Digital Signal Processing

Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

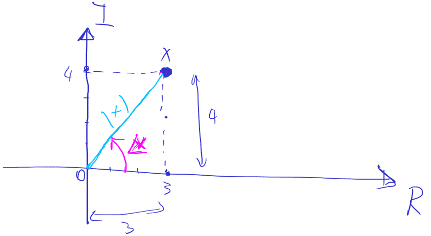
Recap: Complex numbers

- real and imaginary part
- **modulus and phase**
- graphical interpretation
- Euler formula
- modulus and phase of e^{jx}

$$X = \overset{R}{3} + \overset{I}{4j}$$

$$|X| = \sqrt{3^2 + 4^2} = 5$$

$$\angle X = \operatorname{atan}\frac{4}{3} = 0.92$$



$$X = \underbrace{3}_{R} + \underbrace{4j}_{I} = \underbrace{5 \cdot e^{j \cdot 0.92}}_{|X| \cdot e^{j\angle X}} = 5 \cdot \left(\cos(0.92) + j \cdot \sin 0.92\right)$$

$$X = |X| \cdot e^{j\angle X}$$

$$\boxed{e^{j \cdot \text{something}} = \cos(\text{something}) + j \cdot \sin(\text{something})}$$

Definition of Z transform

$$x[n] \xleftrightarrow{Z} X(z)$$

$$\mathcal{Z}\left(x[n]\right) = X(z)$$

- The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

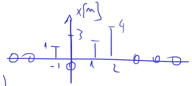
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Notation:

$$\mathcal{Z}\left(x[n]\right) = X(Z)$$

$$x[n] \xleftrightarrow{\text{Z}} X(Z)$$

Definition of Z transform



$01011 \longrightarrow 0\cdot x^0 \oplus 1\cdot x^1 \oplus 0\cdot x^2 \oplus 1\cdot x^3 \oplus 1\cdot x^4$

$$x[n] = \{\dots 0, \underset{z^{1}}{\overset{x[-1]}{1}}, \underset{z^{0}}{\overset{x[0]}{3}}, \underset{z^{-1}}{\overset{x[1]}{2}}, \underset{z^{-2}}{\overset{x[2]}{4}}, 0, 0, \dots \overset{x[n]}{}\underset{z^{-n}}{}\}$$

$$X(z) = 1\cdot z^{1} + 3\cdot \underbrace{z^{0}}_{1} + 2\cdot z^{-1} + 4\cdot z^{-2}$$

- Similar to the Laplace transform for continuous signals
- The Z transform associates **a polynomial** to a signal (think Information Theory class)
- Why?
  - Easy representation of convolution
  - Convolution of two signals = multiplication of polynomials
  - Efficient descriptions of complicated systems with poles and zeros

$x_1 = \{\dots 0, 2, 2, 2, 0, \dots\}$

$x_2 = \{\dots 0, 1, 2, 3, 4, 0 \dots\}$

$x_1 * x_2$

$x_1 \longrightarrow 2\cdot z + 2 + 2\cdot z^{-1}$

$x_2 \longrightarrow 1\cdot z + 2 + 3\cdot z^{-1} + 4\cdot z^{-2}$

$$(2\cdot z + 2 + 2\cdot z^{-1})\cdot(1\cdot z + 2 + 3\cdot z^{-1} + 4\cdot z^{-2}) = 2\cdot z^{2} + 6z + \dots \longleftrightarrow \{2, 6, \dots\}$$

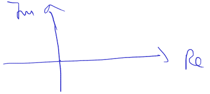
Region of convergence

$X(z) =$

z small

- $X(\cancel{Z})$ is a sum dependent on some variable z (complex number)
- The **Region Of Convergence (ROC)** = the values of z for which the sum is convergent (does not go to $\pm\infty$)

Examples



$x[n] = \{\ldots 0, 1, 2, \underset{\uparrow}{5}, 7, 0, \ldots\}$

$z^{2} \quad z^{1} \quad z^{0}=1 \quad z^{-1}$

$\mathcal{Z}(x[n]) = X(z) = z^{2} + 2\cdot z^{1} + 5 + 7z^{-1}$, ROC : z plane $-\{0, \infty\}$

Exercises:

- Compute Z transform for the following signals:

  $x[n] = 1, 2, 5, 7, 0,$ (with time origin in 1 or in 5)

  $\delta[n]$, $\delta[n-k]$, $\delta[n+k]$

  $x[n] = \left(\frac{1}{2}\right)^{n} u[n]$

$x[n] = \{\ldots 0, \underset{\uparrow}{1}, 2, 5, 7, 0, \ldots\}$

$z^{0} \quad z^{-1} \quad z^{-2} \quad z^{-3}$

$X(z) = 1 + 2\cdot z^{-1} + 5\cdot z^{-2} + 7z^{-3}$, ROC : z plane $-\{0\}$

$\boxed{\delta[n] \xleftrightarrow{Z} 1}$ ROC : z plane

$z^{1} \quad z^{0} \quad z^{-1}$

$\{\ldots 0, \underset{\uparrow}{1}, 0, \ldots\} \xleftrightarrow{Z} 1$

$\boxed{\begin{array}{l} \delta[n-k] \longleftrightarrow z^{-k} \\ \delta[n-3] \longleftrightarrow z^{-3} \end{array}}$

ROC : z plane $-\{0\}$

$z^{0} \quad z^{-1} \quad z^{-2} \quad z^{-3}$

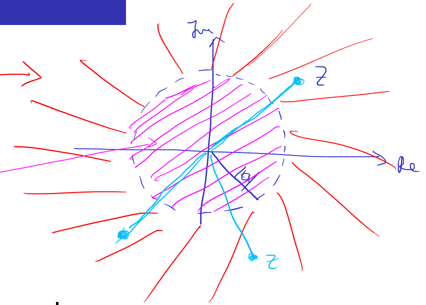
$\{\ldots \underset{\uparrow}{0}, 0, 0, 1, 0 \ldots\}$

$n=3$

Region of convergence

$|z| > |a|$

$|z| < |a|$



- z is a complex number
- Region of convergence (ROC) is displayed as an area in the complex plane (also known as the Z plane)

Region of convergence

- For **finite-support** signals, the ROC is the **whole** Z plane, possibly except 0 or ∞
- For **causal** signals, the ROC is the **outside** of a circle:

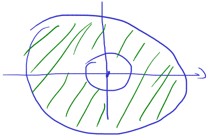
$$|z| > r_1$$

  - e.g. if $|z|$ is big enough, the sum is convergent
- For **anti-causal** signals, the ROC is the **inside** of a circle:

$$|z| < r_2$$

  - e.g. if $|z|$ is small enough, the sum is convergent
- Why circles? Because only modulus of z matters
  - complex numbers on a circle have the same modulus

Region of convergence



- For **bilateral** signals, the ROC is the area **between** two circles:

$$r_1 < |z| < r_2$$

  - bilateral signals have a causal part and an anti-causal part
- For finite-support signals, the two "circles" are of "radius" 0 and ∞
- Two different signals can have the same expression of $X(z)$, but with different ROC!
  - ROC is an essential part in specifying a Z transform
  - it should never be omitted

The Inverse Z Transform

$x[n] \longleftrightarrow X(z)$

- From a purely mathematical point of view, $X(z)$ is a complex function

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Proper definition of inverse transform is based on the theory of complex functions
- Multiply with z^{n-1} and integrate along a contour C inside the ROC:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

The Inverse Z Transform

- The Cauchy integral theorem says that:

$$\frac{1}{2\pi j}\oint_C z^{n-k-1}dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- And therefore:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

- We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

$\mathcal{Z}\{a \cdot x_1[n] + b x_2[n]\} = a \cdot \mathcal{Z}\{x_1[n]\} + b \cdot \mathcal{Z}\{x_2[n]\}$

1. Linearity

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with ROC1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with ROC2, then:

$$\underbrace{ax_1[n] + bx_2[n]}_{x[n]} \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z)$$

and ROC is at least the intersection of ROC1 and ROC2.

Proof: use definition

$x[n]$ = something $[n]$ + else $[n]$

$X(z)$ = Something (z) + Else(z)

Properties of Z transform

2. Shifting in time

If $\underbrace{x[n]}$ $\overset{Z}{\longleftrightarrow}$ $\underbrace{X(z)}$ with ROC, then:

$$\underbrace{x[n-k]} \overset{Z}{\longleftrightarrow} \underbrace{z^{-k}}_{!} X(z)$$

with same ROC, possibly except 0 and ∞.

Proof: by definition

- valid for all k, also for $k < 0$
- delay of 1 sample $= z^{-1}$

$\delta[n] \overset{z}{\longleftrightarrow} 1$

$\delta[n-3] \longleftrightarrow z^{-3} \cdot 1$

$\delta[n+4] \longleftrightarrow z^{4} \cdot 1$

$\delta[n-1] \longleftrightarrow z^{-1} \cdot 1$

$x[n] \longleftrightarrow X(z) = \sum x[n] z^{-n}$

$x[n-3] \longleftrightarrow = \sum_{n} \underbrace{x[n-3]} \cdot z^{-n} = \sum_{n} x[\underbrace{n-3}_{m}] \cdot z^{-\underbrace{(n-3)}_{m}} \cdot z^{-3} = z^{-3} \cdot \underbrace{\sum_{m} x[m] z^{-m}}_{X(z)}$

Properties of Z transform

3. Modulation in time

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$e^{j\omega_0 n}x[n] \overset{Z}{\longleftrightarrow} X\left(e^{-j\omega_0}z\right)$$

with same ROC

Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC $r_1 < |z| < r_2$, then:

$$x[-n] \overset{Z}{\longleftrightarrow} X\left(z^{-1}\right)$$

with ROC $\frac{1}{r_2} < |z| \frac{1}{r_1}$
Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

with same ROC

Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$x[n] - x[n-1] \overset{Z}{\longleftrightarrow} (1 - z^{-1})X(z)$$

with same ROC except $z = 0$.
Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with ROC, then:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \overset{Z}{\longleftrightarrow} \frac{X(z)}{(1 - z^{-1})}$$

with same ROC except $z = 1$.

Proof: $x[n] = y[n] - y[n-1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with ROC, and $x[n]$ is a complex signal, then:

$$x^*[n] \overset{\mathrm{Z}}{\longleftrightarrow} X^*\left(z^*\right)$$

with same ROC except $z = 0$.
Proof: apply definition

Consequence

If $x[n]$ is a real signal, the poles / zeroes are either real or in complex pairs

Properties of Z transform

9. Convolution in time

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with ROC1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with ROC2, then:

$$x[n] = x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X(z) = X_1(z) \cdot X_2(z)$$

and ROC the intersection of ROC1 and ROC2.
Proof: use definition

- **Very important property!**
- Can compute the convolution of two signals via the Z transform

$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$x_2[n] = \left(\frac{1}{3}\right)^n u[n]$

$x_1 * x_2 = ?$

$X_1(z) = \frac{z}{z - \frac{1}{2}}$

$X_2(z) = \frac{z}{z - \frac{1}{3}}$

$X_1(z) \cdot X_2(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$

$\mathcal{Z}^{-1}$

$x_1[n] * x_2[n]$

Properties of Z transform

10. Correlation in time

If $x_1[n] \stackrel{Z}{\longleftrightarrow} X_1(z)$ with ROC1, and $x_2[n] \stackrel{Z}{\longleftrightarrow} X_2(z)$ with ROC2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \stackrel{Z}{\longleftrightarrow} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and ROC the intersection of ROC1 and with the ROC of $X_2(z^{-1})$ (see reflection property)

Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

Properties of Z transform

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

When $z \to \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

III.2. Z transforms which are Rational Functions

Rational functions

- Many Z transforms are in the form of a **rational function**, i.e. a **fraction** where
  - numerator = **polynomial** in z^{-1} or z
  - denominator = **polynomial** in z^{-1} or z

exponent

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

- Example:

$$\boxed{X(z) = \frac{\overset{b_0}{3} + \overset{b_1}{2} z^{-1} + \overset{b_2}{4} z^{-2}}{\underset{a_0}{1} + \underset{a_1}{5} z^{-2} + \underset{a_2}{7} z^{-4}}}$$

$= B(z)$

$=$

$= A(z)$

Poles and zeros

- Any polynomial is completely determined by its **roots** and a **scaling** factor

$$Any\ polynomial(X) = G \cdot (X - x_1) \dots (X - x_n)$$

- The **zeros** of $X(z)$ are the **roots of the numerator** $B(z)$ $\quad z$ for which $B(z) = 0$
- The **poles** of $X(z)$ are the **roots of the denominator** $A(z)$
- The zeros are usually named $z_1, z_2, ...z_M$, and the poles $p_1, p_2, ...p_N$.

Poles and zeros

- The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1)...(z - z_M)}{(z - p_1)...(z - z_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1})...(1 - z_M z^{-1})}{(1 - p_1 z^{-1})...(1 - z_N z^{-1})}$$

- It has:
  - M zeros with finite values
  - N poles with finite values
  - and either N-M zeros in 0, if N $>$ M, or N-M poles in 0, if N $<$ M (trivial poles/zeros)

Poles and zeros

- Example:

$$
\begin{aligned}
X(z) =& \frac{2z^2 + 0.4z - 1}{3z^3 + 2.4z^2 - 3z - 2.4} \quad \cdot z^{-3} \quad \cdot z^{3} \\
=& \frac{2}{3} \cdot \frac{(z-0.3)(z+0.5)}{(z-1)(z+1)(z+0.8)} \\
=& z^{-1} \cdot \frac{(2 + 0.4z^{-1} - 1z^{-2})}{3 + 2.4z^{-1} - 3z^{-2} - 2.4z^{-3}} \\
=& z^{-1} \cdot \frac{2}{3} \cdot \frac{(1-0.3z^{-1})(1+0.5z^{-1})}{(1-z^{-1})(1+z^{-1})(1+0.8z^{-1})}
\end{aligned}
$$

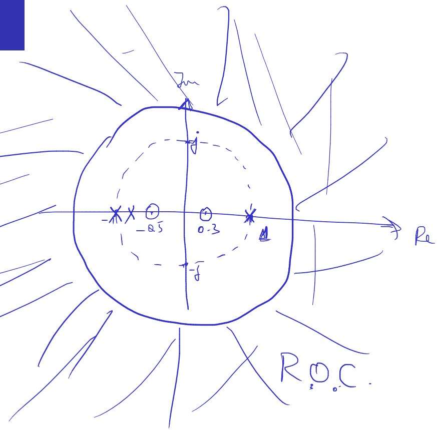
zeros : $\begin{cases} z_1 = 0.3 \\ z_2 = -0.5 \end{cases}$

poles : $\begin{cases} p_1 = 1 \\ p_2 = -1 \\ p_3 = -0.8 \end{cases}$

- Multiple ways of writing same expression

Graphical representation

- The graphical representation of poles and zeros in the complex place is called **the pole-zero plot**
- Graphical: poles = "x", zeros = "0"
- ROC cannot contain poles
- Example: at whiteboard



III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

Inverse Z Transform:

- We have X(z) and the ROC, what is the signal $x[n] =?$

Methods:

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition** (the one we'll actually use)

Partial fraction decomposition

Any rational function

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z{-}2 + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-}2 + \ldots + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$X(z) = c_0 + c_1 z^{-1} + \ldots c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + \ldots \frac{A_N}{z - p_N}$$

- Each pole p_i has a corresponding partial fraction $\frac{A_i}{z-p_i}$
- First terms appear if $M \leq N$
- Based on linearity, we invert each term individually (simple)

Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, **divide numerator to denominator** to obtain the first terms.
   - The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller than denominator
2. In the remaining fraction, **eliminate the negative powers** of z by multiplying with z^N. We want all powers like z^N, not z^{-N}
3. **Divide by** z:

$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$

Procedure for Inverse Z Transform

4. Compute the poles of $\frac{X_1(z)}{z}$ and **decompose in partial fractions**:

$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + \ldots$$

5. **Multiply back with** z:

$$X_1(z) = A_1 \frac{z}{z - p_1} + \ldots$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- If all poles are distinct:

$$A_k = (z - p_k)\frac{X(z)}{z}|_{z=p_k}$$

- If poles are in complex conjugate pairs
  - group the two fractions into a single fraction of degree 2
- If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + ... + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z - p_k)^m \cdot \frac{X(z)}{z}\right]|_{z=p_k}$$

  - example for m = 2

Real signals and complex poles/zeros

$$\frac{-b \pm j\sqrt{\Delta}}{2a}$$

$$X(z) = \frac{3z^2 + 7z - 0.3}{1\cdot z^2 + 0.5z^1 + 0.9}$$

- Consequence of the complex-conjugate property of Z transform:
- A signal $x[n]$ with **real values** can have only:
  - **real-valued** poles or zeroes
  - complex poles and zeroes in **conjugate pairs**, which can be grouped into a single term of degree 2, with real coefficients
- If a Z transform has a complex pole or zero **without** its conjugate pair, then the corresponding signal $x[n]$ is **complex**

$p_1 = -0.7$

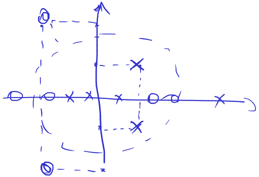
$p_2 = 2.4$

$p_3 = 2.5 + 0.4j$

$p_4 = 2.5 - 0.4j$

$a = -3 - 4j$

complex conjugate: $a^* = -3 + 4j$



Position of poles and signal behavior

$x[n] =$

$X(z) = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \dots$

$a^n \cdot u[n]$

- A rational Z transform $X(z) =$ sum of partial fractions, as we just saw
  - and some initial terms z^k in front
- Each **partial fraction** (pole) generates an **exponential signal**:

  - $a^n u[n]$, or

    - $-a^n u[-n-1]$ $-a^n u[-n-1]$

- For a single partial fraction (one pole only), we will analyze the relation between the position of the pole and the signal in time

Position of poles and signal behaviour - 1 pole

- Consider a single partial fraction with **1 pole** $p_1 = a$:

$$X(z) = C \cdot \frac{z}{z-a}, \quad ROC : |z| > |a|$$

  - Consider only real signals $x[n] \in \mathbb{R}$ —> a is real
  - Consider only causal signals $x[n]$ —> ROC is $|z| > |a|$

- Let's analyze how the corresponding signal looks like
  - use the formulas:

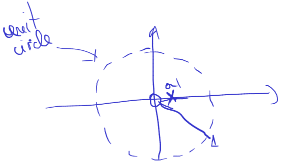
$$x[n] = a^n u[n]$$

$X(z) = \frac{z}{z-a} \longleftrightarrow x[n] = a^n u[n]$

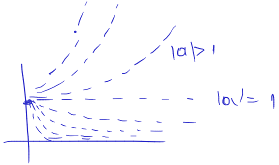
ROC: $|z| > |a|$

$z = 0$

$p = a$

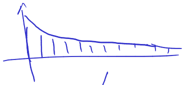


Position of poles and signal behavior - 1 pole

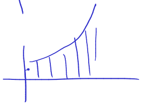


How does the signal look like, depending on the pole value $p_1 = a$:

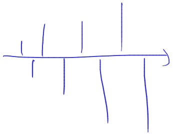
- Pole **inside** the unit circle ($|a| < 1$) = **exponentially decreasing** signal



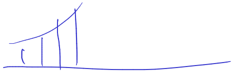
- Pole **outside** the unit circle ($|a| > 1$) = **exponentially increasing** signal



- Pole **exactly on** unit circle ($|a| = 1$)= not increasing, not decreasing, but **constant** signal
- **Negative** pole ($a < 0$) —> **alternating** signal



- **Positive** value ($a > 0$) –> **non-alternating** signal



Position of poles and signal behavior - 1 pole

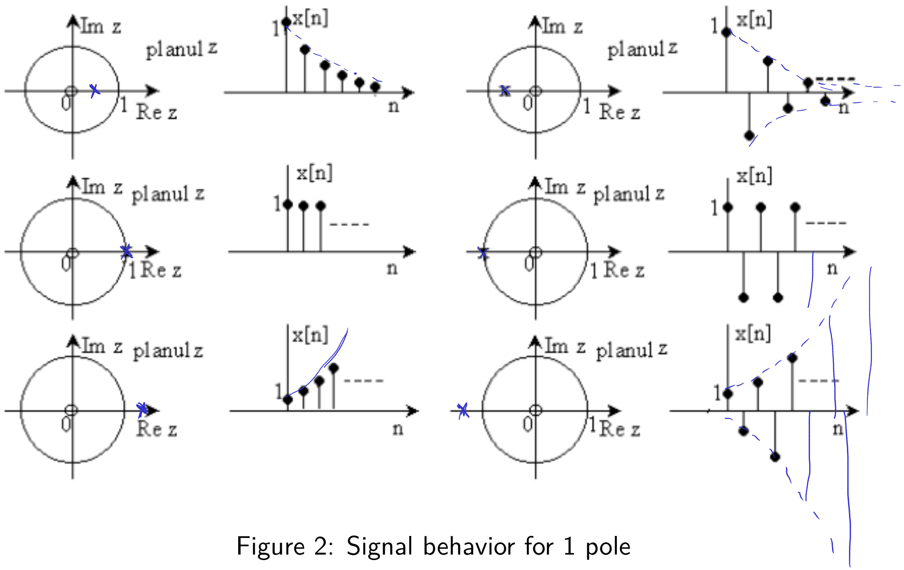


Figure 2: Signal behavior for 1 pole

Position of poles and signal behavior - 1 double pole

- Consider a **double pole** ($p_1 = a$, $p_2 = a$):

$$X(z) = C\frac{az}{(z-a)^2} = C\frac{az^{-1}}{(1-az^{-1})^2}, ROC : |z| > |a|$$

$p_1 = a$
$p_2 = a$

- The corresponding signal is:

$$x[n] = na^n u[n]$$

Effect of double pole in $p_1 = p_2 = a$:

- Pole inside the unit circle ($|a| < 1$) = decreasing signal
- Pole outside the unit circle ($|a| > 1$) = increasing signal
- Pole **exactly on unit** circle ($|a| = 1$) = **increasing signal**
- Negative pole ($a < 0$) = alternating signal
- Positive value ($a > 0$) = non-alternating signal

Position of poles and signal behavior - 1 double pole

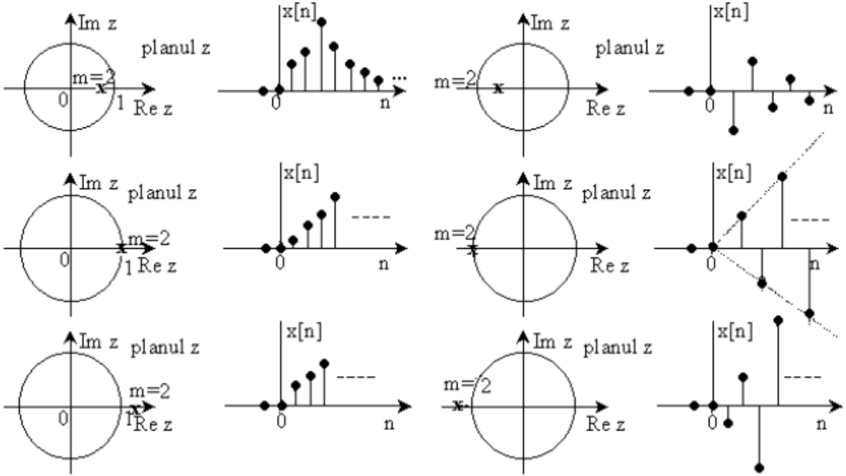
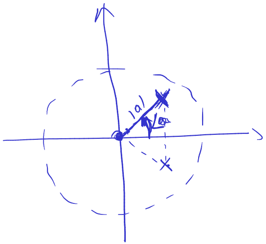


Figure 3: Signal behavior for 1 double pole

Position of poles and signal behavior - conjugate poles

$$X(z) = \frac{z^2}{z^2 + 0.5}$$



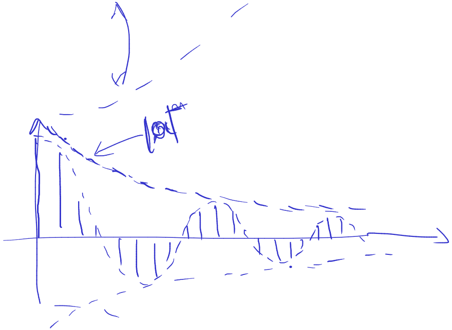
- Consider a **pair of complex conjugate** poles ($p_1 = a$, $p_2 = a^*$):

$$X(z) = \frac{1 - a\cos\omega_0 z^{-1}}{1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}}, ROC : |z| > |a|$$

- The correponding signal is:

$$x[n] = a^n \cos(\omega_0 n) u[n]$$

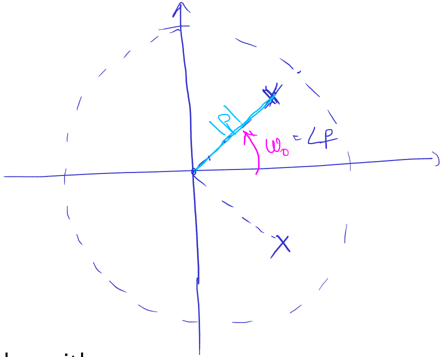
- Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**



Position of poles and signal behavior - conjugate poles

Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**

- **phase** of poles gives the **frequency** of the sinusoidal
- **modulus** of poles gives the **exponential envelope**
  - poles **inside** unit circle = **decreasing** signal
  - poles **outside** unit circle –> **increasing** signal
  - poles **on** unit circle –> **oscillating signal**, constant amplitude, neither increasing nor decreasing



$x[n] = |p|^n \cdot \cos(\omega_0 n)$

$\omega_0 = \angle p$

What if the poles are double?

- poles **on** unit circle –> **increasing** signal
- otherwise, similar to above

Position of poles and signal behavior - conjugate poles

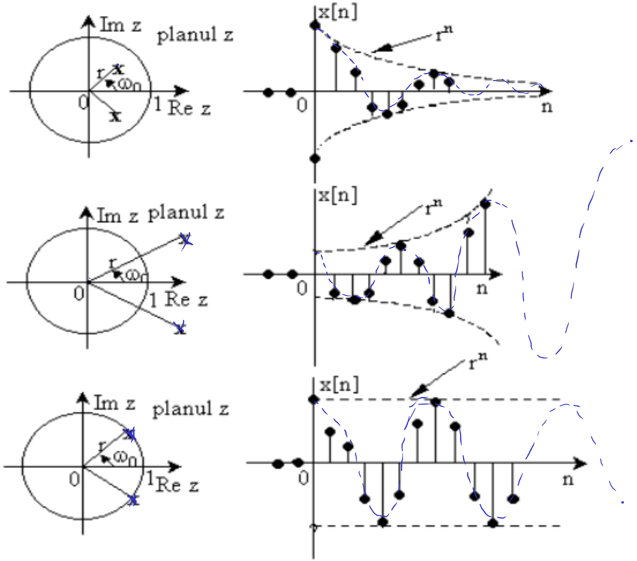


Figure 4: Signal behavior for 1 double pole

Position of poles and signal behavior

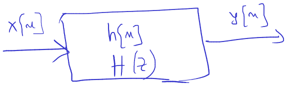
Summary: position of poles and behavior of signal

- A Z transform can be decomposed into **partial fractions**, i.e. separate poles
- Each pole means a separate fraction, means a separate component within the signal
- Conclusions (for real signals, causal):
  - **all poles inside** unit circle = **bounded** signal
    - because all components are exponentially decreasing
  - **simple** poles **on** unit circle = **bounded** signal
    - not increasing to infinity, but also not decreasing
  - **otherwise** = **unbounded** signal
  - poles **closer to 0** = **faster decreasing** signal
  - poles **farther from 0** = **slower decrease** of signal

III.4 LTI systems and the Z Transform

System function of a LTI system

- Consider a LTI system with impulse response $h[n]$



- If we aplpy an input signal $x[n]$, the output is (convolution):

$$\boxed{y[n] = x[n] * h[n]}$$

- In Z transform, **convolution = product** of transforms

$$\boxed{Y(z) = X(z) \cdot H(z)}$$

- **The system function H(z)** of a LTI system = the **Z transform of the impulse response** $h[n]$
- The system function of a LTI system is(you know this from SCS):

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

System function and the system equation

- Reminder: any LTI system has an equation:

$$\boxed{1\cdot} y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\ + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

- which can be rewritten as:

$$y[n] + \sum_{k=1}^{n} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]$$

$1 \cdot Y(z)$ $\quad a_1 \cdot Y(z) \cdot z^{-1}$ $\quad a_2 \cdot Y(z) \cdot z^{-2}$ $\quad \ldots$

$$\rightarrow \quad y[n] + a_1 y[n-1] + a_2 y[n-2] + \ldots + a_N y[n-N] = \\ = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \quad \leftarrow$$

$b_0 \cdot X(z)$ $\quad b_1 \cdot X(z) \cdot z^{-1}$ $\quad \ldots$

System function and the system equation

- The system function $H(z)$ can be written **directly from the equation**
- We apply the Z transform to the whole equation
  - every $y[n-k]$ becomes $z^{-k}Y(z)$
  - every $x[n-k]$ becomes $z^{-k}X(z)$
  - $Y(z)$, $X(z)$ are pulled in front as common factors
- We obtain:

$$Y(z) \cdot \left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z) \cdot \left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}}$$

$$y[n] = -a_1 y[n-1] + b_0 x[n] + b_1 \cdot x[n-1] + b_2 x[n-2]$$

$b_1 = 0$

$$y[n] = 1 \cdot y[n-1] + 0.3\, x[n] + 0.5\, x[n-2]$$

$a_1 = -1$

$$H(z) = \frac{0.3 + 0.5 z^{-2}}{1 - 1 \cdot z^{-1}}$$

System function and the system equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$
$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

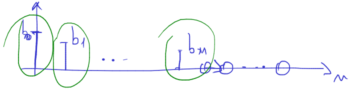
- coefficients b_k of $x[n]$, $x[n-1]$ ... appear at **numerator**
- coefficients a_k of $y[n-1]$, $y[n-2]$ ... appear at **denominator**
  - beware of the sign change of a_k
  - the coefficient of $y[n]$ itself is always $a_0 = 1$

System function of FIR systems

$y[n] = 1 \cdot x[n] + 0.3\, x[n-2] + \ldots$

$H(z) = b_0 + b_1 z^{-1} + \ldots b_M \cdot z^{-M} = \sum_{k=0}^{M} b_k z^{-k}$

$h[n] =$



Particular cases:

- **FIR systems**: when all $a_k = 0$
  - only zeros, no poles ("all-zero system"), no denominator in $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

  - the coefficients b_k are really the impulse response $h[n]$

System function of IIR systems

$H(z) = \frac{\quad}{1 + \dots}$

Particular cases:

- If some $a_k \neq 0$ we have an **IIR system**
  - $H(z)$ has some polynomial at the denominator
  - If ~~denominator~~ numerator is just b_0: **all-pole system**
    - has only poles, no zeros

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

→ no zeros

→ we have poles

  - Or it can be a general IIR system with both poles and zeros (i.e. polynomials both at numerator or denominator)

Stability of a system and H(z)

Reminders from chapter 2:

- **Stable** system = a **bounded input** implies a **bounded output** (BIBO)
- A system is stable if:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \left(\quad \textit{is convergent} \right)$$

Stability of a system and H(z)

$|a \cdot b| = |a| \cdot |b|$

$|a + b| \leq |a| + |b|$

$|\sum \ldots| \leq \sum |\ldots|$

- For a system with system function $H(z)$ we have:

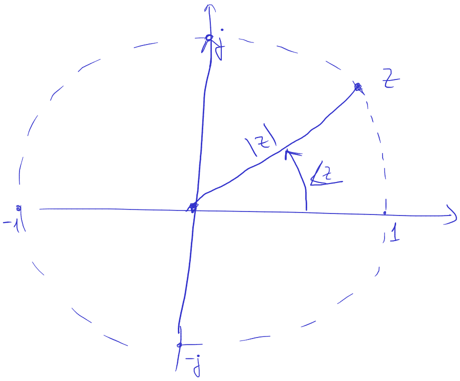
$$|H(z)| = |\underbrace{\sum h[n]z^{-n}}_{\mathcal{Z}\{h[n]\}}| \leq \sum |h[n]| \cdot |z^{-n}|$$

1

- Now let's consider z **on the unit circle**, i.e. $|z| = |z^{-n}| = 1$:

$$|H(z)|\Big|_{|z|=1} \leq \sum |h[n]| \quad < \infty$$

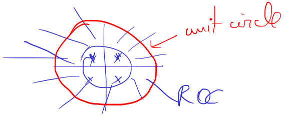
for



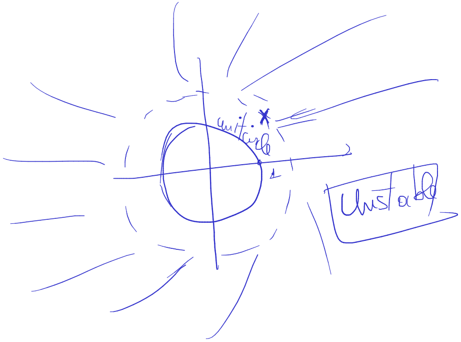
- If the system is **stable**, $\sum |h[n]| < \infty$ (**convergent**), so $|H(z)|\Big|_{|z|=1} < \infty$
  - i.e. the unit circle $|z| = 1$ **is in the ROC** !

Stability of a system and H(z)

- A LTI system is stable if the **unit circle in inside the Region of Convergence** of $H(z)$
  - one can also prove the reciprocal, so there is equivalence



- When the system is also **causal**: $h[n]$ = causal
  - ROC of causal system = exterior of a circle given by the largest pole
  - stable = unit circle inside the ROC
  - therefore stable = all poles **inside** unit circle
- A **causal** LTI system is stable if **all the poles are inside the unit circle**



Stability of a system and H(z)

- Alternative explanation:
  - If one pole is **outside** unit circle, the signal component for that partial fraction will be exponentially **increasing** –> whole signal is **unbounded**

Natural and forced response

- Consider a causal LTI system with initial conditions $= 0$ (IIR)
  - I.C. are relevant for recursive implementations (IIR)
- Consider an input signal:

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{N(z)}{Q(z)}$$

- Consider an impulse response (system function):

$$h[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} H(z) = \frac{B(z)}{A(z)}$$

- Then the output signal is:

$$y[n] = x[n] * h[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} Y(z) = X(z)H(z) = \frac{N(z)B(z)}{Q(z)A(z)}$$

- (Some poles and zeros might simplify, if exactly identical)

Natural and forced response

- Denote the poles of $X(z)$ as q_i and the poles of $H(z)$ as p_i
  - Assume all poles are *simple* (i.e. no multiplicity)
  - Assume all poles $\neq$ all zeros, so no simplification
- The output signal has components dependent on the **input signal** and also of the **system itself**

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

- and $y[n]$ is

poles of the system function $H(z)$

$$y[n] = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u[n]}_{natural\ response} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u[n]}_{forced\ response}$$

poles of the input signal

Natural and forced response

Any output $y[n]$ is the **sum of two signals**:

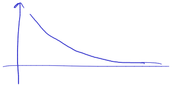
- **Natural response** $y_{nr}[n]$ = the part given by the poles of **the system** itself
- **Forced response** $y_{fr}[n]$ = given by the poles of **the input signal**
- Together they form the **zero-state response** of the system = the output signal when initial conditions are 0

Zero-input response

If there are **non-zero** initial conditions, there is a **third component** as well:

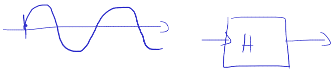
- **Zero-input response** $y_{zi}[n]$ = given by the initial conditions of the system
  - It behaves similarly to the natural response, i.e. depends on the system's poles

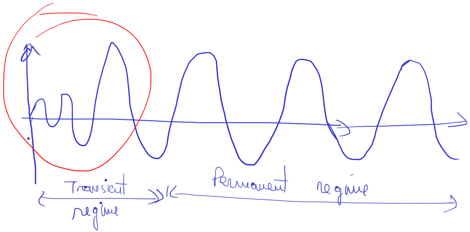
Transient and permanent response



- For a **stable** system, all system poles $|p_k| < 1$
  - therefore, both natural response and zero-input repsonse are made of decreasing exponentials
- For a stable system, the natural response and the zero-input response **die out exponentially**
- Thus, the natural response and the zero-input repsonse are called **transient** response
  - they faded away, usually quickly
- Input signals last indefinitely => the forced response is a **permanent response**

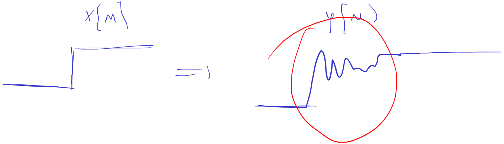
Transient and permanent regime





Operating regimes:

- When the input signal is first applied, and the transient response is present, the system is in **transient regime**
- When the transient response has died out, the system remains in **permanent regime**, where only the input signal determines the output



Example: apply a infinitely long sinusoidal, starting from $n = 0$

- the output has some irregularities at the beginning, due to the natural responses
- afterwards, it becomes perfectly regular