Digital Signal Processing

Chapter V. Digital filtering

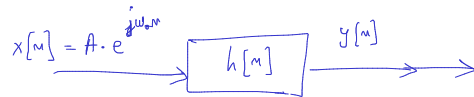# Response of LTI systems to harmonic signals

▶ We consider an LTI system with $h[n]$

▶ Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$

▶ Output signal = convolution

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n}$$

$$= H(\omega_0) \cdot x[n]$$

▶ $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

$x[n] = A \cdot e^{j\omega_0 n}$ → $h[n]$ → $y[n]$

$x[n-k] = A \cdot e^{j\omega_0(n-k)} = A \cdot e^{j\omega_0 n} \cdot e^{-j\omega_0 k}$

$h[n] \underset{DTFT}{\overset{\mathcal{F}}{\longleftrightarrow}} H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

Transfer function

$A \cdot e^{j\omega_a t}$ $\qquad$ $A \cdot H(\omega_a) \cdot e^{j\omega_a t}.$
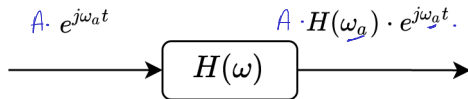
$$\boxed{H(\omega)}$$

Figure 1: Output = a constant $\times$ Input

▶ $H(\omega)$ = Fourier transform of $h[n]$ evaluated for ω = **transfer function**

Eigen-function

$e^{j \omega_0 m}$

- Complex exponential signals are **eigen-functions** (funcții proprii) of LTI systems:
    - output signal = input signal × a (complex) constant

- $H(\omega_0)$ is a constant (complex) that multiplies the input signal
    - Amplitude of input gets multiplies by $|H(\omega_0)|$
    - Phase of input signal is added with $\angle H(\omega_0)$

- Why are sin/cos/exp functions important?
    - If input signal = sum of complex exponential (like coses + sinuses),
    - then output = same sum of complex exponentials, each scaled with some coefficients

$$H(\omega_0) = |H(\omega_0)| \cdot e^{j \angle H(\omega_0)}$$

$$y[m] = H(\omega_0) \cdot A e^{j \omega_0 m} = |H(\omega_0)| \cdot A \cdot e^{j(\omega_0 m + \angle H(\omega_0))}$$

▶ Cosine / sine = sum of two exponentials, via Euler

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}$$

$$\sin(\omega n) = \cos(\omega n - \frac{\pi}{2}) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

▶ System is linear and real =>
  ▶ <u>amplitude</u> is m<u>ultiplied by</u> $|H(\omega_0)|$
  ▶ <u>phase</u> increases by $\angle H(\omega_0)$
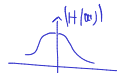
▶ See proof at blackboard

Handwritten notes:

$A \cdot \cos(\omega_0 n) \rightarrow \boxed{h[n] \quad H(\omega)} \rightarrow y[n]$

$\frac{A}{2} e^{j\omega_0 n} + \frac{A}{2} e^{-j\omega_0 n}$

$\frac{A}{2} \cdot H(\omega_0) \cdot e^{j\omega_0 n} + \frac{A}{2} \cdot H(-\omega_0) \cdot e^{-j\omega_0 n}$

$\frac{A}{2} \cdot |H(\omega_0)| \cdot e^{j(\omega_0 n + \angle H(\omega_0))} + \frac{A}{2} \cdot |H(\omega_0)| \cdot e^{-j(\omega_0 n + \angle H(\omega_0))} =$

$= \boxed{A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \angle H(\omega_0))}$

Same for $\sin()$

$H(\omega)$ is an even function:

$H(\omega) = H(-\omega)^*$

$|H(\omega)| = |H(-\omega)|$

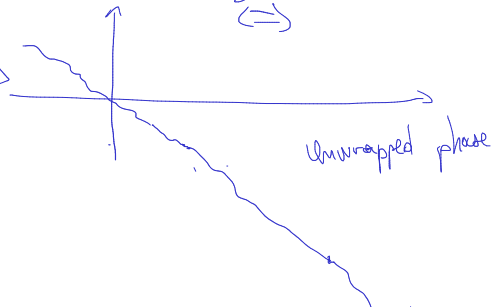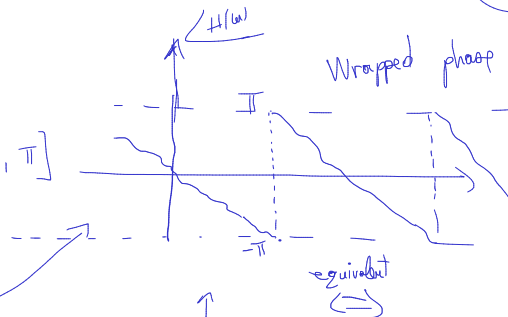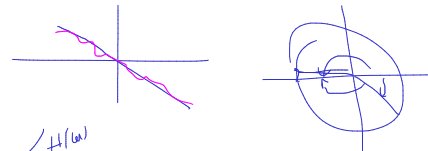$-\angle H(\omega) = \angle H(-\omega)$

Frequency response

- Naming:
  - *or transfer function*
  - $H(\omega) =$ **frequency response** of the system
  - $|H(\omega)| =$ **amplitude response** (or magnitude response)
  - $\angle H(\omega) =$ **phase response**

- Magnitude response is non-negative: $|H(\omega)| \geq 0$

- Phase response is an angle: $\angle H(\omega) \in (-\pi, pi]$

  - Phase response may have jumps of 2π (wrapped phase)
  - Stitching the pieces in a continuous function $=$ phase *unwrapping*
  - Unwrapped phase: continuous function, may go outside interval $(-\pi, pi]$
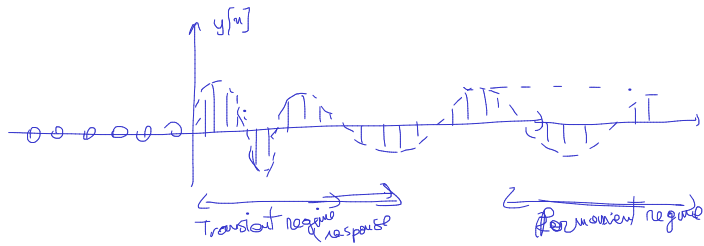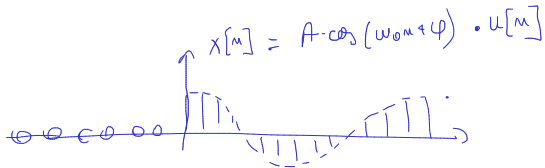 - Example: at blackboard

Permanent and transient response

$$\left.\begin{array}{l} \text{Permanent} \\ \text{response} \end{array}\right\} \begin{array}{l} x[n] = A \cdot \cos(\omega_0 n + \varphi) \;,\; n \in \mathbb{Z} \\ y[n] = A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \varphi + \angle H(\omega_0)) \end{array}$$

$$x[n] = A\cos(\omega_0 n + \varphi)$$

- Warning: $\cos(\omega n)$ does not start at $n = 0$
- The above harmonic signals start at $\underline{n = -\infty}$.
- What's wrong if the signal starts at some time $\underline{n}$?

$$x[n] = A \cdot \cos(\omega_0 n + \varphi) \cdot u[n]$$

$$y[n]$$

Transient regime response   Permanent regime

Permanent and transient response

- ▶ What if the signal starts at some time n?

- ▶ Total response = transient response + permanent response

  - ▶ transient response goes towards 0 as n increases
  - ▶ permanent response = what remains

- ▶ So the above relations are valid only in **permanent regime**

  - ▶ i.e. after the transient regime has passed
  - ▶ i.e. after the transient response has practically vanished
  - ▶ i.e. when the signal started very long ago (from $n = -\infty$)
  - ▶ i.e. when only the permanent response remains in the output signal

- ▶ Example at blackboard
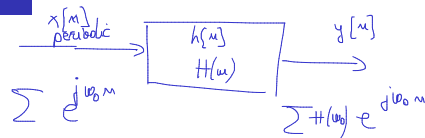
Permanent response of LTI systems to periodic inputs

▶ Consider an input $x[n]$ which is <u>periodic</u> with period N

▶ Then it can be represented as a Fourier series with coefficients c_k:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

▶ Since the system is linear, each component k gets multiplied with $H\left(\frac{2\pi}{N}k\right)$

▶ So the total output is:

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

▶ The output is still <u>periodic</u>, same <u>period</u>, same <u>frequencies</u>

- Consider a general input $x[n]$ (not periodic)
- The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- Output spectrum = Input spectrum $\times$ Transfer function

Response of LTI systems to non-periodic signals

▶ The transfer function $H(\omega)$ "shapes" the spectrum
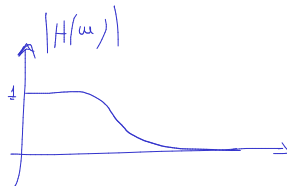
$$Y(\omega) = X(\omega) \cdot H(\omega)$$

▶ In polar form:

  ▶ modulus is multiplied

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$

  ▶ phases is added:

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

- The system **attenuates/amplifies** the input frequencies and **changes their phases**
- $H(\omega) =$ the **transfer function**
- $H(z) =$ the **system function**
- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

ROC

Power spectral density

$$x\{n\} \longmapsto X(\omega)$$

$$|X(\omega)|^2 = S_{xx}(\omega) = \text{Power Spectral Density}$$

$$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2$$

$$S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$$

▶ $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\omega)$

▶ The poles and zeros of $S(\omega)$ come in pairs $(z, 1/z$ and $p, 1/p)$

- LTI systems are also known as **filters** because their transfer function shapes ("filters") the frequencies of the input signals

- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$

- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros
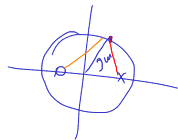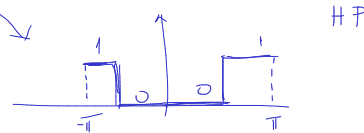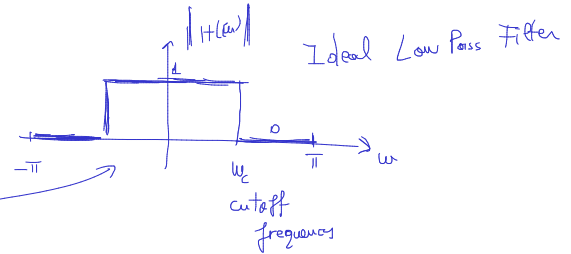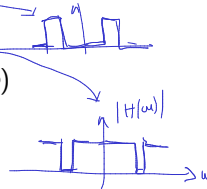
Ideal filters

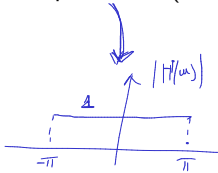- Draw at whiteboard the ideal transfer function of a:
 - low-pass filter
 - high-pass filter
  - band-pass filter
  - band-stop filter
 - all-pass filter (*changes the phase*)

Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
    - i.e. largest power of z or z^{-1}
- Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1})
- Higher order -> better filter transfer function
    - closer to ideal filter
    - more complex to implement
    - more delays (bad)
- Lower order
    - worse transfer function (not close to ideal)
    - simpler, cheaper
    - faster response
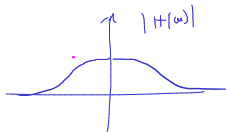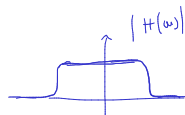
- ▶ Based on geometric method
- ▶ The gain coefficient must be found by separate condition
  - ▶ i.e. specify the desired magnitude response at one frequency
- ▶ Examples at blackboard
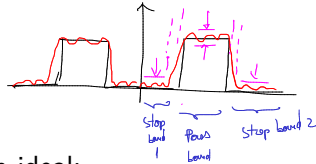
▶ When a filter is non-ideal:

  ▶ non-constant amplitude –> amplitude distortions
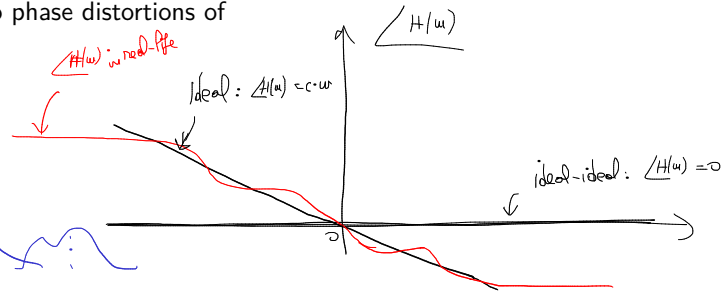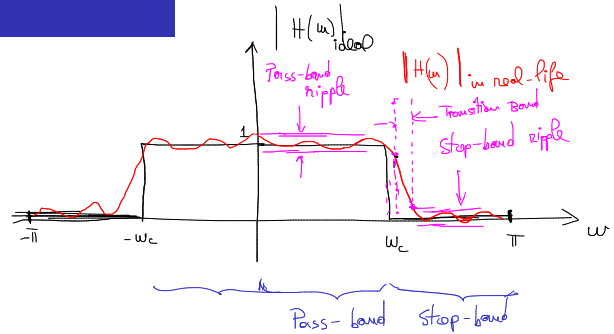 ▶ non-linear phase –> phase distortions

▶ Phase distortions may be tolerated by certain applications

  ▶ e.g. human auditory system is largely insensitive to phase distortions of sounds

How should $\angle H(\omega)$ be in the ideal case:

  1) Ideal-ideal : $\angle H(\omega) = 0 = 0 \cdot \omega$

  2) Ideal : $\angle H(\omega) = linear = c \cdot \omega$

  $\boxed{Linear - phase}$

Effect of system's phase

- What is the effect of system's phase response $\angle H(\omega)$?

- Extra phase = delay
    - different frequencies are delayed differently
    - phase

- **Linear-phase** filter: delays all frequencies with the same amount of time
    - i.e. the whole signal is delayed, but otherwise not distorted
    - otherwise, we get distortions
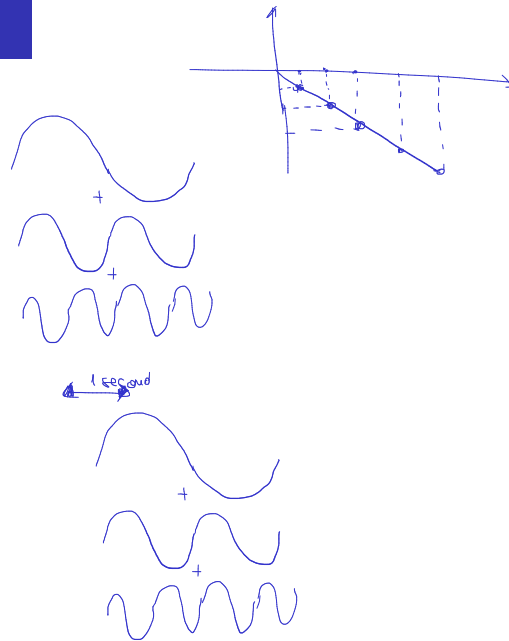
Linear-phase filters

- For a sinusoidal signal, extra phase of 2π = delay of a period $N = \frac{1}{f}$

- To ensure same delay for all frequencies (in time), the phase $\angle H(\omega)$ must be proportional to the frequency

  - draw at blackboard
  - hence the name **linear**

$\dfrac{x[n]}{X(\omega)}$ → | $H(\omega)$ | → $y[n]$

▶ Example: consider the following filter with **linear phase** function:

$$H(\omega) = \underset{\overset{\triangleq}{1}}{C} \cdot e^{-j\omega n_0}$$

$|H(\omega)| = 1$

$\angle H(\omega) = \text{linear} = c \cdot \omega = (-n_0) \cdot \omega$

$\Rightarrow H(\omega) = 1 \cdot e^{-j\omega n_0}$

▶ The output signal is

$$Y(\omega) = X(\omega) \cdot \underset{\overset{=}{1}}{C} \cdot e^{-j\omega n_0}$$

$\overbrace{}^{H(\omega)}$

Fourier Transf. property:

$x[n] \longleftrightarrow X(\omega)$

$x[n-k] \longleftrightarrow e^{-j\omega k} \cdot X(\omega)$

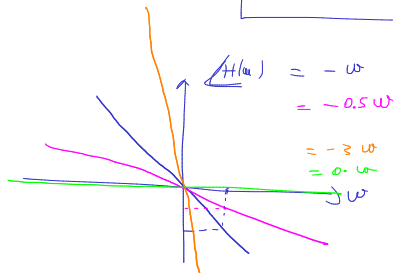$$y[n] = \underset{\overset{=}{1}}{C} \cdot x[n - \boxed{n_0}]$$

▶ <u>Linear phase</u> means **just a delaying** of the input signal
  ▶ Fourier property: $x[n - n_0] \longleftrightarrow X(\omega) e^{-j\omega n_0}$

$\angle H(\omega) = -\omega \Rightarrow y[n] = x[n-1]$

$= -0.5\,\omega \qquad y[n] = x[n = 0.5]$

$= -3\,\omega \qquad y[n] = x[n-3]$

$= 0 \cdot \omega \qquad y[n] = x[n]$

Group delay

- Group delay = The time delay experienced by a component of frequency ω when passing through the filter
    - as opposed to "phase delay" = the phase added by the filter
- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

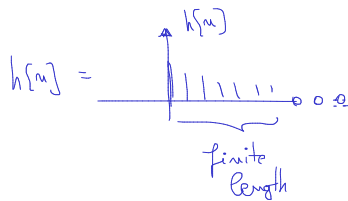- Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

$$\text{IIR}: \quad H(z) = \frac{\sim\sim}{\sim\sim}$$

$$\text{FIR}: \quad H(z) = \frac{\sim\sim}{1}$$

$$h[n] = \quad \text{[graph of } h[n] \text{ with finite length]}$$

What type of filters can have linear phase?

▶ IIR filters cannot have linear phase (no proof provided)

▶ Only FIR filters can have linear phase, and only if they satisfy some symmetry conditions

- Let filter have an impulse response of length M (order is $M - 1$)

- The filter coefficients are $h[0], \ldots h[M - 1]$

- Linear-phase is guaranteed in two cases.

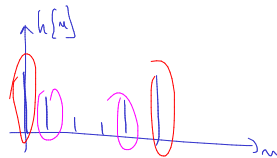 - **Positive symmetry**

    $$h[n] = h[M - 1 - n]$$

  - **Negative symmetry (anti-symmetry)**

    $$h[n] = -h[M - 1 - n]$$

- The delay = the delay of the middle point of the symmetry

$$\frac{M}{2}$$

$h\{0\} = h\{M-1\}$
$h\{1\} = h\{M-2\}$
$h\{2\} = h\{M-3\}$

$h\{M\} = 0.0-0$ { $\quad h[0] \quad h[1] \quad h[2] \quad h[3] \quad h[4] \quad h[5] \qquad h[M-1]$
$\quad 3, \ 2, \ 1, \ 0, \ 4, \ 0, \ 1, \ 2, \ 3, \ 0.00\ldots$

$h\{M\} =$ { $\quad h[e] \qquad h[4] \text{ must be}$
$\quad 3 \ -5 \ 1 \ 4 \ 0 \ -4 \ -1 \ 5 \ -3$

Cases of linear-phase FIR

$$\frac{e^{jx} + e^{-jx}}{2} = \cos(x)$$

Example (1)

$\rightarrow \boxed{h[n]} \rightarrow$

FIR: $h[n] = \dots \bigl[\ \overset{h[0]}{3}\ \overset{h[1]}{-2}\ \overset{h[2]}{1}\ -2\ \overset{h[4]}{3}\ \bigr]\ 0 \dots$

DTFT: $H(\omega) = \sum_n h[n]\, e^{-j\omega n} = 3 \cdot e^{-j\omega \cdot 0} + (-2) \cdot e^{-j\omega} + 1 \cdot e^{-j\omega \cdot 2} + (-2) \cdot e^{-j\omega \cdot 3} + 3 \cdot e^{-j\omega \cdot 4}$

- Proofs at blackboard

1. Positive symmetry, M = odd
2. Positive symmetry, M = <u>even</u>
3. Negative symmetry, M = odd
4. Negative symmetry, M = <u>even</u>

- Check constraints for $H(0)$ and $H(\pi)$
- For what types of filters is each case appropriate?

$$= e^{-j\omega \cdot 2} \cdot \left(3 \cdot e^{j\omega \cdot 2} + (-2) \cdot e^{j\omega} + 1 + (-2)e^{-j\omega} + 3\, e^{-j\omega \cdot 2} \right)$$

$(-2) \cdot 2 \cdot \cos \omega$

$3 \cdot \left(e^{j\,2\omega} + e^{-j\,2\omega} \right) = 6 \cdot \cos(2\omega) \quad \in \mathbb{R}$

$$H(\omega) = \underbrace{\left(1 + 4 \cdot \cos\omega + 6\cos 2\omega \right)}_{|H(\omega)| \ \in \mathbb{R}} \cdot e^{-j\,2\omega} \qquad \angle H(\omega) = -2\omega \ \text{linear} \ (\text{delay by } 2)$$

$$H(\omega) = |H(\omega)| \cdot e^{j\,\angle H(\omega)}$$

$\angle H(\omega)$ linear, delay by 7

- If $h[n]$ longer (e.g. 15 elements): $e^{-j\omega \cdot 0} \dots e^{-j\omega \cdot 14} \Rightarrow e^{-j\omega \cdot 7} \left(e^{+j\omega 7} + \dots + e^{-j\omega 7} \right)$

Zero-phase FIR filters

- Can we avoid delay altogether?

- **Zero-phase** filter = a particular type of linear-phase filter with zero delay
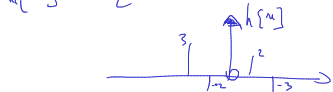
- For a zero-phase filter, the phase response $\angle H(\omega) = 0$

  - (Group) delay = derivative of $\angle H(\omega)$
  - delay 0 $\Leftrightarrow$ flat $\angle H(\omega) \Leftrightarrow \angle H(\omega) = 0$

- Delay is 0 $\Leftrightarrow$ symmetry with respect to $h[0]$

  - the system <u>cannot</u> be causal

Handwritten annotations:

Continued.

$$\frac{e^{jx} - e^{-jx}}{2j} = \sin(x)$$

$$j = e^{j\pi/2}$$

(3) $h[m] = [\,3 \; {}^{\backprime}-2 \; 0 \; 2 \; -3\,]$

$H(\omega) = \sum_m h[m]\, e^{-j\omega m} = 3\cdot e^{-j\omega \cdot 0} + (-2)\cdot e^{-j\omega \cdot 1} + 0 + 2\cdot e^{-j\omega\cdot 3} + (-3)\, e^{-j\omega\cdot 4}$

$= e^{-j\omega\cdot 2}\left(3\cdot e^{+j\omega\cdot 2} + (-2)\cdot e^{+j\omega\cdot 1} + 0 + 2\cdot e^{-j\omega} + (-3)\, e^{-j\omega\cdot 2}\right)$

$(-2)\cdot 2j\cdot \sin\omega$

$3\left(e^{j\omega\cdot 2} - e^{-j\omega\cdot 2}\right) = 3\cdot 2j\cdot \sin(2\omega)$

$H(\omega) = \left(6\sin(2\omega) - 4\sin\omega\right)\cdot j\cdot e^{-j\omega\cdot 2}$

$|H(\omega)|$

$= \left(\right)\cdot e^{j\frac{\pi}{2}}\cdot e^{j(-2\omega + \pi/2)}$

$\angle H(\omega) = \text{linear} = -2\omega + \pi/2$

$H(\omega) = |H(\omega)|\cdot e^{j\angle H(\omega)}$

$h[m] = [\,3\,{}^{\backprime}-2 \; 0 \; 2 \; -3\,]$

$h[-2]\; h[-1]\; h[0]\; h[1]\; h[2]$

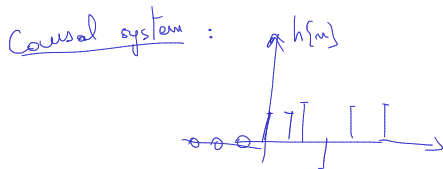# Zero-phase FIR filters

h[n]
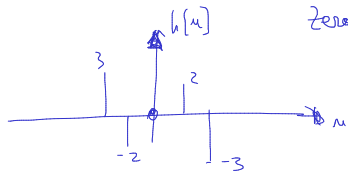
Zero-phase filter is **non-causal**

$\Rightarrow \quad y[n] = 3x[n+2] - 2x[n+1] + 2x[n-1] + \\ - 3x[n-3]$

It cannot run in real time!

- <u>Zero-phase filters</u> must be <u>non-causal</u>
    - left side of $h[n]$ symmetrical to right side of $h[n]$

- For causal, we need to delay $h[n]$ to be wholly on the right side $\Rightarrow$ delay

Causal system :   h[n]

Example

▶ Linear-phase filter (low-pass):
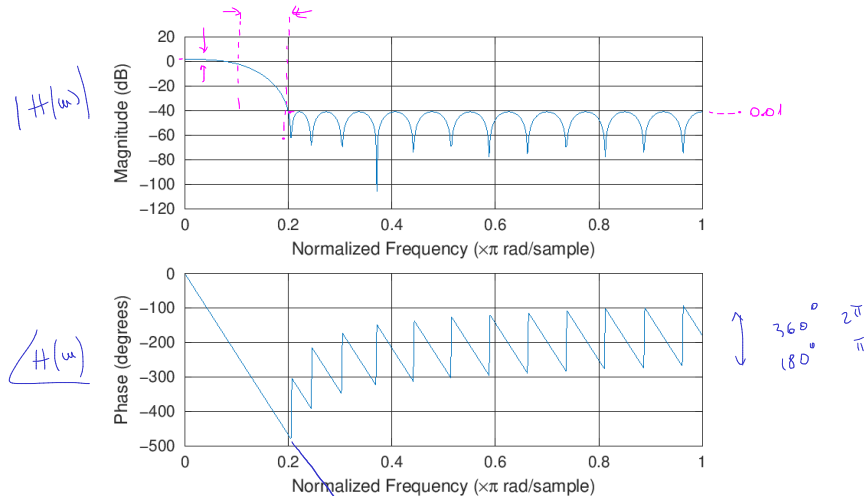
Figure 2: Transfer function of linear-phase filter

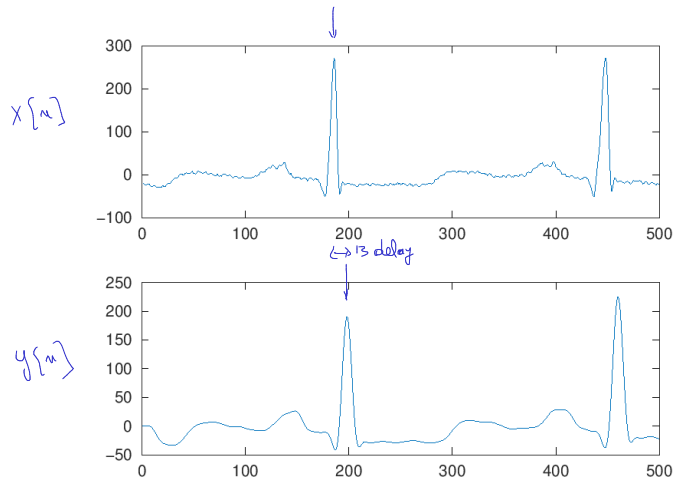# Example

▶ The impulse response (<u>positive symmetry</u>):
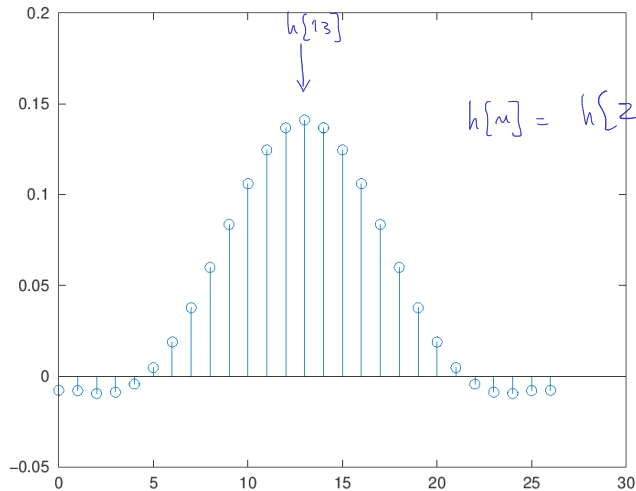
Figure 3: Impulse response of linear-phase filter

Example

▶ ECG signal: original and filtered. Filtering introduces **delay**

Figure 4: Delay introduced by filtering

Example

▶ Solution: <u>zero-phase filter</u> (positive symmetry, and centered in 0):

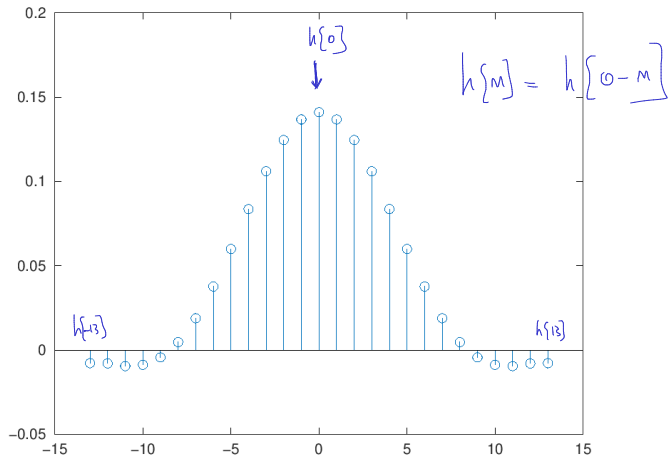▶ But filter is **not causal** anymore

Eliminate the delay

$$h[m] = h[0-m]$$

Figure 5: Impulse response of zero-phase filter
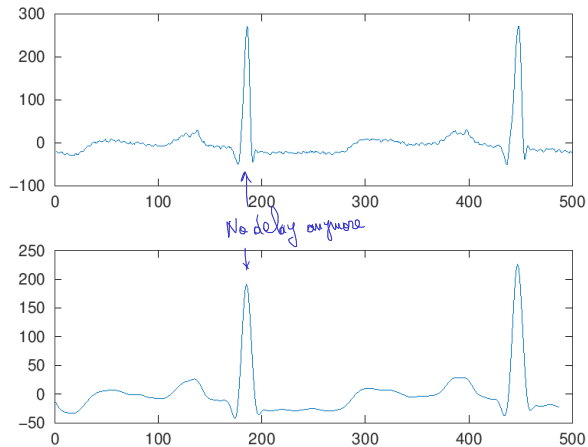
Example

▶ Filtering with zero-phase filter introduces **no delay**

Figure 6: Zero-phase filter introduces no delay

▶ **Digital resonators** = very selective band pass filters
  ▶ poles very close to unit circle
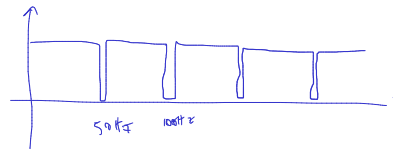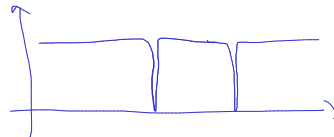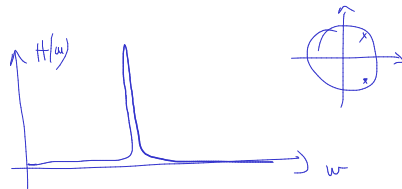 ▶ may have zeros in 0 or at $1/-1$

▶ **Notch filters**
  ▶ have zeros exactly on unit circle
  ▶ will completely reject certain frequencies
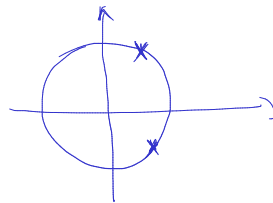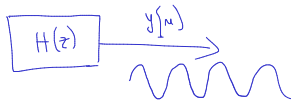 ▶ has additional poles to make the rejection band very narrow

▶ **Comb filters**
  ▶ = periodic notch filters

50 Hz

Digital oscillators

- **Oscillator** = a system which produces an output signal even in absence of input

- Has a pair of complex conjugate poles **exactly on unit circle**

- Example at blackboard

Inverse filters

An inverse filter

$$x[n] \rightarrow \boxed{H(z)} \xrightarrow{y[n]} \boxed{Undo} \xrightarrow{x[n]}$$

▶ Sometimes is necessary to **undo** a filtering

$$H_I(z) = \frac{1}{H(z)}$$

  ▶ e.g. undo attenuation of a signal passed through a channel

▶ Inverse filter: has inverse system function:

$$X(z) \cdot H(z) \cdot \frac{1}{H(z)} = X(z)$$

$$H_I(z) = \frac{1}{H(z)}$$

▶ Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle $->$ unstable

▶ Examples at blackboard