

Digital Signal Processing

## Chapter IV: The Fourier Transform and its applications

## IV.1 Vector spaces of signals (crash course)

- ▶ **Vector space** = a set  $V\{v_i\}$  with the following two properties:
  - ▶ one element + another element = still an element of the same space
  - ▶ a scalar constant  $\times$  an element = still an element of the same space
- ▶ You **can't escape** a vector space by summing or scaling
- ▶ The elements of a vector space are called **vectors**

## Examples of vector spaces

- ▶ Geometric spaces are great intuitive examples:
  - ▶ a line, or the set  $\mathbb{R}$  (one-dimensional)
  - ▶ a plane, or the set  $\mathbb{C}$  (two-dimensional)
  - ▶ 3D space (three-dimensional)
  - ▶ 4D space (four-dimensional, like the spatio-temporal universe)
  - ▶ arrays with  $N$  numbers ( $N$ -dimensional)
  - ▶ space of continuous signals ( $\infty$ -dimensional)
- ▶ The **dimension** of the space = “how many numbers you need in order to specify one element” (informal)
- ▶ A “vector” like in maths = a sequence of  $N$  numbers = a “vector” like in programming
  - ▶ e.g. a point in a plane has two coordinates = a vector of size  $N = 2$
  - ▶ e.g. a point in a 3D-space has three coordinates = a vector of size  $N = 3$

- ▶ Many vector spaces have a fundamental operation: **the (Euclidean) inner product**

- ▶ for **discrete** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$

- ▶ for **continuous** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$

- ▶ \* represents **complex conjugate** (has no effect for real signals)
- ▶ The result is one number (real or complex)
- ▶ Also known as **dot product** or **scalar product** ("produs scalar")

## Inner product

- ▶ Each entry in  $\mathbf{x}$  times the complex conjugate of the one in  $\mathbf{y}$ , all summed
- ▶ For discrete signals, it can be understood as a row  $\times$  column multiplication
- ▶ Discrete vs continuous: just change sum/integral depending on signal type

## Inner product properties

- ▶ Inner product is **linear** in both terms:

$$\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle = \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2 \rangle = \langle \mathbf{x}, \mathbf{y}_1 \rangle + \langle \mathbf{x}, \mathbf{y}_2 \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

## The distance between two vectors

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ **The (Euclidean) distance** between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_N - y_N)^2}$$

- ▶ This distance is the **usual geometric distance** you know from geometry
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ if two vectors have small distance, they are close, they are similar
  - ▶ two vectors with large distance are far away, not similar
  - ▶ two identical vectors have zero distance

## The norm of a vector

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ The **norm** (length) of a vector =  $\text{sqrt}(\text{inner product with itself})$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ The **norm** of a vector is the distance from  $\mathbf{x}$  to point  $\mathbf{0}$ .
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ vector has large norm = has big values, is far from  $\mathbf{0}$
  - ▶ vector has small norm = has small values, is close to  $\mathbf{0}$
  - ▶ vector has zero norm = it is the vector  $\mathbf{0}$
- ▶ Norm of a vector =  $\text{sqrt}(\text{the signal energy})$

## Norm and distance

- ▶ The norm and distance are related
- ▶ The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ Just like in geometry: distance = length of the difference vector

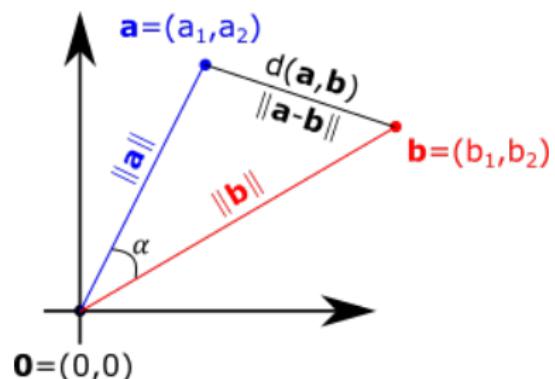


Figure 1: Norm and distance in vector spaces

## Angle between vectors

- ▶ The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

- ▶ is a value between -1 and 1
- ▶ **Otrhogonal vectors** = two vectors with  $\langle x, y \rangle = 0$ 
  - ▶ their angle = 90 deg
  - ▶ in geometric language, the two vectors are **perpendicular**

## Why vector space

- ▶ Why are all these useful?
- ▶ They are a very general **framework** for different kinds of signals
- ▶ We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - ▶ Example in DEDP class: ML decision with 1, 2, N samples

We deal mainly with the following vector spaces:

- ▶ The vector space of all infinitely-long real signals  $x[n]$
- ▶ The vector space of all infinitely-long periodic signals  $x[n]$  with period  $N$ 
  - ▶ for each  $N$  we have a different vector space
- ▶ The vector space of all finite-length signals  $x[n]$  with only  $N$  samples
  - ▶ for each  $N$  we have a different vector space

## Bases

- A **basis** = a set of  $N$  linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \dots + \alpha_N \mathbf{b}^N$$

- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, \dots, \alpha_N)$$

## Bases and coordinate systems

- ▶ Bases are just like **coordinate systems** in a geometric space

- ▶ any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

- ▶ any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶  $N$  = The number of basis elements = The dimension of the space
- ▶ Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

## Bases and coordinate systems

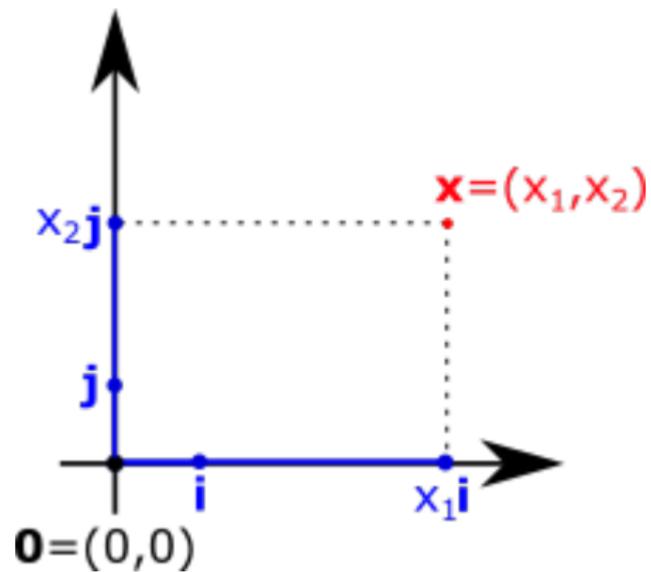


Figure 2: Basis expansion of a vector  $\mathbf{x}$

## Choice of bases

- ▶ There is typically an infinite choice of bases
- ▶ The **canonical basis** = all basis vectors are full of zeros, just with one 1
- ▶ You used it already in an exercise:
  - ▶ any signal  $x[n]$  can be expressed of a sum of  $\delta[n - k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n - 1] + 2\delta[n - 2] + \dots$$

- ▶ the canonical basis is  $B = \{\dots, \delta[n], \delta[n - 1], \delta[n - 2], \dots\}$

## Orthonormal bases

- ▶ An **orthonormal basis** a basis where all elements  $\mathbf{b}^i$  are:

- ▶ orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

- ▶ **normalized** (their norm = 1):

$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- ▶ Example: the canonical basis  $\{\delta[n - k]\}$  is orthonormal:

- ▶  $\langle \delta[n - k], \delta[n - l] \rangle = 0, \forall k \neq l$
  - ▶  $\langle \delta[n - k], \delta[n - k] \rangle = 1, \forall k$

## Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

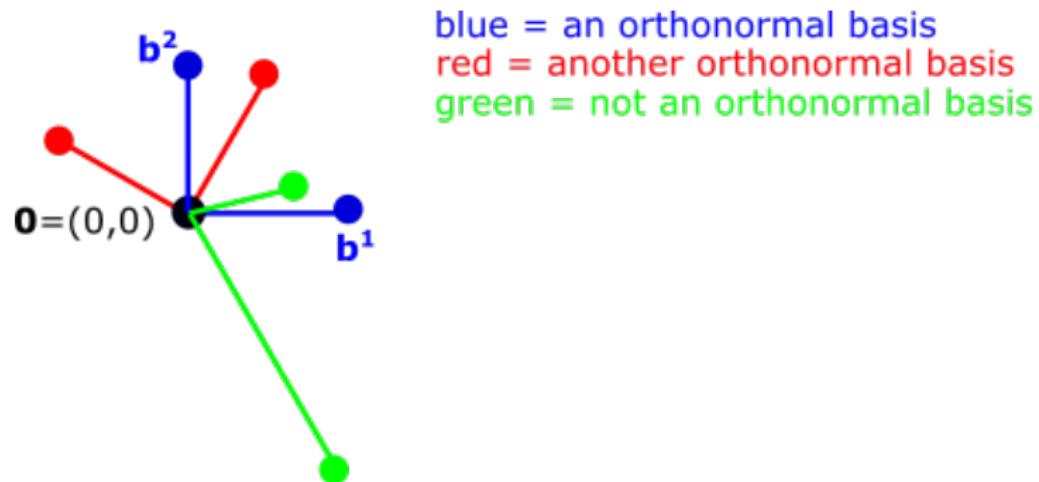


Figure 3: Sample bases in a 2D space

## Basis expansion of a vector

- ▶ Suppose we have an **orthonormal basis**  $B = \{\mathbf{b}^i\}$
- ▶ Suppose we have a vector  $\mathbf{x}$
- ▶ We can write (expand)  $\mathbf{x}$  as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ Question: how to **find** the coefficients  $\alpha_i$ ?

## Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{b}^i \rangle &= \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_i\end{aligned}$$

## Basis expansion of a vector

- ▶ Any vector  $\mathbf{x}$  can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ For orthonormal basis: the coefficients  $\alpha_i$  are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle \mathbf{x}, \mathbf{b}^i \rangle$$

## Why bases

- ▶ How does all this talk about bases help us?
- ▶ To better understand the Fourier transform
- ▶ The signals  $\{e^{j\omega n}\}$  form an **orthonormal basis**
- ▶ The Fourier Transform of a signal  $x$  = finding the coefficients of  $x$  in this basis
- ▶ The Inverse Fourier Transform = expanding  $x$  with the elements of this basis
- ▶ Same **generic** thing every time, only the type of signals differ

## IV.2 Introducing the Fourier Transforms

## Reminder

- ▶ Reminder:

$$e^{jx} = \cos(x) + j \sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

## Why sinusoidal signals

- ▶ Why are sinusoidal signals  $\sin()$  and  $\cos()$  **so prevalent** in signal processing?
- ▶ Answer: because they are combinations of an  $e^{jx}$  and an  $e^{-jx}$
- ▶ Why are these  $e^{jx}$  so special?
- ▶ Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

## Response of LTI systems to harmonic signals

- ▶ Consider an LTI system with  $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal  $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

- ▶ Output signal = input signal  $\times$  a (complex) constant ( $H(\omega_0)$ )

# Eigen-function

- ▶ **Eigen-function** of a system ("funcție proprie") = a function  $f$  which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

- ▶ just like **eigen-vectors** of a matrix (remember algebra):  $A\tilde{\tilde{x}} = \lambda\tilde{\tilde{x}}$
- ▶ we call the "functions" to allow for continuous signals as well
- ▶ Complex exponential signals  $e^{j\omega t}$  (or  $e^{j\omega n}$ ) are **eigen-functions** of Linear and Time Invariant (LTI) systems:
  - ▶ output signal = input signal  $\times$  a (complex) constant

## Representation with respect to eigen-functions (-vectors)

- ▶ We can understand the effect of a LTI system very easily if we **decompose all signals  $x[n]$  as a combination of  $\{e^{j\omega n}\}$**
- ▶ Example: RGB color filter
  - ▶ suppose we have some photographic filters (lenses):
    - ▶ one reduces red to 50%
    - ▶ one reduces green to 25%
    - ▶ one reduces blue to 80%
    - ▶ RGB are eigen-functions of the system: input = 200 Blue, output =  $0.8 * 200$  Blue
    - ▶ what is the output color if input is “pink”?
    - ▶ Answer is easy if we represent all colors in RGB

## Representation with respect to eigen-functions (-vectors)

- ▶ We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of  $\{e^{j\omega n}\}$
- ▶ All vector space theory becomes useful now:
  - ▶  $\{e^{j\omega n}\}$  is an **orthonormal basis**
  - ▶ decomposing signals = finding coefficients  $\alpha_i$
  - ▶ we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

## Discrete-Time Fourier Transform (DTFT)

- ▶ Consider the vector space of **non-periodic infinitely-long signals**
- ▶ This vector space is **infinite-dimensional**
- ▶ The signals  $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$  form an **orthonormal basis**
- ▶ We can expand (almost) any  $x$  in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- ▶ The coefficient of every  $e^{j2\pi fn}$  is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

# Discrete-Time Fourier Transform (DTFT)

## Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- ▶ A signal  $x[n]$  can be written as a linear combination of  $\{e^{j2\pi fn}\}$ ,  $\forall f \in [-\frac{1}{2}, \frac{1}{2}]$ , with some coefficients  $X(f)$

## Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- ▶ The coefficient  $X(f)$  of every  $\{e^{j2\pi fn}\}$  is found using the inner product  $\langle x, e^{j2\pi fn} \rangle$

## Discrete-Time Fourier Transform (DTFT)

- ▶ Alternative form with  $\omega$
- ▶ We can replace  $2\pi f = \omega$ , and  $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_n x[n] e^{-j\omega n}$$

## Discrete-Time Fourier Transform (DTFT)

- ▶ A non-periodic signal  $x[n]$  has a **continuous spectrum**  $X(\omega)$ , with  
 $f \in [-\frac{1}{2}, \frac{1}{2}]$ 
  - ▶ e.g.  $\omega \in [-\frac{1}{2}, \frac{1}{2}]$

# Discrete Fourier Transform (DFT)

- ▶ Consider the vector space of **periodic** signals with **period N**
  - ▶ for some fixed  $N = 2, 3$  or ... etc
- ▶ This is a vector space of **dimension N**
  - ▶ we need N numbers to identify a signal (specify its period)
- ▶ We can consider  $x[n]$  only for **one period**, i.e.  $n = 0, \dots, N - 1$
- ▶ The signals  $\{e^{j2\pi f n}\}, \forall f \in \{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}$  form an **orthonormal basis** with N elements
- ▶ It is a **discrete** set of frequencies:  $f = \frac{k}{N}, \forall k \in \{0, 1, \dots, N - 1\}$

# Discrete Fourier Transform (DFT)

## Inverse Discrete Fourier Transform

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- ▶ A periodic signal  $x[n]$  can be written as a linear combination of  $k$  signals  $\{e^{j2\pi kn/N}\}$ , with some coefficients  $X_k$

## Discrete Fourier Transform

$$X_k = \frac{1}{N} \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- ▶ The coefficient  $X(f)$  of every  $\{e^{j2\pi fn}\}$  is found using the inner product  $\langle x, e^{j2\pi fn} \rangle$

## Discrete Fourier Transform (DFT)

- ▶ A periodic signal  $x[n]$  with period  $N$  has a **discrete spectrum**  $X(\omega)$  composed of only  $N$  frequencies  $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}$
- ▶ Each frequency  $\frac{k}{N}$  has a **coefficient**  $X_k$ 
  - ▶ also written as  $c_k$
  - ▶ The  $N$  coefficients  $X_k$  are the equivalent of  $X(\omega)$
- ▶ It is also known as the “Fourier Series for Discrete Signals”

### IV.3 The Discrete-Time Fourier Transform (DTFT)

## Definition

Definitions (again):

### Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

### Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi f n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

## Basic properties of DTFT

- ▶  $X(\omega)$  is defined only for  $\omega \in [-\pi, \pi]$ 
  - ▶ or  $f \in [-\frac{1}{2}, \frac{1}{2}]$
- ▶  $X(\omega)$  is complex (has  $|X(\omega)|$ ,  $\angle X(\omega)$ )
- ▶ If the signal  $x[n]$  is real,  $X(\omega)$  is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

- ▶ This means:
  - ▶ modulus is even:  $|X(\omega)| = |X(-\omega)|$
  - ▶ phase is odd:  $X(\omega) = -X(-\omega)$

## Expressing as sum of sinusoids

- ▶ Grouping terms with  $e^{j\omega n}$  and  $e^{j(-\omega)n}$  we get:

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \int_{-\pi}^0 X(\omega) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^\pi X(\omega) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_0^\pi (X(\omega) e^{j\omega n} + X(-\omega) e^{j(-\omega)n}) d\omega \\&= \frac{1}{2\pi} \int_0^\pi 2|X(\omega)| (e^{j\omega n + \angle X(\omega)} + e^{-j\omega n - \angle X(\omega)}) d\omega \\&= \frac{1}{2\pi} \int_0^\pi 2|X(\omega)| \cos(\omega n + \angle X(\omega)) d\omega\end{aligned}$$

- ▶ Any signal  $x[n]$  is **a sum of sinusoids with all frequencies**  
 $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$

## Expressing as sum of sinusoids

- ▶ Any signal  $x[n]$  is a **sum of sinusoids with all frequencies**  
 $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$ 
  - ▶ this is the fundamental practical interpretation of the Fourier transform
- ▶ The **modulus**  $|X(\omega)|$  is the **amplitude** of the sinusoids ( $\times 2$ )
  - ▶ for  $\omega = 0$ ,  $|X(\omega = 0)|$  = the DC component
- ▶ The **phase**  $\angle X(\omega)$  gives the initial phase

## 1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

## 2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- ▶ The amplitudes  $|X(\omega)|$  is not affected, shifting in time affects only the phase

# Properties of DTFT

## 3. Modulation in time

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

## 5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- ▶ Not circular convolution, this is the normal convolution

## 6. Product in time

Product in time  $\leftrightarrow$  convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$$

# Properties of DTFT

## Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

## Wiener Khinchin theorem

Autocorrelation of a signal  $\leftrightarrow$  Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

## Parseval theorem

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- ▶ Is true for all orthonormal bases

## IV.4 The Discrete Fourier Transform (DFT)

## Definitions

Definitions (again)

### Inverse Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

### Discrete Fourier Transform (DFT)

$$X_k = \frac{1}{N} \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- ▶ In discrete domain,  $f = \frac{N-k}{N} = \frac{-k}{N}$  (aliasing, we can subtract 1 from  $f$ )
- ▶ We can consider  $X_{N-k}$  as  $X_{-k}$ , due to periodicity
- ▶ Example: a signal with period  $N = 6$  has 6 DFT coefficients
  - ▶ we can call them  $X_0, X_1, X_2, X_3, X_4, X_5$
  - ▶ we have  $X_5 = X_{-1}, X_4 = X_{-2}$
  - ▶ we can also call them  $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$

## Basic Properties of the DFT

- ▶  $X_k$  is complex (has  $|X_k|$ ,  $\angle X_k$ )
- ▶ If the signal  $x[n]$  is real, the coefficients are **even**

$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

- ▶ This means:
  - ▶ modulus is even:  $|X_k| = |X_{-k}|$
  - ▶ phase is odd:  $\angle X_{-k} = -\angle X_k$

## Expressing as sum of sinusoids

- ▶ Grouping terms with  $k$  and  $-k$ :
- ▶ If  $N$  is odd, we have  $X_0$  and pairs  $(X_k, X_{-k})$ :

$$\begin{aligned}x[n] &= X_0 e^{j0n} + \frac{1}{N} \sum_{k=-(N-1)/2}^0 X_k e^{j2\pi kn/N} + \frac{1}{N} \sum_{k=0}^{(N-1)/2} X_k e^{j2\pi kn/N} \\&= X(0) + \frac{1}{N} \sum_{k=0}^{(N-1)/2} (X_k e^{j2\pi kn/N} + X_{-k} e^{-j2\pi kn/N}) \\&= X(0) + \frac{1}{N} \sum_{k=0}^{(N-1)/2} 2|X_k| (e^{j2\pi kn/N + \angle X(k)} + e^{-j2\pi kn/N - \angle X(k)}) \\&= X(0) + \frac{1}{N} \sum_{k=0}^{(N-1)/2} 2|X_k| \cos(2\pi k/N n + \angle X_k)\end{aligned}$$

## Expressing as sum of sinusoids

- ▶ If  $N$  is even, we have  $X_0$  and pairs  $(X_k, X_{-k})$ , with an extra term  $X_{N/2}$  which has no pair
  - ▶ e.g.  $N = 6$ :  $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$
- ▶  $X_{N/2}$  must be a real number
- ▶ The extra term will be  $\frac{1}{N}X_{N/2}e^{j2\pi N/2n/N} = X_{N/2} \cos(n\pi)$
- ▶ Overall:

$$x[n] = X(0) + \frac{1}{N} \sum_{k=0}^{(N-2)/2} 2|X_k| \cos(2\pi k/Nn + \angle X_k) + \frac{1}{N} X_{N/2} \cos(n\pi)$$

- ▶ Any signal  $x[n]$  is a **sum of sinusoids with frequencies**  $f = 0, 1/N, 2/N, \dots, (N-1)/2$  or  $N/2$

## Expressing as sum of sinusoids

- ▶ Any periodic signal  $x[n]$  with period  $N$  is **a sum of N sinusoids with frequencies**  $f = 0, 1/N, 2/N, \dots, (N-1)/2$  or  $N/2$
- ▶ The **modulus**  $|X_k|$  gives the **amplitude** of the sinusoids (sometimes  $\times 2$ )
  - ▶ for  $\omega = 0$ ,  $|X_0|$  = the DC component
  - ▶ when modulus = 0, that frequency has amplitude 0
- ▶ The **phase**  $\angle X_k$  gives the initial phase

## Example

- ▶ Consider a periodic signal  $x[n]$  with period  $N = 6$  and the DFT coefficients:

$$X_k = [15.0000 + 0.0000i, -2.5000 + 3.4410i, -2.5000 + 0.8123i, \\ -2.5000 - 0.8123i, -2.5000 - 3.4410i]$$

Write  $x[n]$  as a sum of sinusoids.

- ▶ Do the same for a periodic signal  $x[n]$  with period  $N = 5$  and the DFT coefficients:

$$X_k = [21.0000 + 0.0000i, -3.0000 + 5.1962i, -3.0000 + 1.7321i, \\ -3.0000 + 0.0000i, -3.0000 - 1.7321i, -3.0000 - 5.1962i]$$

Write  $x[n]$  as a sum of sinusoids.

## 1. Linearity

If the signal  $x_1[n]$  has the DFT coefficients  $\{X_k^{(1)}\}$ , and  $x_2[n]$  has  $\{X_k^{(2)}\}$ ,  
then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$$

Proof: via definition

## 2. Shifting in time

If  $x[n] \leftrightarrow \{X_k\}$ , then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi kn_0/N)} X_k\}$$

Proof: via definition

- ▶ The amplitudes  $|X_k|$  are not affected, shifting in time **affects only the phase**

# Properties of the DFT

## 3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow \{X_{-k}^*\}$$

## 5. Circular convolution

Circular convolution of two signals  $\leftrightarrow$  product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

**Circular convolution** definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- ▶ takes two periodic signals of period N, result is also periodic with period N
- ▶ Example at the whiteboard: how it is computed

## Example

Example (write on slides)

## Circular convolution

- ▶ We are in the vector space of **periodic signals** with period N
- ▶ Linear (e.g. normal) convolution produces a result which is longer periodic with period N
- ▶ Circular convolution takes two sequences of length N and produces another sequence of length N
  - ▶ each sequence is a period of a periodic signal
  - ▶ circular convolution = like a convolution of periodic signals

## 6. Product in time

Product in time  $\leftrightarrow$  circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

## Properties of the DFT

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_0^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum |X_k|^2$$

- ▶ Is true for all orthonormal bases

## Relationship between DTFT and DFT

- ▶ How are DTFT and DFT related?
- ▶ Discrete Time Fourier Transform:
  - ▶ for non-periodical signals
  - ▶ spectrum is continuous
- ▶ Discrete Fourier Transform
  - ▶ for periodical signals
  - ▶ spectrum is discrete
- ▶ Duality: periodic in time  $\leftrightarrow$  discrete in frequency

## Relationship between DTFT and DFT

- ▶ Consider a non-periodic signal  $x[n]$
- ▶ It has a continuous spectrum  $X(\omega)$
- ▶ If we **periodize** it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- ▶ then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k \frac{2\pi}{N})$$

- ▶ The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k / Nn)$$

- ▶ They are **samples** from the continuous  $X(\omega)$  of the non-periodized signal

## Relationship between DTFT and DFT

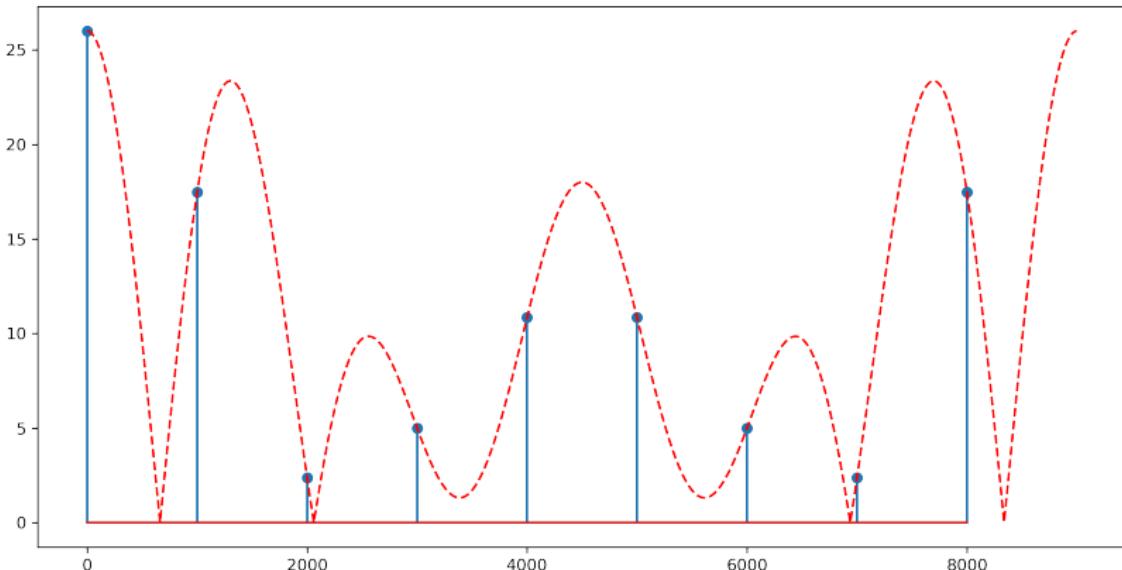
- ▶ Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

- ▶ If we consider a non-periodic  $x[n]$  with infinitely long zeros on either side, we have a continuous spectrum  $X(\omega)$  (DTFT)
- ▶ If we consider that  $x$  is just a period of a periodic signal, we have a discrete spectrum  $X_k$  (DFT)
- ▶ Moreover, the discrete  $X_k$  are just **samples from**  $X(\omega)$ :

$$X_k = X(2\pi k / Nn)$$

## Relationship between DTFT and DFT



$$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$$

- ▶ red line =  $\text{DFT}(x)$  if  $x$  not periodical
  - ▶ actually run as  $\text{fft}(x, 10000)$ ,  $x$  is extended with 9991 zeros
- ▶ blue =  $\text{fft}(x)$

## Relation between DTFT and Z transform

- Z transform:

$$X(z) = \sum_n x[n]z^{-n}$$

- DTFT:

$$X(\omega) = \sum_n x[n]e^{-j\omega n}$$

- DTFT can be obtained from Z transform with

$$z = e^{j\omega}$$

- These  $z = e^{j\omega}$  are **points on the unit circle**

- $|z| = |e^{j\omega}| = 1$  (*modulus*)
- $\angle z = \angle e^{j\omega} = \omega$  (*phase*)

## Relation between DTFT and Z transform

- ▶ Fourier transform = Z transform evaluated **on the unit circle**
  - ▶ if the unit circle is in the convergence region of Z transform
  - ▶ otherwise, equivalence does not hold
- ▶ This is true for most usual signals we work with
  - ▶ some details and discussions are skipped

## Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

► Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

► Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

## Geometric interpretation of Fourier transform

- ▶ For complex numbers:
  - ▶ modulus of  $|a - b|$  = the length of the segment between  $a$  and  $b$
  - ▶ phase of  $|a - b|$  = the angle of the segment from  $b$  to  $a$  (direction is important)
- ▶ So, for a point on the unit circle  $z = e^{j\omega}$ 
  - ▶ modulus  $|X(\omega)|$  is **given by the distances to the zeros and to the poles**
  - ▶ phase  $\angle X(\omega)$  is **given by the angles from the zeros and poles to z**

## Geometric interpretation of Fourier transform

- ▶ Consequences:
  - ▶ when a **pole** is very close to unit circle -> Fourier transform is **large** at this point
  - ▶ when a **zero** is very close to unit circle -> Fourier transform is **small** at this point
- ▶ Examples: . . .

# Geometric interpretation of Fourier transform

- ▶ Simple interpretation for modulus  $|X(\omega)|$ :
  - ▶ Z transform  $X(z)$  is like **a landscape**
    - ▶ **poles = mountains** of infinite height
    - ▶ **zeros = valleys** of zero height
  - ▶ Fourier transform  $X(\omega) = \text{"Walking over this landscape along the unit circle"}$
  - ▶ The height profile of the walk gives the amplitude of the Fourier transform
  - ▶ When close to a mountain  $\rightarrow$  road is high  $\rightarrow$  Fourier transform has large amplitude
  - ▶ When close to a valley  $\rightarrow$  road is low  $\rightarrow$  Fourier transform has small amplitude

## Geometric interpretation of Fourier transform

- ▶ Note:  $X(z)$  might also have a constant  $C$  in front!
  - ▶ It does not appear in pole-zero plot
  - ▶ The value of  $|C|$  and  $\angle C$  must be determined separately
- ▶ This “geometric method” can be applied for phase as well

- ▶ **Duality** properties related to all Fourier transforms
- ▶ Discrete  $\leftrightarrow$  Periodic
  - ▶ **discrete** in time  $\rightarrow$  **periodic** in frequency
  - ▶ **periodic** in time  $\rightarrow$  **discrete** in frequency
- ▶ Continuous  $\leftrightarrow$  Non-periodic
  - ▶ **continuous** in time  $\rightarrow$  **non-periodic** in frequency
  - ▶ **non-periodic** in time  $\rightarrow$  **continuous** in frequency

# Terminology

- ▶ Based on frequency content:
  - ▶ **low-frequency** signals
  - ▶ **mid-frequency** signals (band-pass)
  - ▶ **high-frequency** signals
- ▶ **Band-limited** signals: spectrum is 0 beyond some frequency  $f_{max}$
- ▶ **Bandwidth**  $B$ : frequency interval  $[F_1, F_2]$  which contains 95% of energy
  - ▶  $B = F_2 - F_1$
- ▶ Based on bandwidth  $B$ :
  - ▶ **Narrow-band** signals:  $B \ll$  central frequency  $\frac{F_1+F_2}{2}$
  - ▶ **Wide-band** signals: not narrow-band