Digital Signal Processing

II. Discrete signals and systems

II.1 Discrete signals

Representation

A discrete signal can be represented:

- graphically
- in table form
- as a vector: $x[n] = [..., 0, 0, 1, 3, 4, 5, 0, ...]$
  - an **arrow** indicates the origin of time ($n = 0$).
  - if the arrow is missing, the origin of time is at the first element
  - the dots ... indicate that the value remains the same from that point onwards

Examples: at blackboard

Notation: $x[4]$ represents the value of the fourth sample in the signal $x[n]$

Basic signals

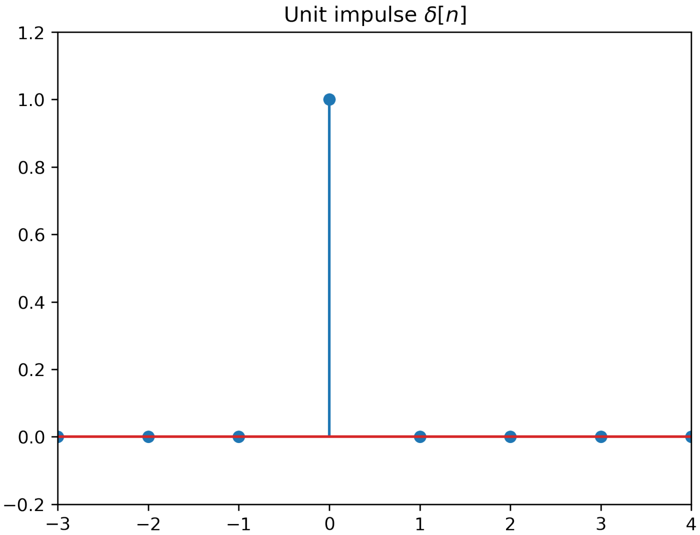
Some elementary signals are presented below.

Unit impulse

Contains a single non-zero value of 1 located at time 0. It is denoted with $\delta[n]$.

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation

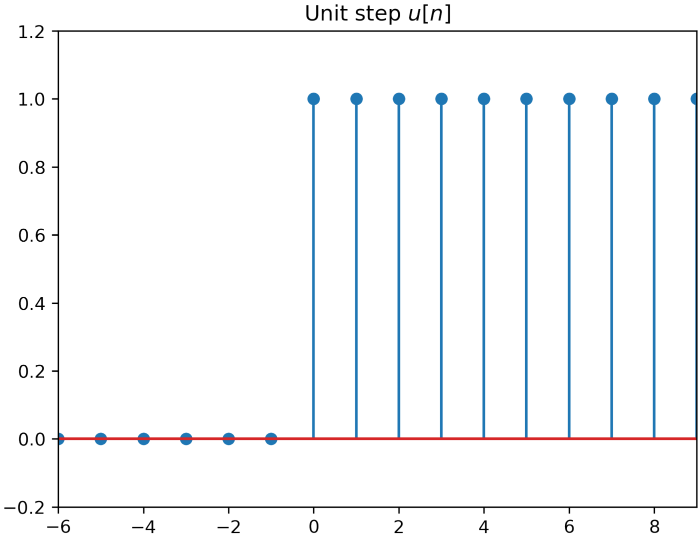


Unit step

It is denoted with $u[n]$.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation

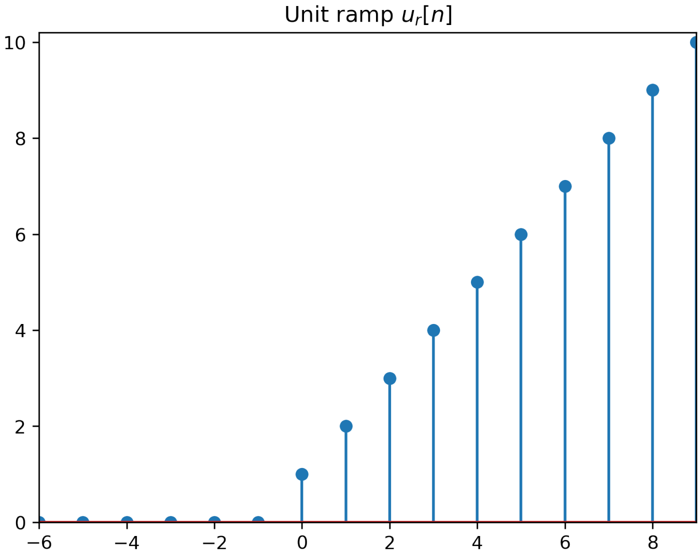


Unit ramp

It is denoted with $u_r[n]$.

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation



Exponential signal

Exponential signal

It does not have a special notation. It is defined by:

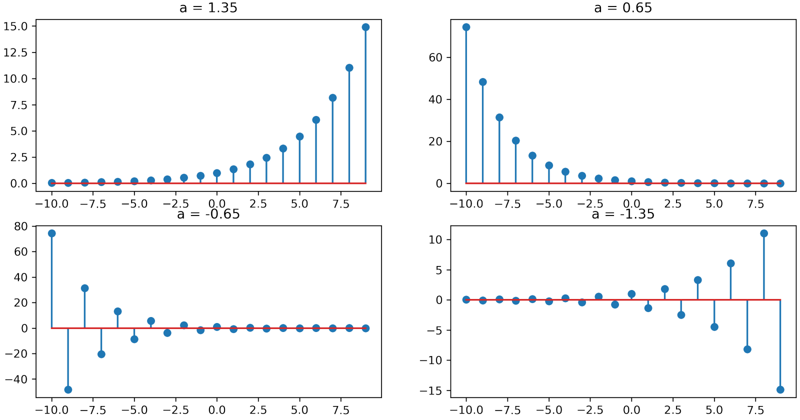
$$x[n] = a^n.$$

a can be a real or a complex number. Here we consider only the case when a is real.

Depending on the value of a, we have four possible cases:

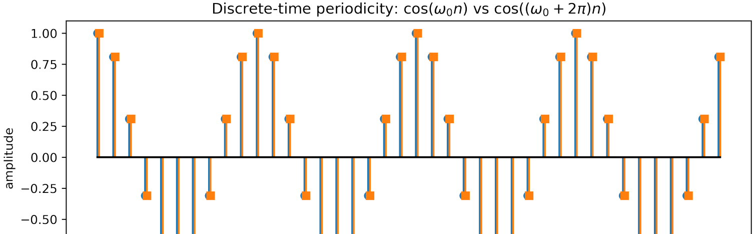
1. $a \geq 1$
2. $0 \leq a < 1$
3. $-1 < a < 0$
4. $a \leq -1$

Representation



Discrete-time sinusoids

- A real sinusoid is defined as
$$x[n] = A\cos(\omega_0 n + \varphi),$$
where ω_0 is the normalized rad/sample frequency and φ the phase.
- Complex exponentials and sinusoids are tightly related:
$$e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n).$$
- Periodicity in discrete time: $e^{j(\omega_0+2\pi k)n} = e^{j\omega_0 n}$, thus frequencies differing by multiples of 2π are indistinguishable. The sinusoid is periodic iff there exist integers N_0, m such that $\omega_0 N_0 = 2\pi m$.



II.2 Types of discrete signals

Signals with finite energy

- The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

- If E is finite, the signal is said to have finite energy.
- Examples:
  - unit impulse has finite energy
  - unit step does not

Connection with DEDP class

- Cross-link with DEDP course:

$$E = \|\mathbf{x} - \mathbf{0}\|^2 = \|\mathbf{x}\|^2$$

- Energy of a signal = **squared Euclidean distance to 0**
  - geometric interpretation: squared length of the segment from 0 to the point $\mathbf{x}$
  - holds for continuous signals as well:

$$E = \|\mathbf{x}\|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Signals with finite power

- The **average power of a discrete signal** is defined as

$$P = \lim_{N\to\infty} \frac{\sum_{n=-N}^{N} |x[n]|^2}{2N+1}.$$

- In other words, the average power is the average energy per sample.
- If P is finite, the signal is said to have finite power.
- Every finite-energy (square-summable) signal has zero average power, $P = 0$. A signal with infinite energy can have finite (e.g., periodic signals) or infinite power.
- Example: unit step has finite power $P = \frac{1}{2}$ under the symmetric averaging definition (proof at blackboard).

Power/energy examples

- $\delta[n]$: $E = 1$, $P = 0$.
- $\cos(\omega_0 n)$: $E = \infty$, $P = \frac{1}{2}$.

Estimated P[delta] $\sim$ 0.000, P[cos] $\sim$ 0.500

Connection with DEDP class

- Average power = temporal average squared value $\overline{X^2}$
  - i.e. average value of the square of samples

Periodic and non-periodic signals

- A signal is called **periodic** if its values repeat after N samples (the **period**):

$$x[n] = x[n+N], \forall n.$$

- The **fundamental period** of a signal is the minimum value of N.
- Periodic signals have infinite energy, and finite power equal to the power of a single period.

Even and odd signals

- A real signal is **even** if it satisfies the following symmetry:

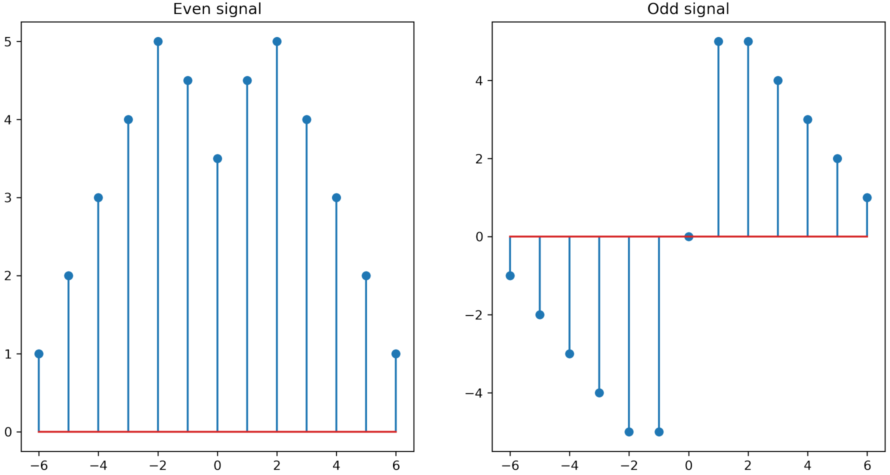
$$x[n] = x[-n], \forall n.$$

- A real signal is **odd** if it satisfies the following anti-symmetry:

$$-x[n] = x[-n], \forall n.$$

- There exist signals which are neither even nor odd.

Even and odd signals: example



Even and odd parts of a signal

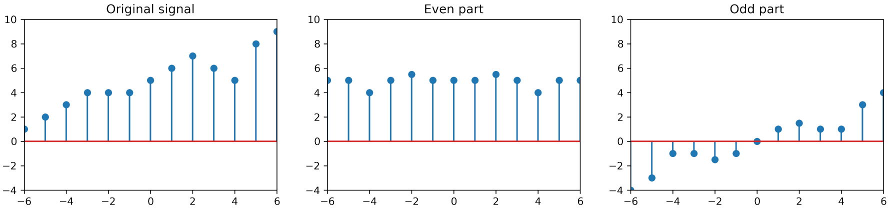
- Every signal can be written as the sum of an even signal and an odd signal:
$$x[n] = x_e[n] + x_o[n]$$
- The even and the odd parts of the signal can be found as follows:

$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

$$x_o[n] = \frac{x[n] - x[-n]}{2}.$$

- Proof: check that $x_e[n]$ is even, $x_o[n]$ is odd, and their sum is $x[n]$

Even and odd parts: example

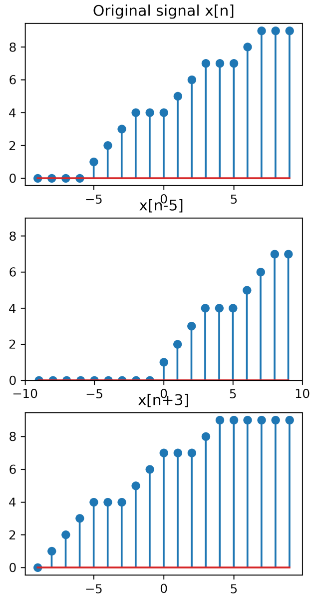


II.3 Basic operations with discrete signals

Time shifting

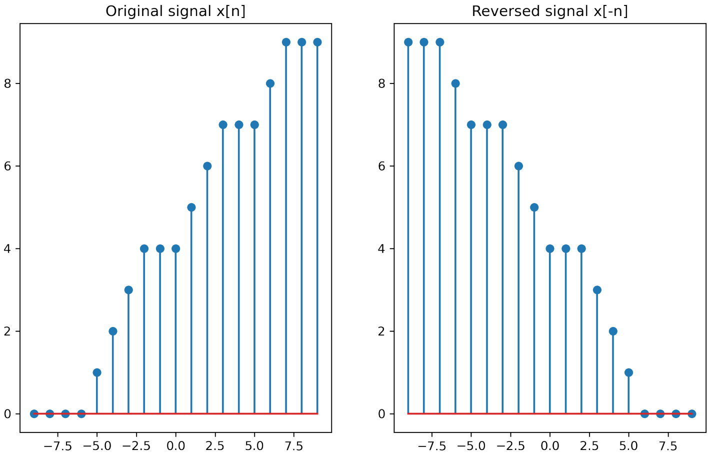
- The signal $x[n-k]$ is $x[n]$ **delayed with k time units**
  - Graphically, $x[n-k]$ is shifted k units to the **right** compared to the original
- The signal $x[n+k]$ is $x[n]$ **anticipated with k time units**
  - Graphically, $x[n+k]$ is shifted k units to the **left** compared to the original signal.

Time shifting: representation



Time reversal

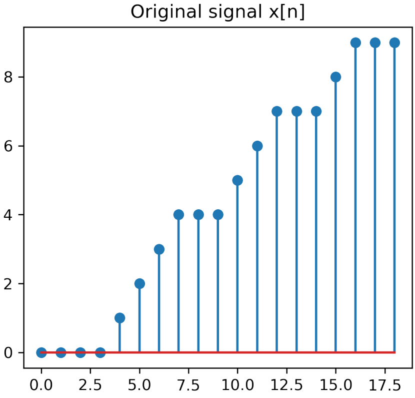
- Changing the variable n to $-n$ produces a signal $x[-n]$ which mirrors $x[n]$.

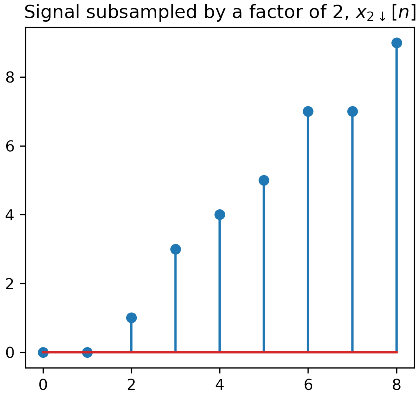


Subsampling

- **Subsampling** by a factor M: keep only one sample out of every M samples of the original signal.
  - Total number of samples is reduced M times

$$x_{M\downarrow}[n] = x[Mn]$$

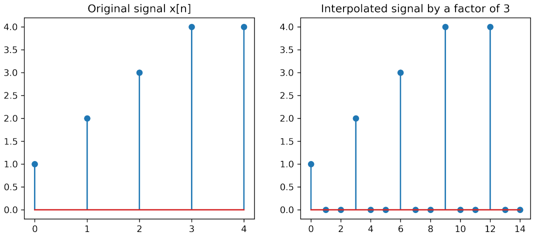




Interpolation

- **Interpolation** by a factor of L adds $(L-1)$ zeros between two samples in the original signal
  - Total number of samples increases L times

$$x_{L\uparrow} = \begin{cases} x[\frac{n}{L}] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$



- Note: Subsampling can introduce aliasing. In practice, apply an anti-alias low-pass filter before downsampling.

Mathematical operations

- A signal $x[n]$ can be **scaled** by a constant A, i.e. each sample is multiplied by A:

$$y[n] = Ax[n].$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **multiplied** by multiplying the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

II.4 Discrete systems

Definition

- **System** = a device or algorithm which produces an **output signal** based on an **input signal**
- We will only consider systems with a single input and a single output
- Figure here: blackboard.
- Common notation:
  - x[n] is the input
  - y[n] is the output
  - H is the system.

Notations

- Notations:

$$y[n] = H\left[x[n]\right]$$

("*the system H applied to the input x[n] produces the output y[n]*")

$$x[n] \stackrel{H}{\rightarrow} y[n]$$

("*the input* $x[n]$ *is transformed by the system H into y[n]*")

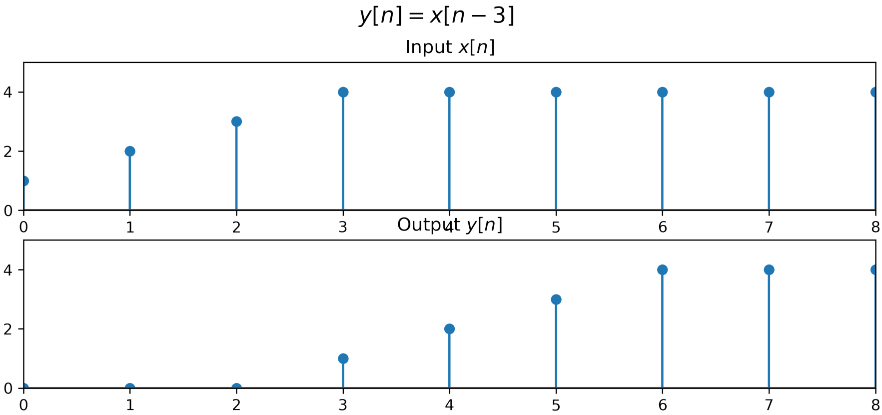
Equations

- Usually, a system is described by the **input-output equation** (or **difference equation**) which explains how $y[n]$ is defined in terms of $x[n]$.

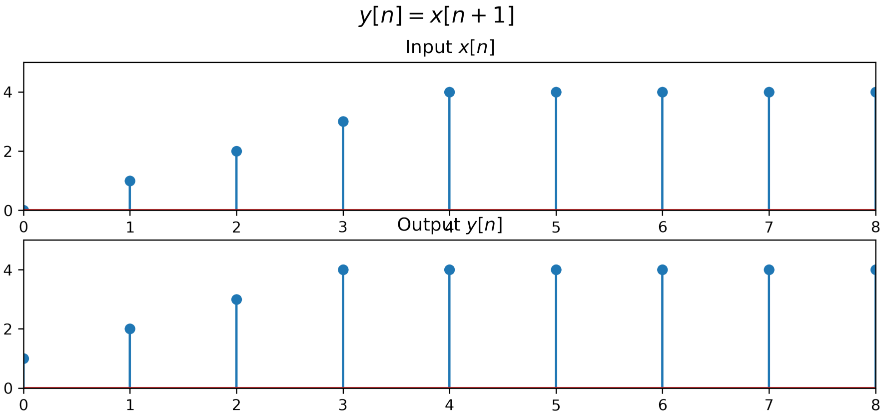
Examples:

1. $y[n] = x[n]$ (the identity system)
2. $y[n] = x[n-3]$
3. $y[n] = x[n+1]$
4. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
5. $y[n] = \max(x[n+1], x[n], x[n-1])$
6. $y[n] = (x[n])^2 + \log_{10} x[n-1]$
7. $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$

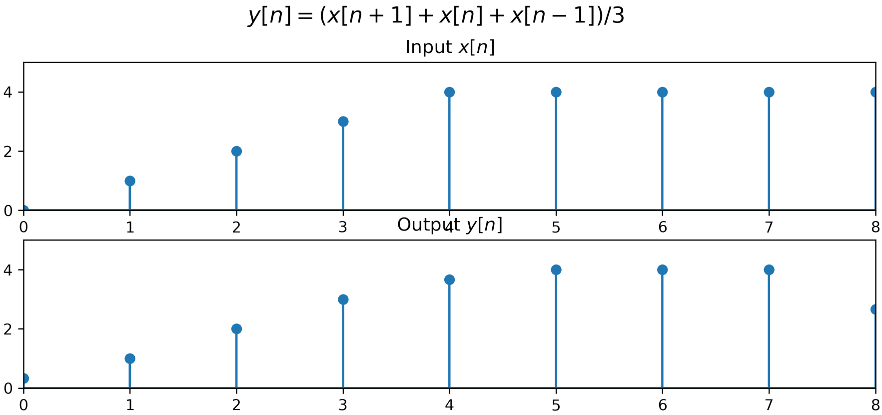
Example



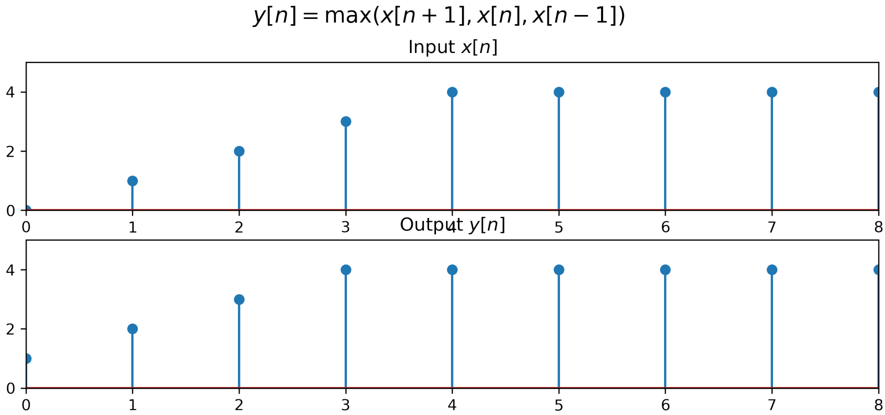
Example



Example

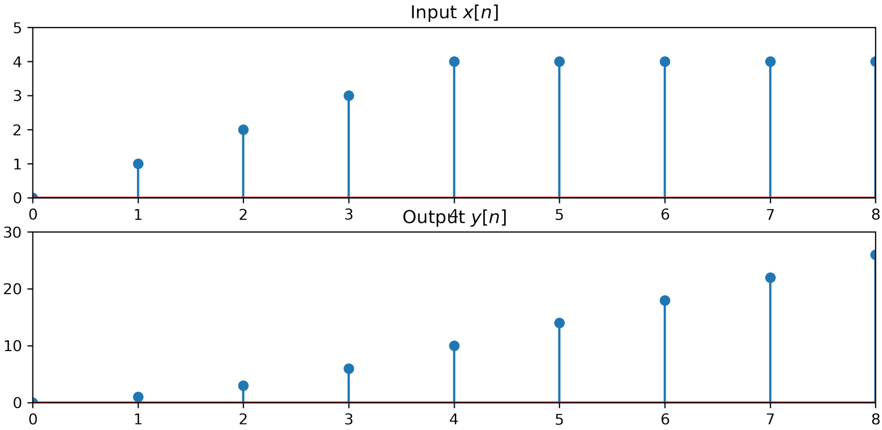


Example



Example

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$$



Recursive systems

- Some systems can/must be written in **recursive form**

$$y[n] = y[n-1] + x[n],$$

- Must always specify **initial conditions**
  - i.e. initial value (e.g. $y[-1] = 2.5$)
  - if not mentioned, assume they are 0 ("relaxed system")
  - they represent the internal state of the system at the starting moment
- For recursive systems, the output signal depends on both the input signal **and** on the initial conditions
  - different initial conditions lead to different outputs, even if input signal is the same
  - a recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ($x[n] = 0$)

Representation of systems

- The operation of a system can be described graphically (see examples on blackboard):
  - summation of two signals
  - scaling of a signal with a constant
  - multiplication of two signals
  - delay element
  - anticipation element
  - other blocks for more complicated math operations

II.5 Classification of discrete systems

Memoryless / systems with memory

- **Memoryless (or static)**: output at time n depends only on the input **from the same moment** n
- Otherwise, the system **has memory (dynamic)**
- Examples:
  - memoryless: $y[n] = (x[n])^3 + 5$
  - with memory: $y[n] = (x[n])^3 + x[n-1]$

Memoryless / systems with memory

- Memory of size N:
  - output at time n $y[n]$ depends only up to the last N inputs, $x[n-N], x[n-(N-1)], ...x[n]$,
  - if N is finite: the system has **finite memory**
  - if $N = \infty$, the system has **infinite memory**
- Examples:
  - finite memory of order 4: $y[n] = x[n] + x[n-2] + x[n-4]$
  - infinite memory: $y[n] = 0.5y[n-1] + 0.8x[n]$
    - recursive systems usually have infinite memory

Memoryless / systems with memory

- An input sample has an effect on the output only for the next N time moments
- For systems with infinite memory, any sample influences **all** the following samples, forever
  - but, if system is stable, the influence gets smaller and smaller

Time-Invariant and Time-Variant systems

- A relaxed system H is **time-invariant** if and only if:

  $$x[n] \xrightarrow{H} y[n]$$

  implies, $\forall x[n], \forall k$, that

  $$x[n-k] \xrightarrow{H} y[n-k]$$

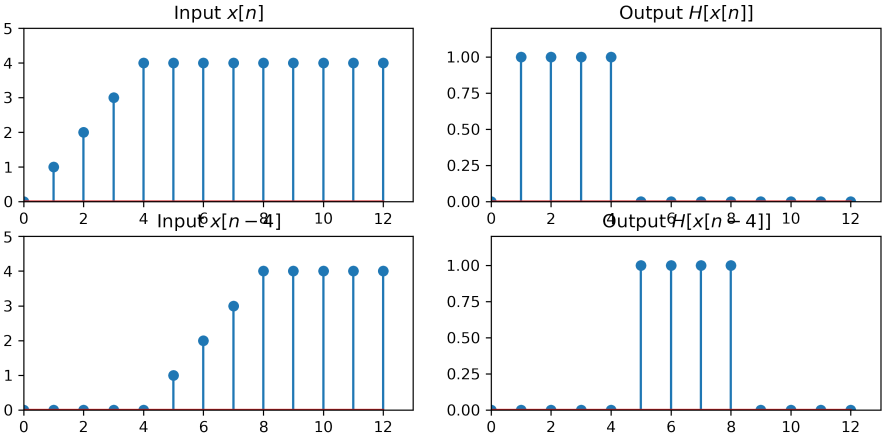
- Delaying the input signal with k will only delay the output with the same amount, otherwise the output is not affected
  - Must be true for all input signals, for all possible delays (positive or negative)
- Otherwise, the system is said to be **time-variant**

Time-Invariant and Time-Variant systems

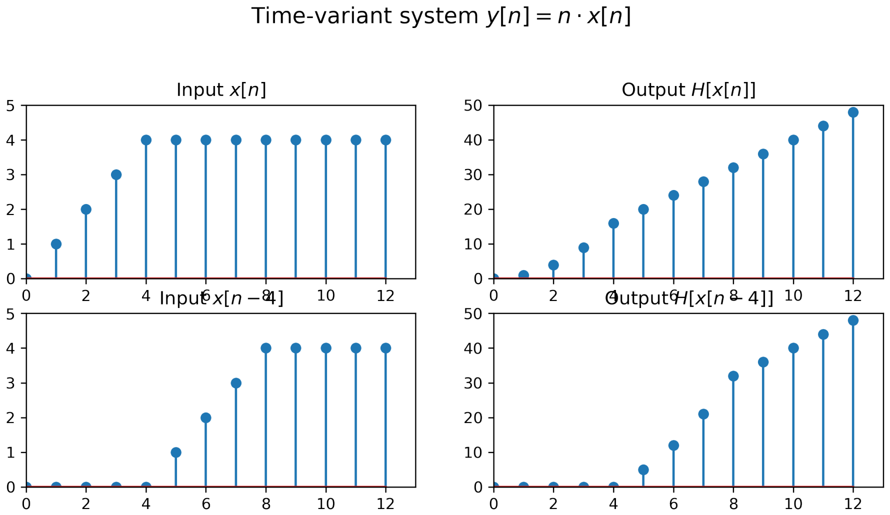
- Examples:
  - $y[n] = x[n] - x[n-1]$ is time-invariant
  - $y[n] = n \cdot x[n]$ is not time-invariant
- A system is time-invariant if it depends on n only through the input signal $x[n]$

Example

Time-invariant system $y[n] = x[n] - x[n-1]$



Another example



Linear and nonlinear systems

- A system H is **linear** if:

$$H\left[ax_1[n] + bx_2[n]\right] = aH\left[x_1[n]\right] + bH\left[x_2[n]\right].$$

- Composed of two parts:
  - Applying the system to a sum of two signals = applying the system to each signal, and adding the results
  - Scaling the input signal with a constant a is the same as scaling the output signal with a
- The same relation will be true for a sum of many signals, not just two

Linear and nonlinear systems

- Advantage of linear systems
  - Complicated input signals can be decomposed into a sum of smaller parts
  - The system can be applied to each part independently
  - Then the results are added back
- Examples:
  - linear system: $y[n] = 3x[n] + 5x[n-2]$
  - nonlinear system: $y[n] = 3(x[n])^2 + 5x[n-2]$

Linear and nonlinear systems

- For a system to be linear, the input samples $x[n]$ must not undergo non-linear transformations.
- **The only transformations** of the input x[n] allowed to take place in a linear system are:
  - scaling (multiplication) with a constant
  - delaying
  - summing different delayed versions of the signal (not summing with a constant)

Causal and non-causal systems

- **Causal**: the output $y[n]$ depends only on the current input $x[n]$ and the past values $x[n-1]$, $x[n-2]\ldots$, but not on the future samples $x[n+1]$, $x[n+2]\ldots$
- Otherwise the system is **non-causal**.
- A causal system can operate in real-time
  - we need only the input samples from the past
  - non-causal systems need samples from the future
- Examples:
  - $y[n] = x[n] - x[n-1]$ is causal
  - $y[n] = x[n+1] - x[n-1]$ is non-causal
  - $y[n] = x[-n]$ is non-causal

Stable and unstable systems

- **Bounded** signal: if there exists a value M such that all the samples of the signal are smaller than M in absolute value

$$x[n] \in [-M, M]$$

$$|x[n]| \leq M$$

- **Stable** system: if for any bounded input signal it produces a bounded output signal
  - not necessarily with the same M
  - known as BIBO (Bounded Input –> Bounded Output)
- In other words: when the input signal has bounded values, the output signal does not go towards ∞ or $-\infty$.

Stable and unstable systems

- Examples:
  - $y[n] = (x[n])^3 - x[n+4]$ is stable
  - $y[n] = \frac{1}{x[n]-x[n-1]}$ is unstable
  - $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$ is unstable

Impulse response of Linear Time-Invariant (LTI) systems

Linear Time-Invariant (LTI) systems

- Notation: An **LTI** system (**L**inear **T**ime-**I**nvariant) is a system which is simultaneously **linear** and **time-invariant**.
- LTI systems have an equation like this:

$$\begin{aligned} y[n] = & - a_1 y[n-1] - a_2 y[n-2] - ... - a_N y[n-N] + \\ & + b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M] \\ = & - \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \end{aligned}$$

  - the above is for causal systems; non-causal can also have [n+k]

The impulse response

- **Impulse response** of a system = output (response) of when the input signal is the impulse $\delta[n]$:

$$h[n] = H(\delta[n])$$

- The impulse response of a LTI system **fully characterizes the system**:
  - based on $h[n]$ we can compute the response of the system to **any** input signal
  - all the properties of LTI systems can be described via characteristics of the impulse response

Signals are a sum of impulses

- Any signal $x[n]$ can be composed as **a sum of scaled and delayed impulses** $\delta[n]$.
- Example:
  $x[n] = \{3, 1, -5, 0, 2\} = 3\delta[n] + \delta[n-1] - 5\,\delta[n-2] + 0 \cdot \delta[n-3] + 2\,\delta[n-4]$
- In general
  $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$
  i.e. a sum of impulses $\delta[n]$, each one delayed with k and scaled with the corresponding value $x[k]$

Convolution

- The response of a LTI system to a sum of impulses, delayed with k and scaled with $x[k]$, **is a sum of impulse responses, delayed with** k **and scaled with** $x[k]$.

$$
\begin{aligned}
y[n] =& H\left(x[n]\right) \\
=& H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) \\
=& \sum_{k=-\infty}^{\infty} x[k]H\left(\delta[n-k]\right) \\
=& \sum_{k=-\infty}^{\infty} x[k]h[n-k].
\end{aligned}
$$

Convolution

- Convolution in short:
  - The input signal is composed of separate impulses
  - Each impulse will generate its own response (LTI)
  - Output signal is the sum of impulse responses, delayed and scaled
- Convolution only applies for LTI systems

Convolution

- This operation = the **convolution** of two signals $x[n]$ and $h[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The response of a LTI system to an input signal x[n] is **the convolution of x[n] with the system's impulse response h[n]**

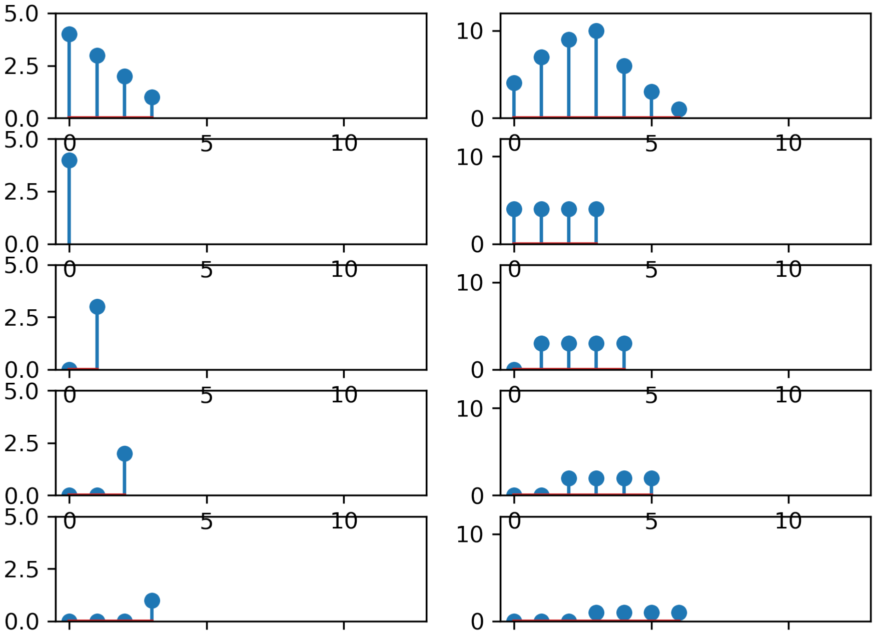
$$y[n] = x[n] * h[n]$$

Convolution

- Convolution is commutative: $x[n] * h[n] = h[n] * x[n]$
  - in equation it doesn't matter which signal has $[k]$ and which with $[n-k]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Example



Interpretation of the convolution equation

The convolution equation can be interpreted in three ways:

1. The output signal $y[n]$ = a sum of a lot of impulse responses $h[n]$, each one delayed by k (hence $[n-k]$) and scaled by $x[k]$
   - one for each sample in the input signal
   - explain at blackboard

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Interpretation of the convolution equation

2. Each output sample $y[n]$ = a **weighted sum** of the input samples around it

$$y[n] = ... + h[2] \cdot x[n-2] + h[1] \cdot x[n-1] + h[0] \cdot x[n] + h[-1] \cdot x[n+1] + ...$$

- If $h[n]$ has finite length (e.g. non-zero only between $h[-2]...h[2]$), then there are only a few terms in the sum
  - Example at blackboard

$$x[n] * h[n] = \mathbf{y[n]} = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Interpretation of the convolution equation

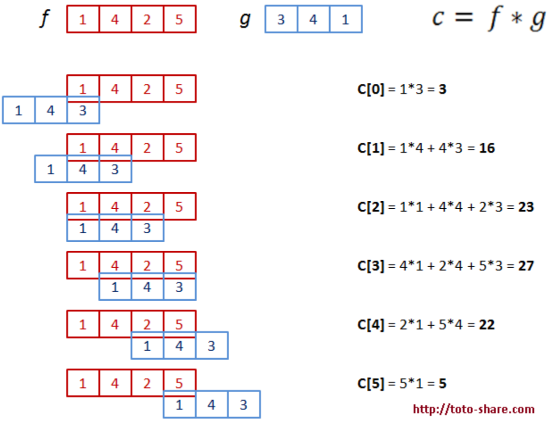


Figure 1: Convolution as weighted sum

- image from http://www.stokastik.in

Interpretation of the convolution equation

- Watch the following:

  https://www.youtube.com/watch?v=ulKbLD6BRJA

Example

The impulse response can be read directly from the system equation (for non-recursive systems):

- Suppose we have the system:

$$y[n] = 3x[n+1] + 5x[n] - 2x[n-1] + 4x[n-2]$$

- What is the impulse response of the system?
- Answer: $h[n] = \{...0, 3, \underset{\uparrow}{5}, -2, 4, 0, ...\}$

Convolution as matrix multiplication

- 3. Linear convolution can be written as multiplication with a **Toeplitz** matrix. Under periodic boundary conditions (circular convolution), the matrix becomes **circulant**.
  - in this example, assuming $h[n]$ is non-zero only from $h[-1]$ to $h[3]$

$$\begin{bmatrix} \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \end{bmatrix} = \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & \dots \\ \dots & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & \dots \\ \dots & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ x_{n+3} \\ \vdots \end{bmatrix}$$

2D convolution

- Convolution can be applied in 2D (for images)
- The input signal = an image $I[x, y]$
- The impulse response (the ***kernel***) = a matrix H[x, y]
- The convolution result:

$$Y[x, y] = I * H = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I[x - i, y - j] \cdot H[i, j]$$

2D convolution

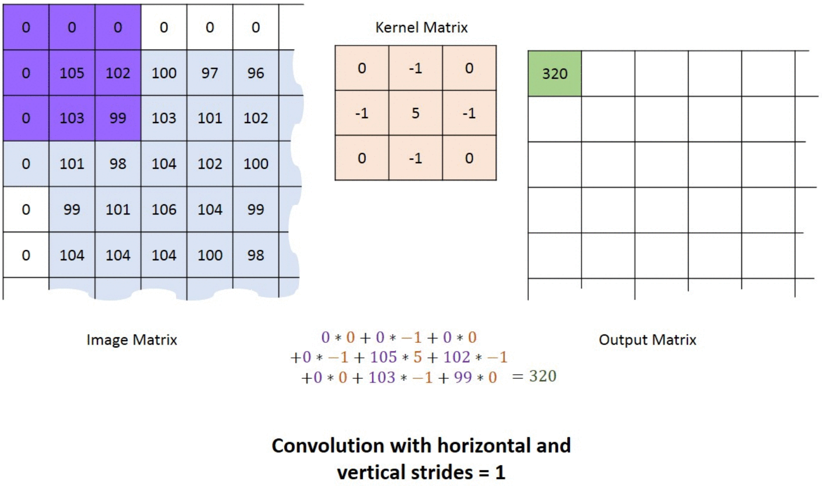


Figure 2: 2D Convolution as weighted sum

- image from http://machinelearningguru.com

2D Convolution

- Watch this:

  http://machinelearninguru.com/computer_vision/basics/convolution/cor

2D Convolution

- Simple image effects with 2D convolutions:
  - the “kernel” $=$ the impulse response $H[x, y]$
- See here: https://en.wikipedia.org/wiki/Kernel_(image_processing)
- What are their 1D counterparts?

Properties of convolution

Basic properties of convolution

- Convolution is **commutative** (the order of the signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

  - Proof: make variable change $(n-k) \rightarrow l$, change all in equation

- Convolution is **associative**:

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

  - (No proof)

Properties of convolution

- The unit impulse is **neutral element** for convolution:

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

.

  - Proof: equation

- Convolution is a **linear operation** (or **distributive**):

$$(\alpha \cdot a[n] + \beta \cdot b[n]) * c[n] = \alpha\,(a[n] * c[n]) + \beta\,(b[n] * c[n])$$

- Proof: by linearity of the corresponding system

Properties of LTI systems expressed with $h[n]$

1. Identity system

- A system with $h[n] = \delta[n]$ produces an response equal to the input, $y[n] = x[n], \forall x[n]$.
- Proof: $\delta[n]$ is neutral element for convolution.

Properties of LTI systems expressed with $h[n]$

2. Series connection is commutative

- LTI systems connected in series can be interchanged in any order
- Proof: by commutativity of convolution.
- LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * ... * h_N[n]$$

Properties of LTI systems expressed with $h[n]$

3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + ... + h_N[n]$$

Properties of LTI systems expressed with $h[n]$

4. Response of LTI systems to unit step

- If the input signal is $u[n]$, the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

.

Properties of LTI systems expressed with $h[n]$

- Proof:
  - The signal $\sum_{k=-\infty}^{n} h[k]$ is a *discrete-time integration* of $h[n]$
  - The unit step $u[n]$ itself is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

  - Therefore the system response to the integral of the impulse = the integral of the system response to the impulse
  - The interchanging of the integration with the system is due to the linearity of the system and is valid for all signals:

$$H\left(\sum_{k=-\infty}^{n} x[k]\right) = \sum_{k=-\infty}^{n} H\left(x[k]\right)$$

Relation between LTI system properties and $h[n]$

Relation between LTI system properties and $h[n]$

- For an LTI system, if we know $h[n]$, we know **everything** about the system
- Therefore, the properties (causal, memory, stability) must be reflected somehow in $h[n]$
  - Not linearity and time-invariance, they must be true, otherwise we wouldn't talk about $h[n]$

1. Causal LTI systems and their $h[n]$

If a LTI system is causal, then

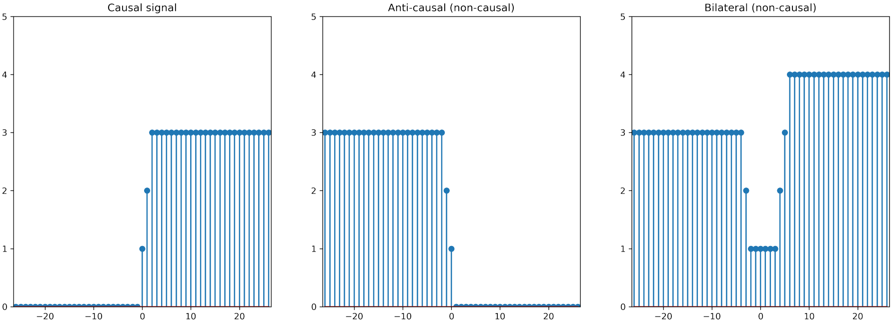
$$h[n] = 0, \forall n < 0$$

- Proof:
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$,
  - $y[n]$ does not depend on $x[n+1], x[n+2], \ldots$
  - it means that these terms are multiplied with 0
  - the value $x[n+1]$ is multiplied with $h[n-(n+1)] = h[-1]$, $x[n+2]$ is multiplied with $h[n-(n+2)] = h[-2]$, and so on
  - Therefore:
  $$h[n] = 0, \forall n < 0$$
  .

Causal signals and causal systems

- A **signal** which is 0 for $n < 0$ is called a **causal** signal
- Otherwise the signal is **non-causal**
- We can say that a **system** is causal if and only if it has a causal **impulse response**
- Further definitions:
  - a signal which is 0 for $n > 0$ is called an **anti-causal** signal
  - a signal which has non-zero values both for some $n > 0$ and for some $n < 0$ (and thus is neither causal nor non-causal) is called **bilateral**

Example



Causal signal
Anti-causal (non-causal)
Bilateral (non-causal)
0
1
2
3
4
5
−20
−10
0
10
20

2. Stable systems and their $h[n]$

- Considering a bounded input signal, $|x[n]| \leq A$, the absolute value of the output is:

$$
\begin{aligned}
|y[n]| &= \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \\
&\leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| \\
&= \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \\
&\leq A \sum_{k=-\infty}^{\infty} |h[n-k]|
\end{aligned}
$$

- Therefore **a LTI system is stable if**

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

.

3. Memoryless systems and their $h[n]$ (Exercise)

Exercises:

- What can we say about the impulse response $h[n]$ of a memoryless system?
- What about a system with finite memory M?

Hint/Answer:

- For an LTI memoryless system, $y[n] = c\,x[n]$, so $h[n] = c\,\delta[n]$.
- For an LTI system with finite memory M (causal), $h[n]$ has finite support (FIR), typically nonzero only for $n \in \{0, 1, \ldots, M\}$.

FIR and IIR systems

Support

- The **support** of a discrete signal = the smallest interval of n such that the signal is 0 everywhere outside the interval.
- Examples: at whiteboard
- Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.

FIR systems

- A **F**inite **I**mpulse **R**esponse (**FIR**) system has an impulse response with **finite support**
  - i.e. the impulse response is 0 outside a certain interval.
  - i.e. $h[n]$ is zero beyond some element $h[M]$
- The system equation for a FIR system:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = h[0]\cdot x[n] + h[1]\cdot x[n-1] + \dots h[M]\cdot x[n-M]$$

  - is non-recursive (depends only on x)
  - goes only up to some term $h[M]x[n-M]$
  - for causal system, starts from $h[0]x[n]$; for non-causal, can start from $h[-k]x[n+k]$

FIR systems

- For a causal FIR system, the output is a **linear combination** of **the last M+1 input samples**
- For non-causal FIR system, some future input samples enter the combination

IIR systems

- An **I**nfinite **I**mpulse **R**esponse (**IIR**) system has an impulse response with **infinite support**
  - i.e. the impulse response never becomes completely 0 forever.
- The output $y[n]$ potentially depends on all the preceding input samples
  - from the convolution equation:

$$\begin{aligned} y[n] &= \sum_{k=0}^{\infty} h[k]x[n-k] \\ &= h[0] \cdot x[n] + h[1] \cdot x[n-1] + \ldots h[M] \cdot x[n-M] + \ldots \textbf{goes on} + \ldots \end{aligned}$$

- An IIR system has infinite memory

IIR systems

- IIR systems must have **recursive** equations:
  - they depend on previous outputs $y[n-1]$ up to $y[n-N]$
  - they also depend on input, going back up to $x[n-k]$
- General equation of an IIR system:

$$\begin{aligned} y[n] = &- a_1 y[n-1] - a_2 y[n-2] - ... - a_N y[n-N] + \\ &+ b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M] \\ = &- \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \end{aligned}$$

  - the impulse response cannot be read explicitly from the equation
  - IIR equations are more general than FIR

General equation of an LTI system

Recap:

- The general equation of an LTI system is:

$$\begin{aligned} y[n] = & -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\ & + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \\ = & -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \end{aligned}$$

- If all $a_i = 0$, it is a FIR system, no $y[n-k]$ term
  - in this case the coefficients $b_k = h[k]$ (impulse response)
- If some $a_i \neq 0$, it is an IIR system
  - impulse response $h[n]$ is infinitely long, is more complicated to find
- Note: if system is non-causal, can start from $x[n+k]$

Initial conditions for recursive systems

- Recursive systems need **initial conditions** (starting values)
  - since they rely on previous outputs
- If initial conditions are all 0, the system is **relaxed**
  - the output depends only on the input signal
- If initial conditions are not zero, the output depends on the input signal **and** the initial conditions

Initial conditions for recursive systems

- The effect of the input signal and the effect of initial conditions are **independent**
  - the system behaves **linear** with respect to them
  - total output = output due to input + output due to initial conditions

Input	Init.Cond.	Output
x[n]	0	$y[n] = y_{zs}[n]$
0	non-zero	$y[n] = y_{zi}[n]$
x[n]	non-zero	$y[n] = y_{zs}[n] + y_{zi}[n]$

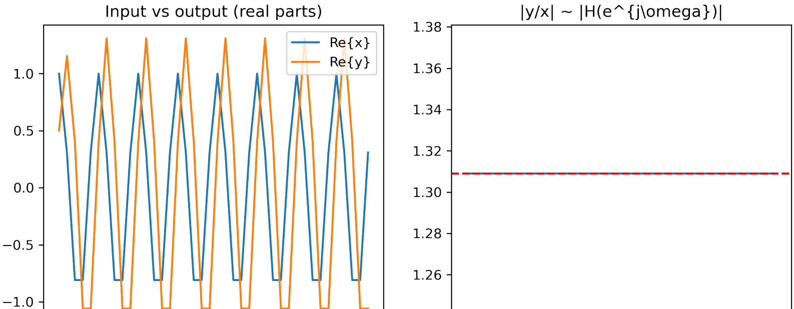
Complex exponentials as LTI eigenfunctions

Complex exponentials as LTI eigenfunctions

- For an LTI system with impulse response $h[n]$, a complex exponential input $x[n] = z^n$ produces an output

$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right)}_{H(z)} z^n.$$

Thus, $e^{j\omega n}$ is an eigenfunction with eigenvalue $H(e^{j\omega})$ (the frequency response).



Moving-average FIR example

Moving-average FIR example

- A 3-point moving average has $h[n] = \frac{1}{3}\{1, 1, 1\}$ and smooths noise.