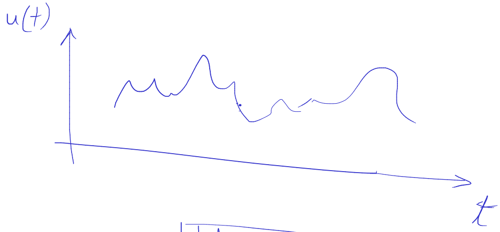
Digital Signal Processing

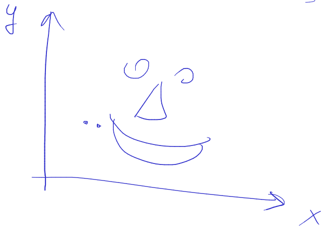
I. Sampling of analog signals

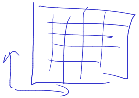
I.1. Analog and Digital Signals

Signals

- Signal = a measurable quantity which varies in time, space or some other variable
- Examples:
  - a voltage which varies in time (1D voltage signal)
  - sound pressure which varies in time (sound signal)
  - intensity of light which varies across a photo (2D image)
- Represented as a mathematical function, e.g. $v(t)$.







Glossary

- Glossary:
  - "e.g." = "*exampli gratia*" (lat.) = "for example" (eng.) = "de exemplu" (rom.)
  - "i.e." = "*id est*" (lat) = "that is" (eng.) = "adică" (rom.)

Signal dimension

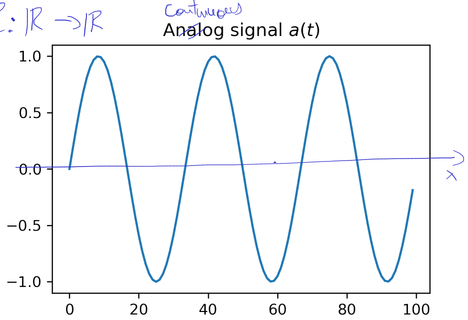
$f(t)$ $s(t)$

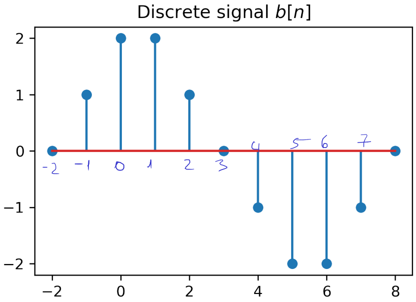
- **Unidimensional** (1D) signal = a function of a single variable
  - Example: a voltage signal $v(t)$ only varies in time.
- **Multidimensional** (2D, 3D ... M-D) signal = a function of a multiple variables
  - Example: intensity of a grayscale image $I(x, y)$ across the surface of a photo
- In these lectures we consider only 1D signals, but the theory is similar

Continuous and discrete signals

- Continuous (analog) signal = function of a continuous variable
  - Signal has a value for possible value of the variable in the defined range
  - The variable may be defined only in a certain range (e.g. $t \in [0, 100]$), but it is a compact range
- Discrete signal = function of a discrete variable
  - Signal has values only at certain discrete values (*samples*)
  - Indexed with natural numbers: $x[-1]$, $x[0]$, $x[1]$ etc.
  - Outside the samples, the signal is **not defined**

$f: \mathbb{R} \to \mathbb{R}$



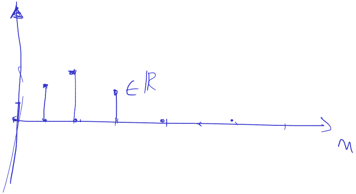


$f: \mathbb{Z} \to \mathbb{R}$

Notation

- We use the following notation:
- Continuous signal $x(t)$
  - Has **round parantheses**, e.g. $x_a(t)$
  - Sometimes has the a subscript
  - The variable is usually t (time)
  - $x(2.3)$ = the value of the signal $a(t)$ at $t = 2.3$
- Discrete signal
  - Has **square brackets**, e.g. $x[n]$ $\boxed{x[n]}$
  - The variables are denoted as n or k (suggest natural numbers)
  - $x[3]$ = the value of the signal $x[n]$ for $n = 3$
  - $x[1.5]$ = does not exist

Signals with continuous and discrete values



- Not only the time can be continuous or discrete
- The signal **values** can also be continuous or discrete
  - Example: signal values stored as 8-bit or 16-bit values
- On digital systems, signals always have discrete values due to finite number precision

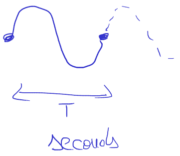
Discrete frequency

- A signal is **periodic** if the values repeat themselves after a certain time (**period**)
- Frequency = inverse of period
- Pulsation ω = 2 * π * frequency
- Continuous signals:
  - Periodic: $x_a(t) = x(t+T)$
  - T is usually measured in seconds (or some other unit)
  - $F = \frac{1}{T}$ is measured in Hz $= \frac{1}{s}$ (Hertz)
- Discrete signals:
  - Periodic: $x[n] = x[n+N]$
  - N **has no unit**, because it is just a number
  - $f = \frac{1}{N}$ **has no unit** also

period

$x(t) = x(t + T)$

$f = \frac{1}{T}$



$f = \frac{1}{T \text{ seconds}}$

$1\ Hz = \frac{1}{1 \text{ second}}$

$T = 0.1$ seconds

$f = \frac{1}{0.1 \text{ seconds}} = 10\ Hz$

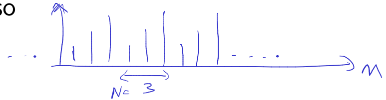
n just a number

$x[0]$, $x[1]$, $x[2]$

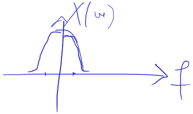
period has no unit

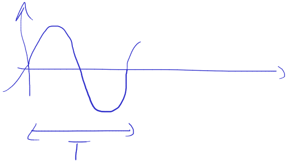
$x[n] = x[n + N]$

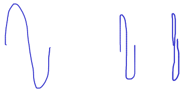
$f = \frac{1}{N} = \frac{1}{3}$



Frequency limits

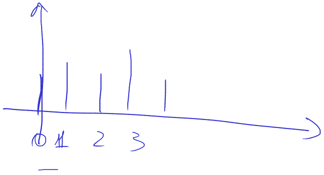




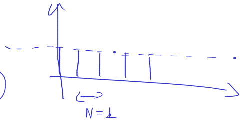


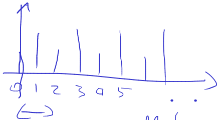
$T \to 0 \Rightarrow f = \frac{1}{T} \to \infty$

- For continuous signals, F can go to ∞
  - Because period T can be $T \to 0$
- For discrete signals, **largest frequency** is $f_{max} = \frac{1}{2}$
  - Smallest period is $N = 2$ (excluding $N = 1$, constant signals)
  - Consequence of using natural numbers to index the samples (x[0], x[1], x[2]...), without any physical unit attached
- For mathematical reasons, we will consider negative frequencies as well (remember SCS) (e.g. $-\omega$)



$N = 1 \Rightarrow \cancel{f = 1}$ $f = 0$





$N = 2$ smallest

$f_{max} = \frac{1}{2}$

$\cos(2\pi f n) = 1$ forever, $f = 0$

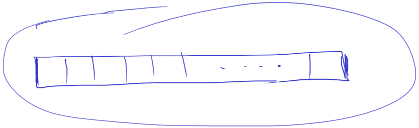
Domain of definition

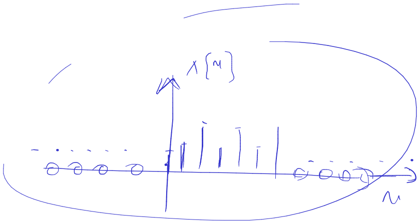
X{

x[n]

- **Finite-length** discrete signals $x[n]$:
  - have only a certain number N of samples (e.g. for $n = 0, 1, ... N-1)
  - they are not defined outside these samples
  - can be represented as a **vector** of numbers (e.g. like in Matlab, C)
- **Infinite-length** discrete signals $x[n]$:
  - e.g. defined for $n = ... -2, -1, 0, 1, 2, ...$ or

$x[n] = \cos(2\pi\, 0.5\, n), \quad n \in \mathbb{Z}$





Vector space of signals

- All signals of a certain length N form a **vector space**
- In mathematics, a vector space = a set V of elements $\{v\}$ (called "vectors") such that:
  - the sum of any two elements from V is still a member of V
  - any vector from V multiplied by a constant is still a member of V
- These properties can easily be verified for signals

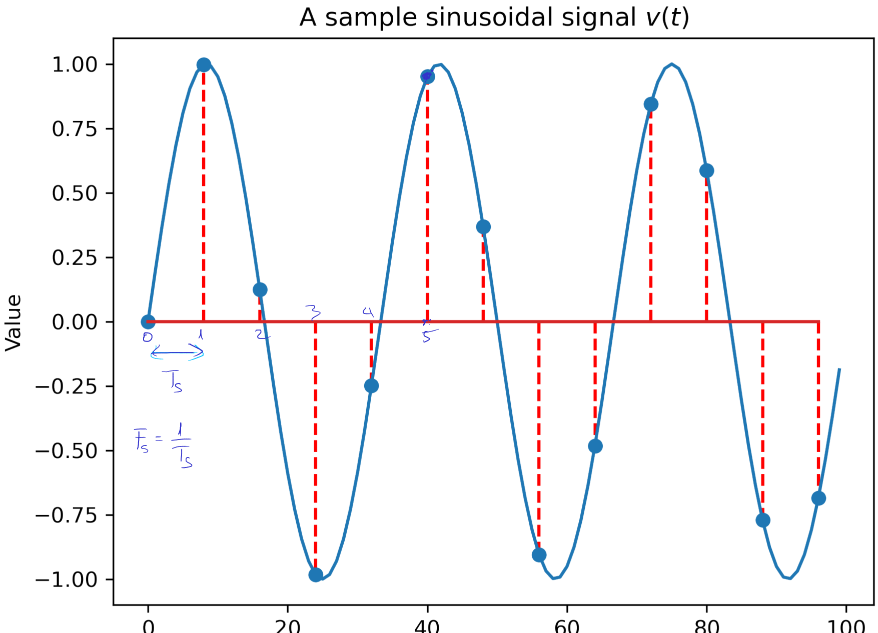
$$\begin{cases} x[n] + y[n] = z[n] \\ 2 \cdot x[n] = z[n] \end{cases}$$

I.2. Sampling

Sampling

- Sampling = Taking the values from an analog signal at certain discrete moments of time, usually periodic
- Distance between two samples = **sampling period** T_s
- **Sampling frequency** $F_s = \frac{1}{T_s}$
- Why sampling?
  - Converts continuous signals to discrete
  - Processing of continuous signals is expensive
  - Processing of discrete signals is cheap (digital devices)
  - Sometimes nothing is lost due to sampling

Graphical example



$x[5] = x_a(5 \cdot T_s)$

$\boxed{x[n] = x_a(n \cdot T_s)}$

Sampling equation

$x_a(t)$ = original signal (continuous)

- Mathematically, it is described by **the sampling equation**:

$$\boxed{x[n] = x_a(n \cdot T_s)}$$

- Produces a discrete signal $x[n]$ from a continuous signal $x_a(t)$
- The n-th value of the discrete signal $x[n]$ is the value of the analog signal $x_a(t)$ taken after n sampling periods, at time $n \cdot T_s$

Sampling of harmonic signals

- Let's sample a cosine: $x_a(t) = \overset{\sin}{cos}(2\pi F t)$

$$
\begin{aligned}
x[n] &= x_a(n T_s) \\
&= cos(2\pi F n T_s) \\
&= cos(2\pi F n \frac{1}{F_s}) \\
&= cos(2\pi \underbrace{\frac{F}{F_s}}_{f} n)
\end{aligned}
$$

$= \text{discrete frequency}$

$x_a(t) = 2 \cdot \cos(2\pi \cdot \overset{F}{1\,MHz} \cdot t)$

Sample with $F_s = 3\,MHz$

$x[n] = 2 \cdot \cos(2\pi \cdot \underset{f}{\frac{1}{3}} \cdot n)$

$f_{max} = \frac{1}{2}$

- Sampling a continuous cosine produces a discrete cosine with **discrete frequency**:

$$f = \frac{F}{F_s}$$

f has no unit

$f = \frac{1}{3} = \frac{1\,\cancel{MHz}}{3\,\cancel{MHz}}$

- Same for sine instead of cosine

Discrete frequency is relative

$$\boxed{f = \frac{F}{F_s}}$$

- Discrete frequency should be understood as a value **relative to the sampling frequency**
- Example: $\boxed{f = \frac{1}{4}}$ means "coming from an analog frequency F which was $\frac{1}{4}$ of the sampling frequency"
  - it could have been a 100Hz signal sampled with 400Hz
  - it could also have been a 3MHz signal sampled with 12MHz

$\Rightarrow f = 1/4$

$= f = 1/4$

$f = \frac{1}{2}$ = "half of the sampling freq. F_s"

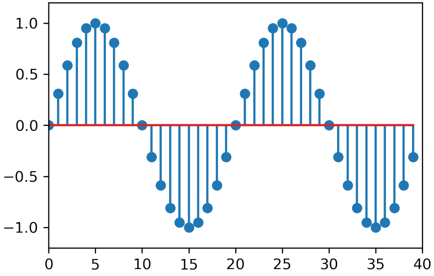
$f = \frac{1}{2} = \frac{F}{F_s}$

$f = \frac{1}{5}$ = "one fifth of the F_s"

False friends

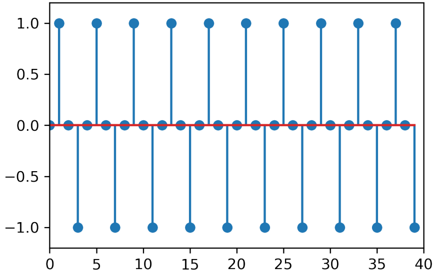
- **Note:** A discrete sinusoidal signal might not *look* sinusoidal, when its frequency is high (close to $\frac{1}{2}$).

Frequency $f = 0.05$



$\sin(2\cdot\pi\ 0.05\cdot n)$

Frequency $f = 0.25$



$\sin(2\cdot\pi\cdot 0.25\cdot n)$

Sampling theorem (Nyquist-Shannon)

The Nyquist-Shannon sampling theorem:

- If a signal $x_a(t)$ that has maximum frequency F_{max} is sampled with a a sampling frequency

$$F_s \geq 2F_{max},$$

then it can be perfectly reconstructed from its samples using the formula:

$$x_a(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot \frac{sin(\pi(F_s t - n))}{\pi(F_s t - n)}.$$

$F_s \geq 2 \cdot F_{max}$

$$\boxed{f \leq \frac{1}{2}}$$

Comments on the sampling theorem

- All the information in the original signal is contained in the samples, provided the sampling frequency is high enough
- It is much easier to process discrete samples instead of analog signals (e.g. using Matlab instead of capacitors :))
- Sampling with $F_s \geq 2F_{max}$ makes the discrete frequency smaller than 1/2

$$f = \frac{F}{F_s} \leq \frac{F_{max}}{F_s} \leq \frac{1}{2}$$

Example of the sampling theorem in action

Sampling theorem in action:

- Humans can only hear sounds up to ~20kHz
- Use sampling rates higher than 40kHz => no quality loss
  - Standardized for CD-Audio: 44100Hz

$F_s = 44.100$ kHz

Aliasing

- http://www.dictionary.com/browse/alias:
  - "alias": a false name used to conceal one's identity; an assumed name
- What happens when the sampling frequency is not high enough?
- Example: $F = 600Hz$ sampled with $F_s = 1000Hz$

$$
\begin{aligned}
x[n] &= x_a(nT_s) \\
&= cos(2\pi 600 n T_s) \\
&= cos(2\pi 600 n \frac{1}{1000}) \\
&= cos(2\pi \underbrace{\frac{6}{10}}_{f} n)
\end{aligned}
$$

- Bad sign: We get a frequency larger than $f_{max} = \frac{1}{2}$

$F = 400\ Hz$

$\cos(2\pi\, 400\, t)$

$\Downarrow$

$\cos(2\pi \frac{4}{10} n)$

$x_a(t) = \cos(2 \cdot \pi \cdot 600 \cdot t)$

$x[n] = \cos(2\pi \cdot \frac{6}{10} \cdot n)$

$= \cos(2\pi \frac{4}{10} n)$

n	$\cos(2\pi \frac{6}{10} n)$		$\cos(2\pi \frac{4}{10} n)$
$n = 0$	1	=	1
$n = 1$	−0.80	=	−0.80
$n = 2$	0.30	=	0.30
⋮			

Funny things with cos() and sin()

- Discrete cos() and sin() have funny properties
- They are **the same** when adding an integer to the frequency:

$$\cos(2\pi(f+\underbrace{k}_{\text{integer}})n) = \cos(2\pi fn + (\underbrace{2kn\pi}_{\text{multiple of } 2\pi})) = \cos(2\pi fn)$$

- So all these discrete frequencies are identical:

$$f = \ldots = -1.4 = -0.4 = 0.6 = 1.6 = 2.6 = 3.6 = \ldots$$

- In addition, negative frequencies can be turned into positive:

$$\cos(2\pi(-f)n) = \cos(2\pi fn)$$

$$\sin(2\pi(-f)n) = -\sin(2\pi fn)$$

n = integer

$x[n] = \cos(2\pi f n) = \cos(2\pi(f+1)n)$

$\cos(2\pi\, 0.3\, n) = \cos(2\pi \cdot (1.3) \cdot n)$

$= \cos(2\pi\, 0.3\, n + 2\pi \cdot 1 \cdot n)$

$= \cos(2\pi \cdot 775.3 \cdot n)$

~~cos(~~

$f =$

$\cos(2\pi\, 0.6\, n)$

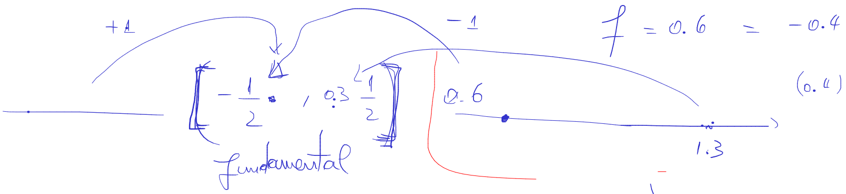
$=$

$\cos(2\pi(-0.4)n)$

$=$

$\cos(2\pi\, 0.4\, n)$

Aliasing

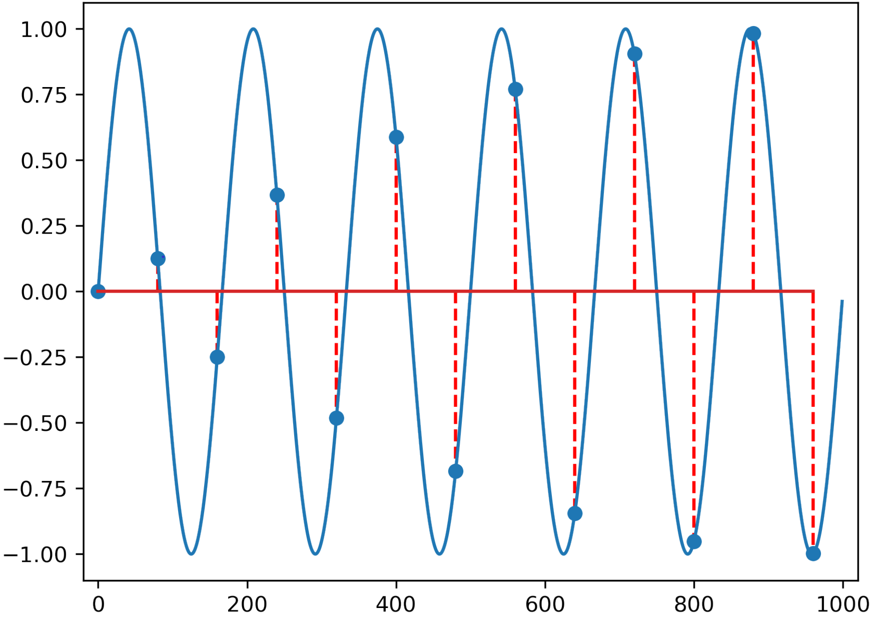


$f = \cancel{0.6} = -0.4$

Aliasing:

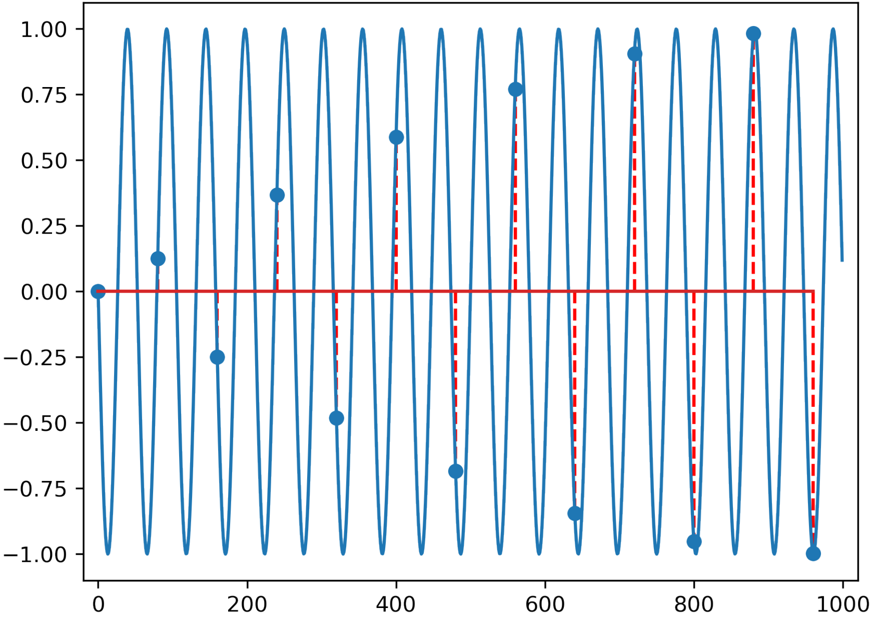
- Every discrete frequency f outside the interval $[-\frac{1}{2}, \frac{1}{2}]$ is **identical** (an "alias") with a frequency from this interval $f_{alias} \in [-\frac{1}{2}, \frac{1}{2}]$
- Just add or subtract 1's to f until the result is in $[-\frac{1}{2}, \frac{1}{2}]$

Aliasing example - low frequency signal



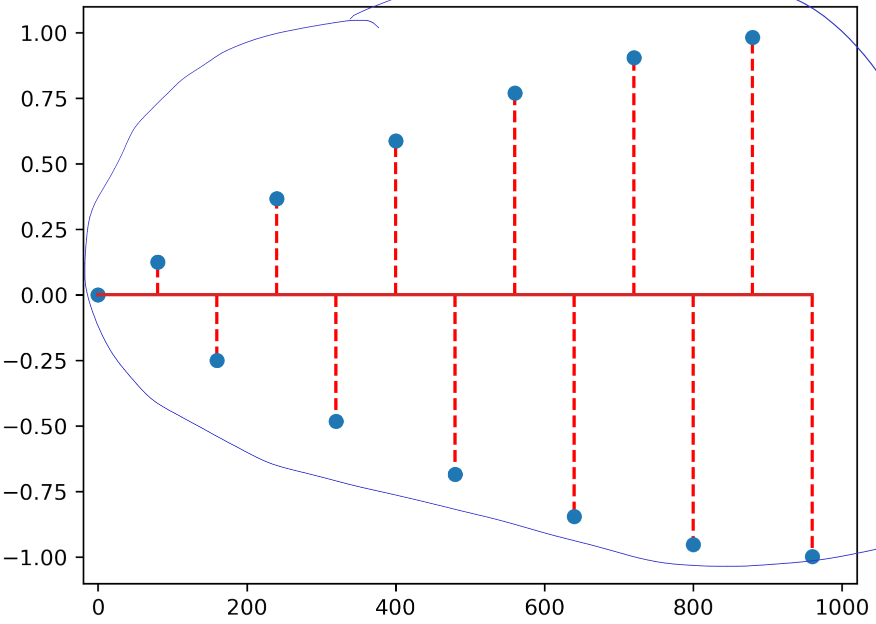
$f = 400\ Hz$

Aliasing example - high frequency signal, same samples



600 Hz

Aliasing example - samples only



?

The problem of aliasing

- Sampling different signals can lead to exactly same samples
- Problem: how to know from what signal did the samples come from? Impossible.
- Example:
  - all these discrete frequencies are identical:

$$f = -0.4 = 0.4 = 0.6 = 1.6 = \ldots$$

  - so if $F_s = 1000Hz$, the original signal could have been any frequency F out of: 400Hz or 600Hz or 1400Hz or 1600Hz or …
  - Exercise: check some of these

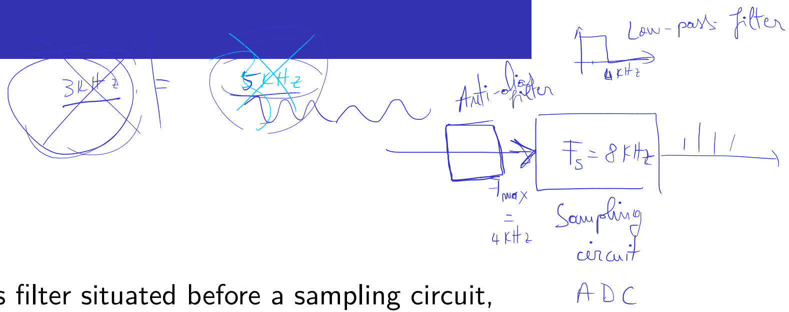
Anti-alias

- Aliasing only affects digital signals (it is caused by sampling)
- Sampling according to Shannon theorem guarantees no aliasing:

$$F_s \geq 2F_{max} \Rightarrow f = \frac{F}{F_{max}} \leq \frac{1}{2}$$

- Better remove from the signal the frequencies larger than $\frac{F_s}{2}$, which will not be sampled correctly, otherwise they will create a false frequency and bring confusion

Anti-alias



- **Anti-alias filter**: a low-pass filter situated before a sampling circuit, rejecting all frequencies $F > \frac{F_s}{2}$ from the signal before sampling
  - Standard practice in the design of processing systems

$$\frac{F_s}{2} = F_{Nyquist}$$

Ideal signal reconstruction from samples

Sampling $t \to \frac{n}{F_s}$

Rec: $n \to t \cdot F_s$

- Reconstruction = opposite of sampling
- Produces a continuous signal from a discrete one

Ideal reconstruction equation:

$$\boxed{x_r(t) = x[\frac{t}{T_s}] = x[t \cdot F_s]}$$

$n \to t \cdot F_s$

- A discrete frequency f becomes $F = f \cdot F_s$

$x[n] = \cos(2\pi\, 0.3\, n)$

$x_r(t) = \cos(2\pi \cdot 0.3 \cdot t \cdot F_s)$

$F_s = 1000$ Hz

$= \cos(2\pi\, 300 \cdot t)$

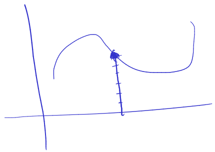
Reconstruction and aliasing

- What value to use for f?
  - we know $f = f + 1 = f + 2 = ...$, which one to use?
- The reconstruction assumes all f are in the interval $[-\frac{1}{2}, \frac{1}{2}]$
  - apply reconstruction equation
  - the resulting signal has all frequencies $F \leq \frac{F_s}{2} = F_N$ (= "the Nyquist frequency")
- **In exercises:** Always bring f in the interval $[-\frac{1}{2}, \frac{1}{2}]$ before reconstruction
- Reconstruction always produces signals with frequencies in $[-\frac{Fs}{2}, \frac{Fs}{2}]$
  - Only signals or components sampled according to the sampling theorem will be reconstructed identically
  - Any other components are replaced with their aliased counterparts

A/D and D/A conversion

write the sample value in binary format

- Sampling + quantization + coding is usually done by an **Analog to Digital Converter (ADC)**
  - It takes an analog signal and produces a sequence of binary-coded values
- Reconstructing an analog signal from numeric samples is done by a **Digital to Analog Converter (DAC)**
  - Usually the reconstruction is not based on sampling theorem equation, which is too complicated, but with simpler empirical solutions
- You have ADCs and DACs for any In or Out audio jack (phone, computer etc)



Signal quantization and coding

- In practice, the amplitudes of the samples are converted to binary representation
- Because of this, the amplitudes are rounded to fixed levels, e.g. 8-bit values (256 distinct levels) , 16-bit values (65536).
- This "rounding" is known as **quantization**
- The "rounding error" is known as **quantization error**
- Converting the value to binary form is known as **coding**
- ADCs handle sampling, quantization and coding simultaneously