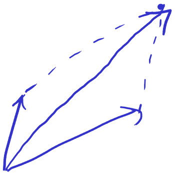
Digital Signal Processing

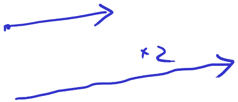
Chapter IV: The Fourier Transform and its applications

IV.1 Vector spaces of signals (crash course)

Vector spaces



- **Vector space** = a set $V = \{v_i\}$ with the following two properties:
  - one element + another element = still an element of the same space
  - a scalar constant $\times$ an element = still an element of the same space
- You **can't escape** a vector space by summing or scaling
- The elements of a vector space are called **vectors**



Examples of vector spaces

- Geometric spaces are great intuitive examples:
  - a line, or the set $\mathbb{R}$ (one-dimensional)
  - a plane, or the set $\mathbb{C}$ (two-dimensional)
  - 3D space (three-dimensional) $\rightarrow$ $\boxed{3 \mid 0 \mid 2}$
  - 4D space (four-dimensional, like the spatio-temporal universe)
  - arrays with N numbers (N-dimensional) $v = \boxed{ \mid \mid \mid \mid \mid }$
  - space of continuous signals (∞-dimensional)

$$\boxed{3 \mid 5} + \boxed{0 \mid 3} = \boxed{3 \mid 8}$$

- The **dimension** of the space = "how many numbers you need in order to specify one element" (informal)
- A "vector" like in maths = a sequence of N numbers = a "vector" like in programming
  - e.g. a point in a plane has two coordinates = a vector of size $N = 2$
  - e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

Inner product

- Many vector spaces have a fundamental operation: **the (Euclidean) inner product**
  - for **discrete** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$
  - for **continuous** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t) \, dt$$
- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** (“produs scalar”)

$$\langle x, y \rangle = \sum_i x_i \cdot y_i^*$$
$$= \text{sum}(x \;.\!*\; y^*)$$
$$= x \cdot y^T$$
$$= x * y';$$

Inner product

- Each entry in $\mathbf{x}$ times the complex conjugate of the one in $\mathbf{y}$, all summed
- For discrete signals, it can be understood as a row $\times$ column multiplication
- Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- Inner product is **linear** in both terms:

$$\langle \mathbf{x_1} + \mathbf{x_2}, \mathbf{y} \rangle = \langle \mathbf{x_1}, \mathbf{y} \rangle + \langle \mathbf{x_2}, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y_1} + \mathbf{y_2} \rangle = \langle \mathbf{x}, \mathbf{y_1} \rangle + \langle \mathbf{x}, \mathbf{y_2} \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x}_{\cancel{1}}, \mathbf{y} \rangle$$

The distance between two vectors

$e = x - y$

- An inner product induces a **norm** and a **distance** function
- **The (Euclidean) distance** between two vectors =

Discrete:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_N - y_N)^2}$$

Continuous:

$$d(x, y) = \sqrt{\int (x(t) - y(t))^2 \, dt}$$

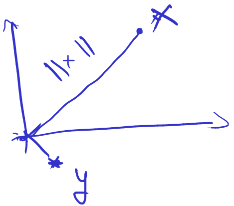
$$d(x, y) = \sqrt{\langle (x-y), (x-y) \rangle}$$

- This distance is the **usual geometric distance** you know from geometry
- It has the exact same intuition like in **normal geometry**:
  - if two vectors have small distance, they are close, they are similar
  - two vectors with large distance are far away, not similar
  - two identical vectors have zero distance

The norm of a vector

- An inner product induces a **norm** and a **distance** function
- The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + ... + x_N^2} = \sqrt{E}$$



- The **norm** of a vector is the distance from **x** to point **0**.
- It has the exact same intuition like in **normal geometry**:
  - vector has large norm = has big values, is far from **0** $\rightarrow$ $x = [203, -79, 7, 25]$
  - vector has small norm = has small values, is close to **0** $\rightarrow$ $y = [-0.3, 2.1, 0.7, 1.9]$
  - vector has zero norm = it is the vector **0**
- Norm of a vector = sqrt(the signal **energy**)

$$E = \sum (x[n])^2$$

Norm and distance

- The norm and distance are related
- The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$$

- Just like in geometry: distance = length of the difference vector

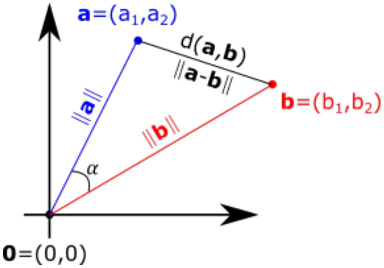
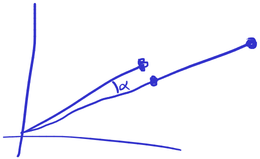


Figure 1: Norm and distance in vector spaces

$x = \begin{bmatrix} x_1, & x_2 \end{bmatrix}$



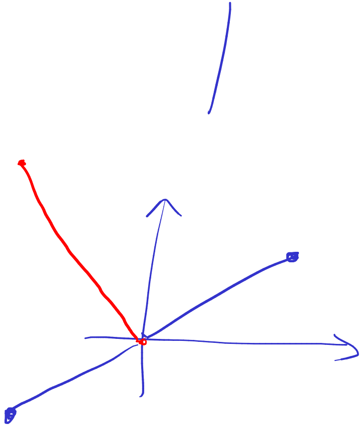
Angle between vectors

- The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - is a value between -1 and 1

- **O~~tr~~**rt**hogonal vectors** $=$ two vectors with $\langle x, y \rangle = 0$
  - their angle $= 90\,\text{deg}$
  - in geometric language, the two vectors are **perpendicular**



Why vector space

- Why are all these useful?
- They are a very general **framework** for different kinds of signals
- We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - Example in DEDP class: ML decision with 1, 2, N samples

Vector spaces in DSP class

We deal mainly with the following vector spaces:

- The vector space of all infinitely-long real signals $x[n]$
- The vector space of all infinitely-long periodic signals $x[n]$ with period N
  - for each N we have a different vector space
- The vector space of all finite-length signals $x[n]$ with only N samples
  - for each N we have a different vector space

Bases

- A **basis** = a set of N linear independent elements from a vector space

dim

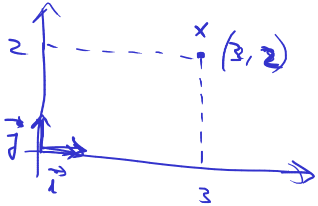
$$B = \{\mathbf{b}^1, \mathbf{b}^2 ... \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\boxed{\mathbf{x} = \alpha_1 \mathbf{b}^{(1)} + \alpha_2 \mathbf{b}^{(2)} + ... + \alpha_N \mathbf{b}^{(N)}}$$

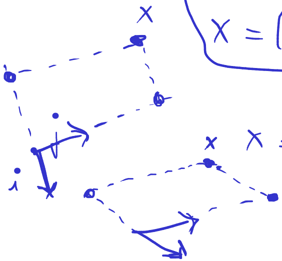
- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, ... \alpha_N)$$



$X = 3 \cdot \vec{i} + 2 \vec{j}$

$X = (3, 2)$



$X = (-2, 3.5)$

something

$X = \text{ceva} \times \vec{i} + \text{altceva} \times \vec{j}$

else

Bases and coordinate systems

- Bases are just like **coordinate systems** in a geometric space
  - any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$$

  - any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1\mathbf{b}^1 + \alpha_2\mathbf{b}^2 + \cdots + \alpha_N\mathbf{b}^N$$

- N = The number of basis elements = The dimension of the space
- Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

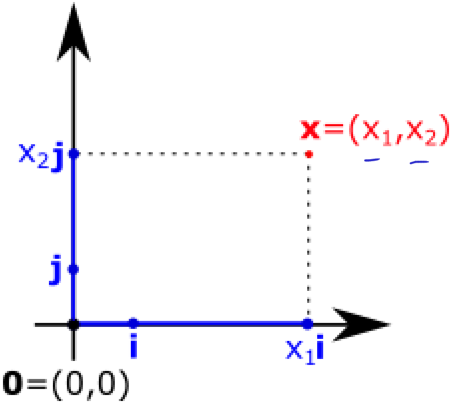
CYMK

K = black

pink = 0.7 R + 0.2 G + 0.1 B (0.7, 0.2, 0.1)

= 0.1 C + 0.1 Y + 0.6 M (0.1, 0.1, 0.6)

Bases and coordinate systems



$$= x_1 \cdot \vec{i} + x_2 \cdot \vec{j}$$

Figure 2: Basis expansion of a vector x

Choice of bases

- There is typically an infinite choice of bases
- The **canonical basis** = all basis vectors are full of zeros, just with one 1
- You used it already in an exercise:
  - any signal $x[n]$ can be expressed of a sum of $\delta[n-k]$

    $$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n-1] + 2\delta[n-2] + \dots$$

  - the canonical basis is $B = \{\dots, \delta[n], \delta[n-1], \delta[n-2], \dots\}$

$x[n] =$

$$x = [2, 3, 1, 5]$$

$$= 2 \cdot [1, 0, 0, 0] + 3 \cdot [0, 1, 0, 0] + 1 \cdot [0, 0, 1, 0] + 5 \cdot [0, 0, 0, 1]$$

Canonical basis

Orthonormal bases

"ortonormala"

- An **orthonormal basis** a basis where all elements $\mathbf{b}^i$ are:
  - orthogonal to each other:

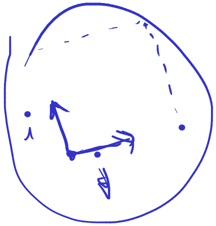
$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

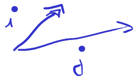
  - **normalized** (their norm $= 1$):

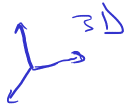
$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

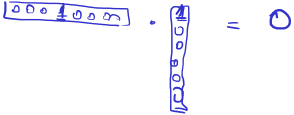
B =

- Example: the canonical basis $\{\delta[n-k]\}$ is orthonormal:
  - $\langle \delta[n-k], \delta[n-l] \rangle = 0, \forall k \neq l$
  - $\langle \delta[n-k], \delta[n-k] \rangle = 1, \forall k$









Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

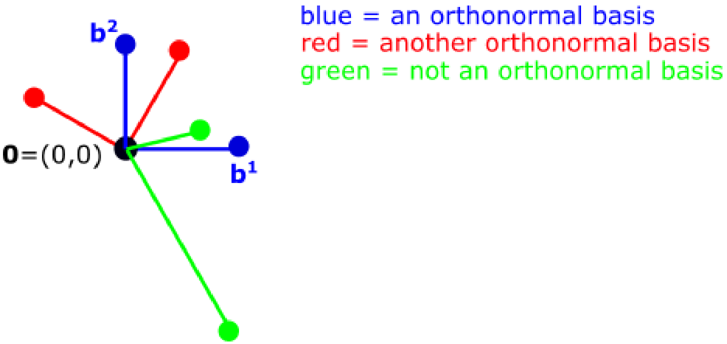


Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- Suppose we have an **orthonormal basis** $B = \{\mathbf{b}^i\}$
- Suppose we have a vector $\mathbf{x}$
- We can write (expand) $\mathbf{x}$ as:

$\alpha_i b^i$

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- Question: how to **find** the coefficients α_i?

Basis expansion of a vector

$$\langle \mathbf{x}_1 + \mathbf{x}_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

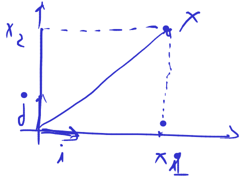
$$\langle \alpha_i \cdot x, y \rangle = \alpha_i \langle x, y \rangle$$

- If the basis is **orthonormal** we have:

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{b}^i \rangle &= \langle \overbrace{\alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N}^{x}, \mathbf{b}^i \rangle \\
&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \alpha_1 \underbrace{\langle \mathbf{b}^1, \mathbf{b}^i \rangle}_{0} + \alpha_2 \underbrace{\langle \mathbf{b}^2, \mathbf{b}^i \rangle}_{0} + \cdots + \alpha_i \cdot \langle b^i, b^i \rangle + \alpha_N \underbrace{\langle \mathbf{b}^N, \mathbf{b}^i \rangle}_{0} \\
&= \alpha_i
\end{aligned}$$

$$\alpha_i = \langle x, b^i \rangle$$

Basis expansion of a vector



- Any vector **x** can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

$\vec{x} = x_1 \cdot \vec{i} + x_2 \cdot \vec{j}$

- For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle \mathbf{x}, \mathbf{b}^i \rangle$$

$\alpha_1 = \langle x, b^1 \rangle$

$\alpha_2 = \langle x, b^2 \rangle$

$\vdots$

$\alpha_N = \langle x, b^N \rangle$

Why bases

$$X[n] = \int X(f)\, e^{j\omega n}\, d\omega$$

T.F. inverse

$$X(f) = \langle x[n], e^{j\omega n} \rangle$$

$$= \sum_n x[n] \cdot e^{-j\omega n} = \text{T.F}$$

- How does all this talk about bases help us?
- To better understand the Fourier transform
- The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- The Fourier Transform of a signal x = finding the coefficients of **x** in this basis
- The Inverse Fourier Transform = expanding **x** with the elements of this basis
- Same **generic** thing every time, only the type of signals differ

IV.2 Introducing the Fourier Transforms

Reminder

- Reminder:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos(x - \frac{\pi}{2})$$

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

Why sinusoidal signals

- Why are sinusoidal signals sin() and cos() **so prevalent** in signal processing?
- Answer: because they are combinations of an e^{jx} and an e^{-jx}
- Why are these e^{jx} so special?
- Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

- Consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\underbrace{x[n-k]} \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

$A \cdot e^{j \cdot \omega_0 n}$

$x[n-k] = Ae^{j\omega_0(n-k)}$

- Output signal = input signal $\times$ a (complex) constant ($H(\omega_0)$)

Eigen-function

- **Eigen-function** of a system ("funcție proprie") = a function f which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

  - just like **eigen-vectors** of a matrix (remember algebra): $A\underset{\approx}{x} = \lambda\underset{\approx}{x}$
  - we call the "functions" to allow for continuous signals as well

$A \cdot v = \lambda \cdot v$

- Complex exponential signals $e^{j\omega t}$ (or $e^{j\omega n}$) are **eigen-functions** of Linear and Time Invariant (LTI) systems:
  - output signal = input signal $\times$ a (complex) constant

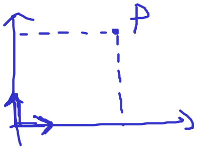
Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system very easily if we **decompose all signals** $x[n]$ **as a combination of** $\{e^{j\omega n}\}$
- Example: RGB color filter
  - suppose we have some photographic filters (lenses):
    - one reduces red to 50%
    - one reduces green to 25%
    - one reduces blue to 80%
    - RGB are eigen-functions of the system: input = 200 Blue, output = 0.8 * 200 Blue
    - what is the output color if input is "pink"?
    - Answer is easy if we represent all colors in RGB

$$\text{pink} = 200R + 50G + 70B$$

$$y = 100R + 12.5G + 56B$$

Representation with respect to eigen-functions (-vectors)



$\vec{P} = 2 \cdot \vec{i} + 3 \cdot \vec{j}$

- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$
- All vector space theory becomes useful now:
  - $\{e^{j\omega n}\}$ is an **orthonormal basis**
  - decomposing signals = finding coefficients α_i
  - we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

Discrete-Time Fourier Transform (DTFT)

- Consider the vector space of **non-periodic infinitely-long signals**
- This vector space is **infinite-dimensional**
- The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis**
- We can expand (almost) any $\mathbf{x}$ in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

$\langle a, b \rangle = \sum a_i b_i^*$

. . . 1 2 0 1 2 3 -2 . . . . .

$\omega = 2\pi f$

$\int x(t) e^{-j\omega t} dt$

T.F.

Discrete-Time Fourier Transform (DTFT)

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- A signal $x[n]$ can be written as a linear combination of $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn} \rangle$

Discrete-Time Fourier Transform (DTFT)

- Alternative form with ω
- We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$\boxed{X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_n x[n] e^{-j\omega n}}$$

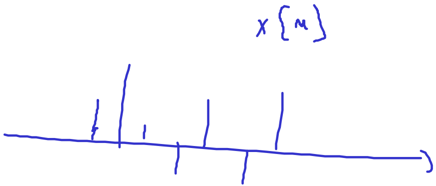
$2\pi f = \omega$

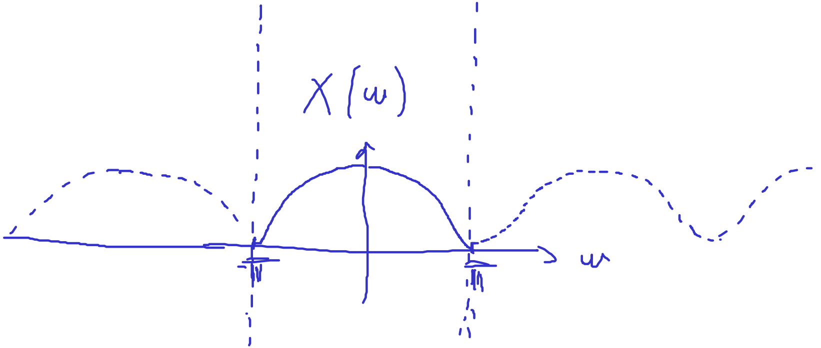
$f = [-\frac{1}{2}, \frac{1}{2}]$

$\omega = [-\pi, \pi]$

Discrete-Time Fourier Transform (DTFT)

- A non-periodic signal $x[n]$ has a **continuous spectrum** $X(\omega)$, with $f \in [-\frac{1}{2}, \frac{1}{2}]$
  - e.g. $\omega \in [-\cancel{\frac{1}{2}}, \cancel{\frac{1}{2}}]$
    $[-\pi, \pi]$





Discrete Fourier Transform (DFT)

- Consider the vector space of **periodic** signals with **period N**
  - for some fixed $N = 2$, 3 or $\ldots$ etc
- This is a vector space of **dimension N**
  - we need N numbers to identify a signal (specify its period)
- We can consider $x[n]$ only for **one period**, i.e. $n = 0, \ldots N-1$
- The signals $\{e^{j2\pi fn}\}, \forall f \in \{0, \frac{1}{N}, \ldots \frac{N-1}{N}\}$ form an **orthonormal basis** with N elements
- It is a **discrete** set of frequencies: $f = \frac{k}{N}, \forall k \in \{0, 1, \ldots N-1\}$

Discrete Fourier Transform (DFT)

$e^{j2\pi fn}$ $f = \frac{k}{N}$ $e^{j2\pi \cdot \frac{k}{N} \cdot n}$

Inverse Discrete Fourier Transform

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- A periodic signal $x[n]$ can be written as a linear combination of ~~k~~ N signals $\{e^{j2\pi kn/N}\}$, with some coefficients X_k

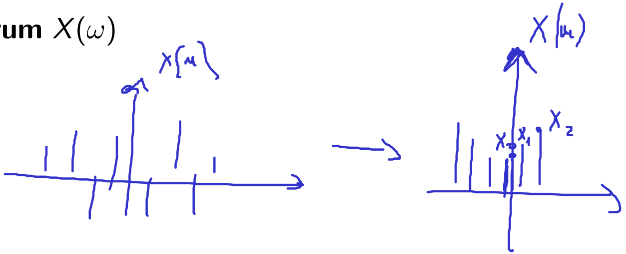
Discrete Fourier Transform

$$X_k = \frac{1}{N}\langle x[n], e^{j2\pi fn}\rangle = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn}\rangle$

Discrete Fourier Transform (DFT)

- A periodic signal $x[n]$ with period N has a **discrete spectrum** $X(\omega)$ composed of only N frequencies $\{0, \frac{1}{N} \dots \frac{N-1}{N}\}$
- Each frequency $\frac{k}{N}$ has a **coefficient** X_k
  - also written as c_k
  - The N coefficients X_k are the equivalent of $X(\omega)$
- It is also known as the "Fourier Series for Discrete Signals"



IV.3 The Discrete-Time Fourier Transform (DTFT)

Definition

Definitions (again):

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=\infty}^{\infty} x[n] e^{-j\underbrace{2\pi f}_{\omega} n}$$

$\omega = 2\pi f$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Basic properties of DTFT

- $X(\omega)$ is defined only for $\omega \in [-\pi, \pi]$
  - or $f \in [-\frac{1}{2}, \frac{1}{2}]$
- $X(\omega)$ is complex (has $|X(\omega)|$, $\angle X(\omega)$)
- If the signal $x[n]$ is real, $X(\omega)$ is **even**

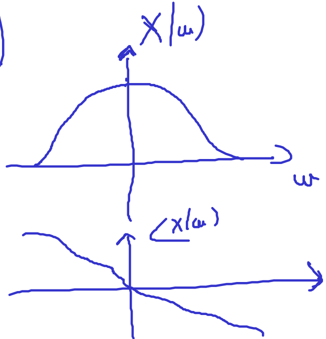
$$x[n] \in \mathbb{R} \rightarrow \boxed{X(-\omega) = X^*(\omega)}$$

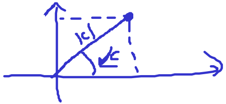
  - This means:
    - modulus is even: $|X(\omega)| = |X(-\omega)|$
    - phase is odd: $X(\omega) = -X(-\omega)$

$$a + jb = c = |c| \cdot e^{j\angle c}$$

$$|c| = \sqrt{a^2 + b^2}$$

$$\angle c = \operatorname{arctg} \frac{b}{a}$$





Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} X(\omega)e^{j\omega n} + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi}(X(\omega)e^{j\omega n} + X(-\omega)e^{j(-\omega)n})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|(e^{j\omega n+\angle X(\omega)} + e^{-j\omega n-\angle X(\omega)})d\omega \\
x[n] &= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|\underbrace{\cos(\omega n + \angle X(\omega))}d\omega
\end{aligned}
$$

$\omega = -\omega$

$e^{j\omega n} + e^{-j\omega n} = 2\cos\omega n$

$f = [0, \frac{1}{2}]$

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

Expressing as sum of sinusoids

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
  - this is the fundamental practical interpretation of the Fourier transform
- The **modulus** $|X(\omega)|$ is the **amplitude** of the sinusoids $(\times\ 2)$
  - for $\omega = 0$, $|X(\omega = 0)|$ = the DC component
- The **phase** $\angle X(\omega)$ gives the initial phase

Properties of DTFT

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

$$x[n] = a \cdot x_1[n] + b \cdot x_2[n] \qquad X(\omega) = \sum_{n} x[n] e^{-j\omega n}$$

$$X(\omega) = a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Properties of DTFT

$$\left| e^{-j\omega n_0} \cdot X(\omega) \right| = \underbrace{\left| e^{-j\omega n_0} \right|}_{1} \cdot \left| X(\omega) \right| = \left| X(\omega) \right|$$

$$c = |c| \cdot e^{j \angle c}$$

$$e^{-j\omega n_0} = \boxed{1} \cdot e^{j \cdot (-\omega n_0)}$$

2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

$$|a \cdot b| = |a| \cdot |b|$$

Properties of DTFT

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of DTFT

5. Convolution

$$\boxed{x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)}$$

- Not circular convolution, this is the normal convolution

Properties of DTFT

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of DTFT

Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

Wiener Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Is true for all orthonormal bases

IV.4 The Discrete Fourier Transform (DFT)

Definitions

DT FT $\neq$ $\boxed{\text{DFT}}$

$x[n] = \dots \underbrace{1\ 2\ 3\ 4}_{N=4}\ 1\ 2\ 3\ 4\ \ 1\ 2\ 3\ 4 \dots\dots$

$e^{j2\pi f n}$, $f = \frac{k}{N}$, $k = \underbrace{0, 1, 2, \dots N-1}_{N}$

Definitions (again)

Inverse Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} = f$$

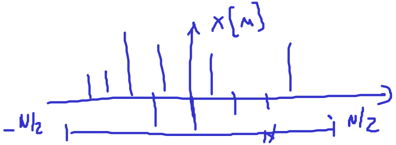
OR

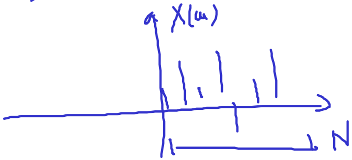
Discrete Fourier Transform (DFT)

$\omega = \frac{2\pi k}{N}$

$$X[k] = X_k = \frac{1}{N}\langle x[n], e^{j2\pi fn}\rangle = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \ \omega$$

, $k = 0, 1, \dots N-1$





Discrete spectrum

Periodicity and notation

$X_k : X_0, X_1, X_2, \ldots X_{N-1}$

$X_9 = X_{-1}$

$f = 0, \frac{1}{N}, \frac{2}{N}, \ldots \frac{N-1}{N}$

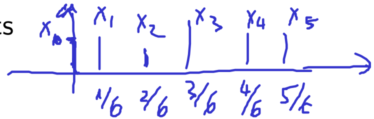
$f = 1/2$

$\boxed{f = 1}$

$f = \frac{9}{10} = \frac{-1}{10}$

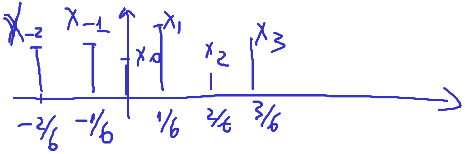
$4 = 2+2$

- In discrete domain, $f = \frac{N-k}{N} = \frac{-k}{N}$ (aliasing, we can subtract 1 from f)
- We can consider X_{N-k} as X_{-k}, due to periodicity
- Example: a signal with period $N = 6$ has 6 DFT coefficients
    - we can call them $X_0, X_1, X_2, X_3, X_4, X_5$
    - we have $X_5 = X_{-1}$, $X_4 = X_{-2}$
    - we can also call them $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$

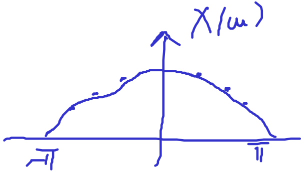


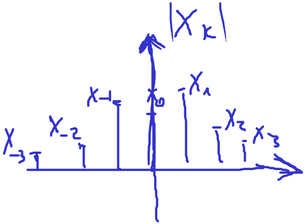
X_4 X_5

$f = \frac{-2}{6}, \frac{-1}{6}, \frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}$



Basic Properties of the DFT





- X_k is complex (has $|X_k|$, $\angle X_k$)
- If the signal $x[n]$ is real, the coefficients are **even**

$$x[n] \in \mathbb{R} \rightarrow \underbrace{\overset{X_{N-k}}{X_{-k}} = X_k^*}_{}$$

  - This means:
    - modulus is even: $|X_k| = |X_{-k}|$
    - phase is odd: $\angle X_{-k} = -\angle X_k$

Expressing as sum of sinusoids

- Grouping terms with k and $-k$:
- If N is odd, we have X_0 and pairs (X_k, X_{-k}):

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X_k e^{j2\pi kn/N} \qquad k=0,\; k=-\frac{N}{2},\; \frac{N}{2}$$

$$X_k : \quad X_{-2},\; X_{-1},\; X_0,\; X_1,\; X_2,$$

$$X_k = |X_k| \cdot e^{j\angle X_k}$$

$$X_{-k} = |X_k| \cdot e^{-j\angle X_k}$$

$$x[n] = \frac{1}{N} X_0 \underbrace{e^{j0n}}_{1} + \frac{1}{N} \sum_{k=-(N-1)/2}^{0} X_k e^{j2\pi kn/N} + \frac{1}{N} \sum_{k=0}^{(N-1)/2} X_k e^{j2\pi kn/N}$$

$$= \frac{1}{N} X(0) + \frac{1}{N} \sum_{k=0}^{(N-1)/2} (X_k e^{j2\pi kn/N} + X_{-k} e^{-j2\pi kn/N})$$

$$= \frac{1}{N} X(0) + \frac{1}{N} \sum_{k=0}^{(N-1)/2} 2|X_k| \underbrace{(e^{j2\pi kn/N + j\angle X(k)} + e^{-j2\pi kn/N - j\angle X(k)})}_{2 \cdot \cos\left(2\pi \frac{kn}{N} + \angle X(k)\right)}$$

$$\frac{e^{j \cdot x} + e^{-j x}}{2} = \cos(x)$$

$$= \underbrace{\frac{1}{N} X(0)}_{\text{D.C.}} + \frac{1}{N} \sum_{k=0}^{(N-1)/2} 2|X_k| \cos(2\pi \underbrace{k/N}_{f=\frac{k}{N}} n + \angle X_k)$$

$$f = \frac{k}{N}$$

$$N = 5 \Rightarrow f = 0, \quad f = \frac{1}{5}, \quad f = \frac{2}{5}$$

Expressing as sum of sinusoids

- If N is even, we have X_0 and pairs (X_k, X_{-k}), with an extra term $X_{N/2}$ which has no pair
  - e.g. N = 6: $X_{-2}, X_{-1}, \overset{\text{DC}}{X_0}, X_1, X_2, X_3$ $\quad k = \frac{N}{2}$
- $X_{N/2}$ must be a real number
- The extra term will be $\frac{1}{N} X_{N/2} e^{j2\pi N/2n/N} = X_{N/2}\cos(n\pi)$ $\qquad f = \frac{1}{2} = \frac{N/2}{N}$
- Overall:

$$2\pi\left(\tfrac{1}{2}\right)n$$

$$x[n] = \frac{1}{N} X(0) + \frac{1}{N}\sum_{k=0}^{(N-2)/2} 2|X_k|\cos(2\pi k/Nn + \angle X_k) + \frac{1}{N}\underbrace{X_{N/2}}_{\in\mathbb{R}}\cos(n\pi)$$

- Any signal $x[n]$ is **a sum of sinusoids with frequencies** $f = 0$, 1/N, 2/N, ... (N-1)/2/N or (N/2)/N $\qquad f = \frac{k}{N}, \; k = 0, 1, 2, \ldots N/2$

Expressing as sum of sinusoids

- Any periodic signal $x[n]$ with period N is **a sum of N/2 sinusoids with frequencies** $f = 0, 1/N, 2/N, \ldots$ (N-1)/2(N) or N/2(N)
- The **modulus** $|X_k|$ gives the **amplitude** of the sinusoids (sometimes $\times$ 2)
  - for $\omega = 0$, $|X_0|$ = the DC component
  - when modulus = 0, that frequency has amplitude 0
- The **phase** $\angle X_k$ gives the initial phase

$N=5: \quad f = 0, \frac{1}{5}, \frac{2}{5}$

$N=9: \quad f = 0, \frac{1}{9}, \frac{2}{9}, \ldots \frac{4}{9}$

$N=20: \quad f = 0, \frac{1}{20}, \frac{2}{20}, \ldots \frac{10}{20}$

$|X_7| = 0 \Rightarrow f = \frac{7}{N}$ is not in our signal

Example

- Consider a periodic signal $x[n]$ with period $N = \cancel{6}$ 5 and the DFT coefficients:

  X_0 X_1 X_2

  X_k = [15.0000 + 0.0000i , -2.5000 + 3.4410i , -2.5000 + 0.8123i , -2.5000 - 0.8123i , -2.5000 - 3.4410i]

  $X_3 = X_{-2}$ $X_4 = X_{-1}$

  Write $x[n]$ as a sum of sinusoids.

- Do the same for a periodic signal $x[n]$ with period $N = \cancel{5}$ 6 and the DFT coefficients:

  X_0 X_1 X_2

  X_k = [21.0000 + 0.0000i , -3.0000 + 5.1962i , -3.0000 + 1.7321i , -3.0000 + 0.0000i , -3.0000 - 1.7321i , -3.0000 - 5.1962i]

  X_3 $X_4 = X_{-2}$ $X_5 = X_{-1}$

  Write $x[n]$ as a sum of sinusoids.

Properties of the DFT

DFT: $x[n] \longleftrightarrow X_k$

1. Linearity

If the signal $x_1[n]$ has the DFT coefficients $\{X_k^{(1)}\}$, and $x_2[n]$ has $\{X_k^{(2)}\}$, then their sum has

$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$

Proof: via definition

Properties of the DFT

$x = [1 \quad 2 \quad 3 \quad 4 \quad 5]$

$x = [4 \quad 5 \quad 1 \quad 2 \quad 3]$

... 4 5 | 1 2 3 4 5 | 1 2 3 ...

... 2 3 | 4 5 1 2 3 | 4 5 1 ...

2. Shifting in time (circular)!

If $x[n] \leftrightarrow \{X_k\}$, then

$$x[n - n_0] \leftrightarrow \{\underbrace{e^{(-j2\pi k n_0/N)}}_{|\cdot| = 1} X_k\}$$

... | 1 2 3 4 5 |

| 4 5 1 2 3 |

Proof: via definition

- The amplitudes $|X_k|$ are not affected, shifting in time **affects only the phase**

Properties of the DFT

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{X^*_{-k}\}$$

Properties of the DFT

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

Circular convolution definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is also periodic with period N
- Example at the whiteboard: how it is computed

$x_1 = [1 \quad 2 \quad 3 \quad 2]$

$x_2 = [2 \quad 2 \quad 4 \quad 4]$

$x_1 \circledast x_2$:

2	2	4	4
8	4	4	8
12	12	6	6
4	8	8	4
~~2~~	~~6~~	~~14~~	~~22~~
26	26	22	22

2 6 26 22 22

Example

Example (write on slides)

Circular convolution

- We are in the vector space of **periodic signals** with period N
- Linear (e.g. normal) convolution produces a result which is longer ~~periodic with period N~~
- Circular convolution takes two sequences of length N and produces another sequence of length N
  - each sequence is a period of a periodic signal
  - circular convolution = like a convolution of periodic signals

Properties of the DFT

6. Product in time

Product in time $\leftrightarrow$ circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

Properties of the DFT

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{0}^{N-1} |x[n]|^2 = \frac{1}{\underset{(N)}{2\pi}} \sum_{k=0}^{N} |X_k|^2$$

- Is true for all orthonormal bases

Relationship between DTFT and DFT

- How are DTFT and DFT related?
- Discrete Time Fourier Transform:
  - for non-periodical signals
  - spectrum is continuous

  $X(f)$

- Discrete Fourier Transform
  - for periodical signals
  - spectrum is discrete

  X_k

- Duality: periodic in time $\leftrightarrow$ discrete in frequency

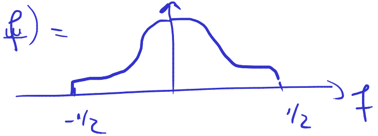
Relationship between DTFT and DFT

- Consider a non-periodic signal $x[n]$

$x[n] = \ldots 0\ 0\ [1\ 2\ 3\ 4\ 5]\ 0\ 0 \ldots$

- It has a continuous spectrum $X(\omega)$

$X(f) =$



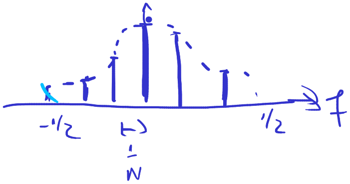
- If we **periodize** it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

$x_N[n] = \ldots 1\ 2\ 3\ 4\ 5\ [1\ 2\ 3\ 4\ 5]\ 1\ 2\ 3\ 4\ 5\ 1 \ldots$

- then the Fourier transform is **discrete** (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k\frac{2\pi}{N})$$



- The coefficients of the Diracs = the DFT coefficients

$$X_k = X(2\pi k/N)$$

- They are **samples** from the continuous $X(\omega)$ of the non-periodized signal

Relationship between DTFT and DFT

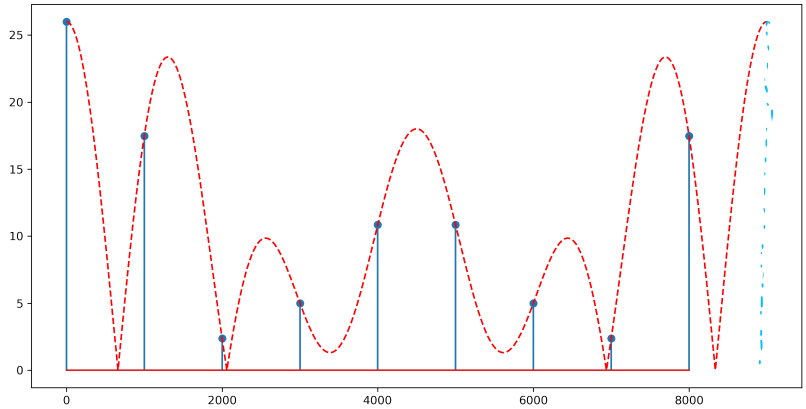
- Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2]$$

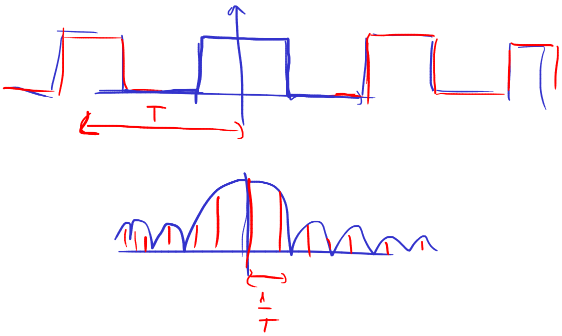
- If we consider a non-periodic $x[n]$ with infinitely long zeros on either side, we have a continuous spectrum $X(\omega)$ (DTFT)
- If we consider that x is just a period of a periodic signal, we have a discrete spectrum X_k (DFT)
- Moreover, the discrete X_k are just **samples from** $X(\omega)$:

$$X_k = X(2\pi k / Nn)$$

Relationship between DTFT and DFT



Scs:



$x = [6, 5, 4, -3, 2, -3, 4, 5, 6]$

- red line = ~~D~~ DT FT(x) if x not periodical
  - actually run as `fft(x, 10000)`, x is extended with 9991 zeros
- blue = `fft(x)` DFT(x)

Relation between DTFT and Z transform

- Z transform:

$$X(z) = \sum_{n} x[n] z^{-n}$$

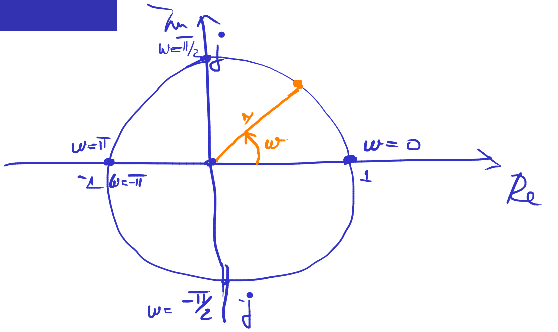
- DTFT:

$$X(\omega) = \sum_{n} x[n] \underbrace{e^{-j\omega}}_{z}{}^{n}$$

- DTFT can be obtained from Z transform with

$$z = e^{j\omega}$$

- These $z = e^{j\omega}$ are **points on the unit circle**
  - $|z| = |e^{j\omega}| = 1(\textit{modulus})$
  - $\angle z = \angle e^{j\omega} = \omega(\textit{phase})$



$z = a + jb$

$z = \underbrace{|z|}_{1} \cdot e^{j\angle z}$

$X(\omega) = X(z)\Big|_{z=e^{j\omega}} = X(z)$ evaluated on the unit circle

Relation between DTFT and Z transform

- Fourier transform = Z transform evaluated **on the unit circle**
  - if the unit circle is in the convergence region of Z transform
  - otherwise, equivalence does not hold
- This is true for most usual signals we work with
  - some details and discussions are skipped

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$z = e^{j\omega}$ $\downarrow$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

!

- Phase:

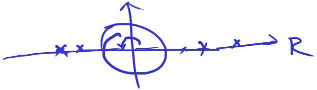
$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

$|a \cdot b| = |a| \cdot |b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\angle a \cdot b = \angle a + \angle b$

$\angle \frac{a}{b} = \angle a - \angle b$

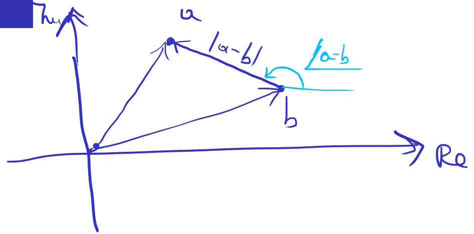


Geometric interpretation of Fourier transform

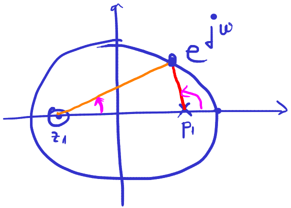
$a = a_1 + a_2 j$

$b = b_1 + b_2 j$

$|a - b| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$



- For complex numbers: a and $b \in \mathbb{C}$
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**



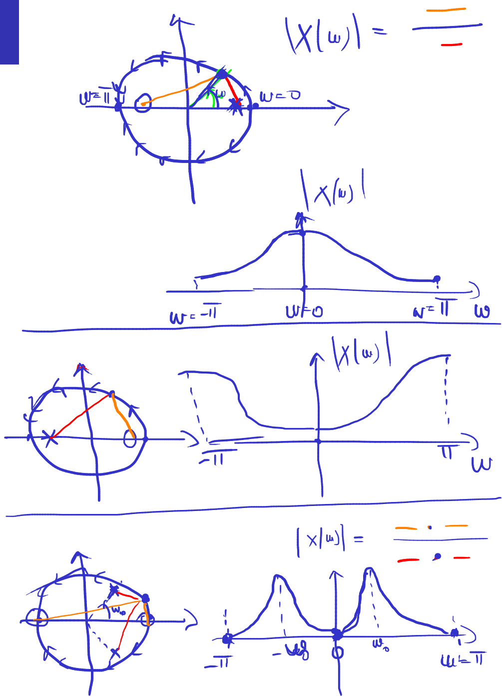
$\angle X(\omega) = \angle - \angle$

Geometric interpretation of Fourier transform

- Consequences:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples: ...

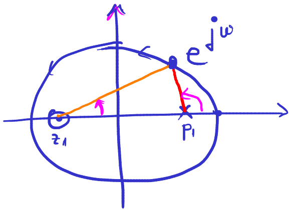
Geometric interpretation of Fourier transform

- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is like **a landscape**
    - **poles = mountains** of infinite height
    - **zeros = valleys** of zero height
  - Fourier transform $X(\omega)$ = "*Walking over this landscape along the unit circle*"
  - The height profile of the walk gives the amplitude of the Fourier transform
  - When close to a mountain –> road is high –> Fourier transform has large amplitude
  - When close to a valley –> road is low –> Fourier transform has small amplitude



Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for phase as well



$$\angle X(\omega) = \angle - \angle$$

$$\angle X(0) = 0$$

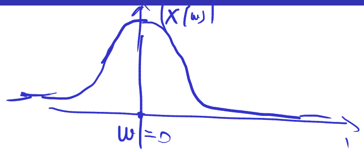
$$\angle X(\pi) = 0$$

Time-frequency duality

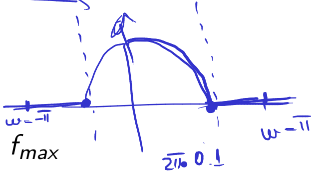
- **Duality** properties related to all Fourier transforms
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Terminology

$\omega = 2\pi f$



- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals



- **Band-limited** signals: spectrum is 0 beyond some frequency f_{max}
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: $B << $ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band