Digital Signal Processing

Chapter V. Digital filtering

Response of LTI systems to harmonic signals

- We consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} A e^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

Response of LTI systems to harmonic signals

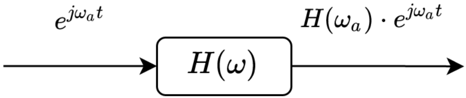

Figure 1: Output = a constant × Input

- $H(\omega)$ = Fourier transform of $h[n]$ evaluated for ω = **transfer function**

Eigen-function

- Complex exponential signals are **eigen-functions** (funcții proprii) of LTI systems:
  - output signal = input signal × a (complex) constant
- $H(\omega_0)$ is a constant that multiplies the input signal
  - Amplitude of input gets multiplies by $|H(\omega_0)|$
  - Phase of input signal is added with $\angle H(\omega_0)$
- Why are sin/cos/exp functions important?
  - If input signal = sum of complex exponential (like coses + sinuses),
  - then output = same sum of complex exponentials, each scaled with some coefficients

Response to cosine and sine

- Cosine / sine = sum of two exponentials, via Euler

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega t}}{2}$$

$$\sin(\omega n) = \cos(\omega n - \frac{\pi}{2})$$

- System is linear and real =>
  - amplitude is multiplied by $|H(\omega_0)|$
  - phase increases by $\angle H(\omega_0)$
- See proof at blackboard

Frequency response

- Naming:
  - $H(\omega)$ = **frequency response** of the system
  - $|H(\omega)|$ = **amplitude response** (or magnitude response)
  - $\angle H(\omega)$ = **phase response**
- Magnitude response is non-negative: $|H(\omega)| \geq 0$
- Phase response is an angle: $\angle H(\omega) \in (-\pi, pi]$
  - Phase response may have jumps of 2π (wrapped phase)
  - Stitching the pieces in a continuous function = phase *unwrapping*
  - Unwrapped phase: continuous function, may go outside interval $(-\pi, pi]$
  - Example: at blackboard

Permanent and transient response

- Warning: $\cos(\omega n)$ does not start at $n = 0$
- The above harmonic signals start at $n = -\infty$.
- What's wrong if the signal starts at some time n?

Permanent and transient response

- What if the signal starts at some time n?
- Total response = transient response + permanent response
  - transient response goes towards 0 as n increases
  - permanent response = what remains
- So the above relations are valid only in **permanent regime**
  - i.e. after the transient regime has passed
  - i.e. after the transient response has practically vanished
  - i.e. when the signal started very long ago (from $n = -\infty$)
  - i.e. when only the permanent response remains in the output signal
- Example at blackboard

Permanent response of LTI systems to periodic inputs

- Consider an input $x[n]$ which is periodic with period N
- Then it can be represented as a Fourier series with coefficients c_k:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- Since the system is linear, each component k gets multiplied with $H\left(\frac{2\pi}{N}k\right)$
- So the total output is:

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- The output is still periodic, same period, same frequencies

Response of LTI systems to non-periodic signals

- Consider a general input $x[n]$ (not periodic)
- The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- Output spectrum = Input spectrum $\times$ Transfer function

Response of LTI systems to non-periodic signals

- The transfer function $H(\omega)$ "shapes" the spectrum

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- In polar form:
  - modulus is multiplied

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$

  - phases is added:

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

Response of LTI systems to non-periodic signals

- The system **attenuates/amplifies** the input frequencies and **changes their phases**
- $H(\omega)$ = the **transfer function**
- $H(z)$ = the **system function**
- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

Power spectral density

- $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- The poles and zeros of $S(\omega)$ come in pairs $(z, 1/z$ and $p, 1/p)$

Digital filters

- LTI systems are also known as **filters** because their transfer function shapes ("filters") the frequencies of the input signals
- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

Ideal filters

- Draw at whiteboard the ideal transfer function of a:
  - low-pass filter
  - high-pass filter
  - band-pass filter
  - band-stop filter
  - all-pass filter (*changes the phase*)

Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
  - i.e. largest power of z or z^{-1}
- Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1})
- Higher order -> better filter transfer function
  - closer to ideal filter
  - more complex to implement
  - more delays (bad)
- Lower order
  - worse transfer function (not close to ideal)
  - simpler, cheaper
  - faster response

Filter design by pole and zero placements

- Based on geometric method
- The gain coefficient must be found by separate condition
  - i.e. specify the desired magnitude response at one frequency
- Examples at blackboard

Filter distortions

- When a filter is non-ideal:
  - non-constant amplitude –> amplitude distortions
  - non-linear phase –> phase distortions
- Phase distortions may be tolerated by certain applications
  - e.g. human auditory system is largely insensitive to phase distortions of sounds

Effect of system's phase

- What is the effect of system's phase response $\angle H(\omega)$?
- Extra phase = delay
  - different frequencies are delayed differently
  - phase
- **Linear-phase** filter: delays all frequencies with the same amount of time
  - i.e. the whole signal is delayed, but otherwise not distorted
  - otherwise, we get distortions

Linear-phase filters

- For a sinusoidal signal, extra phase of 2π = delay of a period $N = \frac{1}{f}$
- To ensure same delay for all frequencies (in time), the phase $\angle H(\omega)$ must be proportional to the frequency
  - draw at blackboard
  - hence the name **linear**

Linear-phase filters

- Example: consider the following filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

- The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

- Linear phase means **just a delaying** of the input signal
  - Fourier property: $x[n - n_0] <-- > X(\omega)e^{-j\omega n_0}$

Group delay

- Group delay = The time delay experienced by a component of frequency ω when passing through the filter
  - as opposed to "phase delay" = the phase added by the filter
- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Linear-phase FIR filters

What type of filters can have linear phase?

- IIR filters cannot have linear phase (no proof provided)
- Only FIR filters can have linear phase, and only if they satisfy some symmetry conditions

Symmetry conditions for linear-phase FIR

- Let filter have an impulse response of length M (order is $M-1$)
- The filter coefficients are $h[0], \ldots\ h[M-1]$
- Linear-phase is guaranteed in two cases
  - **Positive symmetry**
    $$h[n] = h[M-1-n]$$
  - **Negative symmetry (anti-symmetry)**
    $$h[n] = -h[M-1-n]$$
- The delay = the delay of the middle point of the symmetry

Cases of linear-phase FIR

- Proofs at blackboard

1. Positive symmetry, M = odd
2. Positive symmetry, M = even
3. Negative symmetry, M = odd
4. Negative symmetry, M = even

- Check constraints for $H(0)$ and $H(\pi)$
- For what types of filters is each case appropriate?

Proof example

Linear-phase proof for a FIR system with positive symmetry, M = odd

- Only for an example, it is simpler (general case at blackboard)
- Suppose we have a FIR filter with $M = 5$ coefficients:

$$h[n] = \{4, 3, 2, 3, 4\}$$

$$H(z) = 4 + 3z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4}$$

- having positive symmetry (first = last, second = second to last, etc)
- and length $M =$ odd, i.e. one coefficient is alone in the middle

Proof example

- Let's compute $H(\omega)$:

$$
\begin{aligned}
H(\omega) &= \sum_n h[n]e^{-j\omega n} \\
&= 4e^0 + 3e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega} + 4e^{-j4\omega} \\
&= e^{-j2\omega}(4e^{j2\omega} + 3e^{j\omega} + 2 + 3e^{-j1\omega} + 4e^{-j2\omega}) \\
&= e^{-j2\omega}(4e^{j2\omega} + 4e^{-j2\omega} + 3e^{j\omega} + 3e^{-j1\omega} + 2) \\
&= e^{-j2\omega}(4 \cdot 2\cos(2\omega) + 3 \cdot 2\cos(\omega) + 2) \\
&= \underbrace{e^{j\angle H(\omega)}}_{e^{j \cdot phase}} \underbrace{|H(\omega)|}_{real}
\end{aligned}
$$

- The phase is $\angle(H(\omega)) = -2\omega$, a **linear** function
- The phase of the filter is linear

Proof explained

Key points in this proof:

- we pull a common factor, so that the first and last terms have the same exponents, but with opposite signs
- we group first with last term, second with second-to-last:
  - they have same coefficient in front, because of positive symmetry
  - $e^{jx} + e^{-jx} = 2\cos(x) = \textit{real}$
- everything remaining in the right-side paranthesis is a real-valued

Since $H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$, we identify the two terms:

- $|H(\omega)|$ must be the real part in the right-side
- $\angle H(\omega)$ must be the term -2ω, which is a linear function of ω (up to some changes in sign of the real part)

Other cases

Generalizations:

- the filter length can be anything, as long as it has symmetry
- if M is even, there is no single term remaining in the middle, but the proof stays the same
- if we have **negative** symmetry, the terms have opposite signs, and we use:

$$e^{jx} - e^{-jx} = 2j\sin(x) = 2\sin(x) \cdot e^{j\frac{\pi}{2}}$$

Zero-phase FIR filters

- Can we avoid delay altogether?
- **Zero-phase** filter = a particular type of linear-phase filter with zero delay
- For a zero-phase filter, the phase response $\angle H(\omega) = 0$
  - (Group) delay = derivative of $\angle H(\omega)$
  - delay $0 \Leftrightarrow$ flat $\angle H(\omega) \Leftrightarrow \angle H(\omega) = 0$
- Delay is $0 \Leftrightarrow$ symmetry with respect to $h[0]$
  - the system cannot be causal

Zero-phase FIR filters

- Zero-phase filters must be non-causal
  - left side of $h[n]$ symmetrical to right side of $h[n]$
- For causal, we need to delay $h[n]$ to be wholly on the right side $=>$ delay

Example

- Linear-phase filter (low-pass):

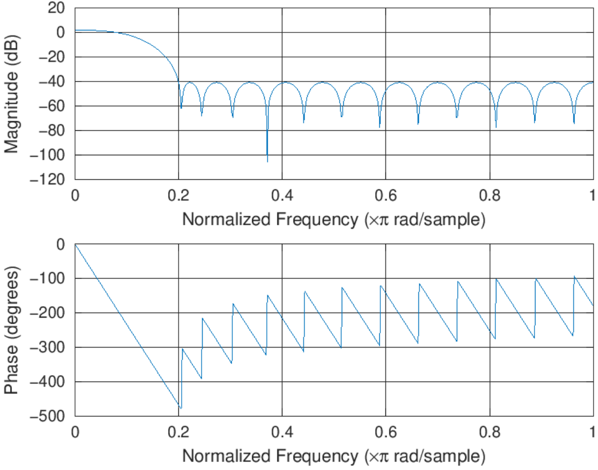

Figure 2: Transfer function of linear-phase filter

Example

- The impulse response (positive symmetry):

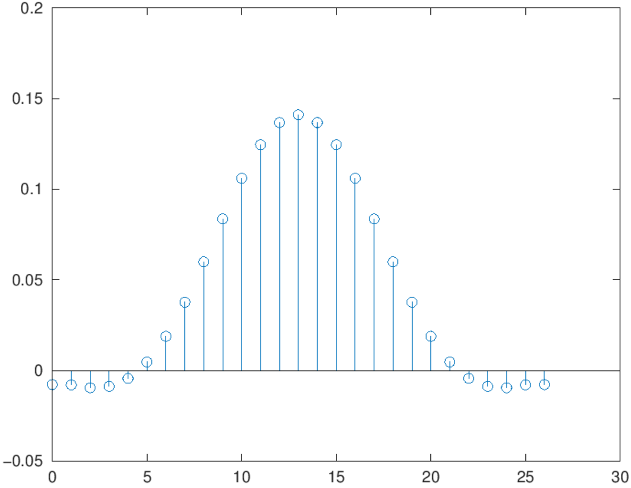

Figure 3: Impulse response of linear-phase filter

Example

- ECG signal: original and filtered. Filtering introduces **delay**

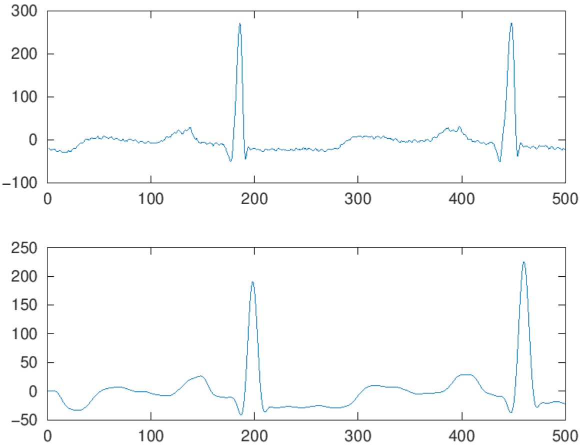

Figure 4: Delay introduced by filtering

Example

- Solution: zero-phase filter (positive symmetry, and centered in 0):
- But filter is **not causal** anymore

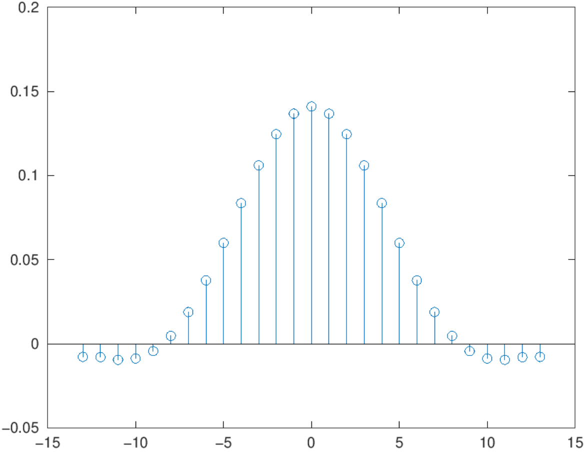

Figure 5: Impulse response of zero-phase filter

Example

- Filtering with zero-phase filter introduces **no delay**

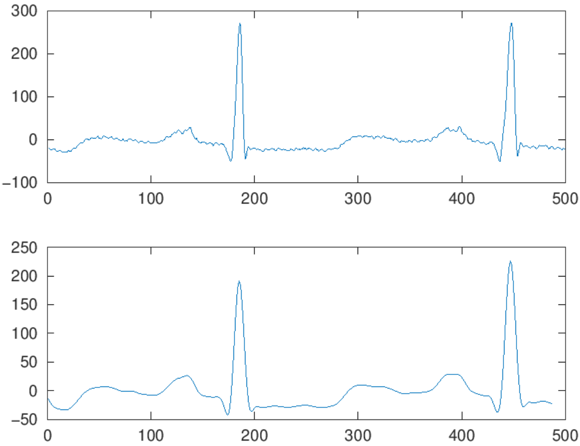

Figure 6: Zero-phase filter introduces no delay

Particular classes of filters

- **Digital resonators** = very selective band pass filters
  - poles very close to unit circle
  - may have zeros in 0 or at 1/-1
- **Notch filters**
  - have zeros exactly on unit circle
  - will completely reject certain frequencies
  - has additional poles to make the rejection band very narrow
- **Comb filters**
  - = periodic notch filters

Digital oscillators

- **Oscillator** $=$ a system which produces an output signal even in absence of input
- Has a pair of complex conjugate poles **exactly on unit circle**
- Example at blackboard

Inverse filters

- Sometimes is necessary to **undo** a filtering
  - e.g. undo attenuation of a signal passed through a channel
- Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle –> unstable
- Examples at blackboard