Digital Signal Processing

II. Discrete signals and systems

II.1 Discrete signals

Representation

A discrete signal can represented:

- graphically
- in table form

  x[0] x[1] x[4]
- as a vector: $x[n] = [\ldots, 0, 0, 1, 3, 4, 5, 0, \ldots]$
  - an **arrow** indicates the origin of time ($n = 0$).
  - if the arrow is missing, the origin of time is at the first element
  - the dots ... indicate that the value remains the same from that point onwards

Examples: at blackboard

Notation: $x[4]$ represents the value of the fourth sample in the signal $x[n]$

Basic signals

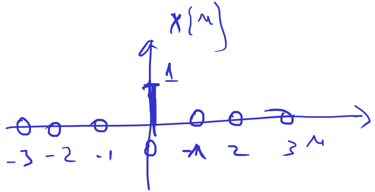
Some elementary signals are presented below.

Unit impulse

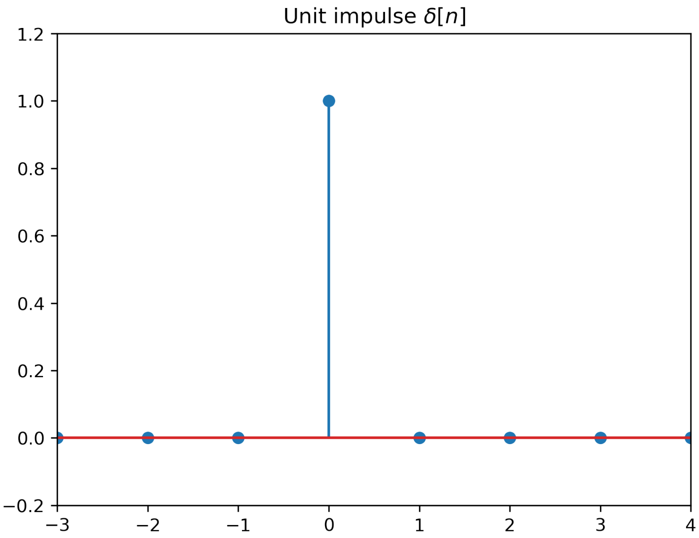
Contains a single non-zero value of 1 located at time 0. It is denoted with $\delta[n]$.

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$$\delta[n] = [\dots 0 \;\; 0 \;\; \underline{1} \;\; 0 \;\; 0 \dots 0]$$



Representation



Unit step

Step (treapta)

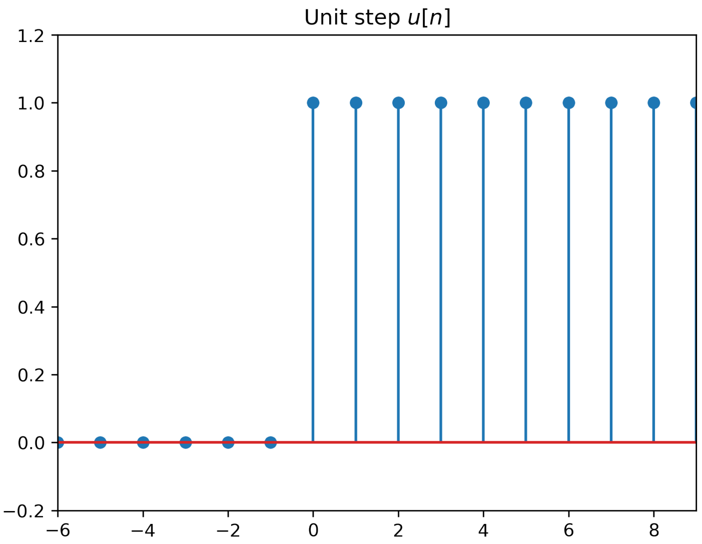
Unit step

It is denoted with $u[n]$.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

$u[n] = [\ldots 0\ 0\ 1\ 1\ 1\ \ldots\ldots]$

Representation



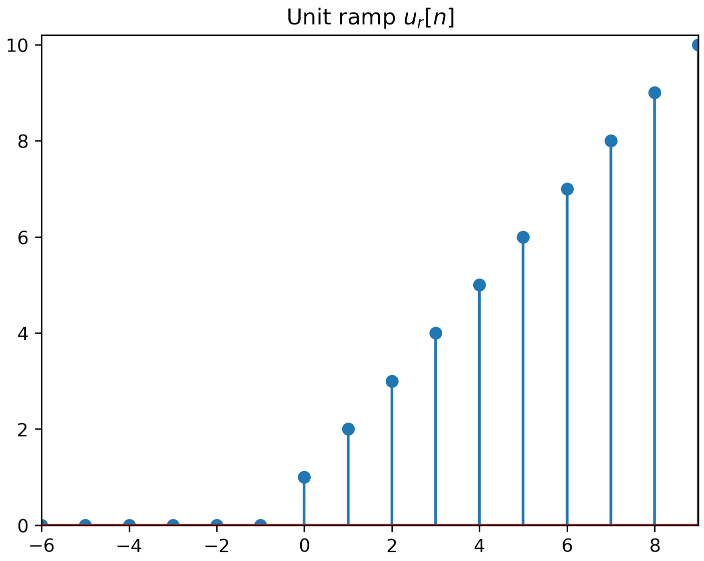
Unit ramp

Unit ramp

It is denoted with $u_r[n]$.

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation



Exponential signal

Exponential signal

It does not have a special notation. It is defined by:

$$x[n] = a^n.$$

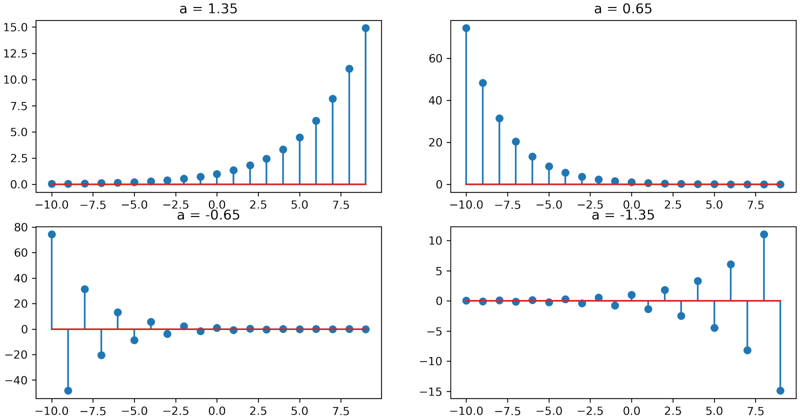
a can be a real or a complex number. Here we consider only the case when a is real.

Depending on the value of a, we have four possible cases:

1. $a \geq 1$
2. $0 \leq a < 1$
3. $-1 < a < 0$
4. $a \leq -1$

$x[n] = a^n$

Representation



II.2 Types of discrete signals

Signals with finite energy

- The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} (x[n])^2.$$

- If E is finite, the signal is said to have finite energy.
- Examples:
  - unit impulse has finite energy
  - unit step does not

Connection with DEDP class

$$x = \quad x_0 \quad x_1 \quad x_2 \cdots \quad x_{50} \cdots$$

$$0 = \quad 0 \quad 0 \quad 0 \cdots \quad 0 \cdots$$

$$d(x, 0) = \sqrt{x_0^2 + x_1^2 + \ldots + x_N}$$

$$\| x - 0 \| = d(x, 0)$$

- Cross-link with DEDP course:

$$E = \|\mathbf{x} - \mathbf{0}\|^2 = \|\mathbf{x}\|^2$$

- Energy of a signal = **squared Euclidean distance to 0**
  - geometric interpretation: squared length of the segment from 0 to the point $\mathbf{x}$
  - holds for continuous signals as well:

$$E = \|\mathbf{x}\|^2 = \int_{-\infty}^{\infty} x^2(t)dt$$

Signals with finite power

- The **average power of a discrete signal** is defined as

$$P = \lim_{N\to\infty} \frac{\sum_{n=-N}^{N}(x[n])^2}{2N+1}.$$

$= \text{media dei } (x[n])^2 = \overline{A^2}$

- In other words, the average power is the average energy per sample.
- If P is finite, the signal is said to have finite power.
- A signal with finite energy has finite power ($P = 0$ if the signal has infinite length). A signal with infinite energy can have finite or infinite power.
- Example: unit step has finite power $P = \frac{1}{2}$ (proof at blackboard).

2N+1 termini

$P = \frac{1}{2}$

$[\ldots 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1 \ldots\ldots] = \lim_{N\to\infty} \frac{\sum_{-N}^{N}(u[n])^2}{2N+1} = \frac{N+1}{2N+1} \to \frac{1}{2}$

Connection with DEDP class

- Average power = (temporal) average squared value $\overline{X^2}$
  - i.e. average value of the square of samples

Periodic and non-periodic signals

- A signal is called **periodic** if its values repeat themselves after a certain time (known as **period**).

$$x[n] = x[n+N]\cancel{)}, \forall \cancel{t} \quad \forall n \quad , N \in \mathbb{Z}$$

- The **fundamental period** of a signal is the minimum value of N.
- Periodic signals have infinite energy, and finite power equal to the power of a single period.

Even and odd signals

- A real signal is **even** if it satisfies the following symmetry:

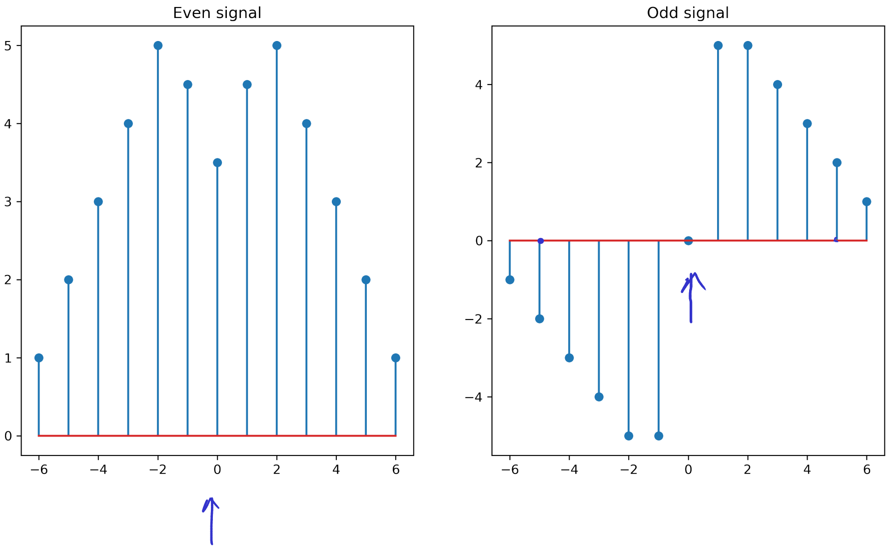
$$x[n] = x[-n], \forall n.$$

- A real signal is **odd** if it satisfies the following anti-symmetry:

$$-x[n] = x[-n], \forall n.$$

- There exist signals which are neither even nor odd.

Even and odd signals: example



Even and odd parts of a signal

- Every signal can be written as the sum of an even signal and an odd signal:

$$x[n] = x_e[n] + x_o[n]$$

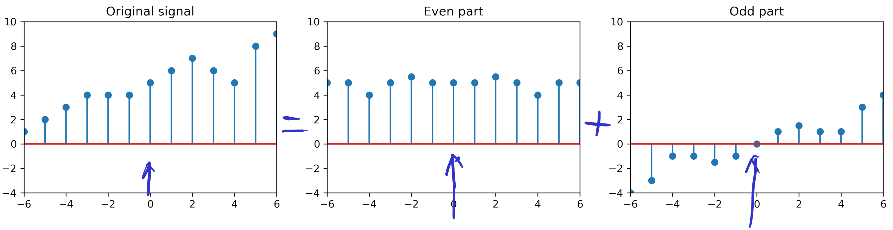
- The even and the odd parts of the signal can be found as follows:

even $$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

odd $$x_o[n] = \frac{x[n] - x[-n]}{2}.$$

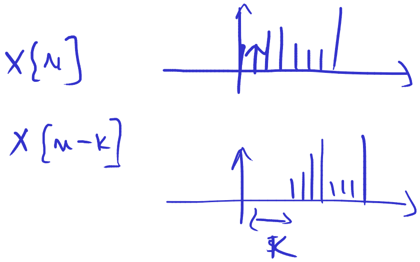
- Proof: check that $x_e[n]$ is even, $x_o[n]$ is odd, and their sum is $x[n]$

Even and odd parts: example



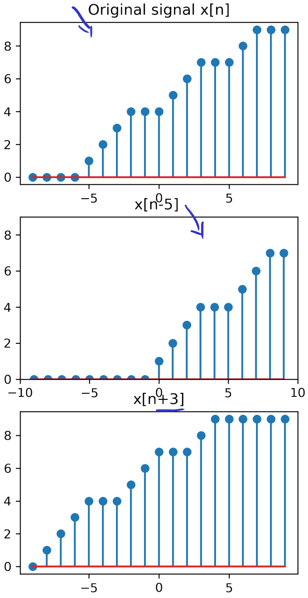
II.3 Basic operations with discrete signals

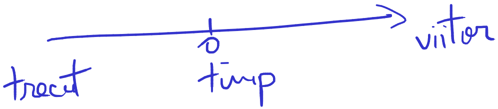
Time shifting



- The signal $x[n-k]$ is $x[n]$ **delayed with** k **time units**
  - Graphically, $x[n-k]$ is shifted k units to the **right** compared to the original
- The signal $x[n+k]$ is $x[n]$ **anticipated with** k **time units**
  - Graphically, $x[n+k]$ is shifted k units to the **left** compared to the original signal.

Time shifting: representation

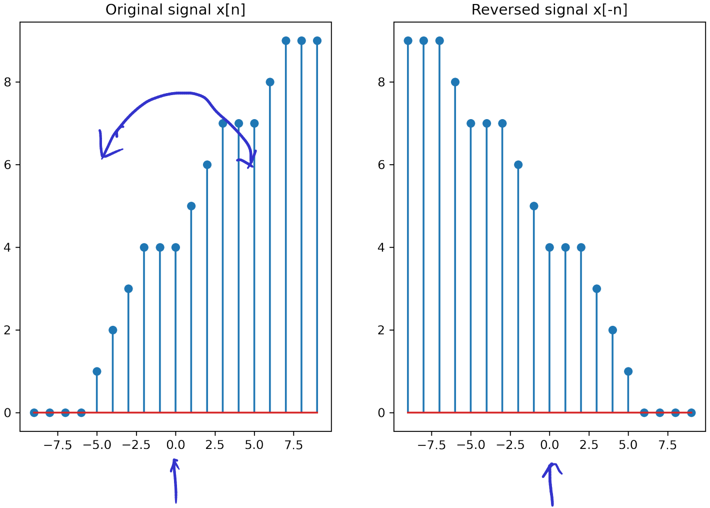




Time reversal „Oglindire"

- Changing the variable n to $-n$ produces a signal $x[-n]$ which mirrors $x[n]$.

$n \longrightarrow -n$

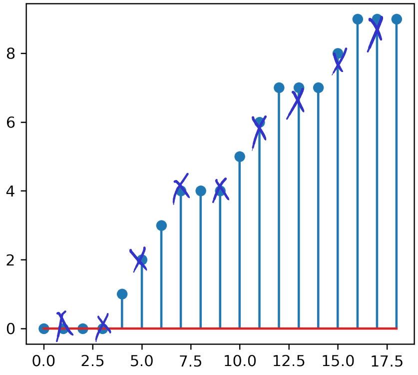


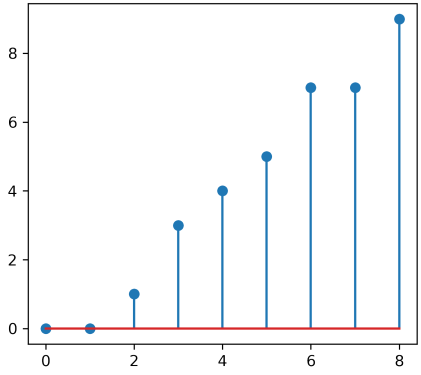
Subsampling = downsampling

- **Subsampling** by a factor of M = keep only 1 sample from every M of the original signal
  - Total number of samples is reduced M times

$$x_{M\downarrow}[n] = x[Mn]$$

$x_{2\downarrow}[n] = x[2n]$



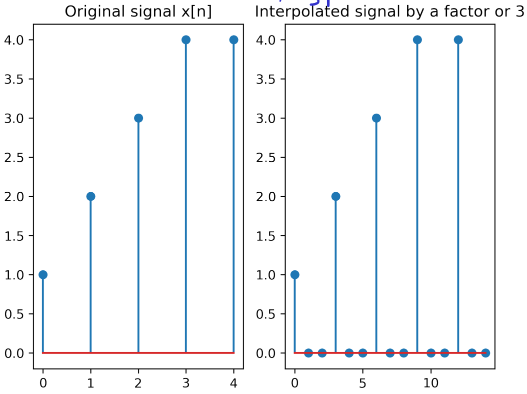


Interpolation "upsampling"

- **Interpolation** by a factor of L adds $(L-1)$ zeros between two samples in the original signal
  - Total number of samples increases L times

$$x_{L\uparrow} = \begin{cases} x[\frac{n}{L}] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

$x[n]$ $\quad$ $x_{3\uparrow}$



Mathematical operations

- A signal $x[n]$ can be **scaled** by a constant A, i.e. each sample is multipled by A:

$$y[n] = A \cdot x[n].$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **multiplied** by multiypling the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

$y[n] = \cos(x[n])$

II.4 Discrete systems

Definition

- **System** = a device or algorithm which produces an **output signal** based on an **input signal**
- We will only consider systems with a single input and a single output
- Figure here: blackboard.
- Common notation:
  - x[n] is the input
  - y[n] is the output
  - H is the system.

Notations

$$y[n] = H\left\{ x[n] \right\}$$

- Notations:

$$y[n] = H\left[x[n]\right]$$

("*the system H applied to the input x[n] produces the output y[n]*")

$$x[n] \xrightarrow{H} y[n]$$

$$x[n] \xrightarrow{H} y[n]$$

("*the input* $x[n]$ *is transformed by the system H into y[n]*")

Equations

- Usually, a system is described by the **input-output equation** (or **difference equation**) which expains how $y[n]$ is defined in terms of $x[n]$.

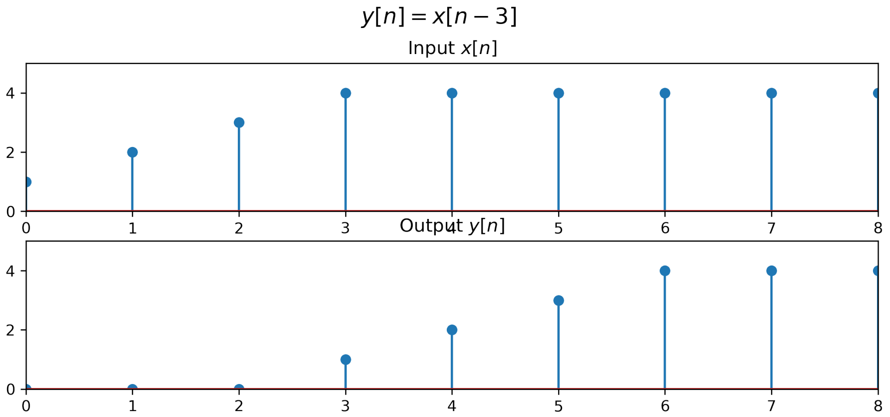
Examples:

1. $y[n] = x[n]$ (the identity system)
2. $y[n] = x[n-3]$
3. $y[n] = x[n+1]$
4. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
5. $y[n] = \max(x[n+1], x[n], x[n-1])$
6. $y[n] = (x[n])^2 + \log_{10} x[n-1]$
7. $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$

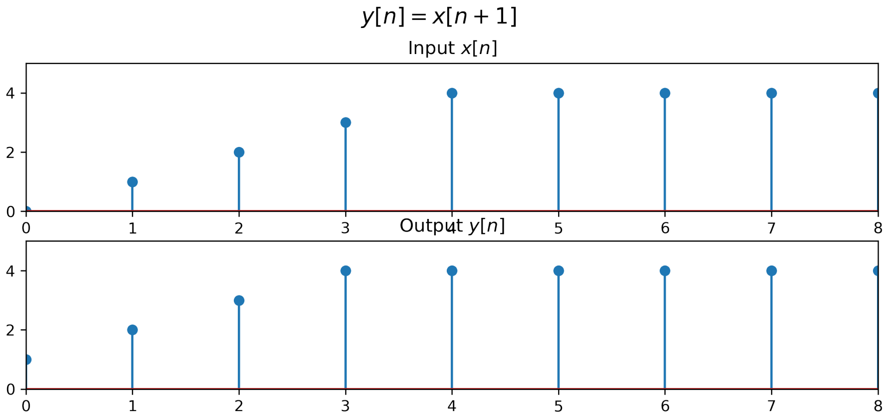
$y[n] = \ldots$

based on $x[n]$, $x[n-1]$ ...

Example

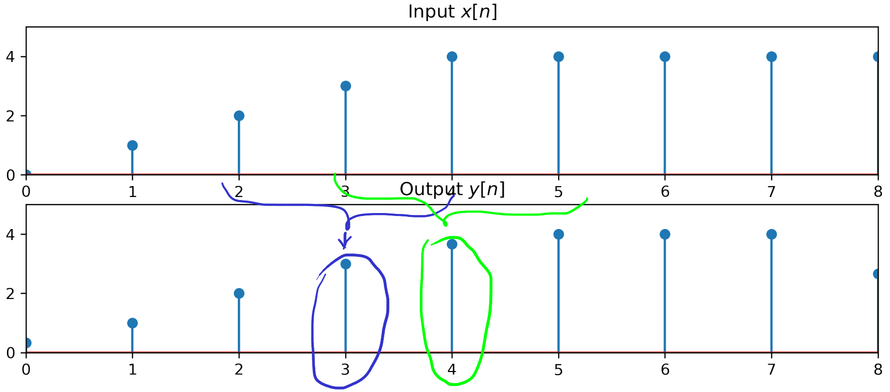


Example

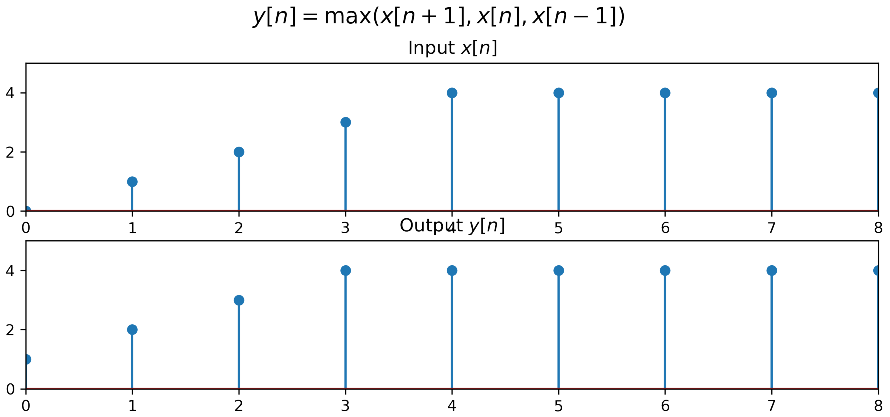


Example

$$y[n] = (x[n+1] + x[n] + x[n-1])/3$$

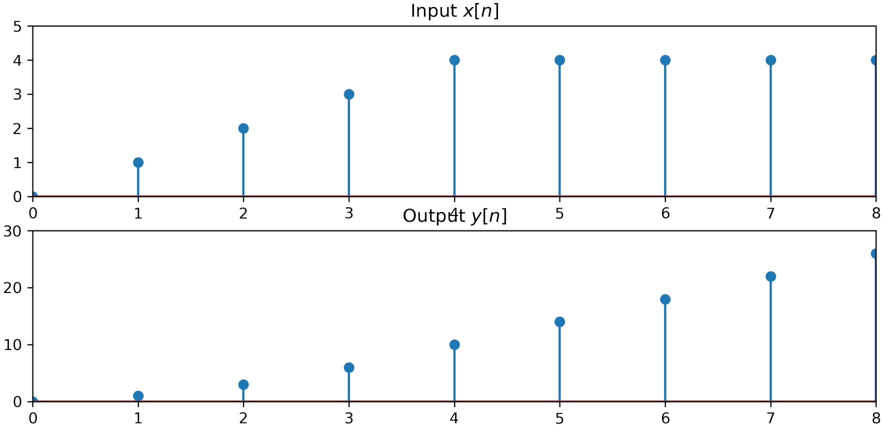


Example



Example

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \dots$$



Recursive systems

- Some systems can/must be written in **recursive form**

$$y[n] = y[n-1] + x[n],$$

Recursiv

- Must always specify **initial conditions**
  - i.e. initial value (e.g. $y[-1] = 2.5$)
  - if not mentioned, assume they are 0 ("relaxed system")
  - they represent the internal state of the system at the starting moment
- For recursive systems, the output signal depends on both the input signal **and** on the initial conditions
  - different initial conditions lead to different outputs, even if input signal is the same
  - a recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ($x[n] = 0$)

$y[-1] = 2.5$

$x[n] = 0$

$y[100] = ?$

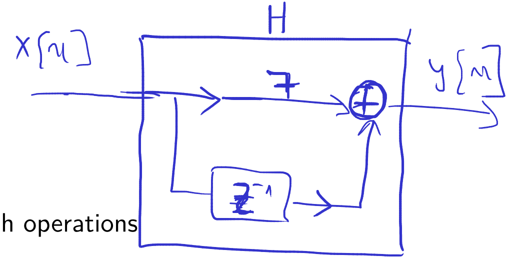
$y[0] = 2.5 + x[0]$

$y[1] =$

$\vdots$

Representation of systems

- The operation of a system can be described graphically (see examples on blackboard):
  - summation of two signals
  - scaling of a signal with a constant
  - multiplication of two signals
  - delay element
  - anticipation element
  - other blocks for more complicated math operations



$$y[n] = 7 \cdot x[n] + x[n-1]$$

II.4 Classification of discrete systems

Memoryless / systems with memory

- **Memoryless (or static)**: output at time n depends only on the input **from the same moment** n
- Otherwise, the system **has memory (dynamic)**
- Examples:
  - memoryless: $y[n] = (x[n])^3 + 5$
  - with memory: $y[n] = (x[n])^3 + x[n-1]$

Memoryless / systems with memory

- Memory of size N:
  - output at time n $y[n]$ depends only up to the last N inputs, $x[n-N], x[n-(N-1)], \ldots x[n]$,
  - if N is finite: the system has **finite memory**
  - if $N = \infty$, the system has **infinite memory**
- Examples:
  - finite memory of order 4: $y[n] = x[n] + x[n-2] + x[n-\underline{4}]$
  - infinite memory: $y[n] = 0.5y[n-1] + 0.8x[n]$
    - recursive systems usually have infinite memory

$x[n-1000]$

Memoryless / systems with memory

- An input sample has an effect on the output only for the next N time moments
- For systems infinite memory, any sample influences **all** the following samples, forever
  - but, if system is stable, the influence gets smaller and smaller

Time-Invariant and Time-Variant systems

- A relaxed system H is **time-invariant** if and only if:

$$x[n] \xrightarrow{H} y[n]$$

implies, $\forall x[n], \forall k$, that

$$x[n-k] \xrightarrow{H} y[n-k]$$

- Delaying the input signal with k will only delay the output with the same amount, otherwise the output is not affected
  - Must be true for all input signals, for all possible delays (positive or negative)
- Otherwise, the system is said to be **time-variant**

Time-Invariant and Time-Variant systems

$$y[n] = x[n] - x[n-1]$$

$$n \longrightarrow (n-k)$$

$$x[n] \xrightarrow{H} y[n]$$

$$x[n-k] \xrightarrow{H} y[n-k] \quad ??$$

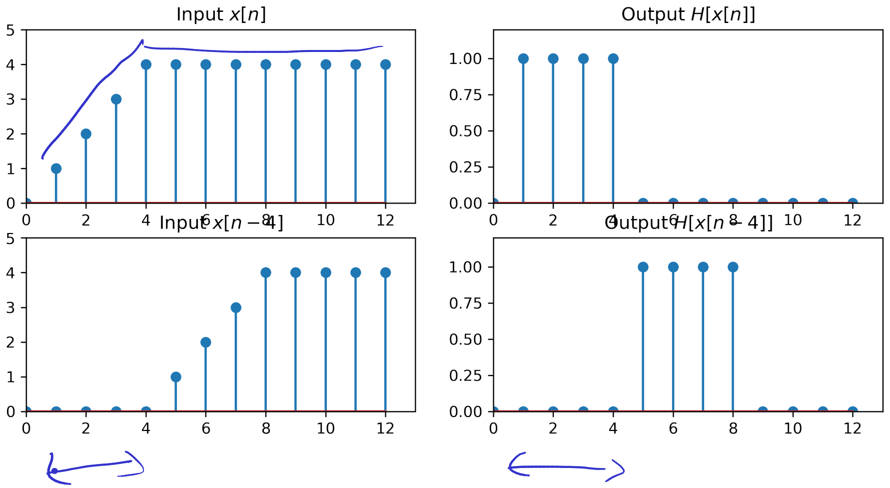
- Examples:
  - $y[n] = x[n] - x[n-1]$ is time-invariant
  - $y[n] = n \cdot x[n]$ is not time-invariant
- A system is time-invariant if it depends on n only through the input signal $x[n]$

$$\begin{cases} H(x[n-k]) = n \cdot x[n-k] \\ y[n-k] = (n-k) \cdot x[n-k] \end{cases} \neq \Rightarrow \text{T.V.}$$

$$\begin{cases} H(x[n-k]) \Rightarrow x[n-k] - x[n-k-1] \\ y[n-k] = x[n-k] - x[n-k-1] \end{cases} \Rightarrow \text{T.I.}$$

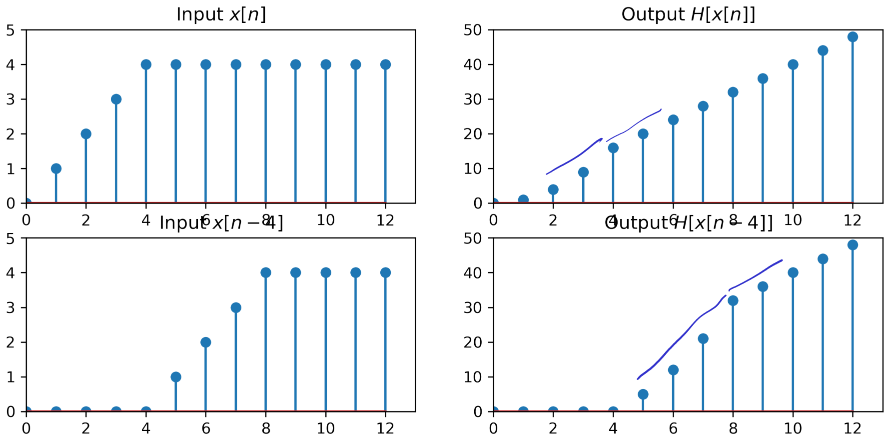
Example

Time-invariant system $y[n] = x[n] - x[n-1]$



Another example

Time-variant system $y[n] = n \cdot x[n]$



Linear and nonlinear systems

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(a \cdot x) = a \cdot f(x) \end{cases}$$

$$\Leftrightarrow$$

$$f(a \cdot x + b \cdot y) = a \cdot f(x) + b \cdot f(y)$$

- A system H is **linear** if:

$$H\left[ax_1[n] + bx_2[n]\right] = aH\left[x_1[n]\right] + bH\left[x_2[n]\right].$$

- Composed of two parts:
  - Applying the system to a sum of two signals = applying the system to each signal, and adding the results
  - Scaling the input signal with a constant a is the same as scaling the output signal with a
- The same relation will be true for a sum of many signals, not just two

Linear and nonlinear systems

$$y[n] = 3x[n] + 5x[n-2]$$

$$x[n] \longrightarrow a \cdot x_1[n] + b \cdot x_2[n]$$

$$H(a \cdot x_1[n] + b \cdot x_2[n]) = 3 \cdot (a \cdot x_1[n] + b \cdot x_2[n]) + 5 \cdot (a \cdot x_1[n-2] + b \cdot x_2[n-2])$$

$$= a \cdot \underbrace{(3 \cdot x_1[n] + 5 \cdot x_1[n-2])}_{H(x_1)} + b \cdot \underbrace{(3 \cdot x_2[n] + 5 \cdot x_2[n-2])}_{H(x_2)}$$

- Advantage of linear systems
  - Complicated input signals can be decomposed into a sum of smaller parts
  - The system can be applied to each part independently
  - Then the results are added back
- Examples:
  - linear system: $y[n] = 3x[n] + 5x[n-2]$ $\rightarrow$ $+ \; a \cdot [n-k]$
  - nonlinear system: $y[n] = 3(x[n])^2 + 5x[n-2]$

$$f(a+b) = f(a) + f(b)$$

$$f(x) = c \cdot x$$

$$(x_1[n] + x_2[n])^2 \neq (x_1[n])^2 + (x_2[n])^2$$

$$\log(x_1 + x_2) \neq \log(x_1) + \log(x_2)$$

Linear and nonlinear systems

$$(x_1 + x_2 + 5) \neq (x_1 + 5) + (x_2 + 5)$$

- For a system to be linear, the input samples $x[n]$ must not undergo non-linear transformations.
- **The only transformations** of the input x[n] allowed to take place in a linear system are:
  - scaling (multiplication) with a constant
  - delaying
  - summing different delayed versions of the signal (not summing with a constant)

Causal and non-causal systems

- **Causal**: the output $y[n]$ depends only on the current input $x[n]$ and the past values $x[n-1]$, $x[n-2]$..., but not on the future samples $x[n+1]$, $x[n+2]$...
- Otherwise the system is **non-causal**.
- A causal system can operate in real-time
  - we need only the input samples from the past
  - non-causal systems need samples from the future
- Examples:
  - $y[n] = x[n] - x[n-1]$ is causal
  - $y[n] = x[n+1] - x[n-1]$ is non-causal
  - $y[n] = x[-n]$ is non-causal

$y[-5] = x[5]$

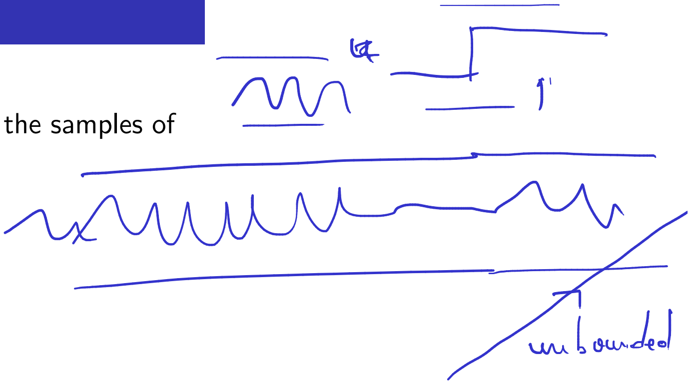
Stable and unstable systems

„Mărginit"

- **Bounded** signal: if there exists a value M such that all the samples of the signal or smaller than M, in absolute values

$$x[n] \in [-M, M]$$

$$|x[n]| \leq M$$



- **Stable** system: if for any bounded input signal it produces a bounded output signal
  - not necessarily with the same M
  - known as BIBO (Bounded Input --> Bounded Output)
- In other words: when the input signal has bounded values, the output signal does not go towards ∞ or $-\infty$.

Stable and unstable systems

$$y[n] = (x[n])^3 - x[n+4]$$

Când $n \to \infty$, $y[n]$ să fie mărginit

- Examples:
  - $y[n] = (x[n])^3 - x[n+4]$ is stable
  - $y[n] = \frac{1}{x[n]-x[n-1]}$ is unstable
  - $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$ is unstable

$$y[n] = \frac{1}{x[n] - x[n-1]}$$

not stable

$$y[n] = y[n-1] + x[n]$$

$$x[n] = 1\ 1\ 1\ 1\ 1\ \ldots$$

$$y[n] = 1\ 2\ 3\ 4\ \ldots$$

Impulse response of Linear Time-Invariant (LTI) systems

Linear Time-Invariant (LTI) systems

- Notation: An **LTI** system (**L**inear **T**ime-**I**nvariant) is a system which is simultaneously **linear** and **time-invariant**.
- LTI systems have an equation like this:

$$\boxed{\begin{aligned} y[n] &= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\ &\quad + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \\ &= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=1}^{M} b_k x[n-k] \end{aligned}}$$

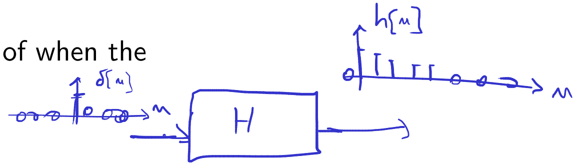
$y[n] = -a_1 \cdot y[n-1] - a_2 \cdot y[n-2] - \ldots$
$\ldots - a_N \cdot y[n-N] \quad +$
$+ b_0 \cdot x[n] + b_1 \cdot x[n-1] + \ldots$
$\ldots + b_M \cdot x[n-M]$

  - the above is for causal systems; non-causal can also have [n+k]

The impulse response

- **Impulse response** of a system = output (response) of when the input signal is the impulse $\delta[n]$:

$$h[n] = H(\delta[n])$$



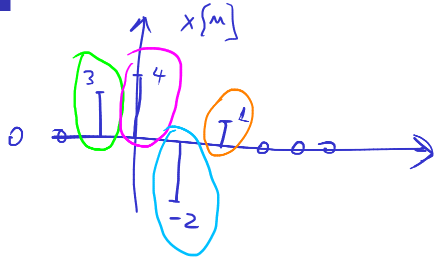
- The impulse response of a LTI system **fully characterizes the system**:
  - based on $h[n]$ we can compute the response of the system to **any** input signal
  - all the properties of LTI systems can be described via characteristics of the impulse response

Signals are a sum of impulses

- Any signal $x[n]$ can be composed as **a sum of scaled and delayed impulses** $\delta[n]$.
- Example:
  $x[n] = \{3, 1, -5, 0, 2\} = 3\delta[n] + \delta[n-1] - 5\delta[n-2] + 2\delta[n-2]$
- In general

$$\underline{x[n]} = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{ampl.}} \delta[n - \underbrace{k}_{\text{delay}}]$$

  i.e. a sum of impulses $\delta[n]$, each one delayed with k and scaled with the corresponding value $x[k]$



$x[n] = 3 \cdot \delta[n+1] + 4 \cdot \delta[n]$

$- 2\,\delta[n-1] + 1 \cdot \delta[n-2]$

Convolution

- The response of a LTI system to a sum of impulses, delayed with k and scaled with $x[k]$, **is a sum of impulse responses, delayed with k and scaled with $x[k]$.**

Linear: $H(a \cdot x_1 + b \cdot x_2) = a \cdot H(x_1) + b \cdot H(x_2)$

$$
\begin{aligned}
y[n] &= H(x[n]) \\
&= H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) = \sum \left(H\left(x[k] \cdot \delta[n-k]\right)\right) = \\
&\overset{\text{Linear}}{=} \sum_{k=-\infty}^{\infty} x[k]H(\delta[n-k]) \\
&\overset{\text{T. Invar.}}{=} \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{ampl.}} h\underbrace{[n-k]}_{\text{delay}}.
\end{aligned}
$$

$H(\delta[n]) = h[n]$

$H(\delta[n-k]) = h[n-k]$

Convolution

- Convolution in short:
  - The input signal is composed of separate impulses
  - Each impulse will generate its own response (LTI)
  - Output signal is the sum of impulse responses, delayed and scaled
- Convolution only applies for LTI systems

Convolution

- This operation = the **convolution** of two signals $x[n]$ and $h[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The response of a LTI system to an input signal x[n] is **the convolution of x[n] with the system's impulse response h[n]**

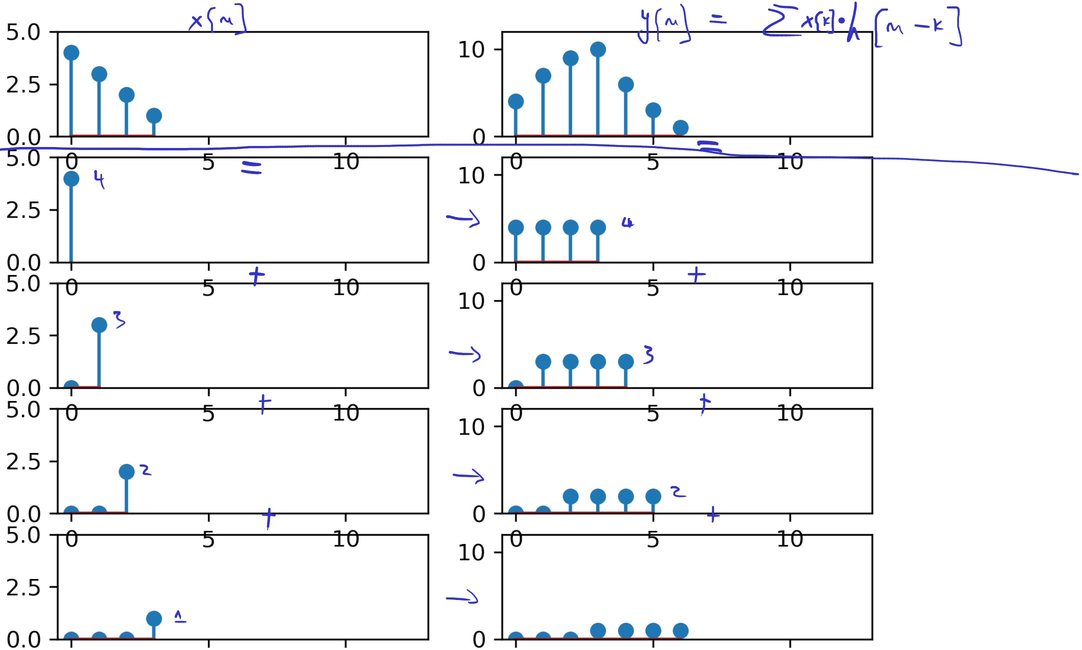
$$y[n] = x[n] * h[n]$$

Convolution

- Convolution is commutative: $x[n] * h[n] = h[n] * x[n]$
  - in equation it doesn't matter which signal has $[k]$ and which with $[n-k]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Example



Interpretation of the convolution equation

The convolution equation can be interpreted in two ways:

1. The output signal $y[n]$ = a sum of a lot of impulse responses $h[n]$, each one delayed by k (hence $[n-k]$) and scaled by $x[k]$
   - one for each sample in the input signal
   - explain at blackboard

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{ampl.}}\mathbf{h}[\mathbf{n} - \underbrace{k}_{\text{delay}}]$$

Interpretation of the convolution equation

2. Each output sample $y[n]$ = a **weighted sum** of the input samples around it

$$y[n] = 2 \cdot x[n] + 3 \cdot x[n-1]$$

example

$$\boxed{y[n]} = \ldots + \underbrace{h[2] \cdot x[n-2]}_{k=2} + \underbrace{h[1] \cdot x[n-1]}_{k=1} + \underbrace{h[0] \cdot x[n]}_{k=0} + \underbrace{h[-1] \cdot x[n+1]}_{k=-1} + \ldots \;\; \ldots\ldots$$

- If $h[n]$ has finite length (e.g. non-zero only between $h[-2] \ldots h[2]$), then there are only a few terms in the sum
  - Example at blackboard

$$x[n] * h[n] = \underbrace{\mathbf{y[n]}} = \sum_{k=-\infty}^{\infty} \underbrace{h[k]}_{\text{coef.}} \underbrace{\mathbf{x[n}}\underbrace{- k]}_{\text{delay}}$$

Example

$$h[n] = \ldots 0 \{ \underset{\uparrow}{1}, 2, 3, \} 0 \ldots$$

with $h[0]$, $h[1]$, $h[2]$ written above the values 1, 2, 3.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$$= h[0] \cdot x[n] + h[1] \cdot x[n-1] + h[2] \cdot x[n-2]$$

$$y[n] = x[n] + 2x[n-1] + 3x[n-2]$$

Interpretation of the convolution equation

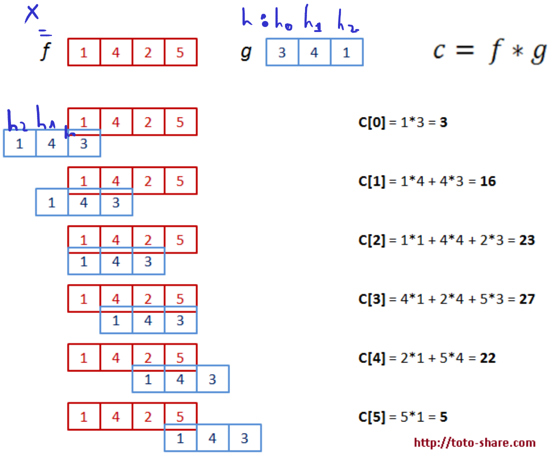
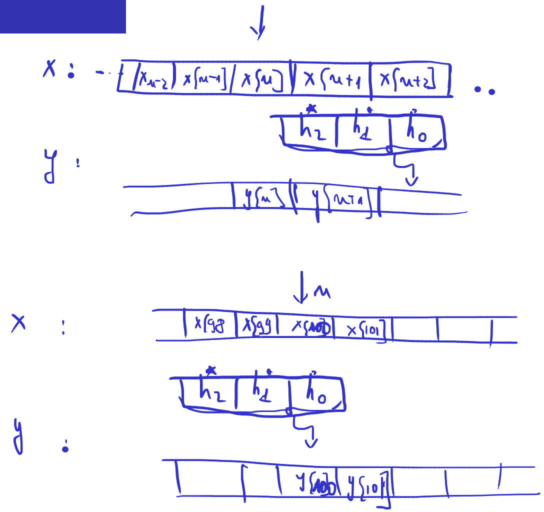


Figure 1: Convolution as weighted sum



- image from http://www.stokastik.in

Interpretation of the convolution equation

- Watch the following:

  https://www.youtube.com/watch?v=ulKbLD6BRJA

Example

The impulse response can be read directly from the system equation (for non-recursive systems):

- Suppose we have the system:

$h[k] \cdot x[n-k]$

$$y[n] = \overset{h[-1]}{3}x[n+1] + \overset{h_0}{5}x[n] \overset{h[1]}{-2}x[n-1] + \overset{h[2]}{4}x[n-2] \quad = \text{convolution}$$

- What is the impulse response of the system?

- Answer: $h[n] = \{\ldots 0, \underset{h[-1]}{3}, \underset{\uparrow}{\underset{h[0]}{5}}, \underset{h[1]}{-2}, \underset{h[2]}{4}, 0, \ldots\}$

Convolution as matrix multiplication

- Convolution can we written as multiplication with a **circulant** (or "**Toeplitz**") matrix
  - in this example, assuming $h[n]$ is non-zero only from $h[-1]$ to $h[3]$

$y[n] =$ $x[n]$

$$
\begin{bmatrix} \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
\dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\
\dots & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & 0 & \dots \\
\dots & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & \dots \\
\dots & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & \dots \\
\dots & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & \dots \\
\dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots
\end{bmatrix}
\cdot
\begin{bmatrix} \vdots \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ x_{n+3} \\ \vdots \end{bmatrix}
$$

2D convolution

- Convolution can be applied in 2D (for images)
- The input signal = an image $I[x, y]$
- The impulse response (the ***kernel***) = a matrix H[x, y]
- The convolution result:

$$Y[x, y] = I * H = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \overset{x}{I}[x-i, y-j] \cdot \overset{h}{H}[i, j]$$

2D convolution

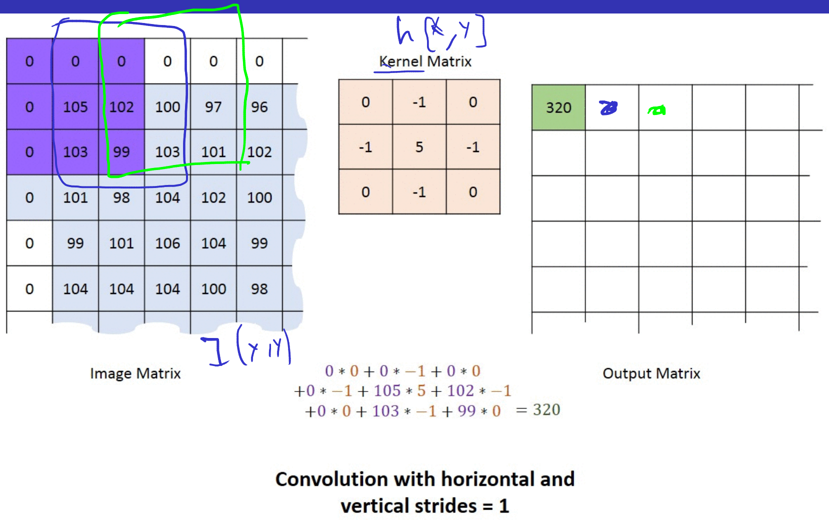


Figure 2: 2D Convolution as weighted sum

- image from http://machinelearninguru.com

2D Convolution

- Watch this:

  http://machinelearninguru.com/computer_vision/basics/convolution/cor

2D Convolution

- Simple image effects with 2D convolutions:
  - the "kernel" $=$ the impulse response $H[x, y]$
- See here: https://en.wikipedia.org/wiki/Kernel_(image_processing)
- What are their 1D counterparts?

Properties of convolution

Basic properties of convolution

- Convolution is **commutative** (the order of the signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

  - Proof: make variable change $(n-k) \rightarrow l$, change all in equation

- Convolution is **associative**:

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

  - (No proof)

Properties of convolution

- The unit impulse is **neutral element** for convolution:

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

.

  - Proof: equation

- Convolution is a **linear operation** (or **distributive**):

$$(\underbrace{\alpha \cdot a[n] + \beta \cdot b[n]}_{x[n]}) * c[n] = \alpha(\cdot a[n] * c[n]) + \beta \cdot (b[n] * c[n])$$

- Proof: by linearity of the corresponding system

$\cancel{a *}$

$(a + b) * c = a * c + b * c$

Properties of LTI systems expressed with $h[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

1. Identity system

- A system with $h[n] = \delta[n]$ produces an response equal to the input, $y[n] = x[n], \forall x[n]$.
- Proof: $\delta[n]$ is neutral element for convolution.

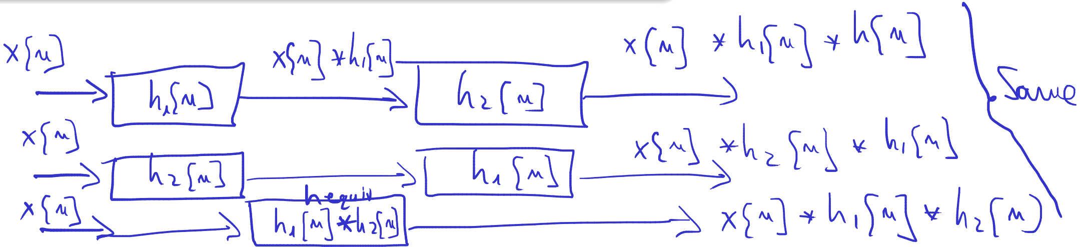
$$\delta[n] \longrightarrow h[n] = \delta[n]$$

Properties of LTI systems expressed with $h[n]$

2. Series connection is commutative

- LTI systems connected in series can be interchanged in any order
- Proof: by commutativity of convolution.
- LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * \ldots * h_N[n]$$

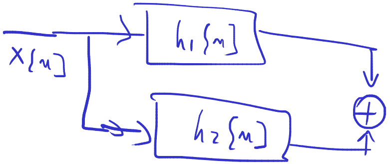


Properties of LTI systems expressed with $h[n]$

3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + \ldots + h_N[n]$$

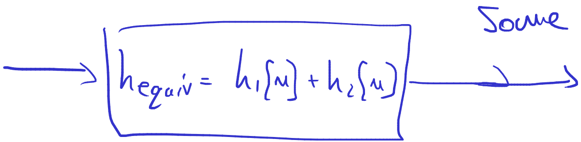


$$x[n] * h_1[n] + x[n] * h_2[n]$$

$$=$$

$$x[n] * (\underbrace{h_1[n] + h_2[n]}_{h_{equiv}})$$

$\Longleftrightarrow$



Properties of LTI systems expressed with $h[n]$

4. Response of LTI systems to unit step

- If the input signal is $u[n]$, the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

.

In | Out
$\delta[n] \longrightarrow h[n]$

$u[n] = \sum_{-\infty}^{n} \delta[k] \longrightarrow \sum_{-\infty}^{n} h[k]$

Properties of LTI systems expressed with $h[n]$

- Proof:
  - The signal $\sum_{k=-\infty}^{n} h[k]$ is a *discrete-time integration* of $h[n]$
  - The unit step $u[n]$ iteslf is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

  - Therefore the system response to the integral of the impulse = the integral of the system response to the impulse
  - The interchanging of the integration with the system is due to the linearity of the system and is valid for all signals:

$$H\left(\sum_{k=-\infty}^{n} x[k]\right) = \sum_{k=-\infty}^{n} H\left(x[k]\right)$$

Relation between LTI system properties and $h[n]$

Relation between LTI system properties and $h[n]$

- For an LTI system, if we know $h[n]$, we know **everything** about the system
- Therefore, the properties (causal, memory, stability) must be reflected somehow in $h[n]$
  - Not linearity and time-invariance, they must be true, otherwise we wouldn't talk about $h[n]$

1. Causal LTI systems and their $h[n]$

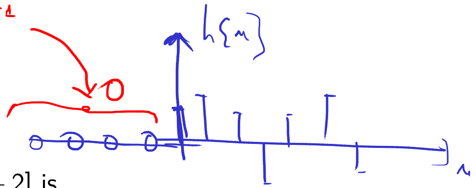
$$y[n] = \cancel{\dots h_{-1}\, x[n+1]} + h_0 \cdot x[n] + h_1 \cdot x[n-1] + \dots$$

$$(h_k \cdot x[n-k])$$

$h_{-2} = 0$ $\quad h_{-1} = 0$

If a LTI system is causal, then

$$h[n] = 0, \forall n < 0$$



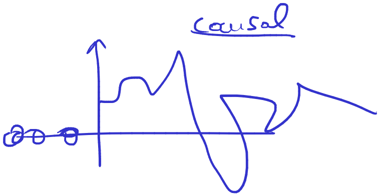
- Proof:
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$,
  - $y[n]$ does not depend on $x[n+1], x[n+2], \dots$
  - it means that these terms are multiplied with 0
  - the value $x[n+1]$ is multiplied with $h[n-(n+1)] = h[-1]$, $x[n+2]$ is multiplied with $h[n-(n+2)] = h[-2]$, and so on
  - Therefore:

$$h[n] = 0, \forall n < 0$$

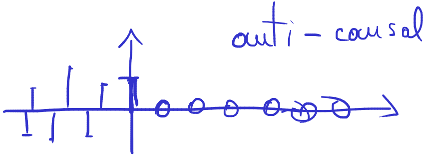
.

Causal signals and causal systems

- A **signal** which is 0 for $n < 0$ is called a **causal** signal

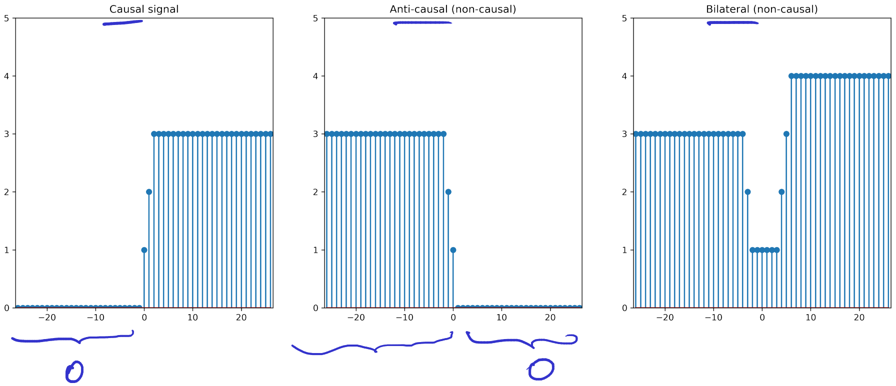


- Otherwise the signal is **non-causal**
- We can say that a **system** is causal if and only if it has a causal **impulse response**
- Further definitions:
  - a signal which 0 for $n > 0$ is called an **anti-causal** signal



  - a signal which has non-zero values both for some $n > 0$ and for some $n < 0$ (and thus is neither causal nor non-causal) is called **bilateral**

Example



Causal signal
Anti-causal (non-causal)
Bilateral (non-causal)
0
1
2
3
4
5
−20
−10
0
10
20
0
0

2. Stable systems and their $h[n]$

- Considering a bounded input signal, $|x[n]| \leq A$, the absolute value of the output is:

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \\
&\leq \sum_{k=-\infty}^{\infty} |x[k] \cdot h[n-k]| \\
&= \sum_{k=-\infty}^{\infty} |x[k]| \cdot |h[n-k]| \\
&\leq A \cdot \underbrace{\sum_{k=-\infty}^{\infty} |h[n-k]|}
\end{aligned}
$$

$$|a+b+c| \leq |a|+|b|+|c|$$

$$|a \cdot b| = |a| \cdot |b|$$

$\rightarrow$ sum $< \infty \Rightarrow |y[n]| \leq$ something (is bounded)

- Therefore **a LTI system is stable if**

$$\boxed{\sum_{k=-\infty}^{\infty} |h[n]| < \infty} \Rightarrow \text{system is stable}$$

Ex: $h[n] = \{1, 3, 7\}$ stable

$h[n] = u[n]$ unstable

.

3. Memoryless systems and their $h[n]$ (Exercise)

$$y[n] = 0 + h_0 \cdot x[n] + 0$$

LTI + memoryless = multipl. by constant

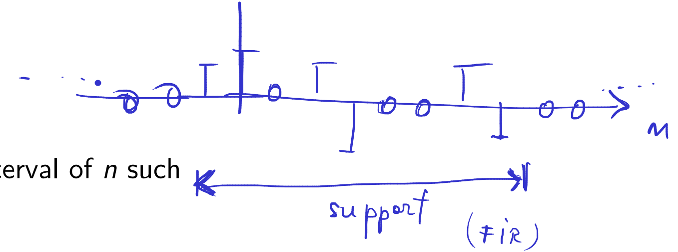
Exercises:

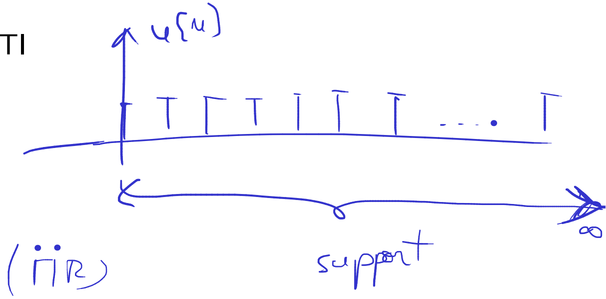
- What can we say about the impulse response $h[n]$ of a memoryless system?
- What about a system with finite memory M?

FIR and IIR systems

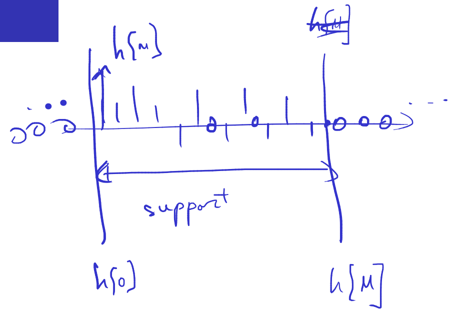
Support

- The **support** of a discrete signal = the smallest interval of n such that the signal is 0 everywhere outside the interval.
- Examples: at whiteboard
- Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.





FIR systems



- A **F**inite **I**mpulse **R**esponse (**FIR**) system has an impulse response with **finite support**
  - i.e. the impulse response is 0 outside a certain interval.
  - i.e. $h[n]$ is zero beyond some element $h[M]$
- The system equation for a FIR system:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = \underbrace{h[0] \cdot x[n] + h[1] \cdot x[n-1] + \ldots h[M] \cdot x[n-M]}_{\text{finite number of terms}}$$

  - is non-recursive (depends only on x)
  - goes only up to some term $h[M]x[n-M]$
  - for causal system, starts from $h[0]x[n]$; for non-causal, can start from $h[-k]x[n+k]$

FIR systems

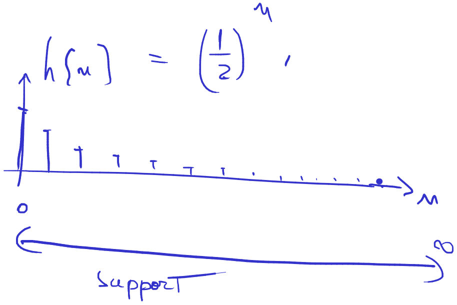
- For a causal FIR system, the output is a **linear combination** of **the last M+1 input samples**
- For non-causal FIR system, some future input samples enter the combination

IIR systems

- An **I**nfinite **I**mpulse **R**esponse (**FIR**) system has an impulse response with **infinite support**
  - i.e. the impulse response never becomes completely 0 forever.
- The output $y[n]$ potentially depends on all the preceding input samples
  - from the convolution equation:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} h[k]x[n-k] \\
&= h[0]\cdot x[n] + h[1]\cdot x[n-1] + \ldots h[M]\cdot x[n-M] + \ldots \textbf{goes on} + \ldots
\end{aligned}
$$

- An IIR system has infinite memory



$x[n - 100000] + \ldots$

IIR systems

- IIR systems must have **recursive** equations:
  - they depend on previous outputs $y[n-1]$ up to $y[n-N]$
  - they also depend on input, going back up to $x[n-k]$
- General equation of an IIR system:

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] +$$
$$+ b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$
$$= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

  - the impulse response cannot be read explicitly from the equation
  - IIR equations are more general than FIR

General equation of an LTI system

Recap:

- The general equation of an LTI system is:

$$\begin{aligned} y[n] &= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\ &\quad + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \\ &= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=1}^{M} b_k x[n-k] \end{aligned}$$

- If all $a_i = 0$, it is a FIR system, no $y[n-k]$ term
  - in this case the coefficients $b_k = h[k]$ (impulse response)
- If some $a_i \neq 0$, it is an IIR system
  - impulse response $h[n]$ is infinitely long, is more complicated to find
- Note: if system is non-causal, can start from $x[n+k]$

Initial conditions for recursive systems

$y[n] = \dots y[n-1]$

- Recursive systems need **initial conditions** (starting values) I.C.: $\boxed{y[-1] = 5}$
  - since they rely on previous outputs
- If initial conditions are all 0, the system is **relaxed**
  - the output depends only on the input signal
- If initial conditions are not zero, the output depends on the input signal **and** the initial conditions

Initial conditions for recursive systems

- The effect of the input signal and the effect of initial conditions are **independent**
  - the system behaves **linear** with respect to them
  - total output = output due to input + output due to initial conditions

	Input	Init.Cond.	Output		
	x[n]	0	$y[n] = y_{zs}[n]$	←	"zero - state"
	0	non-zero	$y[n] = y_{zi}[n]$	←	"zero - input"
BOTH :	x[n]	non-zero	$y[n] = y_{zs}[n] + y_{zi}[n]$	←	BOTH