Digital Signal Processing

II. Discrete signals and systems

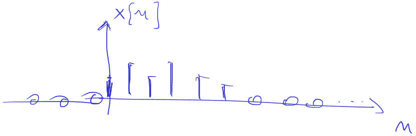
II.1 Discrete signals

Representation

A discrete signal can represented:

- graphically
- in table form
- as a vector: $x[n] = [..., 0, 0, 1, 3, 4, 5, 0, ...]$
  - an **arrow** indicates the origin of time ($n = 0$).
  - if the arrow is missing, the origin of time is at the first element
  - the dots ... indicate that the value remains the same from that point onwards

$x[n] = \cos(2\pi \cdot 0.2 \cdot n)$



$x[n] = \{\ldots 0, \underset{\uparrow\, n=0}{\overset{x[0]}{0}}, \overset{x[1]}{1}, \overset{x[2]}{2}, \overset{x[3]}{3}, 0, \ldots\}$

Examples: at blackboard

Notation: $x[4]$ represents the value of the fourth sample in the signal $x[n]$

Basic signals

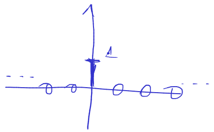
Some elementary signals are presented below.

Unit impulse

Contains a single non-zero value of 1 located at time 0. It is denoted with $\delta[n]$.

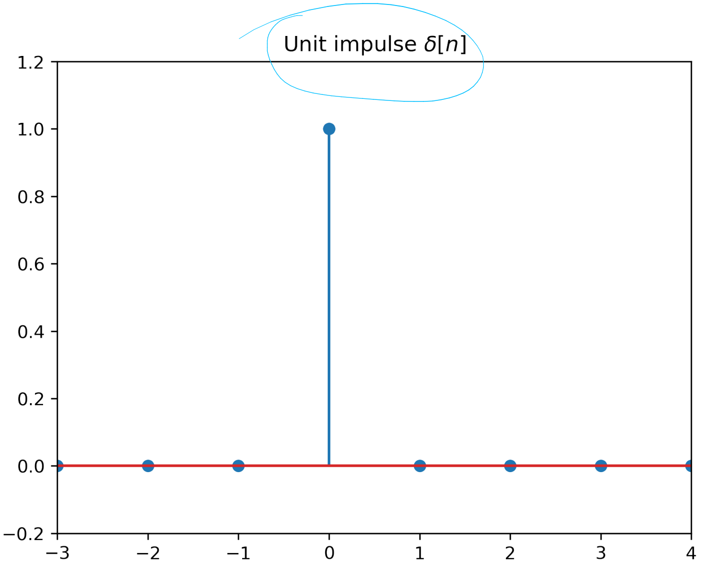
$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$\delta[n] =$



$= \{\dots 0, \underset{\uparrow}{1}, 0, \dots\}$

Representation



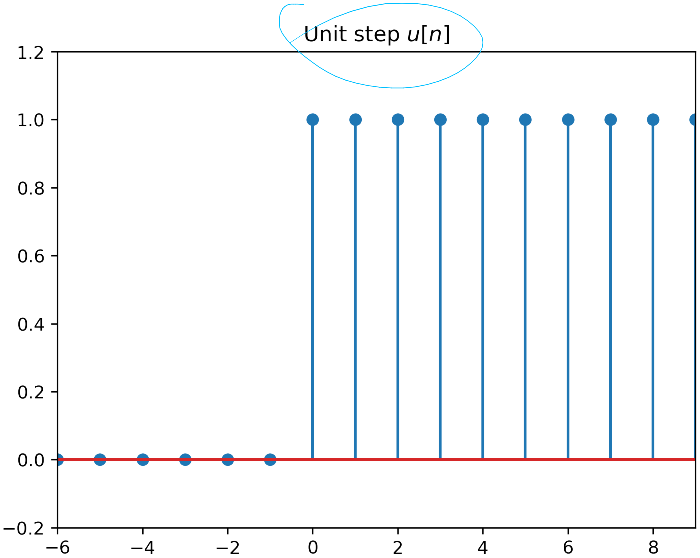
Unit step

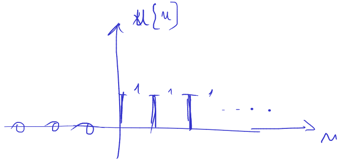
Unit step

It is denoted with $u[n]$.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation





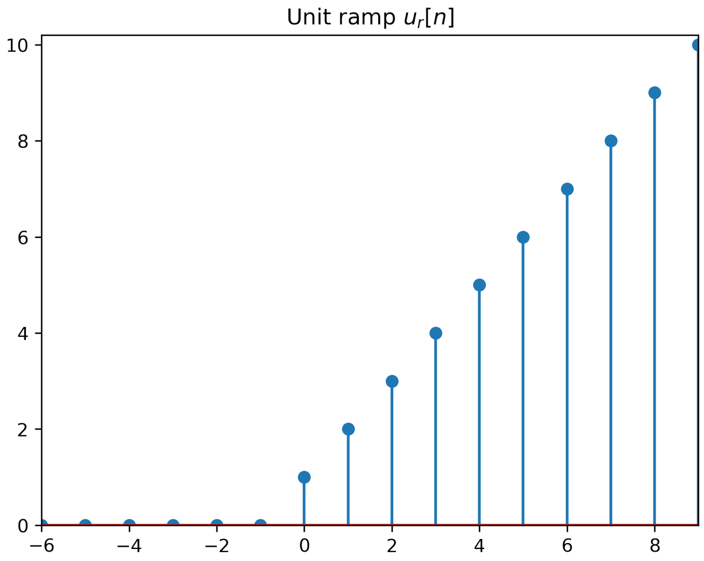
Unit ramp

Unit ramp

It is denoted with $u_r[n]$.

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Representation



Exponential signal

Exponential signal

It does not have a special notation. It is defined by:

$$x[n] = a^n$$

a can be a real or a complex number. Here we consider only the case when a is real.

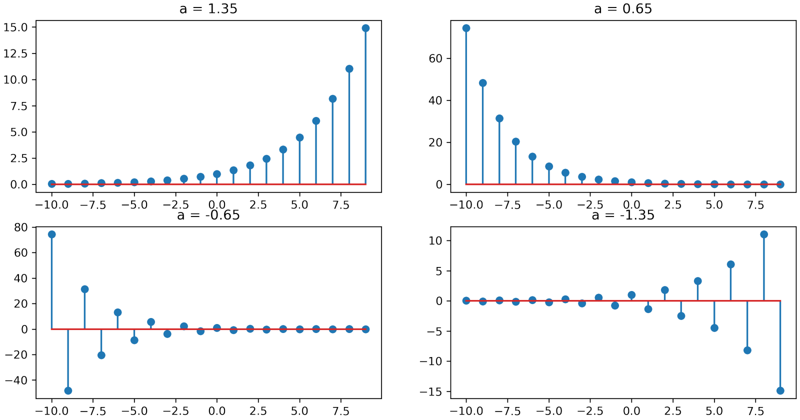
Depending on the value of a, we have four possible cases:

1. $a \geq 1$
2. $0 \leq a < 1$
3. $-1 < a < 0$
4. $a \leq 1$

$x[n] = a^n$

Representation

$a > 1$ $\quad$ $0 < a < 1$



$-1 < a < 0$ $\quad$ $a < -1$

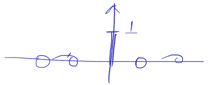
II.2 Types of discrete signals

Signals with finite energy

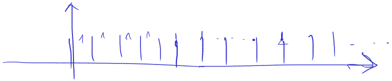
- The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} (x[n])^2.$$

- If E is finite, the signal is said to have finite energy.
- Examples:
  - unit impulse has finite energy
  - unit step does not



$E = 1$



$E = \infty$

$-\infty$

$+\infty$

Connection with DEDP class

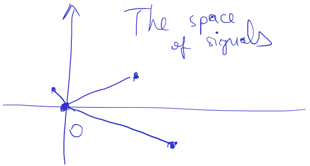
$\|a-b\| = d(a,b)$

- Cross-link with DEDP course:

$$E = \underbrace{\|\mathbf{x} - \mathbf{0}\|^2}_{d(x,0)} = \|\mathbf{x}\|^2$$

- Energy of a signal = **squared Euclidean distance to 0**
  - geometric interpretation: squared length of the segment from 0 to the point **x**
  - holds for continuous signals as well:

$$E = \|\mathbf{x}\|^2 = \int_{-\infty}^{\infty} x^2(t)dt$$



$E = d^2(x, 0)$

$x = (x_1, x_2, x_3)$ N

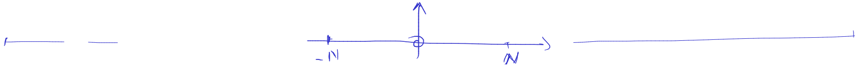
$0 = (0, 0, 0)$

$d(\cancel{x}, 0) = \sqrt{(x_1-0)^2 + (x_2-0)^2 + (x_3-0)^2}$

$= \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots\dots}$

$d(x,0)^2 = \sum_{\text{all}} (x[n])^2 = E$

Signals with finite power



- The **average power of a discrete signal** is defined as

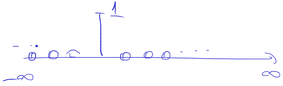
$$P = \lim_{N\to\infty} \frac{\sum_{n=-N}^{N}(x[n])^2}{2N+1}.$$

$= \text{Average value of } (x[n])^2 = \dfrac{\sum \text{all } (x[n])^2}{\text{How many}} = \dfrac{E}{\text{How many}}$

- In other words, the average power is the average energy per sample.

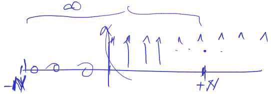
$= \text{the average } (x[n])^2$

- If P is finite, the signal is said to have finite power.

$\delta[n] =$ 

$P = \dfrac{1}{\infty} = 0$

- A signal with finite energy has finite power ($P = 0$ if the signal has infinite length). A signal with infinite energy can have finite or infinite power.

$u[n] =$ 

$P = \dfrac{\infty}{\infty}? =$

- Example: unit step has finite power $P = \frac{1}{2}$ (proof at blackboard).

$P = \lim_{N\to\infty} \dfrac{N+1}{2N+1} = \dfrac{1}{2}$

Connection with DEDP class

- Average power = temporal average squared value $\overline{X^2}$ = the average of the x^2
  - i.e. average value of the square of samples

$x[n] =$

Periodic and non-periodic signals

- A signal is called **periodic** if its values repeat themselves after a certain time (known as **period**).

$$x[n] = x[n + N]), \forall t$$

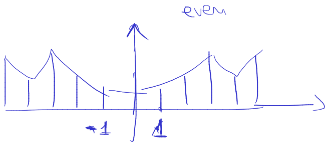
- The **fundamental period** of a signal is the minimum value of N.
- Periodic signals have infinite energy, and finite power equal to the power of a single period.

Even and odd signals

symmetrical!

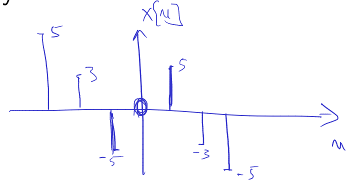
- A real signal is **even** if it satisfies the following symmetry:

$$x[n] = x[-n], \forall n.$$



- A real signal is **odd** if it satisfies the following anti-symmetry:

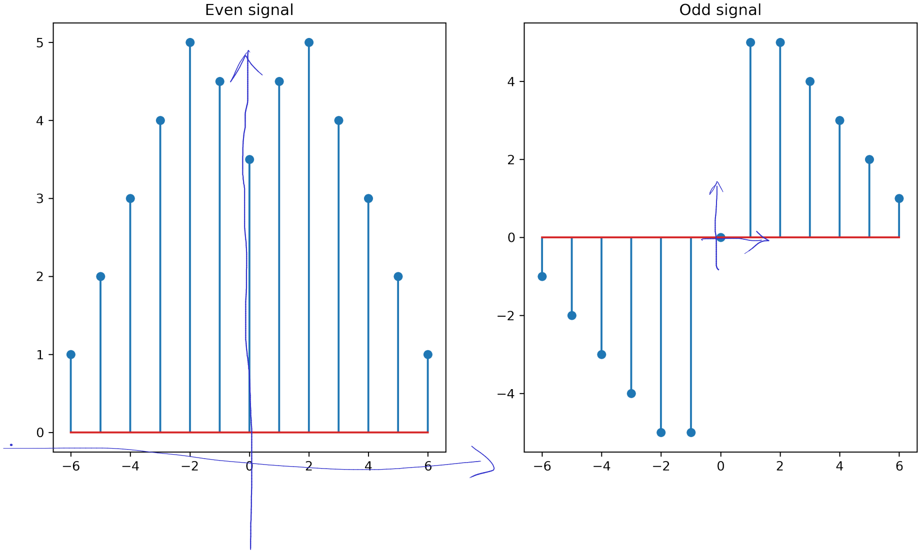
$$-\underbrace{x[n]}_{\text{right}} = \underbrace{x[-n]}_{\text{left}}, \forall n.$$



odd signal

- There exist signals which are neither even nor odd.

Even and odd signals: example



Even and odd parts of a signal

- Every signal can be written as the sum of an even signal and an odd signal:

$$x[n] = \underbrace{x_e[n]}_{\text{even}} + \underbrace{x_o[n]}_{\text{odd}}$$

- The even and the odd parts of the signal can be found as follows:

$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

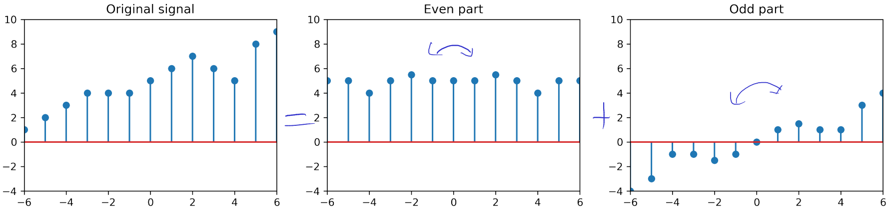
even : $x_e[n] = x_e[-n]$

$$x_o[n] = \frac{x[n] - x[-n]}{2}.$$

odd : $x_o[-n] = -x_o[n]$

- Proof: check that $x_e[n]$ is even, $x_o[n]$ is odd, and their sum is $x[n]$

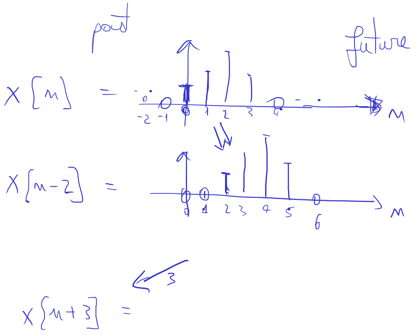
Even and odd parts: example



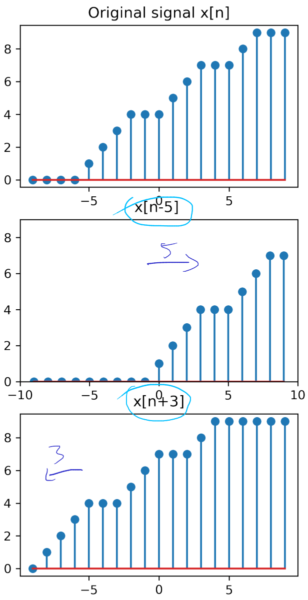
II.3 Basic operations with discrete signals

Time shifting

- The signal $x[n-k]$ is $x[n]$ **delayed with k time units**
  - Graphically, $x[n-k]$ is shifted k units to the **right** compared to the original
- The signal $x[n+k]$ is $x[n]$ **anticipated with k time units**
  - Graphically, $x[n+k]$ is shifted k units to the **left** compared to the original signal.

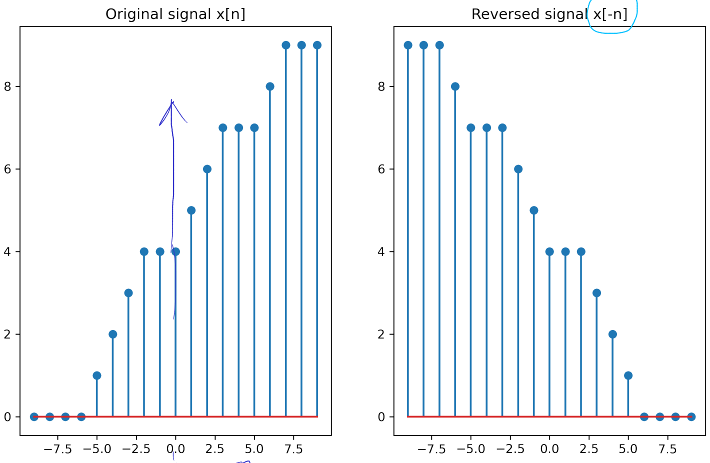


Time shifting: representation



Time reversal

- Changing the variable n to $-n$ produces a signal $x[-n]$ which mirrors $x[n]$.



SWAP

Subsampling

Subesantionare

- **Subsampling** by a factor of M = keep only 1 sample from every M of the original signal
  - Total number of samples is reduced M times

$$x_{M\downarrow}[n] = x[Mn]$$

new signal — old signal

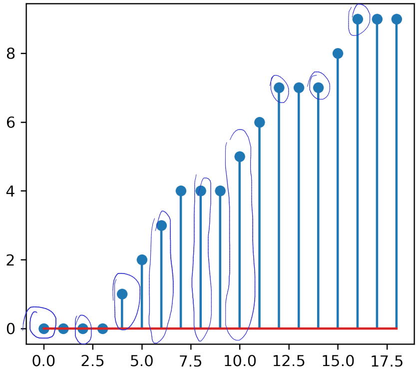
$X = \{6\ -3\ 0\ -1\ 2\ -1\ -1\ 4\ 5\ \ldots\}$

$X_{2\downarrow} = X[2 \cdot n]$

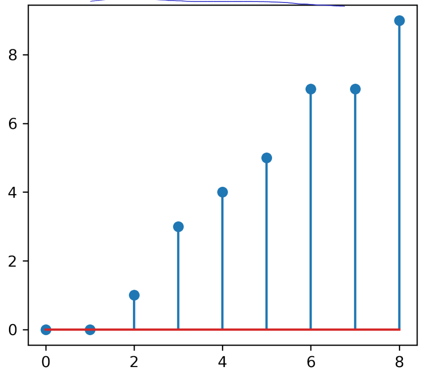
$X_{2\downarrow} = [\ -6\ \ 0\ \ 2\ \ -1\ \ 5\ ]$

[0] [1] [2]

Original signal x[n]



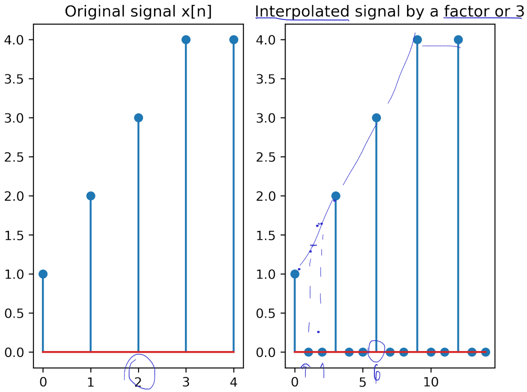
Signal subsampled by a factor of 2, $x_{2\downarrow}[n]$



Interpolation

- **Interpolation** by a factor of L adds $(L-1)$ zeros between two samples in the original signal
  - Total number of samples increases L times

$$x_{L\uparrow} = \begin{cases} x[\frac{n}{L}] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$



$$X_{L\uparrow}[n] = \begin{cases} x\left[\frac{n}{L}\right], & \frac{n}{L} \in \mathbb{N} \\ 0, & \text{rest} \end{cases}$$

new signal

Mathematical operations

- A signal $x[n]$ can be **scaled** by a constant A, i.e. each sample is multipled by A:

$$y[n] = Ax[n].$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **multiplied** by multiypling the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

II.4 Discrete systems

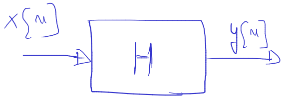
Definition

- **System** = a device or algorithm which produces an **output signal** based on an **input signal**

  y[n]

  x[n]

- We will only consider systems with a single input and a single output
- Figure here: blackboard.
- Common notation:
  - x[n] is the input
  - y[n] is the output
  - H is the system.



Notations

$y[n] = H\{x[n]\}$

- Notations:

$$y[n] = H\left[x[n]\right]$$

("*the system H applied to the input x[n] produces the output y[n]*")

$$x[n] \xrightarrow{H} y[n]$$

("*the input* $x[n]$ *is transformed by the system H into y[n]*")

Equations

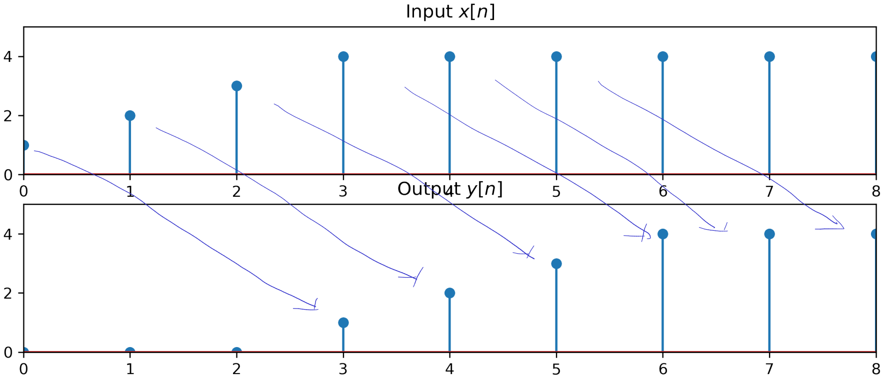
- Usually, a system is described by the **input-output equation** (or **difference equation**) which expains how $y[n]$ is defined in terms of $x[n]$.

Examples:

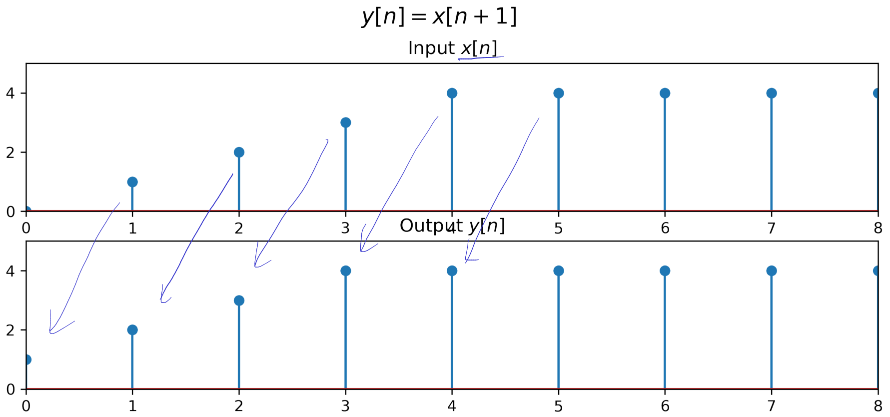
1. $y[n] = x[n]$ (the identity system)
2. $y[n] = x[n-3]$
3. $y[n] = x[n+1]$
4. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
5. $y[n] = \max(x[n+1], x[n], x[n-1])$
6. $y[n] = (x[n])^2 + \log_{10} x[n-1]$
7. $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$

Example

$$\boxed{y[n] = x[n-3]}$$



Example

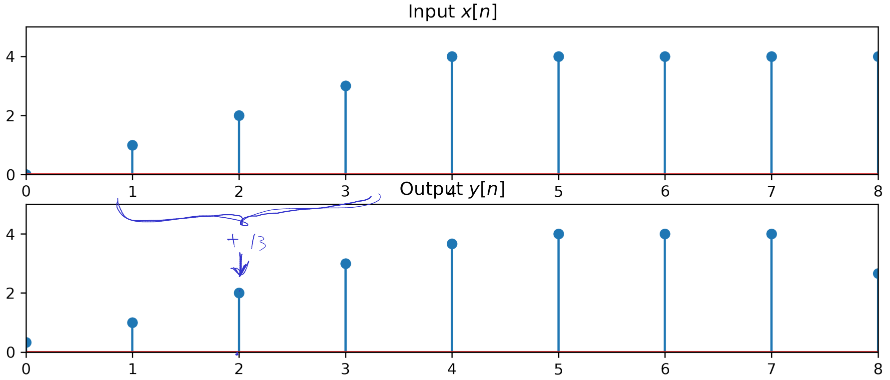


Example

$y[2] = (x[3] + x[2] + x[1])/3$

$y[3] = (x[4] + x[3] + x[2])/3$

$$y[n] = (x[n+1] + x[n] + x[n-1])/3$$

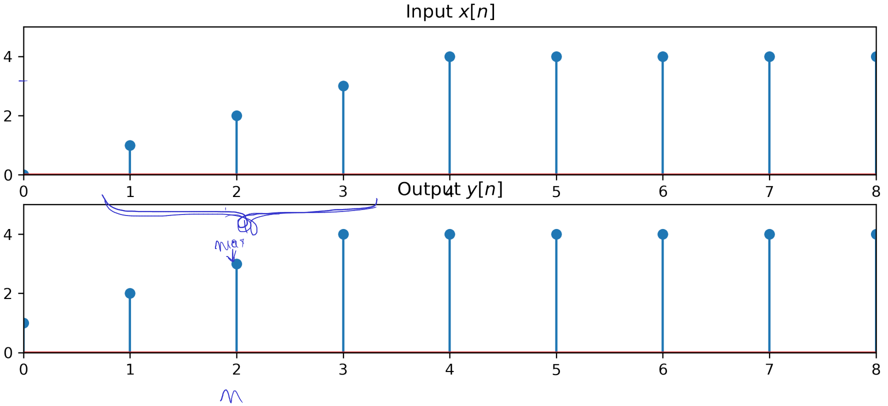


$M = 2$

m

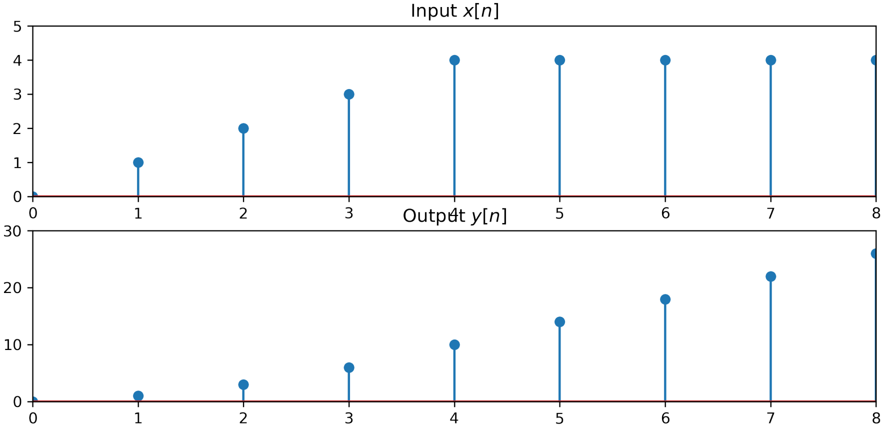
Example

$$y[n] = \max(x[n+1], x[n], x[n-1])$$



Example

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \dots$$



Recursive systems

- Some systems can/must be written in **recursive form**

$$y[n] = y[n-1] + x[n],$$

- Must always specify **initial conditions**
  - i.e. initial value (e.g. $y[-1] = 2.5$)
  - if not mentioned, assume they are 0 ("relaxed system")
  - they represent the internal state of the system at the starting moment
- For recursive systems, the output signal depends on both the input signal **and** on the initial conditions
  - different initial conditions lead to different outputs, even if input signal is the same
  - a recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ($x[n] = 0$)

START

$\boxed{y[-1] = \cancel{2.5}}$ $\cancel{0}$ -33

$y[0] = y[-1] + x[n]$

$y[1] = \dots$

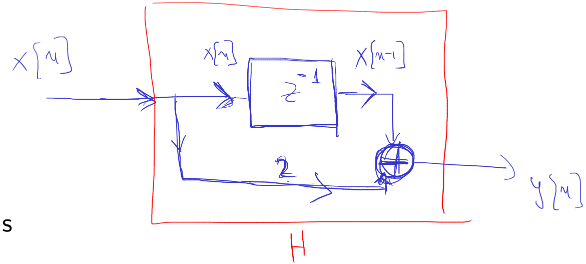
$y[5] = y[4] + \dots$

$y[4] = y[3] + \dots$

$y[3] = y[2] + \dots$

Representation of systems

- The operation of a system can be described graphically (see examples on blackboard):
  - summation of two signals $\rightarrow \oplus \rightarrow$
  - scaling of a signal with a constant
  - multiplication of two signals
  - delay element $\rightarrow \boxed{z^{-1}} \rightarrow$
  - anticipation element $\rightarrow \boxed{z^{+2}} \rightarrow$
  - other blocks for more complicated math operations



$$y[n] = 2 \cdot x[n] + x[n-1]$$

II.4 Classification of discrete systems

Memoryless / systems with memory

fară memorie

- **Memoryless (or static)**: output at time n depends only on the input **from the same moment** n
- Otherwise, the system **has memory (dynamic)**
- Examples:
  - memoryless: $y[n] = (x[n])^3 + 5$
  - with memory: $y[n] = (x[n])^3 + x[n-1]$

Memoryless / systems with memory

- Memory of size N:
  - output at time n $y[n]$ depends only up to the last N inputs, $x[n-N], x[n-(N-1)], \ldots x[n]$,
  - if N is finite: the system has **finite memory**
  - if $N = \infty$, the system has **infinite memory**
- Examples:
  - finite memory of order 4: $y[n] = x[n] + x[n-2] + x[n-4]$ ← $N=4$
  - infinite memory: $y[n] = 0.5y[n-1] + 0.8x[n]$
    - recursive systems usually have infinite memory

$y[n]$

$x[0] \longrightarrow y[1000000]$

Memoryless / systems with memory

- An input sample has an effect on the output only for the next N time moments
- For systems infinite memory, any sample influences **all** the following samples, forever
  - but, if system is stable, the influence gets smaller and smaller

Time-Invariant and Time-Variant systems

- A relaxed system H is **time-invariant** if and only if:

$$x[n] \xrightarrow{H} y[n]$$

implies, $\forall x[n], \forall k$, that

$$x[n-k] \xrightarrow{H} y[n-k]$$

- Delaying the input signal with k will only delay the output with the same amount, otherwise the output is not affected
  - Must be true for all input signals, for all possible delays (positive or negative)
- Otherwise, the system is said to be **time-variant**

$$x[n] \xrightarrow{H} y[n]$$

$$x[n-k] \xrightarrow{H} y[n-k]$$

Time-Invariant and Time-Variant systems

$y[n] = x[n] - x[n-1]$ $\quad x[n] \xrightarrow{H} y[n]$

$x[n-k] \longrightarrow ?$

$y[n-k] \stackrel{?}{=} x[n-k] - x[n-k-1]$

- Examples:
  - $y[n] = x[n] - x[n-1]$ is time-invariant ✓
  - $y[n] = n \cdot x[n]$ is not time-invariant
- A system is time-invariant if it depends on n only through the input signal $x[n]$

$y[n] = n \cdot x[n] \longrightarrow y[n-k] = (n-k) \cdot x[n-k]$

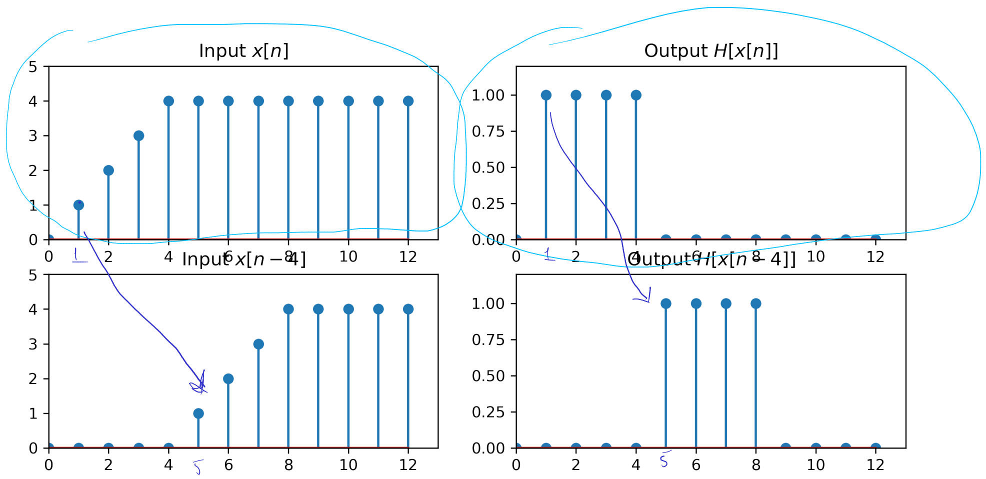
When input is $x[n]$, output is $y[n] = n \cdot x[n]$

When input is $x[n-k]$, output is $n \cdot x[n-k]$

$\neq \Rightarrow$ Not Time Invar.

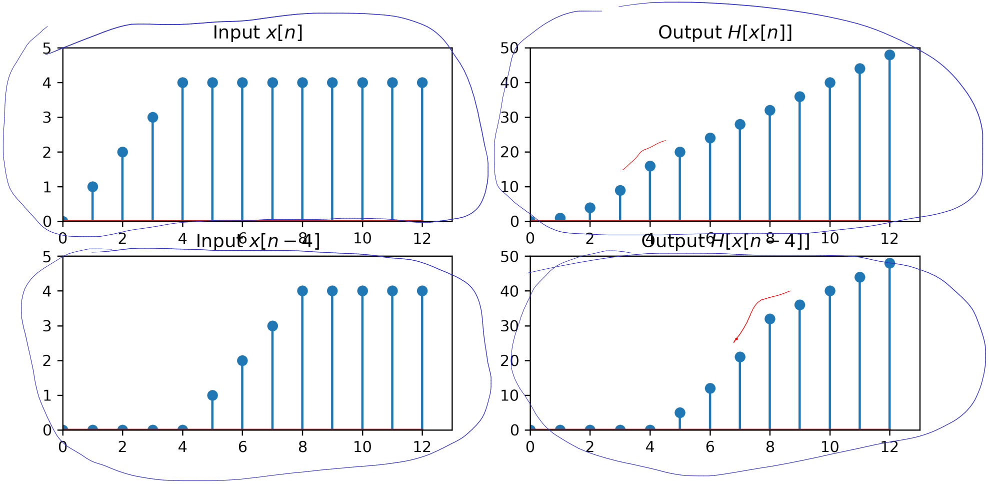
Example

Time-invariant system $y[n] = x[n] - x[n-1]$



Another example

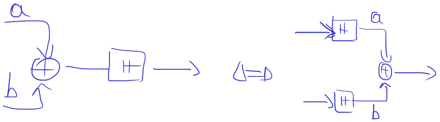
Time-variant system $y[n] = n \cdot x[n]]$



Linear and nonlinear systems

$$H\{a \cdot x_1[n] + b \cdot x_2[n]\} = a \cdot H\{x_1[n]\} + b \cdot H\{x_2[n]\}$$

- A system H is **linear** if:



$$H[ax_1[n] + bx_2[n]] = aH[x_1[n]] + bH[x_2[n]]\,.$$

- Composed of two parts:
  - Applying the system to a sum of two signals = applying the system to each signal, and adding the results
  - Scaling the input signal with a constant a is the same as scaling the output signal with a
- The same relation will be true for a sum of many signals, not just two

Linear and nonlinear systems

- Advantage of linear systems
  - Complicated input signals can be decomposed into a sum of smaller parts
  - The system can be applied to each part independently
  - Then the results are added back
- Examples:
  - linear system: $y[n] = 3x[n] + 5x[n-2]$ ← LINEAR
  - nonlinear system: $y[n] = 3(x[n])^2 + 5x[n-2]$ ← NON-LINEAR

$(\ )^2$

$+5$

$\sqrt{}$

e^x

$\log(\ )$

$\log(\ )$

$\sin(\ )$

Linear and nonlinear systems

- For a system to be linear, the input samples $x[n]$ must not undergo non-linear transformations.
- **The only transformations** of the input x[n] allowed to take place in a linear system are:
  - scaling (multiplication) with a constant
  - delaying
  - summing different delayed versions of the signal (not summing with a constant)

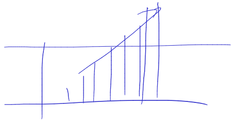
Causal and non-causal systems

- **Causal**: the output $y[n]$ depends only on the current input $x[n]$ and the past values $x[n-1]$, $x[n-2]$..., but not on the future samples $x[n+1]$, $x[n+2]$...
- Otherwise the system is **non-causal**.
- A causal system can operate in real-time
  - we need only the input samples from the past
  - non-causal systems need samples from the future
- Examples:
  - $y[n] = x[n] - x[n-1]$ is causal
  - $y[n] = x[n+1] - x[n-1]$ is non-causal
  - $y[n] = x[-n]$ is non-causal

$y[5] = x[-5]$

$y[-3] = x[3]$ NOT-CAUSAL

Stable and unstable systems

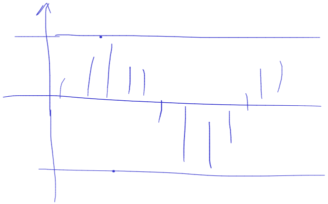


UnBounded

Mărginit

- **Bounded** signal: if there exists a value M such that all the samples of the signal or smaller than M, in absolute values

$$x[n] \in [-M, M]$$



Bounded

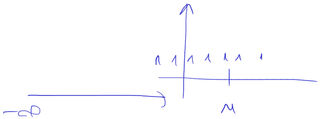
$$|x[n]| \leq M$$

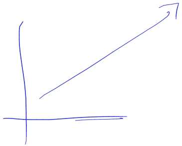
- **Stable** system: if for any bounded input signal it produces a bounded output signal
  - not necessarily with the same M
  - known as BIBO (Bounded Input --> Bounded Output)
- In other words: when the input signal has bounded values, the output signal does not go towards ∞ or $-\infty$.

Stable and unstable systems

- Examples:
  - $y[n] = (x[n])^3 - x[n+4]$ is stable
  - $y[n] = \frac{1}{x[n]-x[n-1]}$ is unstable
  - $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$ is unstable

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$





Impulse response of Linear Time-Invariant (LTI) systems

Linear Time-Invariant (LTI) systems

AND

- Notation: An **LTI** system (**L**inear **T**ime-**I**nvariant) is a system which is simultaneously **linear** and **time-invariant**.
- LTI systems have an equation like this:

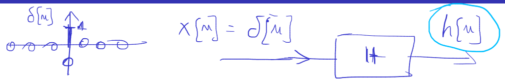
$y[n] = x[n] + x[n-1]$

General template

$$
\begin{aligned}
y[n] &= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\
&\quad + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M] \\
&= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=1}^{M} b_k x[n-k]
\end{aligned}
$$

  - the above is for causal systems; non-causal can also have [n+k]

The impulse response

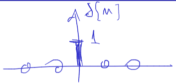


- **Impulse response** of a system = output (response) of when the input signal is the impulse $\delta[n]$:

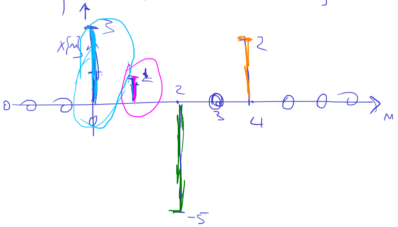
$$h[n] = H(\underbrace{\delta[n]}_{\text{input}})$$

- The impulse response of a LTI system **fully characterizes the system**:
  - based on $h[n]$ we can compute the response of the system to **any** input signal
  - all the properties of LTI systems can be described via characteristics of the impulse response

Signals are a sum of impulses



$x[n] = \{3, 1, -5, 0, 2\}$



- Any signal $x[n]$ can be composed as **a sum of scaled and delayed impulses** $\delta[n]$.
- Example:
  $x[n] = \{3, 1, -5, 0, 2\} = 3\delta[n] + \delta[n-1] - 5\delta[n-2] + 2\delta[n-\cancel{2}\,4]$
- In general

$$x[n] = \sum_{k=-\infty}^{\infty} \underset{\text{amplitude}}{x[k]}\,\underset{\text{delay}}{\delta[n-k]}$$

i.e. a sum of impulses $\delta[n]$, each one delayed with k and scaled with the corresponding value $x[k]$

Convolution

$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$

- The response of a LTI system to a sum of impulses, delayed with k and scaled with $x[k]$, **is a sum of impulse responses, delayed with k and scaled with** $x[k]$.

T.I :

$\delta[n] \rightarrow h[n]$

$\delta[n-k] \rightarrow h[n-k]$

$$
\begin{aligned}
y[n] &= H\left(x[n]\right) \\
&= H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} x[k] H\left(\delta[n-k]\right) \\
\end{aligned}
$$

$$
\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k].}
$$

$= x[n] * h[n]$

$y[n] = H\left(x[n]\right)$

$= H\left(\sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]\right)$

Linear $= \sum_{k=-\infty}^{\infty} H\left(x[k]\cdot\delta[n-k]\right)$

Linear $= \sum_{k=-\infty}^{\infty} x[k]\cdot \underbrace{H\left(\delta[n-k]\right)}_{h[n-k]}$

Time-Invar $= \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$

Convolution

- Convolution in short:
  - The input signal is composed of separate impulses
  - Each impulse will generate its own response (LTI)
  - Output signal is the sum of impulse responses, delayed and scaled
- Convolution only applies for LTI systems

Convolution

- This operation $=$ the **convolution** of two signals $x[n]$ and $h[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The response of a LTI system to an input signal x[n] is **the convolution of x[n] with the system's impulse response h[n]**

$$y[n] = x[n] * h[n]$$

Convolution

$x[n] * h[n] = h[n] * x[n]$

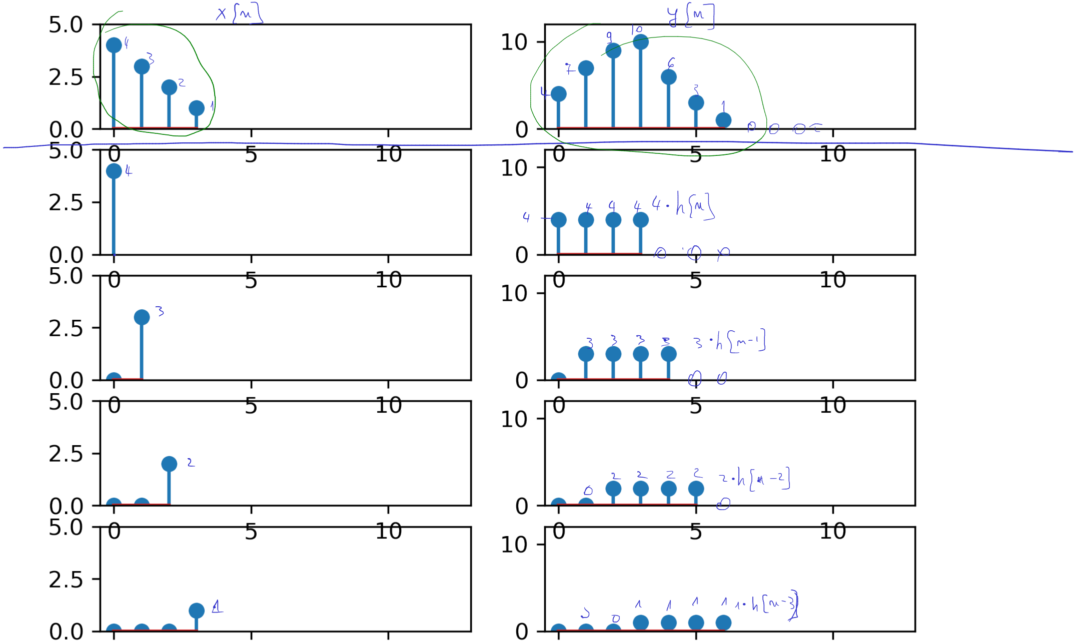
$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$

- Convolution is commutative: $x[n] * h[n] = h[n] * x[n]$
  - in equation it doesn't matter which signal has $[k]$ and which with $[n-k]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Example



Interpretation of the convolution equation

The convolution equation can be interpreted in two ways:

1. The output signal $y[n]$ = a sum of a lot of impulse responses $h[n]$, each one delayed by k (hence $[n-k]$) and scaled by $x[k]$
   - one for each sample in the input signal
   - explain at blackboard

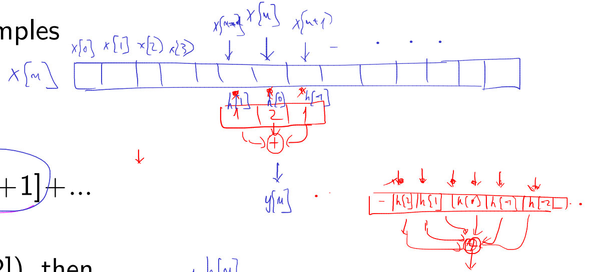
$$y[n] = \ldots + x_0 \cdot h[n] + x_1 h[n-1] + x_2 \cdot h[n-2] + \ldots$$

$k=0$ $k=1$ $k=2$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coef.}} \underbrace{\mathbf{h}[\mathbf{n}-k]}_{\text{Delay}}$$

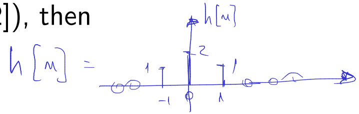
Interpretation of the convolution equation

2. Each output sample $y[n]$ = a **weighted sum** of the input samples around it



$$y[n] = \ldots + \underbrace{h[2] \cdot x[n-2]}_{k=2} + \underbrace{h[1] \cdot x[n-1]}_{k=1} + \underbrace{h[0] \cdot x[n]}_{k=0} + h[-1] \cdot x[n+1] + \ldots$$

- If $h[n]$ has finite length (e.g. non-zero only between $h[-2] \ldots h[2]$), then there are only a few terms in the sum
  - Example at blackboard



$h[n] = \{ \overset{h[-1]}{1}, \overset{h[0]}{2}, \overset{h[1]}{+1} \}$ $0 \ldots 0$

$$x[n] * h[n] = \mathbf{y[n]} = \sum_{k=-\infty}^{\infty} h[k]\mathbf{x[n} - k] = \ldots$$

$y[n] = 1 \cdot x[n-1] + 2 \cdot x[n] + (+1) \cdot x[n+1]$

$y[5] = 1 \cdot x[4] + 2 \cdot x[5] + 1 \cdot x[6]$

$y[6] = 1 \cdot x[5] + 2 \cdot x[6] + 1 \cdot x[7]$

Interpretation of the convolution equation

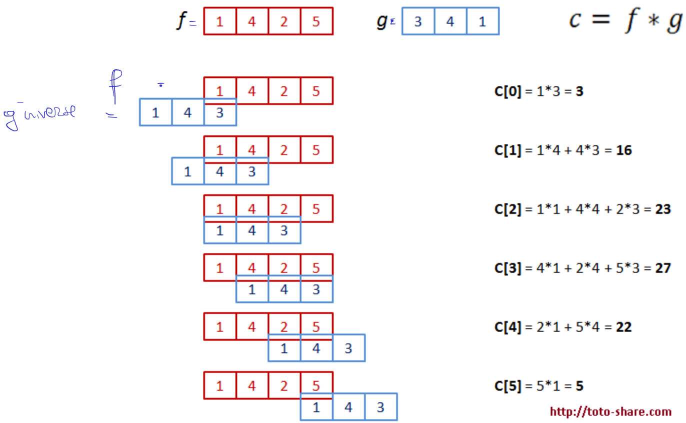


Figure 1: Convolution as weighted sum

- image from http://www.stokastik.in

Interpretation of the convolution equation

- Watch the following:

  https://www.youtube.com/watch?v=ulKbLD6BRJA

Example

$h[n] = \{ \overset{h[-1]}{1}, \overset{h[0]}{2}, \overset{h[1]}{+1} \}$

$y[n] = 1 \cdot x[n-1] + 2\, x[n] + (+1) \cdot x[n+1]$

The impulse response can be read directly from the system equation (for non-recursive systems):

- Suppose we have the system:

$$y[n] = \underset{h[-1]}{3}x[n+1] + \underset{h[0]}{5}x[n] \underset{h[1]}{- 2}x[n-1] + \underset{h[2]}{4}x[n-2]$$

- What is the impulse response of the system?
- Answer: $h[n] = \{...0, 3, \underset{\uparrow}{5}, -2, 4, 0, ...\}$

Convolution as matrix multiplication

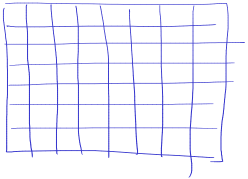
- Convolution can we written as multiplication with a **circulant** (or "**Toeplitz**") matrix
  - in this example, assuming $h[n]$ is non-zero only from $h[-1]$ to $h[3]$

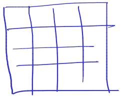
$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$y[n]$ $\qquad$ $x[n]$

$$\begin{bmatrix} \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \end{bmatrix} = \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & \dots \\ \dots & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & \dots \\ \dots & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ x_{n+3} \\ \vdots \end{bmatrix}$$

2D convolution

I = 

H = 

- Convolution can be applied in 2D (for images)
- The input signal = an image $I[x, y]$
- The impulse response (the ***kernel***) = a matrix H[x, y]
- The convolution result:

2D:
$$Y[x,y] = I * H = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I[x-i, y-j] \cdot H[i,j]$$

$m \to x, y$

$k \to i, j$

1D: $y[m] = \; = \sum_{k} x[m-k] \cdot h[k]$

2D convolution

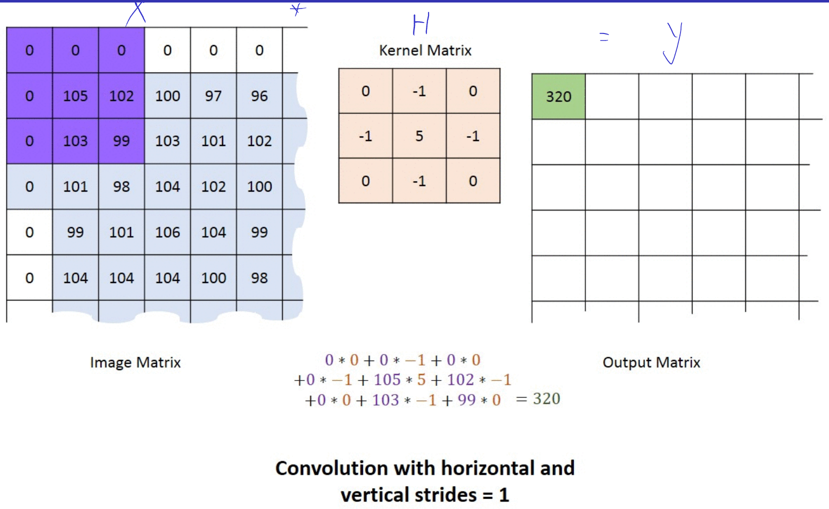


Figure 2: 2D Convolution as weighted sum

- image from http://machinelearninguru.com

2D Convolution

- Watch this:

  http://machinelearninguru.com/computer_vision/basics/convolution/cor

2D Convolution

- Simple image effects with 2D convolutions:
  - the "kernel" $=$ the impulse response $H[x, y]$
- See here: https://en.wikipedia.org/wiki/Kernel_(image_processing)
- What are their 1D counterparts?

Properties of convolution

Basic properties of convolution

- Convolution is **commutative** (the order of the signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

  - Proof: make variable change $(n-k) \to l$, change all in equation

- Convolution is **associative**:

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

  - (No proof)

Properties of convolution

$\delta[n]$ $\quad a \cdot 1 = a$

- The unit impulse is **neutral element** for convolution:

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

$(\cdot)$

  - Proof: equation

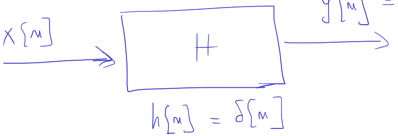
- Convolution is a **linear operation** (or **distributive**): $\quad (a+b) \cdot c = a \cdot c + b \cdot c$

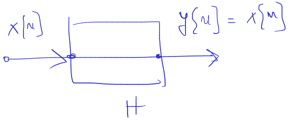
$$\underbrace{(\alpha \cdot a[n] + \beta \cdot b[n])}_{x[n]} * c[n] = \alpha(\cdot a[n] * c[n]) + \beta \cdot (b[n] * c[n])$$

- Proof: by linearity of the corresponding system

Properties of LTI systems expressed with $h[n]$

$y[n] = x[n] * h[n] = x[n]$

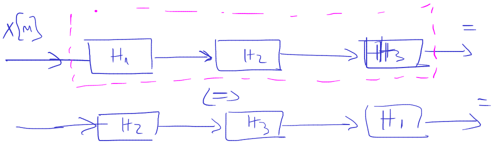




1. Identity system

- A system with $h[n] = \delta[n]$ produces an response equal to the input, $y[n] = x[n], \forall x[n]$.
- Proof: $\delta[n]$ is neutral element for convolution.

Properties of LTI systems expressed with $h[n]$

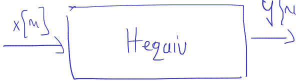


$= x[n] * h_1[n] * h_2[n] * h_3[n]$

$=$

$= x[n] * h_2[n] * h_3[n] * h_1[n]$

$\Longleftrightarrow$



$y[n] = x[n] * h_{equiv}$

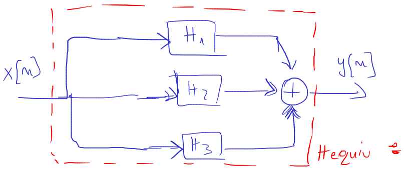
$h_{equiv}[n] = h_1[n] * h_2[n] * h_3[n]$

2. Series connection is commutative

- LTI systems connected in series can be interchanged in any order
- Proof: by commutativity of convolution.
- LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * \ldots * h_N[n]$$

Properties of LTI systems expressed with $h[n]$



$$y[n] = x[n] * h_1[n] + x[n] * h_2[n] + x[n] * h_3[n]$$
$$= x[n] * \left(\underbrace{h_1[n] + h_2[n] + h_3[n]}_{h_{equiv}[n]}\right)$$

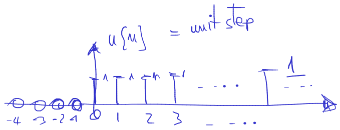
$$h_{equiv}[n] = h_1[n] + h_2[n] + h_3[n]$$

3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + ... + h_N[n]$$

Properties of LTI systems expressed with $h[n]$



4. Response of LTI systems to unit step

- If the input signal is $u[n]$, the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$ = "step response"

.

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot \underline{u[n-k]} = \sum_{k=-\infty}^{n} h[k]$$

$n-k < 0 \Rightarrow 0$

$n-k \geq 0 \Rightarrow 1$

$$= \left(\boxed{\int_{-\infty}^{t} f(t) = F(t)} \right)$$

Properties of LTI systems expressed with $h[n]$

- Proof:
  - The signal $\sum_{k=-\infty}^{n} h[k]$ is a *discrete-time integration* of $h[n]$
  - The unit step $u[n]$ iteslf is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

  - Therefore the system response to the integral of the impulse $=$ the integral of the system response to the impulse
  - The interchanging of the integration with the system is due to the linearity of the system and is valid for all signals:

$$H\left(\sum_{k=-\infty}^{n} x[k]\right) = \sum_{k=-\infty}^{n} H\left(x[k]\right)$$

Relation between LTI system properties and $h[n]$

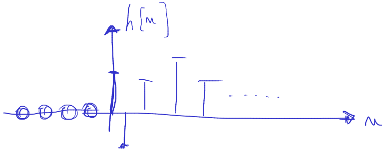
Relation between LTI system properties and $h[n]$

- For an LTI system, if we know $h[n]$, we know **everything** about the system
- Therefore, the properties (causal, memory, stability) must be reflected somehow in $h[n]$
  - Not linearity and time-invariance, they must be true, otherwise we wouldn't talk about $h[n]$

1. Causal LTI systems and their $h[n]$

Causal : $y[n]$ does not depend on $x[n+1] + x[n+2] + \dots$

If a LTI system is causal, then



$x[n+2] \cdot \underbrace{h[-2]}_{=0}$

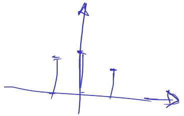
$$h[n] = 0, \forall n < 0$$

$x[n+1] \cdot \underbrace{h[-1]}_{=0}$

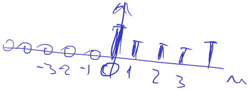
- Proof:
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$,
  - $y[n]$ does not depend on $x[n+1], x[n+2], \dots$
  - it means that these terms are multiplied with 0
  - the value $x[n+1]$ is multiplied with $h[n-(n+1)] = h[-1]$, $x[n+2]$ is multiplied with $h[n-(n+2)] = h[-2]$, and so on
  - Therefore:

$$h[n] = 0, \forall n < 0$$

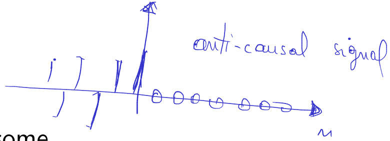
.

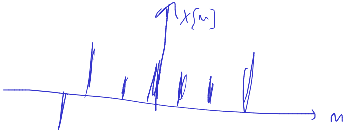


Causal signals and causal systems

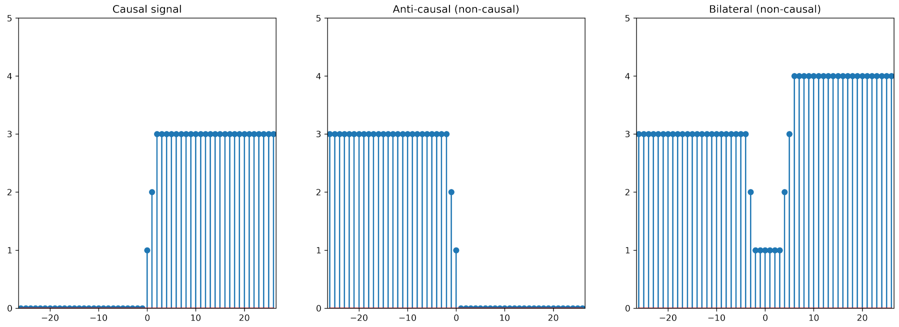


- A **signal** which is 0 for $n < 0$ is called a **causal** signal
- Otherwise the signal is **non-causal**
- We can say that a **system** is causal if and only if it has a causal **impulse response**
- Further definitions:
  - a signal which 0 for $n > 0$ is called an **anti-causal** signal
  - a signal which has non-zero values both for some $n > 0$ and for some $n < 0$ (and thus is neither causal nor non-causal) is called **bilateral**





Example



Causal signal
Anti-causal (non-causal)
Bilateral (non-causal)
5
4
3
2
1
0
−20
−10
0
10
20

2. Stable systems and their $h[n]$

- Considering a bounded input signal, $|x[n]| \leq A$, the absolute value of the output is:

$$
\begin{aligned}
|y[n]| &= \Big|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\Big| \\
&\leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| \\
&= \sum_{k=-\infty}^{\infty} \underbrace{|x[k]|}_{\leq A}|h[n-k]| \\
&\leq A \sum_{k=-\infty}^{\infty} |h[n-k]|
\end{aligned}
$$

$|a+b| \leq |a| + |b|$

$|a \cdot b| = |a| \cdot |b|$

$|y[n]| \leq A \cdot \underbrace{\sum_{k=-\infty}^{\infty} |h[n-k]|}_{\leq B}$

$n-k \to n$

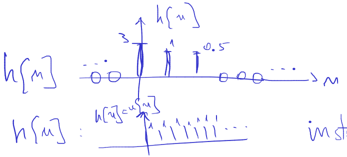
- Therefore **a LTI system is stable if**

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

converge

.

Ex:

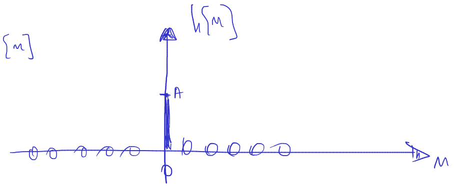


STABIL

instable

3. Memoryless systems and their $h[n]$ (Exercise)

Memoryless : $y[n]$ depinde doar de $x[n]$



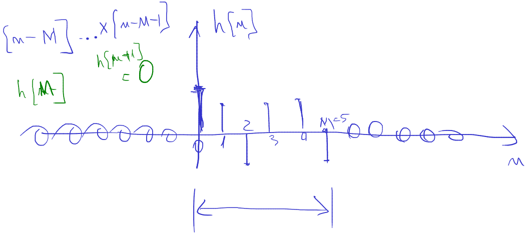
Exercises:

- What can we say about the impulse response $h[n]$ of a memoryless system?
- What about a system with finite memory M?

$$y[n] = \dots x[n] \dots x[n-1] \dots\dots x[n-M] \dots x[n-M-1]$$

$h[0]$ $h[1]$ $h[M]$ $h[M+1] = 0$

$h[k] \cdot x[n-k]$

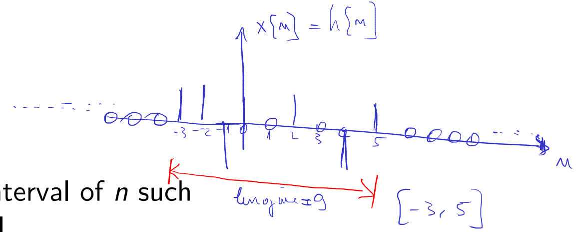


FIR = Finite Impulse Response

IIR = Infinite Impulse Respons

FIR and IIR systems

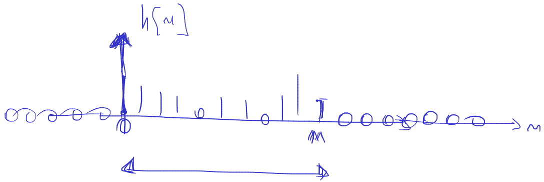
Support



- The **support** of a discrete signal = the smallest interval of n such that the signal is 0 everywhere outside the interval.
- Examples: at whiteboard
- Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.

FIR systems

- A **F**inite **I**mpulse **R**esponse (**FIR**) system has an impulse response with **finite support**
  - i.e. the impulse response is 0 outside a certain interval.
  - i.e. $h[n]$ is zero beyond some element $h[M]$



- The system equation for a FIR system:

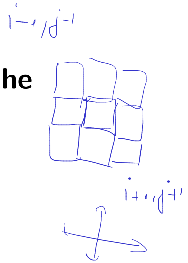
$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = h[0] \cdot x[n] + h[1] \cdot x[n-1] + \ldots h[M] \cdot x[n-M]$$

$$= \sum_{k=0}^{M} \underbrace{b_k}_{= h[k]} \cdot x[n-k]$$

  - is non-recursive (depends only on x)
  - goes only up to some term $h[M] \cdot x[n-M]$
  - for causal system, starts from $h[0]x[n]$; for non-causal, can start from $h[-k]x[n+k]$

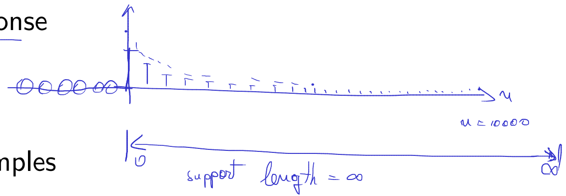
FIR systems

- For a causal FIR system, the output is a **linear combination** of **the last M+1 input samples**
- For non-causal FIR system, some future input samples enter the combination



IIR systems

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



- An **I**nfinite **I**mpulse **R**esponse (**FIR**) system has an impulse response with **infinite support**
  - i.e. the impulse response never becomes completely 0 forever.
- The output $y[n]$ potentially depends on all the preceding input samples
  - from the convolution equation:

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} h[k]x[n-k] \\
&= h[0] \cdot x[n] + h[1] \cdot x[n-1] + \ldots h[M] \cdot x[n-M] + \ldots \textbf{goes on} + \ldots
\end{aligned}
$$

$+ h[10000] \cdot x[n - 10000] + \ldots$

- An IIR system has infinite memory

IIR systems

- IIR systems must have **recursive** equations:
  - they depend on previous outputs $y[n-1]$ up to $y[n-N]$
  - they also depend on input, going back up to $x[n-k]$
- General equation of an IIR system:

$$
\begin{aligned}
y[n] = &- a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\
&+ b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]
\end{aligned}
$$

$$
y[n] = - \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]
$$

  - the impulse response cannot be read explicitly from the equation
  - IIR equations are more general than FIR

$$\boxed{y[n] = \frac{1}{2} \cdot y[n-1] + 1 \cdot x[n]}$$

$$y[n-1] = \frac{1}{2} y[n-2] + x[n-1]$$

$$y[n-2] = \frac{1}{2} y[n-3] + x[n-2]$$

$$\vdots$$

$h[n]$

General equation of an LTI system

Recap:

- The general equation of an LTI system is:

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + \\ + b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

$$\boxed{y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=\cancel{1}0}^{M} b_k x[n-k]}$$

- If all $a_i = 0$, it is a FIR system, no $y[n-k]$ term
  - in this case the coefficients $b_k = h[k]$ (impulse response)
- If some $a_i \neq 0$, it is an IIR system
  - impulse response $h[n]$ is infinitely long, is more complicated to find
- Note: if system is non-causal, can start from $x[n+k]$ $\quad + b_{-1} x[n+1] +$

Initial conditions for recursive systems

$$\boxed{y[n] = \tfrac{1}{2}\cdot y[n-1] + x[n] + x[n-1]}$$

$$\boxed{y[-1] = \cancel{5}\ 55} \rightarrow \begin{array}{l} y[0] = \tfrac{1}{2}\cdot 5 + \dots = \\ y[1] = \tfrac{1}{2}\cdot y[0] + \dots \end{array}$$

- Recursive systems need **initial conditions** (starting values)
  - since they rely on previous outputs
- If initial conditions are all 0, the system is **relaxed**
  - the output depends only on the input signal
- If initial conditions are not zero, the output depends on the input signal **and** the initial conditions

Initial conditions for recursive systems

- The effect of the input signal and the effect of initial conditions are **independent**
  - the system behaves **linear** with respect to them
  - total output = output due to input + output due to initial conditions

$y[n]$

Input	Init.Cond.	Output
x[n]	0	$y[n] = y_{zs}[n]$
0	non-zero	$y[n] = y_{zi}[n]$
x[n]	non-zero	$y[n] = y_{zs}[n] + y_{zi}[n]$

$y_{zero\text{-}state}[n] = y[n]$ if init cond. are 0

$y_{zero\text{-}input}[n] = y[n]$ if input $x[n] = 0$