Chapter IV: The Fourier Transform and its applications

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IV.1 Vector spaces of signals (crash course)

Vector spaces

- **Vector space** $=$ a set $V\{v_i\}$ with the following two properties:
  - one element $+$ another element $=$ still an element of the same space
  - a scalar constant $\times$ an element $=$ still an element of the same space
- You **can't escape** a vector space by summing or scaling
- The elements of a vector space are called **vectors**

Examples of vector spaces

- Geometric spaces are great intuitive examples:
  - a line, or the set $\mathbb{R}$ (one-dimensional)
  - a plane, or the set $\mathbb{C}$ (two-dimensional)
  - 3D space (three-dimensional)
  - 4D space (four-dimensional, like the spatio-temporal universe)
  - arrays with N numbers (N-dimensional)
  - space of continuous signals (∞-dimensional)
- The **dimension** of the space = "how many numbers you need in order to specify one element" (informal)
- A "vector" like in maths = a sequence of N numbers = a "vector" like in programming
  - e.g. a point in a plane has two coordinates = a vector of size $N = 2$
  - e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

Inner product

- Many vector spaces have a fundamental operation: **the (Euclidean) inner product**
  - for **discrete** signals:
$$\langle \vec{x}, \vec{y} \rangle = \sum_i x_i y_i^*$$
  - for **continuous** signals:
$$\langle \vec{x}, \vec{y} \rangle = \int x(t) y^*(t)$$
- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** ("produs scalar")

Inner product

- Each entry in $\vec{x}$ times the complex conjugate of the one in $\vec{y}$, all summed
- For discrete signals, it can be understood as a row $\times$ column multiplication
- Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- Inner product is **linear** in both terms:

$$\langle \vec{x}_1 + \vec{x}_2, \vec{y} \rangle = \langle \vec{x}_1, \vec{y} \rangle + \langle \vec{x}_2, \vec{y} \rangle$$

$$\langle c \cdot \vec{x}, \vec{y} \rangle = c \cdot \langle \vec{x}_1, \vec{y} \rangle$$

$$\langle \vec{x}, \vec{y_1} + \vec{y_2} \rangle = \langle \vec{x}, \vec{y_1} \rangle + \langle \vec{x}, \vec{y_2} \rangle$$

$$\langle \vec{x}, c \cdot \vec{y} \rangle = c^* \cdot \langle \vec{x}_1, \vec{y} \rangle$$

The distance between two vectors

- An inner product induces a **norm** and a **distance** function
- **The (Euclidean) distance** between two vectors =

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_N - y_N)^2}$$

- This distance is the **usual geometric distance** you know from geometry
- It has the exact same intuition like in **normal geometry**:
  - if two vectors have small distance, they are close, they are similar
  - two vectors with large distance are far away, not similar
  - two identical vectors have zero distance

The norm of a vector

- An inner product induces a **norm** and a **distance** function
- The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- The **norm** of a vector is the distance from $\vec{x}$ to point $\vec{0}$.
- It has the exact same intuition like in **normal geometry**:
  - vector has large norm = has big values, is far from $\vec{0}$
  - vector has small norm = has small values, is close to $\vec{0}$
  - vector has zero norm = it is the vector $\vec{0}$
- Norm of a vector = sqrt(the signal **energy**)

Norm and distance

- The norm and distance are related
- The distance between $\vec{a}$ and $\vec{b}$ = norm (length) of their difference

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- Just like in geometry: distance = length of the difference vector

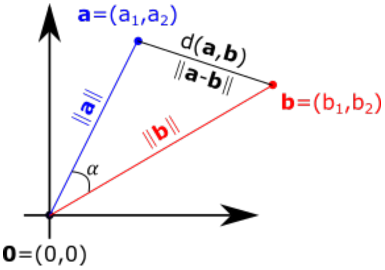


Figure 1: Norm and distance in vector spaces

Angle between vectors

- The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - is a value between -1 and 1
- **Otrhogonal vectors** $=$ two vectors with $\langle x, y \rangle = 0$
  - their angle $= 90\deg$
  - in geometric language, the two vectors are **perpendicular**

Why vector space

- Why are all these useful?
- They are a very general **framework** for different kinds of signals
- We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - Example in DEDP class: ML decision with 1, 2, N samples

Vector spaces in DSP class

We deal mainly with the following vector spaces:

- The vector space of all infinitely-long real signals $x[n]$
- The vector space of all infinitely-long periodic signals $x[n]$ with period N
  - for each N we have a different vector space
- The vector space of all finite-length signals $x[n]$ with only N samples
  - for each N we have a different vector space

Bases

- A **basis** $=$ a set of N linear independent elements from a vector space

$$B = \{\vec{b}^1, \vec{b}^2 ... \vec{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + ... + \alpha_N \vec{b}^N$$

- The vector is defined by these coefficients:

$$\vec{x} = (\alpha_1, \alpha_2, ... \alpha_N)$$

Bases and coordinate systems

- Bases are just like **coordinate systems** in a geometric space
  - any point is expressed w.r.t. a coordinate system

$$\vec{x} = x_1\vec{i} + x_2\vec{j}$$

  - any vector is expressed w.r.t. a basis

$$\vec{x} = \alpha_1\vec{b}^1 + \alpha_2\vec{b}^2 + \cdots + \alpha_N\vec{b}^N$$

- N = The number of basis elements = The dimension of the space
- Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

Bases and coordinate systems

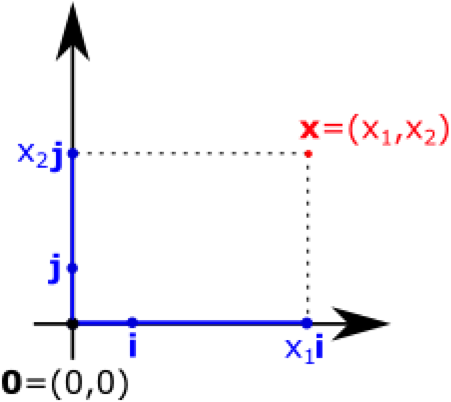


Figure 2: Basis expansion of a vector x

Choice of bases

- There is typically an infinite choice of bases
- The **canonical basis** = all basis vectors are full of zeros, just with one 1
- You used it already in an exercise:
  - any signal $x[n]$ can be expressed of a sum of $\delta[n-k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n-1] + 2\delta[n-2] + \dots$$

  - the canonical basis is $B = \{\dots, \delta[n], \delta[n-1], \delta[n-2], \dots\}$

Orthonormal bases

- An **orthonormal basis** a basis where all elements $\vec{b}^i$ are:
  - orthogonal to each other:

$$\langle \vec{b}^i, \vec{b}^j \rangle = 0, \forall i \neq j$$

  - **normalized** (their norm $= 1$):

$$||\vec{b}^i|| = \sqrt{\langle \vec{b}^i, \vec{b}^i \rangle} = 1, \forall i$$

- Example: the canonical basis $\{\delta[n-k]\}$ is orthonormal:
  - $\langle \delta[n-k], \delta[n-l] \rangle = 0, \forall k \neq l$
  - $\langle \delta[n-k], \delta[n-k] \rangle = 1, \forall k$

Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

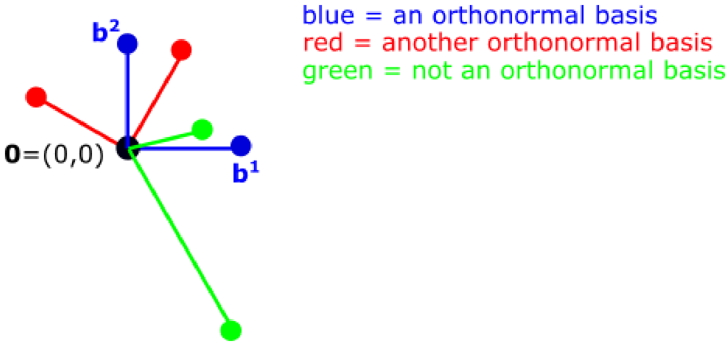


Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- Suppose we have an **orthonormal basis** $B = \{\vec{b}^i\}$
- Suppose we have a vector $\vec{x}$
- We can write (expand) $\vec{x}$ as:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \cdots + \alpha_N \vec{b}^N$$

- Question: how to **find** the coefficients α_i?

Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}
\langle \vec{x}, \vec{b}^i \rangle =& \langle \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \cdots + \alpha_N \vec{b}^N, \vec{b}^i \rangle \\
&= \langle \alpha_1 \vec{b}^1, \vec{b}^i \rangle + \langle \alpha_2 \vec{b}^2, \vec{b}^i \rangle + \cdots + \langle \alpha_N \vec{b}^N, \vec{b}^i \rangle \\
&= \alpha_1 \langle \vec{b}^1, \vec{b}^i \rangle + \alpha_2 \langle \vec{b}^2, \vec{b}^i \rangle + \cdots + \alpha_N \langle \vec{b}^N, \vec{b}^i \rangle \\
&= \alpha_i
\end{aligned}$$

Basis expansion of a vector

- Any vector $\vec{x}$ can be written as:

$$\vec{x} = \alpha_1 \vec{b}^1 + \alpha_2 \vec{b}^2 + \cdots + \alpha_N \vec{b}^N$$

- For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle, \vec{x}, \vec{b}^i \rangle$$

Why bases

- How does all this talk about bases help us?
- To better understand the Fourier transform
- The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- The Fourier Transform of a signal $x =$ finding the coefficients of $\vec{x}$ in this basis
- The Inverse Fourier Transform $=$ expanding $\vec{x}$ with the elements of this basis
- Same **generic** thing every time, only the type of signals differ

IV.2 Introducing the Fourier Transforms

Reminder

- Reminder:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos(x - \frac{\pi}{2})$$

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

Why sinusoidal signals

- Why are sinusoidal signals $\sin()$ and $\cos()$ **so prevalent** in signal processing?
- Answer: because they are combinations of an e^{jx} and an e^{-jx}
- Why are these e^{jx} so special?
- Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

- Consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} A e^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- Output signal = input signal $\times$ a (complex) constant $(H(\omega_0))$

Eigen-function

- **Eigen-function** of a system ("funcție proprie") = a function f which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

  - just like **eigen-vectors** of a matrix (remember algebra): $A\mathbb{v} = \lambda\mathbb{v}$
  - we call the "functions" to allow for continuous signals as well

- Complex exponential signals $e^{j\omega t}$ (or $e^{j\omega n}$) are **eigen-functions** of Linear and Time Invariant (LTI) systems:

  - output signal = input signal $\times$ a (complex) constant

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system very easily if we **decompose all signals $x[n]$ as a combination of $\{e^{j\omega n}\}$**
- Example: RGB color filter
  - suppose we have some photographic filters (lenses):
    - one reduces red to 50%
    - one reduces green to 25%
    - one reduces blue to 80%
    - RGB are eigen-functions of the system: input = 200 Blue, output = 0.8 * 200 Blue
    - what is the output color if input is "pink"?
    - Answer is easy if we represent all colors in RGB

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$
- All vector space theory becomes useful now:
  - $\{e^{j\omega n}\}$ is an **orthonormal basis**
  - decomposing signals $=$ finding coefficients α_i
  - we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

Discrete-Time Fourier Transform (DTFT)

- Consider the vector space of **non-periodic infinitely-long signals**
- This vector space is **infinite-dimensional**
- The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis**
- We can expand (almost) any $\vec{x}$ in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

Discrete-Time Fourier Transform (DTFT)

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- A signal $x[n]$ can be written as a linear combination of $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \vec{x}, e^{j2\pi fn} \rangle$

Discrete-Time Fourier Transform (DTFT)

- Alternative form with ω
- We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

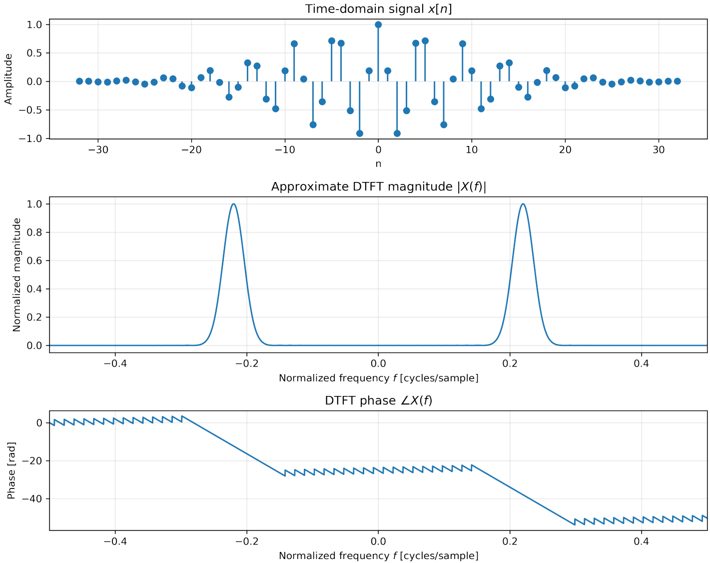
$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_{n} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform (DTFT)

- A non-periodic signal $x[n]$ has a **continuous spectrum** $X(\omega)$, with $f \in [-\frac{1}{2}, \frac{1}{2}]$
  - e.g. $\omega \in [-\pi, \pi]$

DTFT example

- A non-periodic, finite energy signal and its DTFT



Discrete Fourier Transform (DFT)

- Consider the vector space of **periodic** signals with **period N**
  - for some fixed $N = 2$, 3 or ... etc
- This is a vector space of **dimension N**
  - we need N numbers to identify a signal (specify its period)
- We can consider $x[n]$ only for **one period**, i.e. $n = 0, \dots N-1$
- The signals $\{e^{j2\pi fn}\}, \forall f \in \{0, \frac{1}{N}, \dots \frac{N-1}{N}\}$ form an **orthonormal basis** with N elements
- It is a **discrete** set of frequencies: $f = \frac{k}{N}, \forall k \in \{0, 1, \dots N-1\}$

Discrete Fourier Transform (DFT)

Inverse Discrete Fourier Transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- A periodic signal $x[n]$ can be written as a linear combination of k signals $\{e^{j2\pi kn/N}\}$, with some coefficients X_k

Discrete Fourier Transform

$$X_k = \langle x[n], e^{j2\pi kn/N}\rangle = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

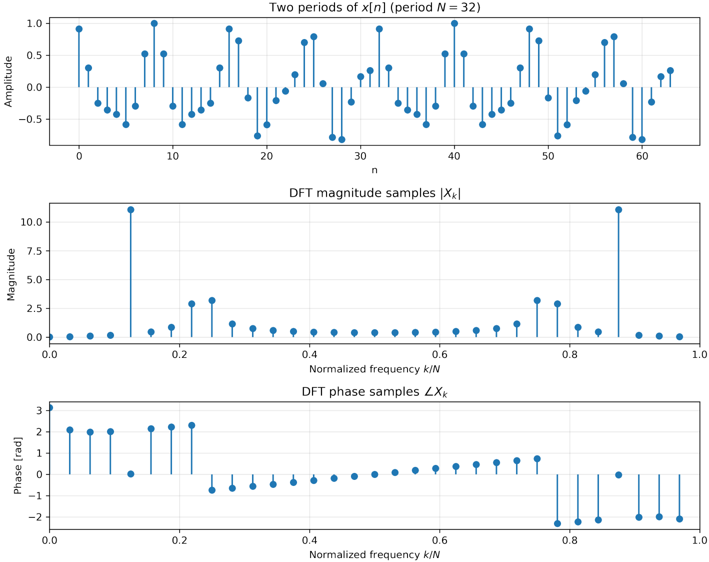
- Each coefficient X_k of every $\{e^{j2\pi kn/N}\}$ is found using the inner product $\langle \vec{x}, e^{j2\pi kn/N}\rangle$

Discrete Fourier Transform (DFT)

- A periodic signal $x[n]$ with period N has a **discrete spectrum** $X(\omega)$ composed of only N frequencies $\{0, \frac{1}{N} \dots \frac{N-1}{N}\}$
- Each frequency $\frac{k}{N}$ has a **coefficient** X_k
  - also written as c_k
  - The N coefficients X_k are the equivalent of $X(\omega)$
- It is also known as the "Fourier Series for Discrete Signals"

DFT example

- A periodic signal and its DFT coefficients



IV.3 The Discrete-Time Fourier Transform (DTFT)

Definition

Definitions (again):

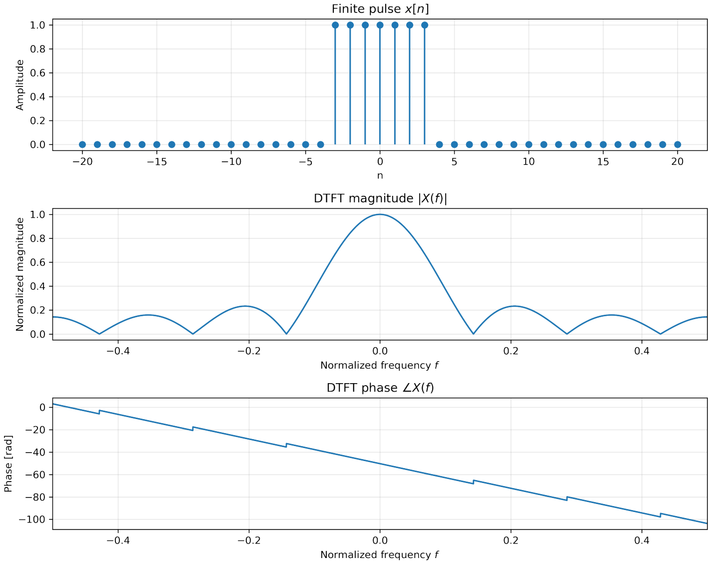
Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

DTFT example



- DTFT of a short rectangular sequence spreads energy over all

Basic properties of DTFT

- $X(\omega)$ is **defined** only for $\omega \in [-\pi, \pi]$
  - or $f \in [-\frac{1}{2}, \frac{1}{2}]$
- $X(\omega)$ is **complex** (has $|X(\omega)|$, $\angle X(\omega)$)
- If the signal $x[n]$ is real, $X(\omega)$ is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

  - This means:
    - modulus is even: $|X(\omega)| = |X(-\omega)|$
    - phase is odd: $\angle X(\omega) = -\angle X(-\omega)$

Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n} + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (X(\omega)e^{j\omega n} + X(-\omega)e^{j(-\omega)n})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|(e^{j\omega n+\angle X(\omega)} + e^{-j\omega n-\angle X(\omega)})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|\cos(\omega n + \angle X(\omega))d\omega
\end{aligned}
$$

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

Expressing as sum of sinusoids

- The DTFT shows that any signal $x[n]$ is **a "sum" of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
  - this is the fundamental practical interpretation of the Fourier transform
  - not really a sum, because we have an integral
- The **modulus** $|X(\omega)|$ gives the **amplitude** of the sinusoids ($\times$ 2)
  - for $\omega = 0$, $|X(\omega = 0)| =$ the DC component
- The **phase** $\angle X(\omega)$ gives the initial phase

Properties of DTFT

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

Properties of DTFT

2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

Properties of DTFT

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of DTFT

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- Not circular convolution, this is the normal convolution

Properties of DTFT

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of DTFT

Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega) X_2^{\star}(\omega)$$

Wiener Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Is true for all orthonormal bases

IV.4 The Discrete Fourier Transform (DFT)

Definitions

Definitions (again):

Inverse Discrete Fourier Transform (DFT)

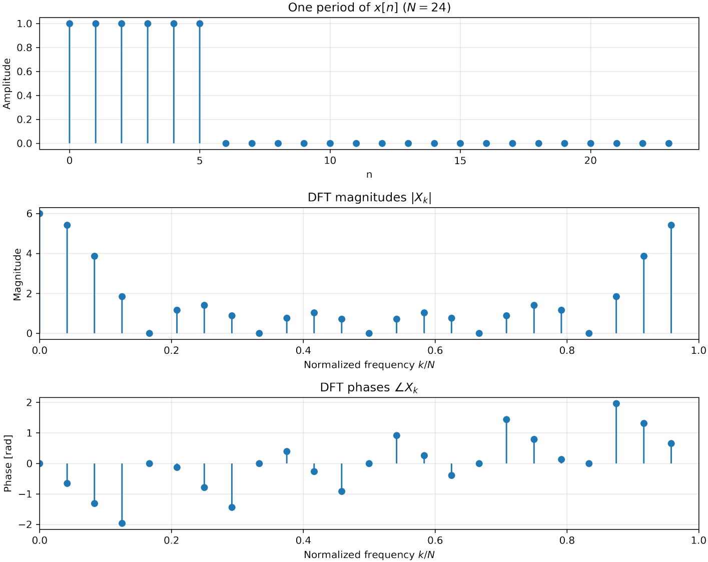
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Discrete Fourier Transform (DFT)

$$X_k = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- DFT is defined for periodical signals with period N
- there are exactly N terms in each sum

DFT example



- $N = 24$

Definitions

To remember:

- DFT: takes a vector with N elements ($x[n]$), produces a vector with N elements (X_k)
  - for this reason, we can compute it e.g. with Matlab
- DTFT: takes a vector with ∞ elements ($x[n]$), produces a continuous function ($X(\omega)$) between $[-\pi, \pi]$

Periodicity and notation

- DFT has only N coefficients X_k, each X_k corresponding to a frequency $f = \frac{k}{N}$
- In frequency domain, $f = \frac{N-k}{N} = \frac{-k}{N}$ (aliasing, we can always add/subtract 1 from f)
- So we can consider X_{N-k} as X_{-k}, due to periodicity
- Example: a signal with period $N = 6$ has 6 DFT coefficients
  - we can call them X_0, X_1, X_2, X_3, X_4, X_5
  - we have $X_5 = X_{-1}$, $X_4 = X_{-2}$
  - we can also call them X_{-2}, X_{-1}, X_0, X_1, X_2, X_3

Basic Properties of the DFT

- Has only N coefficients X_k
- X_k are **complex** (has $|X_k|$, $\angle X_k$)
- If the signal $x[n]$ is real, the coefficients are **even**

$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

  - This means:
    - modulus is even: $|X_k| = |X_{-k}|$
    - phase is odd: $\angle X_{-k} = -\angle X_k$

Expressing as sum of sinusoids, N = odd

- Grouping terms with k and $-k$:
- If N is odd, we have X_0 and pairs (X_k, X_{-k}):

$$
\begin{aligned}
x[n] &= \frac{1}{N}X_0 e^{j0n} + \frac{1}{N}\sum_{k=-(N-1)/2}^{-1} X_k e^{j2\pi kn/N} + \frac{1}{N}\sum_{k=1}^{(N-1)/2} X_k e^{j2\pi kn/} \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2} \left(X_k e^{j2\pi kn/N} + X_{-k} e^{-j2\pi kn/N}\right) \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=1}^{(N-1)/2} |X_k|\left(e^{j2\pi kn/N + \angle X(k)} + e^{-j2\pi kn/N - \angle X(\omega)}\right) \\
&= \frac{1}{N}X_0 + \frac{1}{N}\sum_{k=0}^{(N-1)/2} 2|X_k|\cos(2\pi k/Nn + \angle X_k)
\end{aligned}
$$

- A sum of sinusoids with frequencies up to $1/2$

Expressing as sum of sinusoids, N = even

- If N is even, we have X_0 and pairs $(X_k,\ X_{-k})$, with an extra term $X_{N/2}$ which has no pair
  - e.g. N = 6: $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$
- The term with no pair, $X_{N/2}$, must be a real number, because $X_{N/2} = X^*_{-N/2} = X^*_{N/2}$
- The extra term will be $\frac{1}{N} X_{N/2} e^{j2\pi N/2n/N} = X_{N/2}\cos(n\pi)$
- Overall:

$$x[n] = \frac{1}{N} X_0 + \frac{1}{N} \sum_{k=0}^{(N-2)/2} 2|X_k| \cos(2\pi k/Nn + \angle X_k) + \frac{1}{N} X_{N/2} \cos(n\pi$$

- A sum of sinusoids with frequencies up to 1/2

Expressing as sum of sinusoids

- DFT says that any periodic signal $x[n]$, with period N, is **a sum of N sinusoids with frequencies**:

  $$f = 0, \frac{1}{N}, \frac{2}{N}, ... \text{ up to } \frac{N-1}{2} \text{ or } \frac{N}{2}$$

  (not exceeding 1/2)
- The **modulus** $|X_k|$ gives the **amplitude** of the sinusoids (sometimes $\times$ 2)
  - for $\omega = 0$, $|X_0| =$ the DC component
  - when modulus = 0, that frequency has amplitude 0
- The **phase** $\angle X_k$ gives the initial phase

Example

- Consider a periodic signal $x[n]$ with period $N = 5$ and the DFT coefficients:

  $X_k =$ [21.0000 + 0.0000i , -3.0000 + 5.1962i , -3.0000 + 1.7321i , -3.0000 - 1.7321i , -3.0000 - 5.1962i]

  Write $x[n]$ as a sum of sinusoids.

- Do the same for a periodic signal $x[n]$ with period $N = 6$ and the DFT coefficients:

  $X_k =$ [15.0000 + 0.0000i , -2.5000 + 3.4410i , -2.5000 + 0.8123i , -2.5000 + 0.0000i , -2.5000 - 0.8123i , -2.5000 - 3.4410i]

  Write $x[n]$ as a sum of sinusoids.

Properties of the DFT

1. Linearity

If the signal $x_1[n]$ has the DFT coefficients $\{X_k^{(1)}\}$, and $x_2[n]$ has $\{X_k^{(2)}\}$, then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$$

Proof: via definition

Properties of the DFT

2. Shifting in time

If $x[n] \leftrightarrow \{X_k\}$, then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi k n_0/N)} X_k\}$$

Proof: via definition

- The amplitudes $|X_k|$ are not affected, shifting in time **affects only the phase**

Properties of the DFT

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{X^*_{-k}\}$$

Properties of the DFT

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

Circular convolution definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is also periodic with period N
- How it is computed: at the whiteboard; see solved exercises

Circular convolution

- We are in the vector space of **periodic signals** with period N
- Linear (e.g. normal) convolution produces a result which is longer periodic with period N
- Circular convolution takes two sequences of length N and produces another sequence of length N
  - each sequence is a period of a periodic signal
  - circular convolution = like a convolution of periodic signals

Properties of the DFT

6. Product in time

Product in time $\leftrightarrow$ circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

Properties of the DFT

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{0}^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum |X_k|^2$$

- Is true for all orthonormal bases

Plot / sketch DTFT and DFT of various signals

Let's plot / sketch DTFT and DFT of various signals

DTFT of:

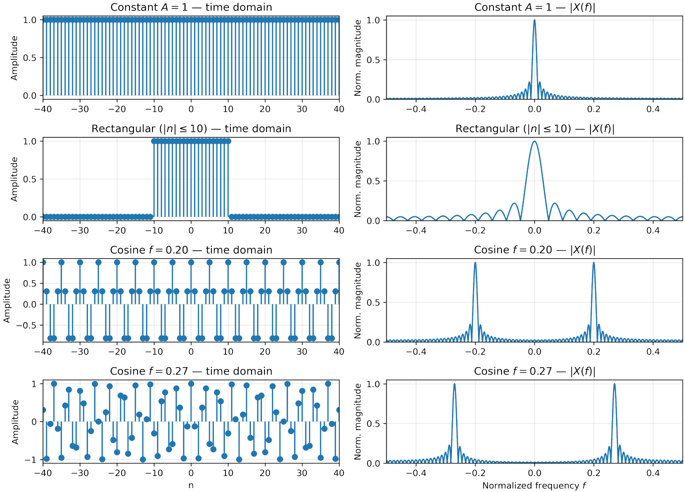
- a constant signal $x[n] = A$
- a rectangular signal $x[n] = A$ between $-\tau$ and τ, 0 elsewhere
- a cosine of frequency precisely $f = k/N$
- a cosine of frequency not $f = k/N$

DFT, with N=20, of:

- a constant signal $x[n] = A$
- a rectangular signal $x[n] = A$ between $-\tau$ and τ, 0 elsewhere
- a cosine of frequency precisely $f = k/N$
- a cosine of frequency not $f = k/N$

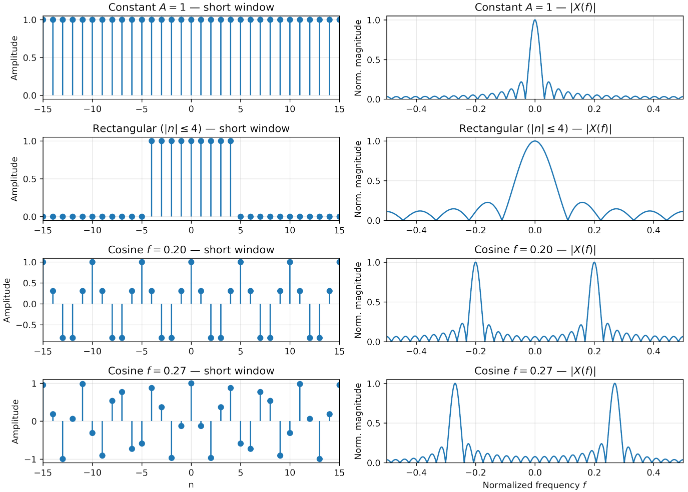
DTFT examples

- Low-frequency, high frequency signals
- Effect of window length?



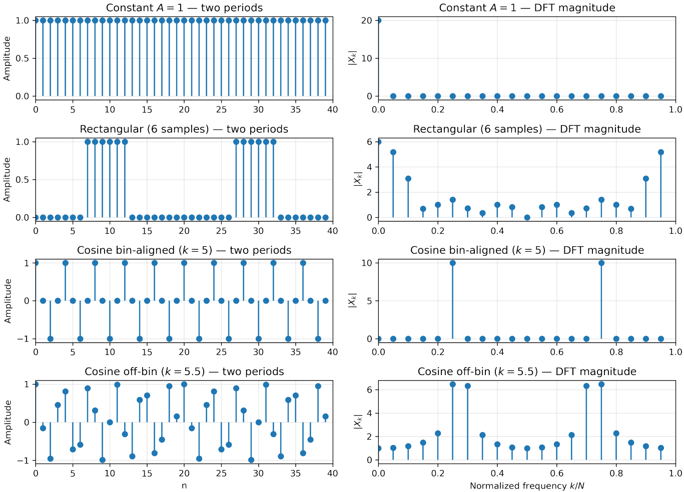
DTFT examples — shorter signals

- Shorter windows = wider main lobes. Longer windows = narrower main lobes. What is the problem with wide main lobes?



DFT examples

- Spectral leakage: what happens when the ends of the signal do not match?



Comments

- The rectangular window is responsible for the sinc-shaped DTFT and DFT.
- Wider windows = narrower main lobes (better), closer to an ideal Dirac delta in frequency.
- **Spectral leakage**: the DFT has a discrete spectrum, so it assumes that the signal is periodic. If the ends of your period do not match, we have discontinuities, which cause high frequency components even if they don't seem to be there in the original signal.

DFT matrix

- The DFT is equivalent to a matrix multiplication which maps an N-sample column vector $\mathbf{x}$ to another N-sample vector $\mathbf{X}$ with the DFT coefficients :

$$\mathbf{X} = \mathbf{W}_N \mathbf{x}, \qquad \mathbf{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

- The inverse transform IDFT uses the inverse matrix $\mathbf{W}_N^{-1}$, which happens to be the conjugate transpose of the same matrix, $\mathbf{W}_N^{-1} = \mathbf{W}_N^H$:

$$\mathbf{x} = \mathbf{W}_N^H \mathbf{X}$$

so the DFT/IDFT pair is completely described by the entries of $\mathbf{W}_N$.

DFT matrix (continued)

- The normalized DFT matrix is

$$\mathbf{W}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot(N-1)} \\ W_N^{1\cdot 0} & W_N^{1\cdot 1} & \cdots & W_N^{1\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & \cdots & W_N^{(N-1)\cdot(N-1)} \end{bmatrix},$$

- Everything is based on the element $W_N^{kn} = e^{-j2\pi \frac{k}{N} n}$ ($k =$ row index, $n =$ column index)
- The rows/columns are orthonormal, $\mathbf{W}_N^H \mathbf{W}_N = \mathbf{I}$, so $\mathbf{W}_N^{-1} = \mathbf{W}_N^H$
- Any other discrete transform (DCT, Walsh–Hadamard etc.) can be expressed with a similar matrix; in signal processing, any linear transformation is just matrix multiplication.
- There might be small variations depending on whether we have $\frac{1}{\sqrt{N}}$ at both DFT and IDFT, or just put $\frac{1}{N}$ just for the IDFT

DFT matrix multiplication

- Naive implementation of DFT, IDFT: use matrix multiplication with W, W^{-1}
- Number of multiplications necessary for a vector of length N is N^2
- In the world of algorithms, the **computational complexity** of an algorithm = number of multiplications necessary, depending on some variable N
  - only the dominant term matters, no coefficient, e.g $O(N^2)$ not $7.3N^2 + 4N$
- Naive DFT has computation complexity $\mathcal{O}(N^2)$
  - this is prohibitively large: when N increases 10x, computation increases 100x

FFT

- The Fast Fourier Transform (FFT) algorithm $=$ a fast algorithm for computing the DFT, exploiting the particular nature (symmetries) in the DFT matrix
- FFT computational complexity: $\mathcal{O}(N \log_2(N))$
- Exercise: for $N = 1024$, how much faster is FFT compared to naive DFT multiplication?
- Invention and adoption of FFT ($\sim$'60s, Cooley & Tukey) $=$ "the birth of Digital Signal Processing"

Other transforms

- In the world of discrete signals, there are many signal transforms possible, and many of them can be expressed as matrix multiplications, just like the DFT.
- Transform = expressing a N-dimensional vector x as a linear combination of a set of N basis vectors
- How:
  1. Put the N vectors of the basis as columns in a matrix A
  2. Solve the system $x = AX$ (inverse transform)
  3. Which means $X = A^{-1}x$ (forward transform)
- Why:
  - compression: the discrete cosine transform is the basis for lossy JPEG image compression
  - ...

Example

- Consider the exercise from Week 2:

$$x[n] = \{..., 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1, 1, 0, ...\}$$

  Write the expression of $x[n]$ based on the signal $u[n]$.

  Solve this in Matlab using a matrix approach

Solution

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

- Solve in Matlab:

```
x = [1/3, 2/3, 1, 1, 1, 1, 0]';
A = tril(ones(7,7));
a = linsolve(A, x)
```

Another example: Haar transform

- The Haar transform uses orthonormal basis vectors that are "square" waves. The first three Haar basis vectors ($N = 4$) are:
- Example: Find Haar transform of signal $x = [1/3, 2/3, 1, 1]$

$$\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

```
% Haar transform of a short signal
H = (1/2)*[ 1  1  1  0;
            1  1 -1  0;
            1 -1  0  1;
            1 -1  0 -1];
x = [1/3; 2/3; 1; 1];
c = linsolve(H, x)
```

Another example: JPEG

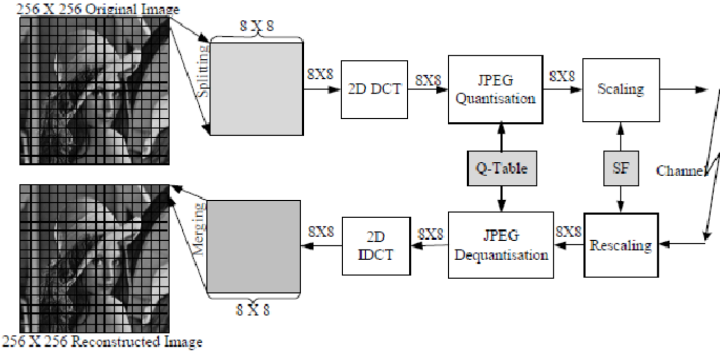


Figure 4: JPEG compression with DCT transform

- image from: JPEG Picture Compression Using Discrete Cosine Transform, N. K. More, S. Dubey, 2012

Another example: JPEG (cont'd)

- Each 8x8 image block is a vector in a 64 dimensional space
- Each 8x8 image block is decomposed into 64 basis vectors

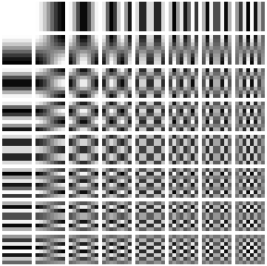


Figure 5: 8x8 DCT basis vectors

- Result: 64 coefficients, but many are small, negligible, quantizable $=>$ compression
- image from: Wikipedia

Relationship between DTFT and DFT

- How are DTFT and DFT related?
- Discrete Time Fourier Transform (DTFT):
  - for non-periodical signals
  - spectrum is continuous
- Discrete Fourier Transform (DFT):
  - for periodical signals
  - spectrum is discrete
- Duality: periodic in time $\leftrightarrow$ discrete in frequency
- The Diracs of the DFT are samples from the continuous DTFT of a single period of the signal

Relationship between DTFT and DFT

- Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

- If we consider a x surrounded by infinitely long zeros ($x[n]$ non-periodical), we have a continuous spectrum $X(\omega)$ (DTFT)

$$x = [...0, 6, 3, -4, 2, 0, 1, 2, 0, ...] \leftrightarrow X(\omega)$$

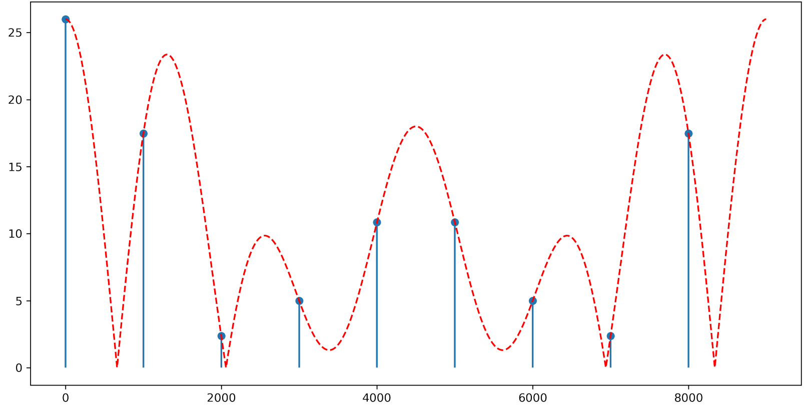
- If we consider that x is surrounded by repeating the sequences ($x[n]$ periodical), we have a discrete spectrum X_k (DFT)

$$x = [..., -4, 2, 0, 1, 2, 0, 6, 3, -4, 2, 0, 1, 2, 0, 6, 3, -4, ...] \leftrightarrow X_k$$

- The discrete X_k are just **samples from** $X(\omega)$, at frequencies k/N:

$$X_k = X(2\pi(k/N)n)$$

Relationship between DTFT and DFT



$$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$$

- red line = DTFT of x (assuming surrounded by zeros)
  - (actually run as `fft(x, 10000)`, x is extended with 9991 zeros)
- blue = DFT of x (assumes periodic)= `fft(x)`

Relationship between DTFT and DFT

- Consider a non-periodic signal $x[n]$
- It has a continuous spectrum $X(\omega)$
- If we **periodize** it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

- then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k\frac{2\pi}{N})$$

- The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k/Nn)$$

- They are **samples** from the continuous $X(\omega)$ of the non-periodized signal

Study case

```
n = 0:99;
f = 0.015;
x = cos(2*pi*f*n)
plot(abs(fft(x)))
```

Discuss:

1. Why is the spectrum not just 2 Diracs, like a normal `cos()`?
2. FFT assumes periodicity. Are there boundary problems?
3. What is the role of the rectangular window?
4. What happens if we run `fft(x, 10000)` instead of `fft(x)`?

Signal windowing and frequency resolution

- When you have finite-length cosine vector x, you have just a part of your signal
- The true signal x, infinitely long, is actually multiplied with a rectangular window $w[n]$

$$x = cos(2\pi f n) \cdot w[n]$$

- Multiplication in time = convolution in frequency

  The spectrum of $x = x[n] \cdot w[n]$ is Diracs * $W(\omega)$
- Instead of Dirac, you have $W(\omega)$'s:
  - wide peak
  - secondary lobes

Signal windowing and frequency resolution

- Working with a piece of a signal **always** distorts a signal
  - every Dirac is "smudged" into a $W(\omega)$
- This is unavoidable
- If we have a segment of a cosine, in the DTFT (continuous) we never see Diracs, but $W(\omega)$
  - The longer the piece, the better

Signal windowing and frequency resolution

- So what's the problem if we see $W(\omega)$ instead of Diracs?
- **Frequency resolution** = the ability to distinguish between closely spaced frequency components in a signal
- Having $W(\omega)$ instead of a Dirac is bad because it **masks** the surrounding region
  - a two close Diracs with similar height are be impossible to differentiate, because of the wide central lobe
  - a Dirac further away but smaller is impossible to differentiate, because of secondary lobes
- Analyzing a short segment of a signal leads to **low resolution in frequency**
- Analyzing a longer segment leads to **higher resolution in frequency**
- Frequency resolution is proportional to the length of the signal

Signal windowing and frequency resolution

- We can change the window $w[n]$
  - Recangular window
  - Hamming window
  - Hann window
  - ...
- What they do: trade narrow peak vs small secondary lobes
  - Rectangular window: widest peak, smallest secondary lobes
  - Other windows : narrower peak, higher secondary lobes
- What they do: attenuate endings, to reduce boundary problems

Signal windowing and frequency resolution

- Remember: every time we work with a piece of a signal (e.g. we process an audio file in pieces of 1024 samples), we are applying windowing
- Even the rectangular window is still a window
- If you need to compute the spectrum, know that is is affected
- Always consider replacing the rectangular window with another one, if you use `fft()` or other frequency-based operations

STFT and Spectrogram

- How to analyze the frequency of a signal whose frequency components change in time (e.g. like a musical song)?
- Short-Time Fourier Transform (STFT) = a technique for analyzing the frequency content of local sections of a signal as it changes over time.
- STFT divides a longer time signal into shorter segments of equal length and then computes the Fourier Transform separately on each short segment.
  - Split the signal into pieces (e.g. 1024-samples long)
  - Compute the spectrum of every piece (e.g. `fft()`)
  - Display the resulting sequence of spectra = "spectrogram"

STFT and Spectrogram

- The STFT is a **time-frequency representation** of a signal
- 2-Dimesional: time and frequency
- Examples: https://en.wikipedia.org/wiki/Spectrogram

STFT: Time and frequency resolution

- Imagine you look at the spectrogram of a music piece, and you want to pinpoint the moment where the bass guitar starts to play a chord of 100Hz
- Do a STFT and look for the moment where you see a high spectrum atoun 100Hz
- If the segments are short:
  - good time resolution
  - poor frequency resolution
- If the segments are long:
  - poor time resolution
  - high frequency resolution

STFT: Time and frequency resolution

Time-frequency Uncertainty Principle:

- you cannot have very good time resolution and very good frequency resolution simultaneously

STFT: other issues

- Other issues with STFT
  - can change the window type, to alleviate boundary problems / artefacts
  - allow some overlap between segments (e.g. 10%)

How to compute the DTFT

- The DFT is computed with `fft(x)`
- How to compute the DTFT?
- You can't. You need to surround x with infinitely long zeros
- You can only surround it with many zeros, but still finite
- Do this with `fft(x, 100000)` (DFT in N=100000 points)
  - x is surrounded with zeros unil total length $= 100000$
  - it's still just `fft()`, so DFT, so you have points and not the full continuous function
  - but it's many many points

How to compute the DTFT

- Computing the `fft()` in N=100000 points is unrelated with frequency resolution
- Frequency resolution is dependent on actual length of x
- Windowing changes the Diracs into $W(\omega)$'s
- `fft()` in N points is just taking N points from the resulting continuous spectrum

Relation between DTFT and Z transform

- Z transform:

$$X(z) = \sum_n x[n]z^{-n}$$

- DTFT:

$$X(\omega) = \sum_n x[n]e^{-j\omega n}$$

- DTFT can be obtained from Z transform with

$$z = e^{j\omega}$$

- These $z = e^{j\omega}$ are **points on the unit circle**
  - $|z| = |e^{j\omega}| = 1(modulus)$
  - $\angle z = \angle e^{j\omega} = \omega(phase)$

Relation between DTFT and Z transform

- Fourier transform = Z transform evaluated **on the unit circle**
  - if the unit circle is in the convergence region of Z transform
  - otherwise, equivalence does not hold
- This is true for most usual signals we work with
  - some details and discussions are skipped

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N$$

Geometric interpretation of Fourier transform

- For complex numbers:
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

Geometric interpretation of Fourier transform

- Consequences:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples: ...

Geometric interpretation of Fourier transform

- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is like **a landscape**
    - **poles = mountains** of infinite height
    - **zeros = valleys** of zero height
  - Fourier transform $X(\omega)$ = "*Walking over this landscape along the unit circle*"
  - The height profile of the walk gives the amplitude of the Fourier transform
  - When close to a mountain –> road is high –> Fourier transform has large amplitude
  - When close to a valley –> road is low –> Fourier transform has small amplitude

Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for phase as well

Time-frequency duality

- **Duality** properties related to all Fourier transforms
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Terminology

- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals
- **Band-limited** signals: spectrum is 0 beyond some frequency f_{max}
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: $B << $ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band