properties of DTFT

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of DTFT

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- Not circular convolution, this is the normal convolution

Properties of DTFT

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of DTFT

Correlation theorem

$$r_{x_1x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

Wiener Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Is true for all orthonormal bases

IV.4 The Discrete Fourier Transform (DFT)

Definitions

Definitions (again):

Inverse Discrete Fourier Transform (DFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Discrete Fourier Transform (DFT)

$$X_k = \langle x[n], e^{j2\pi fn}\rangle = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

- DFT is defined for periodical signals with period N
- there are exactly N terms in each sum

Definitions

To remember:

- DFT: takes a vector with N elements ($x[n]$), produces a vector with N elements (X_k)
  - for this reason, we can compute it e.g. with Matlab
- DTFT: takes a vector with ∞ elements ($x[n]$), produces a continuous function ($X(\omega)$) between $[-\pi, -pi]$

Periodicity and notation

- DTFT has only N coefficients X_k, each X_k corresponding to a frequency $f = \frac{k}{N}$
- In discrete domain, $f = \frac{N-k}{N} = \frac{-k}{N}$ (aliasing, we can always add/subtract 1 from f)
- So we can consider X_{N-k} as X_{-k}, due to periodicity
- Example: a signal with period $N = 6$ has 6 DFT coefficients
    - we can call them X_0, X_1, X_2, X_3, X_4, X_5
    - we have $X_5 = X_{-1}$, $X_4 = X_{-2}$
    - we can also call them X_{-2}, X_{-1}, X_0, X_1, X_2, X_3

Basic Properties of the DFT

- Has only N coefficients X_k
- X_k are **complex** (has $|X_k|$, $\angle X_k$)
- If the signal $x[n]$ is real, the coefficients are **even**

$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

  - This means:
    - modulus is even: $|X_k| = |X_{-k}|$
    - phase is odd: $\angle X_{-k} = -\angle X_k$

Expressing as sum of sinusoids, N = odd

- Grouping terms with k and $-k$:
- If N is odd, we have X_0 and pairs (X_k, X_{-k}):

$$
\begin{aligned}
x[n] &= \frac{1}{N}X_0 e^{j0n} + \frac{1}{N}\sum_{k=-(N-1)/2}^{-1} X_k e^{j2\pi kn/N} + \frac{1}{N}\sum_{k=1}^{(N-1)/2} X_k e^{j2\pi kn/N} \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2} (X_k e^{j2\pi kn/N} + X_{-k} e^{-j2\pi kn/N}) \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2} |X_k|(e^{j2\pi kn/N + \angle X(k)} + e^{-j2\pi kn/N - \angle X(\omega)}) \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=0}^{(N-1)/2} 2|X_k|\cos(2\pi k/Nn + \angle X_k)
\end{aligned}
$$

- A sum of sinusoids with frequencies up to $1/2$

Expressing as sum of sinusoids, N = even

- If N is even, we have X_0 and pairs $(X_k,\ X_{-k})$, with an extra term $X_{N/2}$ which has no pair
  - e.g. N = 6: $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$
- The term with no pair, $X_{N/2}$, must be a real number, because $X_{N/2} = X^*_{-N/2} = XN/2^*$
- The extra term will be $\frac{1}{N} X_{N/2} e^{j2\pi N/2n/N} = X_{N/2} \cos(n\pi)$
- Overall:

$$x[n] = \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=0}^{(N-2)/2} 2|X_k|\cos(2\pi k/Nn + \angle X_k) + \frac{1}{N}X_{N/2}\cos(n\pi)$$

- A sum of sinusoids with frequencies up to $1/2$

Expressing as sum of sinusoids

- DFT says that any periodic signal $x[n]$, with period N, is **a sum of N sinusoids with frequencies**:

$$f = 0, \frac{1}{N}, \frac{2}{N}, \ldots \text{ up to } \frac{N-1}{2} \text{ or } \frac{N}{2}$$

  (not exceeding $1/2$)

- The **modulus** $|X_k|$ gives the **amplitude** of the sinusoids (sometimes $\times$ 2)
  - for $\omega = 0$, $|X_0| =$ the DC component
  - when modulus $= 0$, that frequency has amplitude 0
- The **phase** $\angle X_k$ gives the initial phase

Example

- Consider a periodic signal $x[n]$ with period $N = 6$ and the DFT coefficients:

  X_k = [15.0000 + 0.0000i , -2.5000 + 3.4410i , -2.5000 + 0.8123i , -2.5000 - 0.8123i , -2.5000 - 3.4410i]

  Write $x[n]$ as a sum of sinusoids.

- Do the same for a periodic signal $x[n]$ with period $N = 5$ and the DFT coefficients:

  X_k = [21.0000 + 0.0000i , -3.0000 + 5.1962i , -3.0000 + 1.7321i , -3.0000 + 0.0000i , -3.0000 - 1.7321i , -3.0000 - 5.1962i]

  Write $x[n]$ as a sum of sinusoids.

DFT matrix

- The DFT (and the inverse IDFT) is equivalent with a matrix multiplication:
  - on whiteboard
- In the world of discrete signals, there are many signal transforms possible, and many of them can be expressed as matrix multiplications, just like the DFT.

Properties of the DFT

1. Linearity

If the signal $x_1[n]$ has the DFT coefficients $\{X_k^{(1)}\}$, and $x_2[n]$ has $\{X_k^{(2)}\}$, then their sum has
$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$
Proof: via definition

Properties of the DFT

2. Shifting in time

If $x[n] \leftrightarrow \{X_k\}$, then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi k n_0/N)} X_k\}$$

Proof: via definition

- The amplitudes $|X_k|$ are not affected, shifting in time **affects only the phase**

Properties of the DFT

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{X^*_{-k}\}$$

Properties of the DFT

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

Circular convolution definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is also periodic with period N
- Example at the whiteboard: how it is computed

Example

Example (write on slides)

Circular convolution

- We are in the vector space of **periodic signals** with period N
- Linear (e.g. normal) convolution produces a result which is longer periodic with period N
- Circular convolution takes two sequences of length N and produces another sequence of length N
  - each sequence is a period of a periodic signal
  - circular convolution = like a convolution of periodic signals

Properties of the DFT

6. Product in time

Product in time $\leftrightarrow$ circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

Properties of the DFT

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{0}^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum |X_k|^2$$

- Is true for all orthonormal bases

Plot / sketch DTFT and DFT of various signals

Let's plot / sketch DTFT and DFT of various signals

DTFT of:

- a constant signal $x[n] = A$
- a rectangular signal $x[n] = A$ between $-\tau$ and τ, 0 elsewhere
- a cosine of frequency $f = 1/3$

DFT, with N=20, of:

- a constant signal $x[n] = A$
- a rectangular signal $x[n] = A$ between $-\tau$ and τ, 0 elsewhere
- a cosine of frequency $f = 1/3$

DFT matrix

- The DFT (and the inverse IDFT) is equivalent with a matrix multiplication:

  DFT:
  $$\mathbf{X} = \mathbf{W}_N \mathbf{x}$$

  IDFT:
  $$\mathbf{x} = \mathbf{W}_N^T \mathbf{X}$$

  where
  $$\mathbf{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix},$$

DFT matrix (continued)

$$\mathbf{W}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot (N-1)} \\ W_N^{1\cdot 0} & W_N^{1\cdot 1} & \cdots & W_N^{1\cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & \cdots & W_N^{(N-1)\cdot (N-1)} \end{bmatrix}$$

with an element $W_N^{kn} = e^{-j2\pi \frac{k}{N} n}$ (k = row index, n = column index)

- there might be small variations depending on whether we have $\frac{1}{\sqrt{N}}$ at both DFT and IDFT, or just put $\frac{1}{N}$ just for the IDFT
- note that for IDFT we have $W^{-1} = W^T$ (orthonormal basis)

DFT matrix multiplication

- Naive implementation of DFT, IDFT: use matrix multiplication with W, W^{-1}
- Number of multiplications necessary for a vector of length N is N^2
- In the world of algorithms, the **computational complexity** of an algorithm = number of multiplications necessary, depending on some variable N
  - only the dominant term matters, no coefficient, e.g $O(N^2)$ not $7.3N^2 + 4N$
- Naive DFT has computation complexity $\mathcal{O}(N^2)$
  - prohibitively large

FFT

- The Fast Fourier Transform (FFT) algorithm = a fast algorithm for computing the DFT, exploiting the particular nature (symmetries) in the DFT matrix
- FFT computational complecity: $\mathcal{O}(N \log_2(N))$
- Exercise: for $N = 1024$, how much faster is FFT compared to naive DFT multiplication?
- Invention and adoption of FFT ($\sim$'60s, Cooley & Tukey) = "the birth of Digital Signal Processing"

Other transforms

- In the world of discrete signals, there are many signal transforms possible, and many of them can be expressed as matrix multiplications, just like the DFT.
- Transform = expressing a N-dimensional vector x as a linear combination of a set of N basis vectors
- How:
  1. Put the N vectors of the basis as columns in a matrix A
  2. Solve the system $x = AX$ (inverse transform)
  3. Which means $X = A^{-1}x$ (forward transform)
- Why:
  - compression: the discrete cosine transform is the basis for JPEG image compression
  - ...

Relationship between DTFT and DFT

- How are DTFT and DFT related?
- Discrete Time Fourier Transform:
  - for non-periodical signals
  - spectrum is continuous
- Discrete Fourier Transform
  - for periodical signals
  - spectrum is discrete
- Duality: periodic in time $\leftrightarrow$ discrete in frequency

Relationship between DTFT and DFT

- Consider a non-periodic signal $x[n]$
- It has a continuous spectrum $X(\omega)$
- If we **periodize** it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k\frac{2\pi}{N})$$

- The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k/Nn)$$

- They are **samples** from the continuous $X(\omega)$ of the non-periodized signal

Relationship between DTFT and DFT

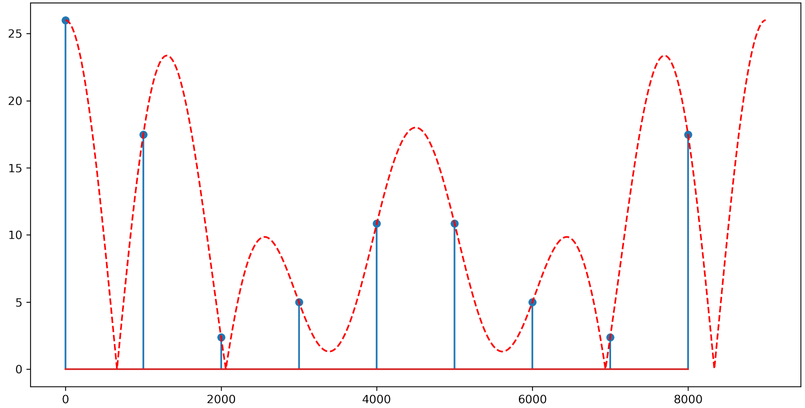
- Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

- If we consider a non-periodic $x[n]$ with infinitely long zeros on either side, we have a continuous spectrum $X(\omega)$ (DTFT)
- If we consider that x is just a period of a periodic signal, we have a discrete spectrum X_k (DFT)
- Moreover, the discrete X_k are just **samples from** $X(\omega)$:

$$X_k = X(2\pi k/Nn)$$

Relationship between DTFT and DFT



$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$

- red line = DFT(x) if x not periodical
  - actually run as `fft(x, 10000)`, x is extended with 9991 zeros
- blue = `fft(x)`

Relation between DTFT and Z transform

- Z transform:
$$X(z) = \sum_n x[n] z^{-n}$$
- DTFT:
$$X(\omega) = \sum_n x[n] e^{-j\omega n}$$
- DTFT can be obtained from Z transform with
$$z = e^{j\omega}$$
- These $z = e^{j\omega}$ are **points on the unit circle**
  - $|z| = |e^{j\omega}| = 1(modulus)$
  - $\angle z = \angle e^{j\omega} = \omega(phase)$

Relation between DTFT and Z transform

- Fourier transform = Z transform evaluated **on the unit circle**
  - if the unit circle is in the convergence region of Z transform
  - otherwise, equivalence does not hold
- This is true for most usual signals we work with
  - some details and discussions are skipped

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

Geometric interpretation of Fourier transform

- For complex numbers:
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

Geometric interpretation of Fourier transform

- Consequences:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples: . . .

Geometric interpretation of Fourier transform

- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is like **a landscape**
    - **poles = mountains** of infinite height
    - **zeros = valleys** of zero height
  - Fourier transform $X(\omega) =$ "*Walking over this landscape along the unit circle*"
  - The height profile of the walk gives the amplitude of the Fourier transform
  - When close to a mountain –> road is high –> Fourier transform has large amplitude
  - When close to a valley –> road is low –> Fourier transform has small amplitude

Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for phase as well

Time-frequency duality

- **Duality** properties related to all Fourier transforms
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Terminology

- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals
- **Band-limited** signals: spectrum is 0 beyond some frequency f_{max}
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: $B <<$ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band