

Digital Signal Processing

## Chapter IV: The Fourier Transform and its applications

## IV.1 Vector spaces of signals (crash course)

- ▶ **Vector space** = a set  $V\{v_i\}$  with the following two properties:
  - ▶ one element + another element = still an element of the same space
  - ▶ a scalar constant  $\times$  an element = still an element of the same space
- ▶ You **can't escape** a vector space by summing or scaling
- ▶ The elements of a vector space are called **vectors**

## Examples of vector spaces

- ▶ Geometric spaces are great intuitive examples:
  - ▶ a line, or the set  $\mathbb{R}$  (one-dimensional)
  - ▶ a plane, or the set  $\mathbb{C}$  (two-dimensional)
  - ▶ 3D space (three-dimensional)
  - ▶ 4D space (four-dimensional, like the spatio-temporal universe)
  - ▶ arrays with  $N$  numbers ( $N$ -dimensional)
  - ▶ space of continuous signals ( $\infty$ -dimensional)
- ▶ The **dimension** of the space = “how many numbers you need in order to specify one element” (informal)
- ▶ A “vector” like in maths = a sequence of  $N$  numbers = a “vector” like in programming
  - ▶ e.g. a point in a plane has two coordinates = a vector of size  $N = 2$
  - ▶ e.g. a point in a 3D-space has three coordinates = a vector of size  $N = 3$

- ▶ Many vector spaces have a fundamental operation: **the (Euclidean) inner product**

- ▶ for **discrete** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$

- ▶ for **continuous** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$

- ▶ \* represents **complex conjugate** (has no effect for real signals)
- ▶ The result is one number (real or complex)
- ▶ Also known as **dot product** or **scalar product** ("produs scalar")

## Inner product

- ▶ Each entry in  $\mathbf{x}$  times the complex conjugate of the one in  $\mathbf{y}$ , all summed
- ▶ For discrete signals, it can be understood as a row  $\times$  column multiplication
- ▶ Discrete vs continuous: just change sum/integral depending on signal type

## Inner product properties

- ▶ Inner product is **linear** in both terms:

$$\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle = \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2 \rangle = \langle \mathbf{x}, \mathbf{y}_1 \rangle + \langle \mathbf{x}, \mathbf{y}_2 \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

## The distance between two vectors

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ **The (Euclidean) distance** between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_N - y_N)^2}$$

- ▶ This distance is the **usual geometric distance** you know from geometry
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ if two vectors have small distance, they are close, they are similar
  - ▶ two vectors with large distance are far away, not similar
  - ▶ two identical vectors have zero distance

## The norm of a vector

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ The **norm** (length) of a vector =  $\text{sqrt}(\text{inner product with itself})$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ The **norm** of a vector is the distance from  $\mathbf{x}$  to point  $\mathbf{0}$ .
- ▶ It has the exact same intuition like in **normal geometry**:
  - ▶ vector has large norm = has big values, is far from  $\mathbf{0}$
  - ▶ vector has small norm = has small values, is close to  $\mathbf{0}$
  - ▶ vector has zero norm = it is the vector  $\mathbf{0}$
- ▶ Norm of a vector =  $\text{sqrt}(\text{the signal energy})$

## Norm and distance

- ▶ The norm and distance are related
- ▶ The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ Just like in geometry: distance = length of the difference vector

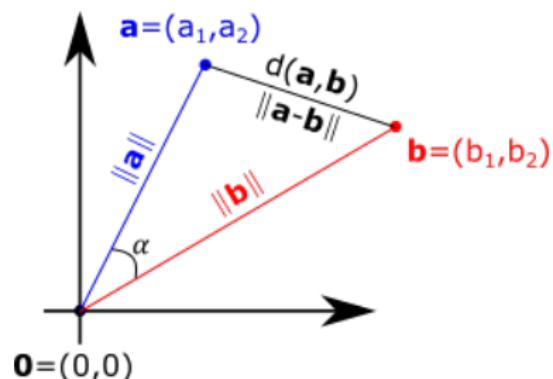


Figure 1: Norm and distance in vector spaces

## Angle between vectors

- ▶ The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

- ▶ is a value between -1 and 1
- ▶ **Otrhogonal vectors** = two vectors with  $\langle x, y \rangle = 0$ 
  - ▶ their angle = 90 deg
  - ▶ in geometric language, the two vectors are **perpendicular**

## Why vector space

- ▶ Why are all these useful?
- ▶ They are a very general **framework** for different kinds of signals
- ▶ We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - ▶ Example in DEDP class: ML decision with 1, 2, N samples

We deal mainly with the following vector spaces:

- ▶ The vector space of all infinitely-long real signals  $x[n]$
- ▶ The vector space of all infinitely-long periodic signals  $x[n]$  with period  $N$ 
  - ▶ for each  $N$  we have a different vector space
- ▶ The vector space of all finite-length signals  $x[n]$  with only  $N$  samples
  - ▶ for each  $N$  we have a different vector space

## Bases

- A **basis** = a set of  $N$  linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \dots + \alpha_N \mathbf{b}^N$$

- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, \dots, \alpha_N)$$

## Bases and coordinate systems

- ▶ Bases are just like **coordinate systems** in a geometric space

- ▶ any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

- ▶ any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶  $N$  = The number of basis elements = The dimension of the space
- ▶ Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

## Bases and coordinate systems

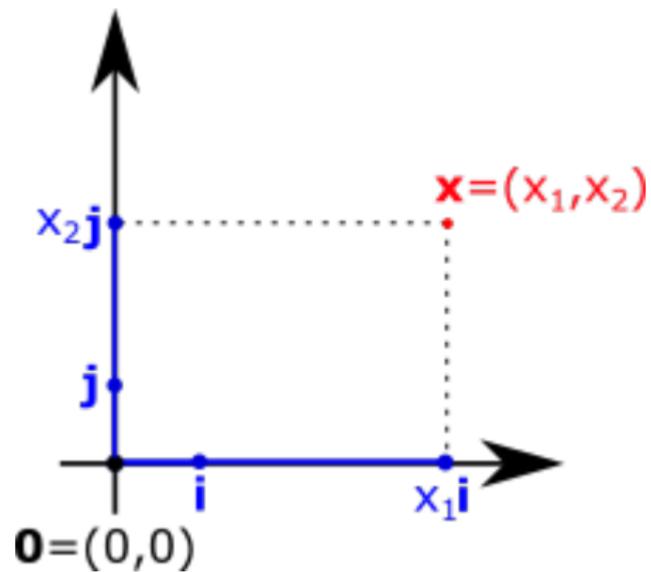


Figure 2: Basis expansion of a vector  $\mathbf{x}$

## Choice of bases

- ▶ There is typically an infinite choice of bases
- ▶ The **canonical basis** = all basis vectors are full of zeros, just with one 1
- ▶ You used it already in an exercise:
  - ▶ any signal  $x[n]$  can be expressed of a sum of  $\delta[n - k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n - 1] + 2\delta[n - 2] + \dots$$

- ▶ the canonical basis is  $B = \{\dots, \delta[n], \delta[n - 1], \delta[n - 2], \dots\}$

## Orthonormal bases

- ▶ An **orthonormal basis** a basis where all elements  $\mathbf{b}^i$  are:

- ▶ orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

- ▶ **normalized** (their norm = 1):

$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- ▶ Example: the canonical basis  $\{\delta[n - k]\}$  is orthonormal:

- ▶  $\langle \delta[n - k], \delta[n - l] \rangle = 0, \forall k \neq l$
  - ▶  $\langle \delta[n - k], \delta[n - k] \rangle = 1, \forall k$

## Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

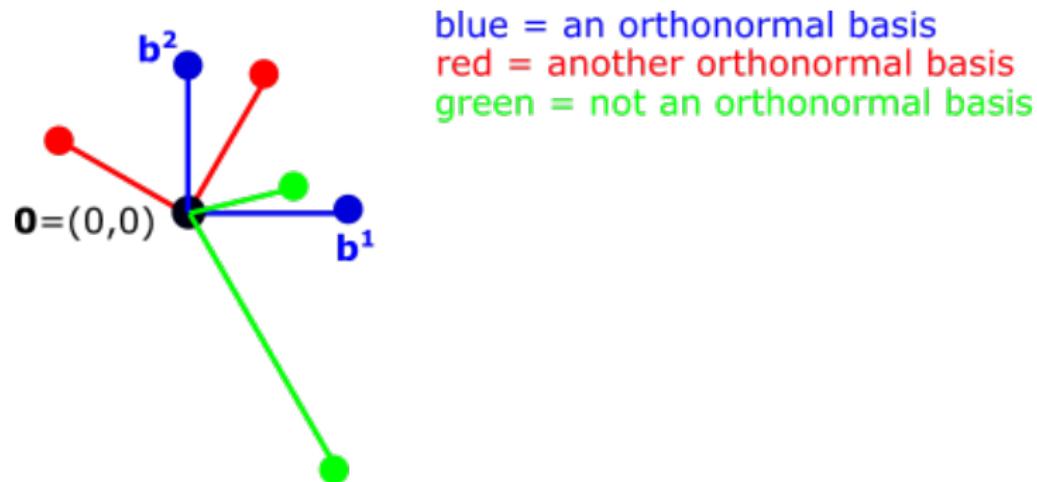


Figure 3: Sample bases in a 2D space

## Basis expansion of a vector

- ▶ Suppose we have an **orthonormal basis**  $B = \{\mathbf{b}^i\}$
- ▶ Suppose we have a vector  $\mathbf{x}$
- ▶ We can write (expand)  $\mathbf{x}$  as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ Question: how to **find** the coefficients  $\alpha_i$ ?

## Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{b}^i \rangle &= \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_i\end{aligned}$$

## Basis expansion of a vector

- ▶ Any vector  $\mathbf{x}$  can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ For orthonormal basis: the coefficients  $\alpha_i$  are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle \mathbf{x}, \mathbf{b}^i \rangle$$

## Why bases

- ▶ How does all this talk about bases help us?
- ▶ To better understand the Fourier transform
- ▶ The signals  $\{e^{j\omega n}\}$  form an **orthonormal basis**
- ▶ The Fourier Transform of a signal  $x$  = finding the coefficients of  $x$  in this basis
- ▶ The Inverse Fourier Transform = expanding  $x$  with the elements of this basis
- ▶ Same **generic** thing every time, only the type of signals differ

## IV.2 The Fourier Transform

## Reminder

► Reminder:

$$e^{jx} = \cos(x) + j \sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

## Why sinusoidal signals

- ▶ Why are sinusoidal signals  $\sin()$  and  $\cos()$  so prevalent in signal processing?
- ▶ Answer: because they are combinations of an  $e^{jx}$  and an  $e^{-jx}$
- ▶ Why are these  $e^{jx}$  so special?
- ▶ Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

## Response of LTI systems to harmonic signals

- ▶ Consider an LTI system with  $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal  $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

- ▶ Output signal = input signal  $\times$  a (complex) constant ( $H(\omega_0)$ )

# Eigen-function

- ▶ Complex exponential signals are **eigen-functions** (“funcții proprii”) of LTI systems:
  - ▶ output signal = input signal  $\times$  a (complex) constant
- ▶ We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of  $\{e^{j\omega n}\}$
- ▶ Example: RGB color filter (listen)
- ▶ All vector space theory becomes useful now:
  - ▶  $\{e^{j\omega n}\}$  is an **orthonormal basis**
  - ▶ decomposing signals = finding coefficients  $\alpha_i$
  - ▶ we know how to do this with the inner product

## Fourier Transform of Discrete Signals

- ▶ Consider signal  $\mathbf{x} = x[n]$  in the vector space of infinitely-long signals
- ▶ The signals  $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$  form an **orthonormal basis** in this space
- ▶ We can expand (almost) any  $\mathbf{x}$  in this basis
- ▶ There are an infinite number of terms, use an integral:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_i} e^{j2\pi fn} df$$

- ▶ The coefficient of every  $e^{j2\pi fn}$  is found by inner product:

$$\alpha_i = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

## Inverse Fourier Transform of Discrete Signals

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- ▶ Any  $x[n]$  can be written as a linear combination of  $\{e^{j2\pi fn}\}$ ,  $\forall f \in [-\frac{1}{2}, \frac{1}{2}]$ , with some coefficients  $X(f)$

## Fourier Transform of Discrete Signals

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

- ▶ The coefficient  $X(f)$  of every  $\{e^{j2\pi fn}\}$  is found using the inner product  $\langle x, e^{j2\pi fn} \rangle$

## Fourier Transform of Discrete Signals

- We can replace  $2\pi f = \omega$ , and  $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(f) = \langle x[n], e^{j\omega n} \rangle = \sum_n x[n] e^{-j\omega n}$$

## Basic properties of Fourier transform

- ▶  $X(\omega)$  is a complex function
- ▶ It is **periodical** with period  $2\pi$

$$X(\omega + 2\pi) = X(\omega)$$

- ▶ If the signal  $x[n]$  is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

- ▶ This means:
  - ▶ modulus is even:  $|X(\omega)| = |X(-\omega)|$
  - ▶ phase is odd:  $X(\omega) = -X(-\omega)$

## Parseval theorem

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- ▶ Is true for all orthonormal bases

## Expressing as sum of sinusoids

- Grouping terms with  $e^{j\omega n}$  and  $e^{j(-\omega)n}$  we get:

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} X(\omega) e^{j\omega n} d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} (X(\omega) e^{j\omega n} + X(\omega) e^{j(-\omega)n}) d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} 2|X(\omega)| (e^{j\omega n + \angle X(\omega)} + e^{-j\omega n - \angle X(\omega)}) d\omega \\&= \frac{1}{2\pi} \int_0^{\pi} 2|X(\omega)| \cos(\omega n + \angle X(\omega)) d\omega\end{aligned}$$

- Any signal  $x[n]$  is **a sum of sinusoids with all frequencies**  
 $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$

## Expressing as sum of sinusoids

- ▶ Any signal  $x[n]$  is a **sum of sinusoids with all frequencies**  
 $f \in [0, \frac{1}{2}]$ , or  $\omega \in [0, \pi]$ 
  - ▶ this is the fundamental practical interpretation of the Fourier transform
- ▶ The **modulus**  $|X(\omega)|$  is the **amplitude** of the sinusoids ( $\times 2$ )
  - ▶ for  $\omega = 0$ ,  $|X(\omega = 0)|$  = the DC component
- ▶ The **phase**  $\angle X(\omega)$  gives the initial phase

## Power spectral density

- ▶ The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- ▶ Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- ▶ Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- ▶ Energy over one period is

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

### IV.3 The Fourier Series

#### IV.4 The Discrete Fourier Transform

## IV.2 Frequency analysis of discrete signals

## Fourier series of discrete periodical signals

- A discrete signal of period  $N$ :

$$x[n] = x[n + N]$$

- Can always be decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, \dots, N-1$$

- Finding the coefficients  $c_k$ :

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

## Comparison with analog Fourier series

- ▶ Compared to analog signals:
  - ▶ consider fundamental frequency  $f_0 = 1/N$
  - ▶ only  $N$  terms, with frequencies  $k \cdot f_0$ :
    - ▶  $0, f_0, 2f_0, \dots, (N-1)f_0$
  - ▶ only  $N$  distinct coefficients  $c_k$
  - ▶ the  $N$  coefficients  $c_k$  can be chosen like  $-\frac{N}{2} < k \leq \frac{N}{2} \Rightarrow$  the frequencies span the range  $-1/2 \dots 1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

## Basic properties of Fourier coefficients

1. Signal is **discrete**  $\rightarrow$  coefficients are **periodic** with period N

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi(k+N)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

2. If signal is real  $x[n] \in \mathbb{R}$ , the coefficients are **even**:

- ▶  $c_k^* = c_{-k}$
  - ▶  $|c_k| = |c_{-k}|$
  - ▶  $\angle c_k = \angle c_{-k}$
- ▶ Together with periodicity:
- ▶  $|c_k| = |c_{-k}| = |c_{N-k}|$
  - ▶  $\angle c_k = -\angle c_{-k} = -\angle c_{N-k}$

## Expressing as sum of sinusoids

- ▶ Grouping terms with  $c_k$  and  $c_{-k}$  we get

$$x[n] = c_0 + 2 \sum_{k=1}^L |c_k| \cos\left(2\pi \frac{k}{N} + \angle c_k\right)$$

where  $L = N/2$  or  $L = (N - 1)/2$  depending if  $N$  is even or odd

- ▶ Signal = DC value + a finite sum of sinusoids with frequencies  $kf_0$ 
  - ▶  $|c_k|$  give the amplitudes ( $\times 2$ )
  - ▶  $\angle c_k$  give the phases

## Power spectral density

- ▶ The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- ▶ Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- ▶ Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- ▶ Energy over one period is

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

## 2018-2019 Exam

### 2018-2019 Exam

- ▶ Properties of Fourier series: only 1, 2, and 5

## 1. Linearity

If the signal  $x_1[n]$  has the Fourier series coefficients  $\{c_k^{(1)}\}$ , and  $x_2[n]$  has  $\{c_k^{(2)}\}$ , then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot c_k^{(1)} + b \cdot c_k^{(2)}\}$$

Proof: via definition

## 2. Shifting in time

If  $x[n] \leftrightarrow \{c_k\}$ , then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi kn_0/N)} c_k\}$$

Proof: via definition

- ▶ The amplitudes  $|c_k|$  are not affected, shifting in time affects only the phase

## 3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{c_{k-k_0}\}$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow \{c_{-k}^*\}$$

## 5. Circular convolution

Circular convolution of two signals  $\leftrightarrow$  product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot c_k^{(1)} \cdot c_k^{(2)}\}$$

Circular convolution:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- ▶ takes two periodic signals of period N, result is also periodic with period N
- ▶ Example at the whiteboard: how it is computed

## 6. Product in time

Product in time  $\leftrightarrow$  circular convolution of Fourier series coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} c_m^{(1)} c_{(k-m)_N}^{(2)} = c_k^{(1)} \otimes c_k^{(2)}$$

## Fourier transform of discrete non-periodical signals

- ▶ Non-periodical signals contain all frequencies, not only the multiples of  $f_0$
- ▶ The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- ▶ The inverse Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

# Comparison

- ▶ Compared with the Fourier transform of analog signals
  - ▶ sum instead of integral in Fourier transform
  - ▶ spectrum is only in range:

$$\omega \in [-\pi, \pi]$$

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- ▶ Compared with the Fourier series of discrete periodical signals
  - ▶ general  $\omega$  instead of  $2\pi kf_0$
  - ▶ spectrum is continuous, not discrete
  - ▶ integral, not sum in inverse Fourier transform

## Parseval theorem

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- ▶ Proof: on whiteboard
- ▶ The **energy spectral density** gives the energy contained for each frequency

$$S_{xx}(\omega) = |X(\omega)|^2$$

## Basic properties of Fourier transform

- ▶ It is **periodical** with period  $2\pi$

$$X(\omega + 2\pi) = X(\omega)$$

- ▶ If the signal  $x[n]$  is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X^*(\omega) = X(-\omega)$$

- ▶ This means

- ▶ modulus is even:  $|X(\omega)| = |X(-\omega)|$
- ▶ phase is odd:  $X(\omega) = -X(-\omega)$

## Relation between Fourier series and Fourier transform

- ▶ If apply Fourier transform to periodical discrete signals,  $X(\omega)$  contains Diracs
- ▶ The Diracs are at frequencies  $kf_0$ , just like the Fourier series
- ▶ The value of an impulse = the coefficient  $c_k$  of the Fourier series
- ▶ **The Fourier series  $\approx$  the Fourier transform of periodic signals**
  - ▶ we directly compute the coefficients  $c_k$  of the impulses in the spectrum

- ▶ Definition of Fourier transform = Z transform with:

$$z = e^{j\omega}$$

- ▶  $e^{j\omega}$  = points on the unit circle
- ▶ Fourier transform = Z transform evaluated **on the unit circle**
  - ▶ if the unit circle is in the convergence region of Z transform
  - ▶ otherwise, equivalence does not hold
- ▶ This is true for most usual signals we work with
  - ▶ there are exceptions, but they are outside the scope of this class

## 2018-2019 Exam

### 2018-2019 Exam

- ▶ Properties of Fourier transform: only 1, 2, 5, and Parseval theorem

## 1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

## 2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- ▶ The amplitudes  $|X(\omega)|$  is not affected, shifting in time affects only the phase

# Properties of Fourier transform

## 3. Modulation in time

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

## 5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- ▶ Not circular convolution, this is the normal convolution

## 6. Product in time

Product in time  $\leftrightarrow$  convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$$

# Properties of Fourier transform

## Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

## Wiener Khinchin theorem

Autocorrelation of a signal  $\leftrightarrow$  Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

## Properties of Fourier transform

### Parseval theorem

Energy is the same when computed in the time or frequency domain

$$\sum |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

## Relationship of Fourier transform and Fourier series

- ▶ How are they related?
  - ▶ Fourier transform: for non-periodical signals
  - ▶ Fourier series: for periodical series
- ▶ Duality: periodic in time  $\leftrightarrow$  discrete in frequency
- ▶ If we **periodize** a signal  $x[n]$  by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- ▶ then the Fourier transform w is discrete (made of Diracs):

$$X_N(\omega) = 2\pi c_k \delta(\omega - k \frac{2\pi}{N})$$

- ▶ The coefficients of the Diracs = exactly the Fourier series coefficients

## Relationship of Fourier transform and Fourier series

- ▶ So, Fourier transform can be considered for both periodic and non-periodic signals
- ▶ Fourier transform for periodic signals = discrete (sum of Diracs with some coefficients)
  - ▶ Diracs at frequencies  $f_0 = 1/N$  and its multiples
- ▶ Fourier series for periodic signals = gives the coefficients of the Diracs directly
  - ▶ it just omits to write the Diracs explicitly in the equation

## Relation of Fourier transform and Z transform

- ▶ Fourier transform:  $X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$
- ▶ Z transform:  $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
- ▶ **Fourier transform = Z transform for  $z = e^{j\omega}$**
- ▶  $z = e^{j\omega}$  means **evaluated on the unit circle**:
  - ▶  $|z| = |e^{j\omega}| = 1$  (*modulus*)
  - ▶  $\angle z = \angle e^{j\omega} = \omega$  (*phase*)
- ▶ Conditions:
  - ▶ unit circle must be in the Convergence Region of Z transform
  - ▶ some signals can have Fourier transform even though unit circle not in CR
- ▶ If signal has pole on unit circle  $\rightarrow$  Dirac (infinite) in Fourier transform
  - ▶ e.g.  $u[n]$
  - ▶ some signals are non-convergent on unit circle, but have Fourier transform (e.g.  $u[n]$ )

## Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

► Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

► Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

## Geometric interpretation of Fourier transform

- ▶ For complex numbers:
  - ▶ modulus of  $|a - b|$  = the length of the segment between  $a$  and  $b$
  - ▶ phase of  $|a - b|$  = the angle of the segment from  $b$  to  $a$  (direction is important)
- ▶ So, for a point on the unit circle  $z = e^{j\omega}$ 
  - ▶ modulus  $|X(\omega)|$  is **given by the distances to the zeros and to the poles**
  - ▶ phase  $\angle X(\omega)$  is **given by the angles from the zeros and poles to z**

## Geometric interpretation of Fourier transform

- ▶ Consequences:
  - ▶ when a **pole** is very close to unit circle → Fourier transform is **large** at this point
  - ▶ when a **zero** is very close to unit circle → Fourier transform is **small** at this point
- ▶ Examples: . . .

# Geometric interpretation of Fourier transform

- ▶ Simple interpretation for modulus  $|X(\omega)|$ :
  - ▶ Z transform  $X(z)$  is a “*landscape*”
    - ▶ poles = mountains of infinite height
    - ▶ zeros = valleys of zero height
  - ▶ Fourier transform  $X(\omega) = \text{“Walking over this landscape along the unit circle”}$  → the heights give the Fourier transform
  - ▶ When close to a mountain → road is high → Fourier transform has large amplitude
  - ▶ When close to a valley → road is low → Fourier transform has small amplitude
- ▶ Enough to sketch the Fourier transform for signals with few poles/zeros

## Geometric interpretation of Fourier transform

- ▶ Note:  $X(z)$  might also have a constant  $C$  in front!
  - ▶ It does not appear in pole-zero plot
  - ▶ The value of  $|C|$  and  $\angle C$  must be determined separately
- ▶ This “geometric method” can be applied for both modulus and phase

- ▶ **Duality** properties related to Fourier transform/series
- ▶ Discrete  $\leftrightarrow$  Periodic
  - ▶ **discrete** in time  $\rightarrow$  **periodic** in frequency
  - ▶ **periodic** in time  $\rightarrow$  **discrete** in frequency
- ▶ Continuous  $\leftrightarrow$  Non-periodic
  - ▶ **continuous** in time  $\rightarrow$  **non-periodic** in frequency
  - ▶ **non-periodic** in time  $\rightarrow$  **continuous** in frequency

# Frequency-based classification of signals

- ▶ Based on frequency content:
  - ▶ **low-frequency** signals
  - ▶ **mid-frequency** signals (band-pass)
  - ▶ **high-frequency** signals
- ▶ **Band-limited** signals: spectrum is 0 over some frequency  $f_{max}$
- ▶ **Time-limited** signals: signal value is 0 outside some time interval
- ▶ **Bandwidth**  $B$ : frequency interval  $[F_1, F_2]$  which contains 95% of energy
  - ▶  $B = F_2 - F_1$
- ▶ Based on bandwidth  $B$ :
  - ▶ **Narrow-band** signals:  $B \ll$  central frequency  $\frac{F_1+F_2}{2}$
  - ▶ **Wide-band** signals: not narrow-band