Digital Signal Processing

Chapter IV: The Fourier Transform and its applications

IV.1 Vector spaces of signals (crash course)

Vector spaces

- **Vector space** = a set $V\{v_i\}$ with the following two properties:
  - one element + another element = still an element of the same space
  - a scalar constant $\times$ an element = still an element of the same space
- You **can't escape** a vector space by summing or scaling
- The elements of a vector space are called **vectors**

Examples of vector spaces

- Geometric spaces are great intuitive examples:
  - a line, or the set $\mathbb{R}$ (one-dimensional)
  - a plane, or the set $\mathbb{C}$ (two-dimensional)
  - 3D space (three-dimensional)
  - 4D space (four-dimensional, like the spatio-temporal universe)
  - arrays with N numbers (N-dimensional)
  - space of continuous signals (∞-dimensional)
- The **dimension** of the space = "how many numbers you need in order to specify one element" (informal)
- A "vector" like in maths = a sequence of N numbers = a "vector" like in programming
  - e.g. a point in a plane has two coordinates = a vector of size $N = 2$
  - e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

Inner product

- Many vector spaces have a fundamental operation: **the (Euclidean) inner product**
  - for **discrete** signals:
  $$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$
  - for **continuous** signals:
  $$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$
- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** (“produs scalar”)

Inner product

- Each entry in $\mathbf{x}$ times the complex conjugate of the one in $\mathbf{y}$, all summed
- For discrete signals, it can be understood as a row $\times$ column multiplication
- Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- Inner product is **linear** in both terms:

$$\langle \mathbf{x_1} + \mathbf{x_2}, \mathbf{y} \rangle = \langle \mathbf{x_1}, \mathbf{y} \rangle + \langle \mathbf{x_2}, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y_1} + \mathbf{y_2} \rangle = \langle \mathbf{x}, \mathbf{y_1} \rangle + \langle \mathbf{x}, \mathbf{y_2} \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

The distance between two vectors

- An inner product induces a **norm** and a **distance** function
- **The (Euclidean) distance** between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_N - y_N)^2}$$

- This distance is the **usual geometric distance** you know from geometry
- It has the exact same intuition like in **normal geometry**:
  - if two vectors have small distance, they are close, they are similar
  - two vectors with large distance are far away, not similar
  - two identical vectors have zero distance

The norm of a vector

- An inner product induces a **norm** and a **distance** function
- The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- The **norm** of a vector is the distance from $\mathbf{x}$ to point $\mathbf{0}$.
- It has the exact same intuition like in **normal geometry**:
  - vector has large norm = has big values, is far from $\mathbf{0}$
  - vector has small norm = has small values, is close to $\mathbf{0}$
  - vector has zero norm = it is the vector $\mathbf{0}$
- Norm of a vector = sqrt(the signal **energy**)

Norm and distance

- The norm and distance are related
- The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$$

- Just like in geometry: distance = length of the difference vector

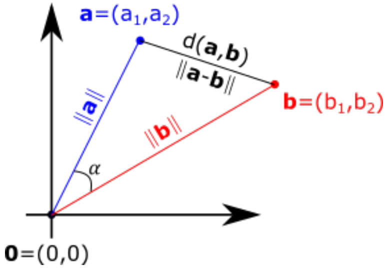


Figure 1: Norm and distance in vector spaces

Angle between vectors

- The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - is a value between -1 and 1
- **Otrhogonal vectors** $=$ two vectors with $\langle x, y \rangle = 0$
  - their angle $= 90\,\text{deg}$
  - in geometric language, the two vectors are **perpendicular**

Why vector space

- Why are all these useful?
- They are a very general **framework** for different kinds of signals
- We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - Example in DEDP class: ML decision with 1, 2, N samples

Vector spaces in DSP class

We deal mainly with the following vector spaces:

- The vector space of all infinitely-long real signals $x[n]$
- The vector space of all infinitely-long periodic signals $x[n]$ with period N
  - for each N we have a different vector space
- The vector space of all finite-length signals $x[n]$ with only N samples
  - for each N we have a different vector space

Bases

- A **basis** = a set of N linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2 ... \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + ... + \alpha_N \mathbf{b}^N$$

- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, ... \alpha_N)$$

Bases and coordinate systems

- Bases are just like **coordinate systems** in a geometric space
  - any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$$

  - any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1\mathbf{b}^1 + \alpha_2\mathbf{b}^2 + \cdots + \alpha_N\mathbf{b}^N$$

- N = The number of basis elements = The dimension of the space
- Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

Bases and coordinate systems

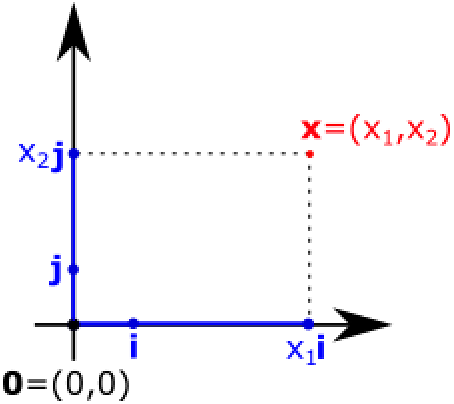


Figure 2: Basis expansion of a vector x

Choice of bases

- There is typically an infinite choice of bases
- The **canonical basis** = all basis vectors are full of zeros, just with one 1
- You used it already in an exercise:
  - any signal $x[n]$ can be expressed of a sum of $\delta[n-k]$

    $$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n-1] + 2\delta[n-2] + \dots$$

  - the canonical basis is $B = \{..., \delta[n], \delta[n-1], \delta[n-2], \dots\}$

Orthonormal bases

- An **orthonormal basis** a basis where all elements $\mathbf{b}^i$ are:
  - orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

  - **normalized** (their norm $= 1$):

$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- Example: the canonical basis $\{\delta[n-k]\}$ is orthonormal:
  - $\langle \delta[n-k], \delta[n-l] \rangle = 0, \forall k \neq l$
  - $\langle \delta[n-k], \delta[n-k] \rangle = 1, \forall k$

Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

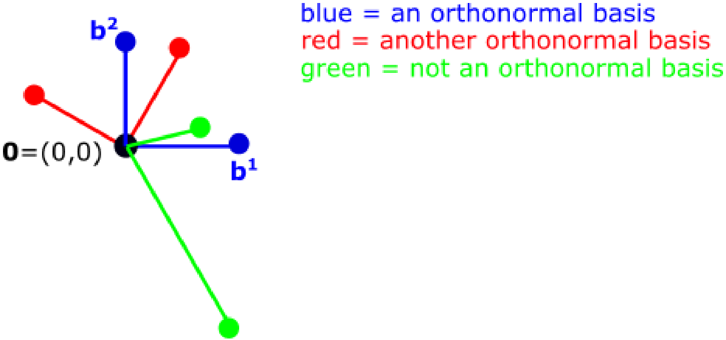


Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- Suppose we have an **orthonormal basis** $B = \{\mathbf{b}^i\}$
- Suppose we have a vector $\mathbf{x}$
- We can write (expand) $\mathbf{x}$ as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- Question: how to **find** the coefficients α_i?

Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{b}^i \rangle =& \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \alpha_i
\end{aligned}
$$

Basis expansion of a vector

- Any vector **x** can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle, \mathbf{x}, \mathbf{b}^i \rangle$$

Why bases

- How does all this talk about bases help us?
- To better understand the Fourier transform
- The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- The Fourier Transform of a signal x = finding the coefficients of $\mathbf{x}$ in this basis
- The Inverse Fourier Transform = expanding $\mathbf{x}$ with the elements of this basis
- Same **generic** thing every time, only the type of signals differ

IV.2 Introducing the Fourier Transforms

Reminder

- Reminder:

$$\boxed{e^{jx}} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos(x - \frac{\pi}{2})$$

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

$e^{jx} = \cos(x) + j \cdot \sin(x)$

Why sinusoidal signals

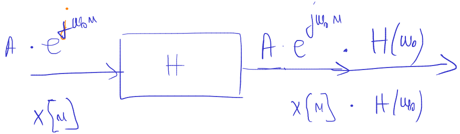
- Why are sinusoidal signals sin() and cos() **so prevalent** in signal processing?
- Answer: because they are combinations of an e^{jx} and an e^{-jx}
- Why are these e^{jx} so special?
- Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

$x[n] = A \cdot e^{j \cdot \omega_0 \cdot n}$

$x[n-k] = A \cdot e^{j\omega_0 (n-k)} = \underbrace{A \cdot e^{j\omega_0 n}}_{x[n]} \cdot e^{-j\omega_0 k}$

- Consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution



$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \cdot \underbrace{Ae^{j\omega_0 n}}_{x[n]} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- $\boxed{\text{Output signal = input signal} \times \text{a (complex) constant } (H(\omega_0))}$

Eigen-function

$\lambda \cdot v = A\, v$

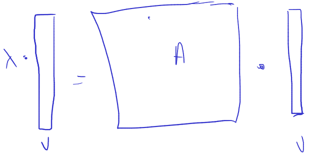
- **Eigen-function** of a system ("funcție proprie") = a function f which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

$A \cdot v = \lambda \cdot v$

$\boxed{A \cdot v = \lambda \cdot v}$

???



  - just like **eigen-vectors** of a matrix (remember algebra): $A\succapprox = \lambda\succapprox$
  - we call the "functions" to allow for continuous signals as well

- Complex exponential signals $e^{j\omega t}$ (or $e^{j\omega n}$) are **eigen-functions** of Linear and Time Invariant (LTI) systems:
  - output signal = input signal $\times$ a (complex) constant

$H \quad e^{j \cdot \omega_0 \cdot n} \longrightarrow H_1 \cdot e^{j \omega_0 n}$

ω $\quad$ ω_1, ω_2

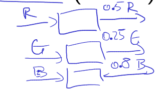
H_2

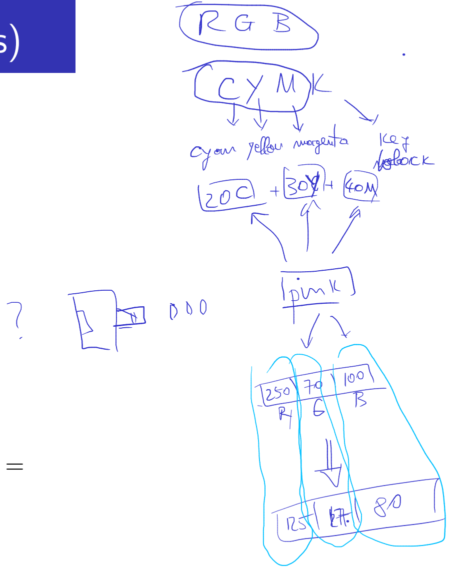
H_3

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system very easily if we **decompose all signals** $x[n]$ **as a combination of** $\{e^{j\omega n}\}$
- Example: RGB color filter
  - suppose we have some photographic filters (lenses):
    - one reduces red to 50%
    - one reduces green to 25%
    - one reduces blue to 80%
    - RGB are eigen-functions of the system: input = 200 Blue, output = 0.8 * 200 Blue
    - what is the output color if input is "pink"?
    - Answer is easy if we represent all colors in RGB

=>





Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$
- All vector space theory becomes useful now:
  - $\{e^{j\omega n}\}$ is an **orthonormal basis**
  - decomposing signals = finding coefficients α_i
  - we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

$$x[n] = \sum_\omega e^{j\omega n}$$

$$= A_1 e^{j\omega_0 n} + A_2 e^{j\omega_1 n} + \dots$$

$$\Big\downarrow H$$

$$y[n] = H_0 A_1 \cdot e^{j\omega_0 n} + H_1 A_2 e^{j\omega_1 n} + \dots$$

Discrete-Time Fourier Transform (DTFT)

- Consider the vector space of **non-periodic infinitely-long signals**
- This vector space is **infinite-dimensional**
- The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis**
- We can expand (almost) any $\mathbf{x}$ in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

Discrete-Time Fourier Transform (DTFT)

$x[n]$

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- A signal $x[n]$ can be written as a linear combination of $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn} \rangle$

$\left(\frac{1}{2\pi} d\omega = df\right)$ $\left(d\omega = 2\pi \cdot df\right)$

$$x[n] = \int_{\boxed{-1/2}}^{\boxed{1/2}} X(f) \cdot e^{j \overset{\omega}{2\pi f} n} df$$

!

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j \overset{\omega}{2\pi f} n}$$

SCS:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$x[n] \longleftrightarrow X\binom{\omega}{f}$

Discrete-Time Fourier Transform (DTFT)

- Alternative form with ω
- We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

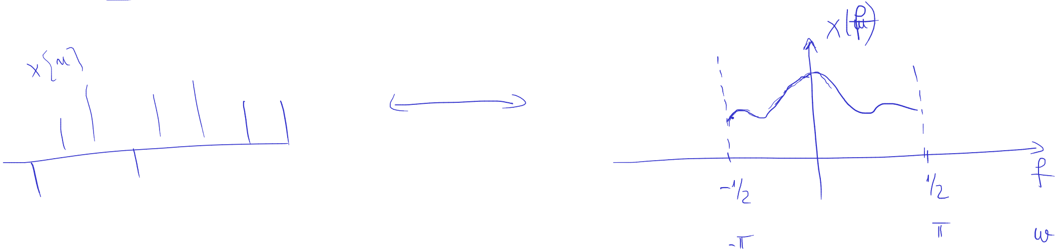
$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} \underline{d\omega}$$

$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_{n} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform (DTFT)

- A non-periodic signal $x[n]$ has a **continuous spectrum** $X(\omega)$, with $f \in [-\frac{1}{2}, \frac{1}{2}]$
  - e.g. $\omega \in [-\frac{1}{2}, \frac{1}{2}]$

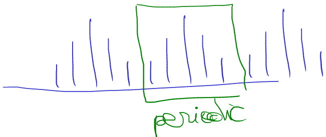
    $-\pi \quad \pi$



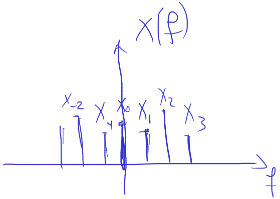
Discrete Fourier Transform (DFT)

$x[n] =$ periodic, per $= N$ $\longleftrightarrow$ $f = \{0, \frac{1}{N}, \dots \frac{N-1}{N}\}$

- Consider the vector space of **periodic** signals with **period N**
  - for some fixed $N = 2, 3$ or $\dots$ etc
- This is a vector space of **dimension N**
  - we need N numbers to identify a signal (specify its period)
- We can consider $x[n]$ only for **one period**, i.e. $n = 0, \dots N-1$
- The signals $\{e^{j2\pi fn}\}, \forall f \in \{0, \frac{1}{N}, \dots \frac{N-1}{N}\}$ form an **orthonormal basis** with N elements
- It is a **discrete** set of frequencies: $f = \frac{k}{N}, \forall k \in \{0, 1, \dots N-1\}$



$\longleftrightarrow$



Discrete Fourier Transform (DFT)

$X(f)$ $X[f]$

Inverse Discrete Fourier Transform

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

$$x[n] = \sum_{k=0}^{N-1} X_k \cdot e^{j2\pi \overset{\overset{\omega}{f}}{\frac{k}{N}} n}$$

- A periodic signal $x[n]$ can be written as a linear combination of k signals $\{e^{j2\pi kn/N}\}$, with some coefficients X_k

$$\left(x[n] = \int_{-1/2}^{1/2} X(f) \cdot e^{j2\pi f n} \right)$$

Discrete Fourier Transform

$$\boxed{X_k = \frac{1}{N}\langle x[n], e^{j2\pi fn}\rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}}$$

$$X_k = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \overset{f}{\frac{k}{N}} n}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn}\rangle$

Discrete Fourier Transform (DFT)

- A periodic signal $x[n]$ with period N has a **discrete spectrum** $X(\omega)$ composed of only N frequencies $\{0, \frac{1}{N} \ldots \frac{N-1}{N}\}$
- Each frequency $\frac{k}{N}$ has a **coefficient** X_k
  - also written as c_k
  - The N coefficients X_k are the equivalent of $X(\omega)$
- It is also known as the "Fourier Series for Discrete Signals"

IV.3 The Discrete-Time Fourier Transform (DTFT)

Definition

Definitions (again):

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=\infty}^{\infty} x[n] e^{-j2\pi fn}$$

Basic properties of DTFT

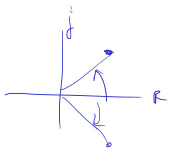
- $X(\omega)$ is defined only for $\omega \in [-\pi, \pi]$
  - or $f \in [-\frac{1}{2}, \frac{1}{2}]$
- $X(\omega)$ is complex (has $|X(\omega)|$, $\angle X(\omega)$)
- If the signal $x[n]$ is real, $X(\omega)$ is **even**

$$x[n] \in \mathbb{R} \to X(-\omega) = X^*(\omega)$$

conjugated

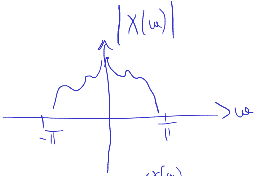
- This means:
  - modulus is even: $|X(\omega)| = |X(-\omega)|$
  - phase is odd: $X(\omega) = -X(-\omega)$

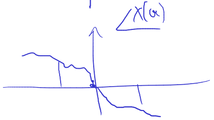
$a = |a| \cdot e^{j \angle a}$



$X(\omega) \in \mathbb{C} \Rightarrow |X(\omega)|, \angle X(\omega)$

$f(x)^* = f(-x)$





Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n} + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \left(X(\omega)e^{j\omega n} + X(-\omega)e^{j(-\omega)n}\right)d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \cancel{2}|X(\omega)|\left(e^{j\omega n+j\angle X(\omega)} + e^{-j\omega n-j\angle X(\omega)}\right)d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2\underbrace{|X(\omega)|}_{A}\cos(\omega n + \underbrace{\angle X(\omega)}_{\text{phase}})d\omega
\end{aligned}
$$

$\omega \to -\omega$

$2\cdot\cos(\omega n + \angle X(\omega))$

$f = 0, \dots, \frac{1}{2}$

$X(\omega)$

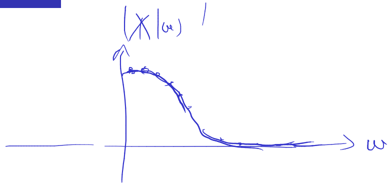
- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(f)\, e^{j2\pi f n}\, d\omega = \int_{-2\pi}^{0} + \int_{0}^{2\pi}$$

$$\begin{cases} X(\omega) = |X(\omega)|\cdot e^{j\angle X(\omega)} \\ X(-\omega) = |X(\omega)|\cdot e^{-j\angle X(\omega)} \end{cases}$$

$$\boxed{\frac{e^{jx} + e^{-jx}}{2} = \cos(x)}$$

Expressing as sum of sinusoids



- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
  - this is the fundamental practical interpretation of the Fourier transform
- The **modulus** $|X(\omega)|$ is the **amplitude** of the sinusoids ($\times$ 2)
  - for $\omega = 0$, $|X(\omega = 0)|$ = the DC component
- The **phase** $\angle X(\omega)$ gives the initial phase

Properties of DTFT

$x[n] \longleftrightarrow X(z)$

$x[n] \longleftrightarrow X(f)$

$x_1[n] \longleftrightarrow X_1(\omega)$

$x_2[n] \longleftrightarrow X_2(\omega)$

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

$$X(\omega) = \sum \underbrace{x[n]}_{x_1 + x_2} e^{j\omega n}$$

Properties of DTFT

$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$

$x[n-k] \longleftrightarrow z^{-k} \cdot X(z)$

$z = e^{j\omega}$ $\Rightarrow$

$x[n] \xleftrightarrow{\mathcal{F}} X(\omega)$

$x[n-k] \xleftrightarrow{\mathcal{F}} e^{-j\omega k} \cdot X(\omega)$

z

$a = |a| \cdot e^{j\angle a}$

$a \cdot e^{j \cdot \text{something}} = |a| \cdot e^{j \cdot (\angle a + \text{something})}$

2. Shifting in time

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

$$\sum_{n=-\infty}^{\infty} x[n-k] \cdot e^{-j\omega n} \underset{n-k=m}{=} \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega(m+k)} = e^{-j\omega k} \cdot \underbrace{\sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega m}}_{X(\omega)}$$

Properties of DTFT

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of DTFT

$x_1[n] * x_2[n] \longleftrightarrow X_1(z) \cdot X_2(z)$

$\Downarrow z = e^{j\omega}$

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- ~~Not circular convolution~~, this is the normal convolution

Properties of DTFT

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of DTFT

Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega) X_2(-\omega)$$

Wiener-Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \overset{\mathcal{F}}{\leftrightarrow} S_{xx}(\omega) = |X(\omega)|^2$$

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega = \int_{-1/2}^{1/2} |X(f)|^2 \, df$$

- Is true for all orthonormal bases

IV.4 The Discrete Fourier Transform (DFT)

Definitions (again)

Inverse Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi \overbrace{kn/N}^{f}}$$

$\frac{k}{N} = f$

$k = 0 \ldots 6$

$k = -2 \ldots 3$

$$\overbrace{f = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \ldots \frac{N-1}{N}}^{N \text{ terms}}$$

$$e^{-j 2\pi \cdot \overbrace{\frac{k}{N}}^{f} \cdot n}$$

$N = 7 \Rightarrow 0, \frac{1}{7}, \frac{2}{7}, \ldots \frac{6}{7}$

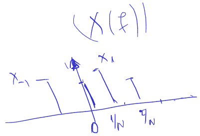
Discrete Fourier Transform (DFT)

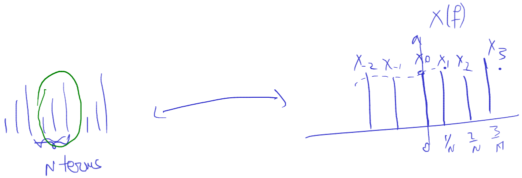
$$X_k = \frac{1}{N}\langle x[n], e^{j2\pi fn}\rangle = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$f = \frac{6}{7} = \frac{-1}{7}$

$f = \frac{5}{7} = \frac{-2}{7}$

$f = \frac{4}{7} = \frac{-3}{7}$





Periodicity and notation

- In discrete domain, $\boxed{f = \frac{N-k}{N} = \frac{-k}{N}}$ (aliasing, we can subtract 1 from f)
- We can consider $\boxed{X_{N-k}}$ as $\boxed{X_{-k}}$, due to periodicity
- Example: a signal with period $N = 6$ has 6 DFT coefficients
  - we can call them $X_0, X_1, X_2, X_3, X_4, X_5$ $\longrightarrow$ $\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$
  - we have $X_5 = X_{-1}$, $X_4 = X_{-2}$
  - we can also call them $\underset{X_4}{\underset{=}{X_{-2}}}, \underset{X_5}{\underset{=}{X_{-1}}}, X_0, X_1, X_2, X_3$

$f = \frac{5}{6} = \frac{-1}{6}$ $\quad X_5 = X_{-1}$

$f = \frac{4}{6} = \frac{-2}{6}$

Basic Properties of the DFT

$X_k = |X_k| \cdot e^{j \angle X_k}$

- X_k is complex (has $|X_k|$, $\angle X_k$)
- If the signal $x[n]$ is real, the coefficients are **even**

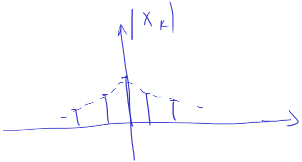
$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

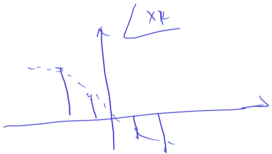
$X_{-k} = X_k^* \Rightarrow |X_{-k}| = |X_k|$

$\angle X_{-k} = -\angle X_k$

$X_{-k} = X_{N-k}$

  - This means:
    - modulus is even: $|X_k| = |X_{-k}|$
    - phase is odd: $\angle X_{-k} = -\angle X_k$





Expressing as sum of sinusoids

- Grouping terms with k and $-k$:
- If N is odd, we have X_0 and pairs (X_k, X_{-k}):

$$x[n] = \frac{1}{N}\cdot\sum_{k=0}^{N-1} X_k e^{j2\pi\frac{k}{N}\cdot n}$$

$$X_{N-1} = X_{-1} \qquad X_{N-2} = X_{-2}$$

$$x[n] = X_0 e^{j0n} + \frac{1}{N}\sum_{k=-(N-1)/2}^{-1} X_k e^{j2\pi kn/N} + \frac{1}{N}\sum_{k=1}^{(N-1)/2} X_k e^{j2\pi kn/N}$$

$$= X(0) + \frac{1}{N}\sum_{k=0}^{(N-1)/2}\left(X_k e^{j2\pi kn/N} + X_{-k}e^{-j2\pi kn/N}\right)$$

$$= X(0) + \frac{1}{N}\sum_{k=0}^{(N-1)/2} |X_k|\left(e^{j(2\pi kn/N+\angle X(k))} + e^{-j(2\pi kn/N+\angle X(\omega))}\right)$$

$$\boxed{= \frac{1}{N}X(0) + \frac{1}{N}\sum_{k=0}^{(N-1)/2} 2|X_k|\cos(2\pi k/Nn + \angle X_k)}$$

D.C.

$$f = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \dots \frac{N-1}{2}/N$$

$$e^{jx} + e^{-jx} = 2\cos(x)$$

Example:

$$N = 5$$

$$x[n] = \frac{1}{N}\cdot\sum_{k=0}^{4} X_k \cdot e^{j2\pi\frac{k}{5}\cdot n}$$

$$= \underbrace{X_0\cdot e^{j2\pi\frac{0}{5}n}}_{X_0} + X_1\cdot e^{j2\pi\frac{1}{5}n} + X_2\cdot e^{j2\pi\frac{2}{5}n} + X_{-2}\cdot e^{j2\pi\left(\frac{-2}{5}\right)n} + X_{-1}\cdot e^{j2\pi\left(\frac{-1}{5}\right)n}$$

$$X_1 = |X_1|\cdot e^{j\angle X_1} \quad \cdot e^{j2\pi\frac{1}{5}n}$$

$$X_{-1} = |X_1|\cdot e^{-j\angle X_1} \quad \cdot e^{j2\pi\frac{-1}{5}n}$$

$$|X_1|\cdot\left(\underbrace{e^{j\left(\angle X_1 + 2\pi\frac{1}{5}n\right)} + e^{-j\left(\angle X_1 + 2\pi\frac{1}{5}n\right)}}_{2\cos\left(2\pi\frac{1}{5}n + \angle X_1\right)}\right)$$

Expressing as sum of sinusoids

- If <u>N is even</u>, we have X_0 and pairs (X_k, X_{-k}), with an extra term $X_{N/2}$ which has no pair
  - e.g. N = 6: $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$
- $X_{N/2}$ must be a real number
- The extra term will be $\frac{1}{N}X_{N/2}e^{j2\pi N/2n/N} = X_{N/2}\cos(n\pi)$
- Overall:

$$\boxed{x[n] = \underbrace{\tfrac{1}{N}\cancel{X(0)}}_{D.C.} + \frac{1}{N}\sum_{k=0}^{(N-2)/2} 2|X_k|\cos(2\pi \underbrace{k/N}_{f} n + \angle X_k) + \frac{1}{N}X_{N/2}\cos(\underbrace{2\pi \tfrac{1}{2}}_{f=\frac{3}{6}} n)}$$

- Any signal $x[n]$ is **<u>a sum of sinusoids with frequencies</u>** $f = 0$, 1/N, 2/N, ... (N-1)/2 or N/2

$N = 6$

$X_0, X_1, X_2, X_3, X_4 = X_{-2}, X_5 = X_{-1}$

$X_0 = DC$

$\frac{1}{N} \cdot 2 |X_1| \cdot \cos(f = \frac{1}{6}, \angle X_1)$

$\frac{1}{N} \cdot 2 \cdot |X_2| \cdot \cos(f = \frac{2}{6}, \angle X_2)$

$\frac{1}{N} \cdot |X_3| \cdot \cos(f = \frac{3}{6}, \angle X_3)$

Expressing as sum of sinusoids

- Any periodic signal $x[n]$ with period N is **a sum of ~~N~~ sinusoids with frequencies** $f = 0, 1/N, 2/N, \ldots$ (N-1)/2 or N/2
- The **modulus** $|X_k|$ gives the **amplitude** of the sinusoids (sometimes $\times$ 2)
  - for $\omega = 0$, $|X_0|$ = the DC component
  - when modulus = 0, that frequency has amplitude 0
- The **phase** $\angle X_k$ gives the initial phase

$f = \frac{k}{N}$

$N = 6 \longrightarrow f = 0$, $f = \frac{1}{6}$, $f = \frac{2}{6}$, $f = \frac{3}{6}$

$N = 5 \longrightarrow f = 0$, $f = \frac{1}{5}$, $f = \frac{2}{5}$

Example

- Consider a periodic signal $x[n]$ with period $N = \cancel{6}$ 5 and the DFT coefficients:

  X_k = [15.0000 + 0.0000i , -2.5000 + 3.4410i , -2.5000 + 0.8123i , -2.5000 - 0.8123i , -2.5000 - 3.4410i]

  X_0 X_1 X_2 X_3 X_4

  Write $x[n]$ as a sum of sinusoids.

- Do the same for a periodic signal $x[n]$ with period $N = \cancel{5}$ 6 and the DFT coefficients:

  X_k = [21.0000 + 0.0000i , -3.0000 + 5.1962i , -3.0000 + 1.7321i , -3.0000 + 0.0000i , -3.0000 - 1.7321i , -3.0000 - 5.1962i]

  Write $x[n]$ as a sum of sinusoids.

$X_1 = X_4^* = X_{-1}$

Properties of the DFT

$x_1[n] \longleftrightarrow \{X_k^{(1)}\}$

$x_2[n] \longleftrightarrow \{X_k^{(2)}\}$

1. Linearity

If the signal $x_1[n]$ has the DFT coefficients $\{X_k^{(1)}\}$, and $x_2[n]$ has $\{X_k^{(2)}\}$, then their sum has

$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$

Proof: via definition

$$\underbrace{a \cdot x_1[n] + b \cdot x_2[n]}_{x_s[n]} \longleftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$$

$$X_k = \sum_{k=0}^{N-1} \underbrace{x_s[n]}_{a \cdot x_1 + b \cdot x_2} \cdot e^{-j \cdots} = a \cdot \sum x_1[n] e^{\cdots} + b \cdot \sum x_2[n] e^{\cdots} = a \cdot X_k^{(1)} + b \cdot X_k^{(2)}$$

Properties of the DFT

$|a| \cdot e^{j\angle a} \cdot e^{j\,\mathrm{cava}} = |a| \cdot e^{j(\angle a + \mathrm{cava})}$

$x[n] \longleftrightarrow \{X_k\}$

$x[n - n_0] \longleftrightarrow \{ e^{-j2\pi k\frac{n_0}{N}} \cdot X_k \}$

2. Shifting in time

If $x[n] \leftrightarrow \{X_k\}$, then

$|\cdot| = 1 \qquad \angle -2\pi k \frac{n_0}{N}$

$$\underbrace{x[n-n_0]}_{x'[n]} \leftrightarrow \{e^{(-j2\pi k n_0/N)} X_k\}$$

Proof: via definition

- The amplitudes $|X_k|$ are not affected, shifting in time **affects only the phase**

$\mathcal{Z}: \longrightarrow z^{-n_0}$

$z = e^{j\omega} = e^{j2\pi f} = e^{j2\pi\frac{k}{N}}$

$x[n-n_0] \longleftrightarrow \underbrace{e^{-j2\pi\frac{k}{N}n_0}}_{z} \cdot X_k$

$\sum x'[n] \cdot e^{-j2\pi kn/N} = \sum_n x[n-n_0] \cdot e^{-j2\pi kn/N} = \underbrace{\sum_m x[m] \cdot e^{-j2\pi k\frac{m}{N}}}_{X_k} \cdot e^{-j2\pi k\frac{n_0}{N}} = e^{-j2\pi k\frac{n_0}{N}} \cdot X_k$

$n - n_0 = m$

$n = m + n_0$

$\underbrace{1}_{|\cdot|} \cdot e^{-j\cdots} =$

Properties of the DFT

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{X^*_{-k}\}$$

Properties of the DFT

periodic with period N

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$\circledast$

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

Circular convolution definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is also periodic with period N
- Example at the whiteboard: how it is computed

Example

Linear convolution

$x_1 = \ldots 0, 1, 2, 3, 0, \ldots$

$x_2 = \ldots 0, 4, 4, 4, 4, 0, \ldots$

$x_1 * x_2 =$

4	4	4	4			
	8	8	8	8		
		12	12	12	12	

$\ldots 0, \underbrace{4, 12, 24, 24, 20, 12}_{6}, 0, \ldots$

Example (write on slides)

$x_1 = \text{per} \ldots, \underbrace{1, 2, 3, 4}_{N}, \ldots \text{periodic}$

$x_2 = \ldots, \underbrace{5, 4, 3, 2}_{N}, \ldots \text{periodic} \ldots$

$x_1 \circledast x_2$

5, 4, 3, 2	5,	4,	3,	2,	5, 4, 3 2		
10, 8, 6	4,	10,	8,	6,	4,		
15, 9	12,	6,	15,	9,	12,	6	
20	16,	12,	8	20,	16,	12,	8
periodic ...	37	32	34	37	periodic		

$\underbrace{37 \quad 32 \quad 34 \quad 37}_{N}$

Circular convolution

- We are in the vector space of **periodic signals** with period N
- Linear (e.g. normal) convolution produces a result which is longer periodic with period N
- Circular convolution takes two sequences of length N and produces another sequence of length N
  - each sequence is a period of a periodic signal
  - circular convolution = like a convolution of periodic signals

Properties of the DFT

6. Product in time

Product in time $\leftrightarrow$ circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

Properties of the DFT

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{0}^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum_{k=0}^{N} |X_k|^2$$

- Is true for all orthonormal bases

Relationship between DTFT and DFT

- How are DTFT and DFT related?
- Discrete Time Fourier Transform:
  - for non-periodical signals
  - spectrum is continuous

f

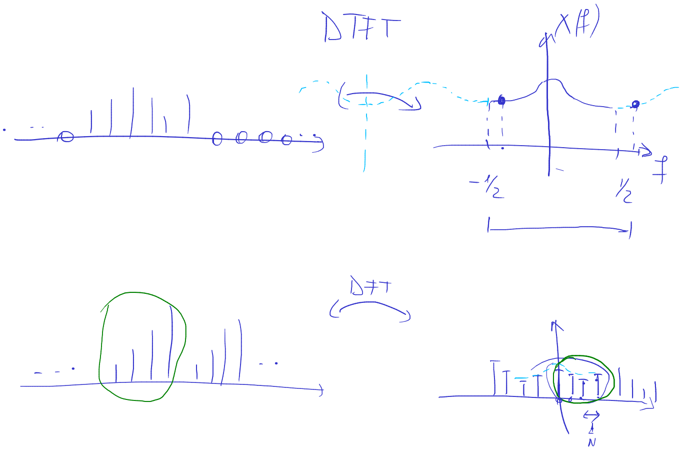
- Discrete Fourier Transform
  - for periodical signals
  - spectrum is discrete

$f = \frac{k}{N}$

- Duality: periodic in time $\leftrightarrow$ discrete in frequency

discrete in time $\leftrightarrow$ periodic in frequency

Both $\leftrightarrow$ Both



Relationship between DTFT and DFT

- Consider a non-periodic signal $x[n]$
- It has a continuous spectrum $X(\omega)$
- If we **periodize** it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k\frac{2\pi}{N})$$

- The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k/Nn)$$

- They are **samples** from the continuous $X(\omega)$ of the non-periodized signal

Relationship between DTFT and DFT

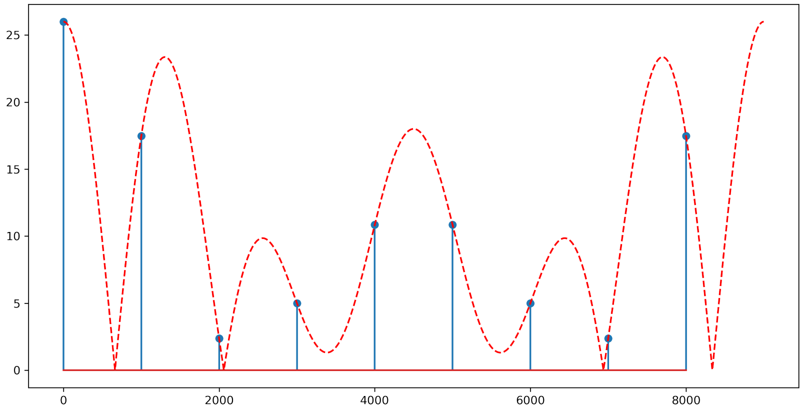
- Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

- If we consider a non-periodic $x[n]$ with infinitely long zeros on either side, we have a continuous spectrum $X(\omega)$ (DTFT)
- If we consider that x is just a period of a periodic signal, we have a discrete spectrum X_k (DFT)
- Moreover, the discrete X_k are just **samples from** $X(\omega)$:

$$X_k = X(2\pi k/Nn)$$

Relationship between DTFT and DFT



$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$

- red line = DFT(x) if x not periodical
  - actually run as `fft(x, 10000)`, x is extended with 9991 zeros
- blue = `fft(x)`

Relation between DTFT and Z transform

- Z transform:

$$X(z) = \sum_n x[n]z^{-n}$$

- DTFT:

$$X(\omega) = \sum_n x[n]e^{-j\omega n}$$

- DTFT can be obtained from Z transform with

$$\boxed{z = e^{j\omega}}$$

- These $z = e^{j\omega}$ are **points on the unit circle**
  - $|z| = |e^{j\omega}| = 1(modulus)$
  - $\angle z = \angle e^{j\omega} = \omega(phase)$

Relation between DTFT and Z transform

- Fourier transform = Z transform evaluated **on the unit circle**
  - if the unit circle is in the convergence region of Z transform
  - otherwise, equivalence does not hold
- This is true for most usual signals we work with
  - some details and discussions are skipped

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1)\cdots(e^{j\omega} - z_M)}{(e^{j\omega} - p_1)\cdots(e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1|\cdots|e^{j\omega} - z_M|}{|e^{j\omega} - p_1|\cdots|e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

Geometric interpretation of Fourier transform

- For complex numbers:
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

Geometric interpretation of Fourier transform

- Consequences:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples: . . .

Geometric interpretation of Fourier transform

- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is like **a landscape**
    - **poles = mountains** of infinite height
    - **zeros = valleys** of zero height
  - Fourier transform $X(\omega) =$ "*Walking over this landscape along the unit circle*"
  - The height profile of the walk gives the amplitude of the Fourier transform
  - When close to a mountain –> road is high –> Fourier transform has large amplitude
  - When close to a valley –> road is low –> Fourier transform has small amplitude

Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for phase as well

Time-frequency duality

- **Duality** properties related to all Fourier transforms
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Terminology

- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals
- **Band-limited** signals: spectrum is 0 beyond some frequency f_{max}
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: $B <<$ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band