

Digital Signal Processing

Chapter V. Digital filtering

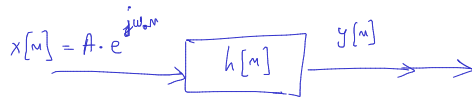
Response of LTI systems to harmonic signals

- ▶ We consider an LTI system with $h[n]$

- ▶ Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$

- ▶ Output signal = convolution

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] \underline{x[n-k]} \\ &= \sum_{k=-\infty}^{\infty} \left[h[k] e^{-j\omega_0 k} \right] \underline{Ae^{j\omega_0 n}} \\ &= \underline{H(\omega_0)} \cdot x[n] \end{aligned}$$



$$x[n-k] = A \cdot e^{j\omega_0(n-k)} = \underline{A \cdot e^{j\omega_0 n} \cdot e^{-j\omega_0 k}}$$

$$h[n] \xleftrightarrow[\text{DTFT}]{\mathcal{F}} H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

Transfer function

- ▶ $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

Response of LTI systems to harmonic signals

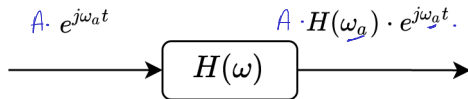


Figure 1: Output = a constant \times Input

- $H(\omega) =$ Fourier transform of $h[n]$ evaluated for $\omega =$ **transfer function**

Eigen-function

$$e^{j\omega_0 n}$$

- ▶ Complex exponential signals are eigen-functions (funcții proprii) of LTI systems:

- ▶ output signal = input signal \times a (complex) constant

- ▶ $H(\omega_0)$ is a complex constant that multiplies the input signal

- ▶ Amplitude of input gets multiplied by $|H(\omega_0)|$
 - ▶ Phase of input signal is added with $\angle H(\omega_0)$

- ▶ Why are sin/cos/exp functions important?

- ▶ If input signal = sum of complex exponential (like cosines + sines),
 - ▶ then output = same sum of complex exponentials, each scaled with some coefficients

$$H(\omega_0) = |H(\omega_0)| \cdot e^{j\angle H(\omega_0)}$$
$$y[n] = H(\omega_0) \cdot A e^{j\omega_0 n} = \underbrace{|H(\omega_0)| \cdot A}_{\text{amplitude}} \cdot e^{j(\omega_0 n + \angle H(\omega_0))}$$

Response to cosine and sine

- Cosine / sine = sum of two exponentials, via Euler

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}$$

$$\sin(\omega n) = \cos\left(\omega n - \frac{\pi}{2}\right) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

- System is linear and real \Rightarrow

- amplitude is multiplied by $|H(\omega_0)|$
- phase increases by $\angle H(\omega_0)$

- See proof at blackboard

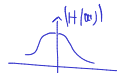
$$\begin{aligned}
 & \boxed{A \cdot \cos(\omega_0 n)} \rightarrow \boxed{\frac{h[n]}{H(\omega)}} \rightarrow y[n] \\
 & \frac{A}{2} e^{j\omega_0 n} + \frac{A}{2} e^{-j\omega_0 n} \rightarrow \frac{A}{2} H(\omega_0) e^{j\omega_0 n} + \frac{A}{2} H(-\omega_0) e^{-j\omega_0 n} \\
 & = \frac{A}{2} |H(\omega_0)| e^{j(\omega_0 n + \angle H(\omega_0))} + \frac{A}{2} |H(\omega_0)| e^{-j(\omega_0 n + \angle H(\omega_0))} \\
 & = \boxed{A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \angle H(\omega_0))} \\
 & \text{Same for sine!}
 \end{aligned}$$

$H(\omega)$ is an even function:

$$H(\omega) = H(-\omega)$$

$$|H(\omega)| = |H(-\omega)|$$

$$-\angle H(\omega) = \angle H(-\omega)$$



Frequency response

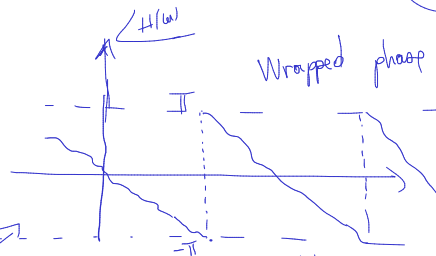
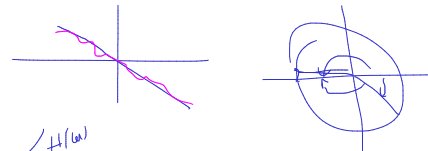
► Naming:

- $H(\omega)$ = *or transfer function* **frequency response** of the system
- $|H(\omega)|$ = **amplitude response** (or magnitude response)
- $\angle H(\omega)$ = **phase response**

► Magnitude response is non-negative: $|H(\omega)| \geq 0$

► Phase response is an angle: $\angle H(\omega) \in (-\pi, \pi]$

- Phase response may have jumps of 2π (wrapped phase)
- Stitching the pieces in a continuous function = phase unwrapping
- Unwrapped phase: continuous function, may go outside interval $(-\pi, \pi]$
- Example: at blackboard



Permanent and transient response

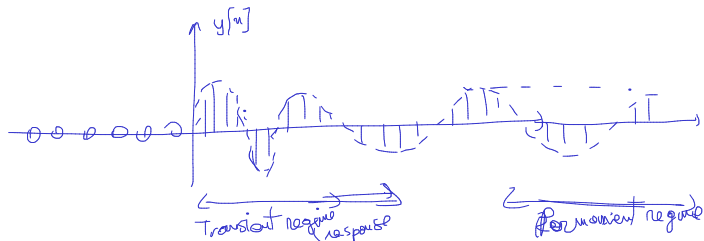
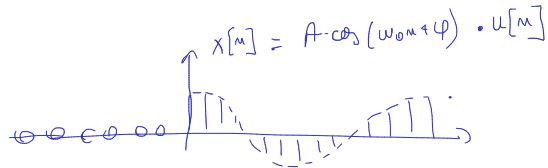
Permanent response

$$x[n] = A \cdot \cos(\omega_0 n + \varphi), \quad n \in \mathbb{Z}$$

$$y[n] = A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \varphi + \angle H(\omega_0))$$



- ▶ Warning: $\cos(\omega n)$ does not start at $n = 0$
- ▶ The above harmonic signals start at $n = -\infty$.
- ▶ What's wrong if the signal starts at some time n ?



Permanent and transient response

- ▶ What if the signal starts at some time n ?
- ▶ Total response = transient response + permanent response
 - ▶ transient response goes towards 0 as n increases
 - ▶ permanent response = what remains
- ▶ So the above relations are valid only in permanent regime
 - ▶ i.e. after the transient regime has passed
 - ▶ i.e. after the transient response has practically vanished
 - ▶ i.e. when the signal started very long ago (from $n = -\infty$)
 - ▶ i.e. when only the permanent response remains in the output signal
- ▶ Example at blackboard

Permanent response of LTI systems to periodic inputs

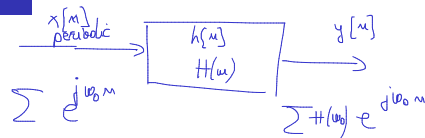
- ▶ Consider an input $x[n]$ which is periodic with period N
- ▶ Then it can be represented as a Fourier series with coefficients c_k :

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \cancel{c_k} e^{j2\pi kn/N}$$

- ▶ Since the system is linear, each component k gets multiplied with $H\left(\frac{2\pi}{N}k\right)$
- ▶ So the total output is:

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} \cancel{c_k} H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ The output is still periodic, same period, same frequencies



Response of LTI systems to non-periodic signals

- ▶ Consider a general input $x[n]$ (not periodic)
- ▶ The output = input convolution with impulse response

$$\underline{y[n]} = \underline{x[n]} * \underline{h[n]}$$

$$\underline{Y(\omega)} = \underline{X(\omega)} \cdot \underline{H(\omega)}$$

- ▶ Output spectrum = Input spectrum \times Transfer function

Response of LTI systems to non-periodic signals

- ▶ The transfer function $H(\omega)$ “shapes” the spectrum

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

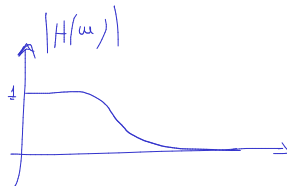
- ▶ In polar form:

- ▶ modulus is multiplied

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$

- ▶ phases is added:

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$



Response of LTI systems to non-periodic signals

- ▶ The system **attenuates/amplifies** the input frequencies and **changes their phases**
- ▶ $H(\omega)$ = the transfer function
- ▶ $H(z)$ = the system function
- ▶ $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

Roc

Power spectral density

$$x\{n\} \mapsto X(\omega) \\ |X(\omega)|^2 = S_{xx}(\omega) = \text{Power Spectral Density}$$

$$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2$$

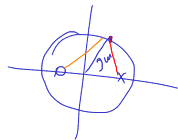
$$S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$$

$$\blacktriangleright S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\omega)$$

\blacktriangleright The poles and zeros of $S(\omega)$ come in pairs $(z, 1/z$ and $p, 1/p)$

Digital filters

- ▶ LTI systems are also known as **filters** because their transfer function shapes (“filters”) the frequencies of the input signals
- ▶ The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- ▶ Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros



$$\frac{\text{red vector}}{\text{blue vector}} = |H(\omega)|$$

Ideal filters

► Draw at whiteboard the ideal transfer function of a:

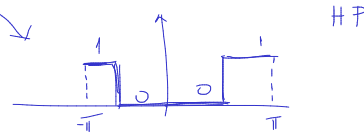
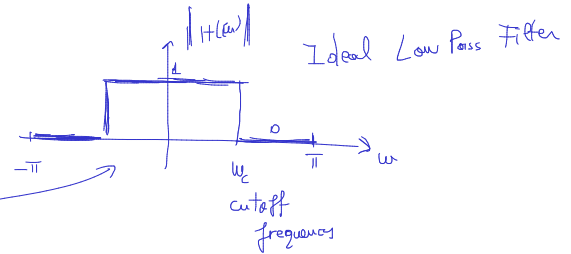
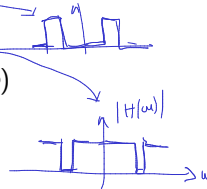
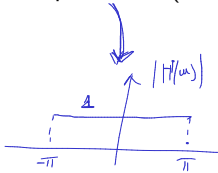
► low-pass filter

► high-pass filter


► band-pass filter

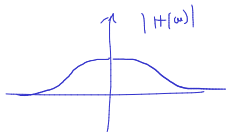
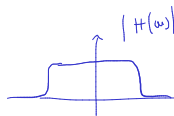
► band-stop filter

► all-pass filter (*changes the phase*)



Filter order

- ▶ The order of a filter = maximum degree in numerator or denominator of $H(z)$
 - ▶ i.e. largest power of z or z^{-1}
- ▶ Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1}) 
- ▶ Higher order -> better filter transfer function
 - ▶ closer to ideal filter
 - ▶ more complex to implement
 - ▶ more delays (bad)
- ▶ Lower order
 - ▶ worse transfer function (not close to ideal)
 - ▶ simpler, cheaper
 - ▶ faster response



Filter design by pole and zero placements

- ▶ Based on geometric method
- ▶ The gain coefficient must be found by separate condition
 - ▶ i.e. specify the desired magnitude response at one frequency
- ▶ Examples at blackboard

Filter distortions

- ▶ When a filter is non-ideal:
 - ▶ non-constant amplitude \rightarrow amplitude distortions
 - ▶ non-linear phase \rightarrow phase distortions
- ▶ Phase distortions may be tolerated by certain applications
 - ▶ e.g. human auditory system is largely insensitive to phase distortions of sounds

Effect of system's phase

- ▶ What is the effect of system's phase response $\angle H(\omega)$?
- ▶ Extra phase = delay
 - ▶ different frequencies are delayed differently
 - ▶ phase
- ▶ **Linear-phase** filter: delays all frequencies with the same amount of time
 - ▶ i.e. the whole signal is delayed, but otherwise not distorted
 - ▶ otherwise, we get distortions

Linear-phase filters

- ▶ For a sinusoidal signal, extra phase of 2π = delay of a period $N = \frac{1}{f}$
- ▶ To ensure same delay for all frequencies (in time), the phase $\angle H(\omega)$ must be proportional to the frequency
 - ▶ draw at blackboard
 - ▶ hence the name **linear**

Linear-phase filters

- ▶ Example: consider the following filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

- ▶ The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

- ▶ Linear phase means **just a delaying** of the input signal

- ▶ Fourier property: $x[n - n_0] \longleftrightarrow X(\omega)e^{-j\omega n_0}$

Group delay

- ▶ Group delay = The time delay experienced by a component of frequency ω when passing through the filter
 - ▶ as opposed to “phase delay” = the phase added by the filter
- ▶ **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- ▶ Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Linear-phase FIR filters

What type of filters can have linear phase?

- ▶ IIR filters cannot have linear phase (no proof provided)
- ▶ Only FIR filters can have linear phase, and only if they satisfy some symmetry conditions

Symmetry conditions for linear-phase FIR

- ▶ Let filter have an impulse response of length M (order is $M - 1$)
- ▶ The filter coefficients are $h[0], \dots, h[M - 1]$
- ▶ Linear-phase is guaranteed in two cases

- ▶ **Positive symmetry**

$$h[n] = h[M - 1 - n]$$

- ▶ **Negative symmetry (anti-symmetry)**

$$h[n] = -h[M - 1 - n]$$

- ▶ The delay = the delay of the middle point of the symmetry

Cases of linear-phase FIR

- ▶ Proofs at blackboard
- 1. Positive symmetry, $M = \text{odd}$
- 2. Positive symmetry, $M = \text{even}$
- 3. Negative symmetry, $M = \text{odd}$
- 4. Negative symmetry, $M = \text{even}$
- ▶ Check constraints for $H(0)$ and $H(\pi)$
- ▶ For what types of filters is each case appropriate?

Zero-phase FIR filters

- ▶ Can we avoid delay altogether?
- ▶ **Zero-phase** filter = a particular type of linear-phase filter with zero delay
- ▶ For a zero-phase filter, the phase response $\angle H(\omega) = 0$
 - ▶ (Group) delay = derivative of $\angle H(\omega)$
 - ▶ delay 0 \Leftrightarrow flat $\angle H(\omega) \Leftrightarrow \angle H(\omega) = 0$
- ▶ Delay is 0 \Leftrightarrow symmetry with respect to $h[0]$
 - ▶ the system cannot be causal

Zero-phase FIR filters

- ▶ Zero-phase filters must be non-causal
 - ▶ left side of $h[n]$ symmetrical to right side of $h[n]$
- ▶ For causal, we need to delay $h[n]$ to be wholly on the right side => delay

Example

- ▶ Linear-phase filter (low-pass):

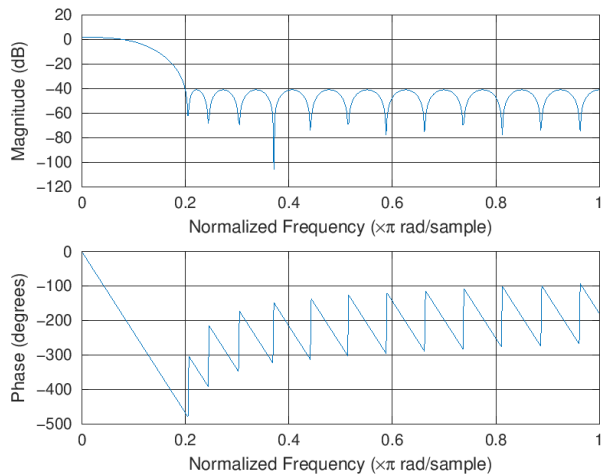


Figure 2: Transfer function of linear-phase filter

Example

- The impulse response (positive symmetry):

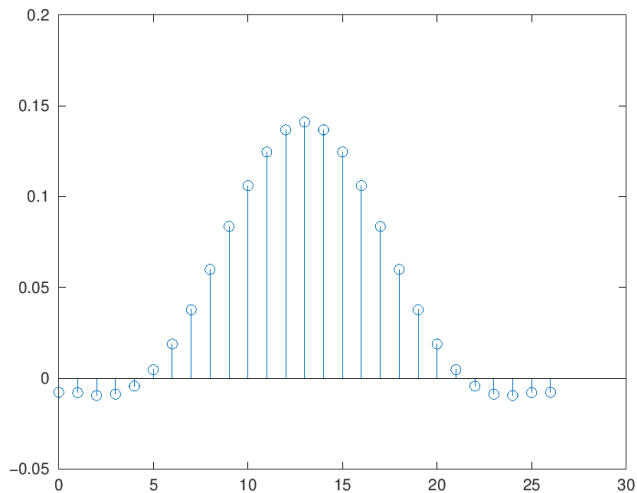


Figure 3: Impulse response of linear-phase filter

Example

- ECG signal: original and filtered. Filtering introduces **delay**

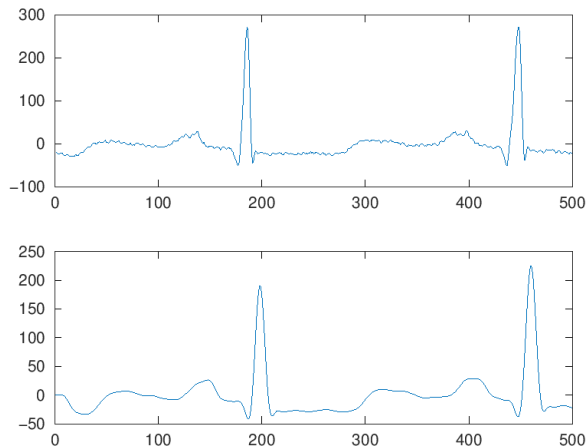


Figure 4: Delay introduced by filtering

Example

- ▶ Solution: zero-phase filter (positive symmetry, and centered in 0):
- ▶ But filter is **not causal** anymore

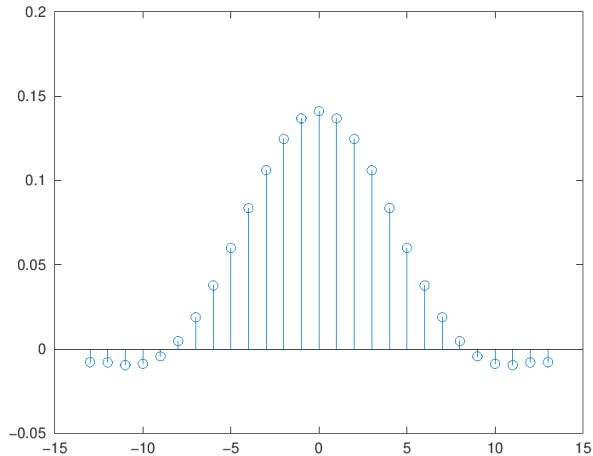


Figure 5: Impulse response of zero-phase filter

Example

- Filtering with zero-phase filter introduces **no delay**

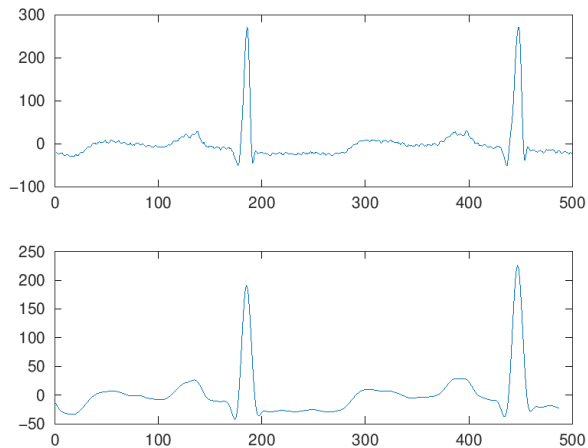


Figure 6: Zero-phase filter introduces no delay

Particular classes of filters

- ▶ **Digital resonators** = very selective band pass filters
 - ▶ poles very close to unit circle
 - ▶ may have zeros in 0 or at $1/-1$
- ▶ **Notch filters**
 - ▶ have zeros exactly on unit circle
 - ▶ will completely reject certain frequencies
 - ▶ has additional poles to make the rejection band very narrow
- ▶ **Comb filters**
 - ▶ = periodic notch filters

Digital oscillators

- ▶ **Oscillator** = a system which produces an output signal even in absence of input
- ▶ Has a pair of complex conjugate poles **exactly on unit circle**
- ▶ Example at blackboard

Inverse filters

- ▶ Sometimes is necessary to **undo** a filtering
 - ▶ e.g. undo attenuation of a signal passed through a channel
- ▶ Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- ▶ Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle \rightarrow unstable
- ▶ Examples at blackboard