

Digital Signal Processing

Chapter IV: The Fourier Transform and its
applications

IV.1 Vector spaces of signals (crash course)

Vector spaces

- ▶ **Vector space** = a set $V\{v_i\}$ with the following two properties:
 - ▶ one element + another element = still an element of the same space
 - ▶ a scalar constant \times an element = still an element of the same space
- ▶ You **can't escape** a vector space by summing or scaling
- ▶ The elements of a vector space are called **vectors**

Examples of vector spaces

► Geometric spaces are great intuitive examples:

- ▶ a line, or the set \mathbb{R} (one-dimensional)
- ▶ a plane, or the set \mathbb{C} (two-dimensional)
- ▶ 3D space (three-dimensional)
- ▶ 4D space (four-dimensional, like the spatio-temporal universe)
- ▶ arrays with N numbers (N -dimensional)
- ▶ space of continuous signals (∞ -dimensional)
- ▶ The **dimension** of the space = “how many numbers you need in order to specify one element” (informal)
- ▶ A “vector” like in maths = a sequence of N numbers = a “vector” like in programming
 - ▶ e.g. a point in a plane has two coordinates = a vector of size $N = 2$
 - ▶ e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

► Many vector spaces have a fundamental operation: **the (Euclidean) inner product**

- for **discrete** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$

- for **continuous** signals:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$

- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** ("produs scalar")

Inner product

- ▶ Each entry in x times the complex conjugate of the one in y , all summed
- ▶ For discrete signals, it can be understood as a row \times column multiplication
- ▶ Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- ▶ Inner product is **linear** in both terms:

$$\langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle = \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2 \rangle = \langle \mathbf{x}, \mathbf{y}_1 \rangle + \langle \mathbf{x}, \mathbf{y}_2 \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x}_1, \mathbf{y} \rangle$$

The distance between two vectors

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ The (Euclidean) distance between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_N - y_N)^2}$$

- ▶ This distance is the **usual geometric distance** you know from geometry
- ▶ It has the exact same intuition like in **normal geometry**:
 - ▶ if two vectors have small distance, they are close, they are similar
 - ▶ two vectors with large distance are far away, not similar
 - ▶ two identical vectors have zero distance

The norm of a vector

- ▶ An inner product induces a **norm** and a **distance** function
- ▶ The **norm** (length) of a vector = $\text{sqrt}(\text{inner product with itself})$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ The **norm** of a vector is the distance from \mathbf{x} to point $\mathbf{0}$.
- ▶ It has the exact same intuition like in **normal geometry**:
 - ▶ vector has large norm = has big values, is far from $\mathbf{0}$
 - ▶ vector has small norm = has small values, is close to $\mathbf{0}$
 - ▶ vector has zero norm = it is the vector $\mathbf{0}$
- ▶ Norm of a vector = $\text{sqrt}(\text{the signal energy})$

Norm and distance

- ▶ The norm and distance are related
- ▶ The distance between \mathbf{a} and \mathbf{b} = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

- ▶ Just like in geometry: distance = length of the difference vector

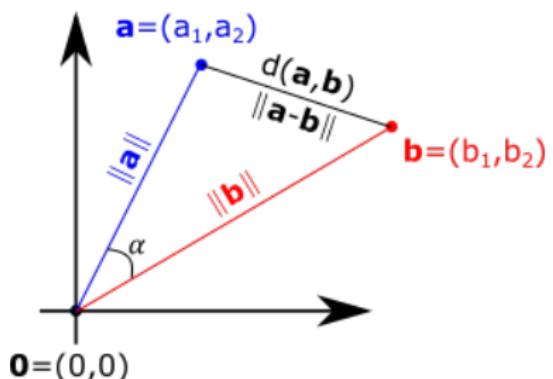


Figure 1: Norm and distance in vector spaces

Angle between vectors

- ▶ The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

- ▶ is a value between -1 and 1
- ▶ **Otrhogonal vectors** = two vectors with $\langle x, y \rangle = 0$
 - ▶ their angle = 90 deg
 - ▶ in geometric language, the two vectors are **perpendicular**

Why vector space

- ▶ Why are all these useful?
- ▶ They are a very general **framework** for different kinds of signals
- ▶ We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
 - ▶ Example in DEDP class: ML decision with 1, 2, N samples

We deal mainly with the following vector spaces:

- ▶ The vector space of all infinitely-long real signals $x[n]$
- ▶ The vector space of all infinitely-long periodic signals $x[n]$ with period N
 - ▶ for each N we have a different vector space
- ▶ The vector space of all finite-length signals $x[n]$ with only N samples
 - ▶ for each N we have a different vector space

- ▶ A **basis** = a set of N linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^N\}$$

- ▶ Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \dots + \alpha_N \mathbf{b}^N$$

- ▶ The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, \dots, \alpha_N)$$

Bases and coordinate systems

- ▶ Bases are just like **coordinate systems** in a geometric space

- ▶ any point is expressed w.r.t. a coordinate system

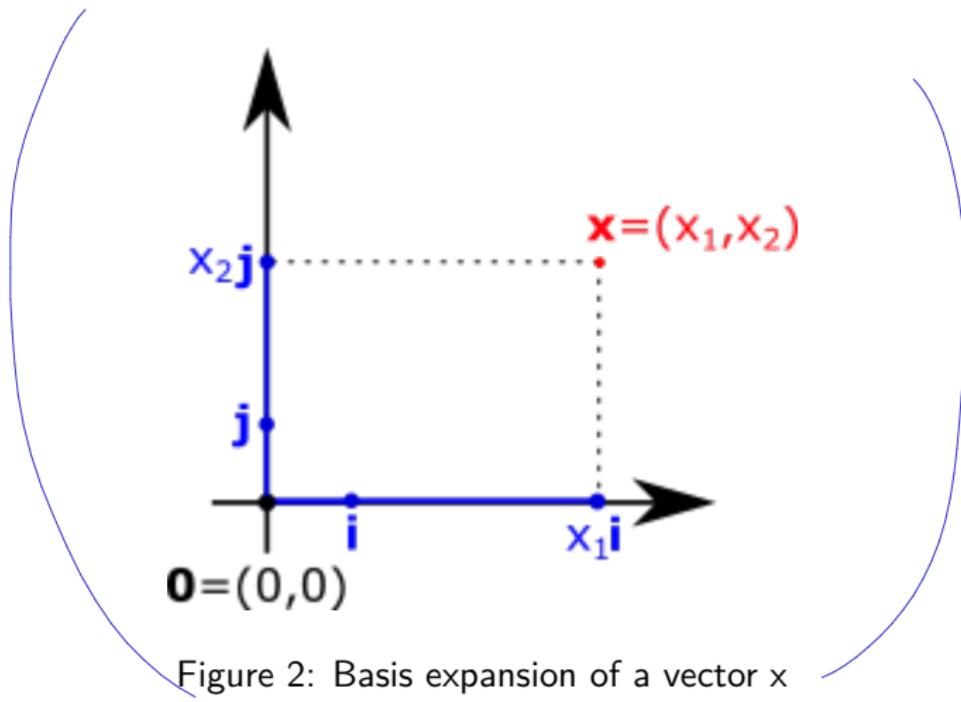
$$\mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

- ▶ any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ N = The number of basis elements = The dimension of the space
- ▶ Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

Bases and coordinate systems



Choice of bases

- ▶ There is typically an infinite choice of bases
- ▶ The **canonical basis** = all basis vectors are full of zeros, just with one 1
- ▶ You used it already in an exercise:
 - ▶ any signal $x[n]$ can be expressed of a sum of $\delta[n - k]$

$$\{\dots, 3, 6, 2, \dots\} = \dots + 3\delta[n] + 6\delta[n - 1] + 2\delta[n - 2] + \dots$$

- ▶ the canonical basis is $B = \{\dots, \delta[n], \delta[n - 1], \delta[n - 2], \dots\}$

Orthonormal bases

- ▶ An **orthonormal basis** a basis where all elements \mathbf{b}^i are:

- ▶ orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

- ▶ **normalized** (their norm = 1):

$$\|\mathbf{b}^i\| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- ▶ Example: the canonical basis $\{\delta[n - k]\}$ is orthonormal:

- ▶ $\langle \delta[n - k], \delta[n - l] \rangle = 0, \forall k \neq l$
 - ▶ $\langle \delta[n - k], \delta[n - k] \rangle = 1, \forall k$

Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

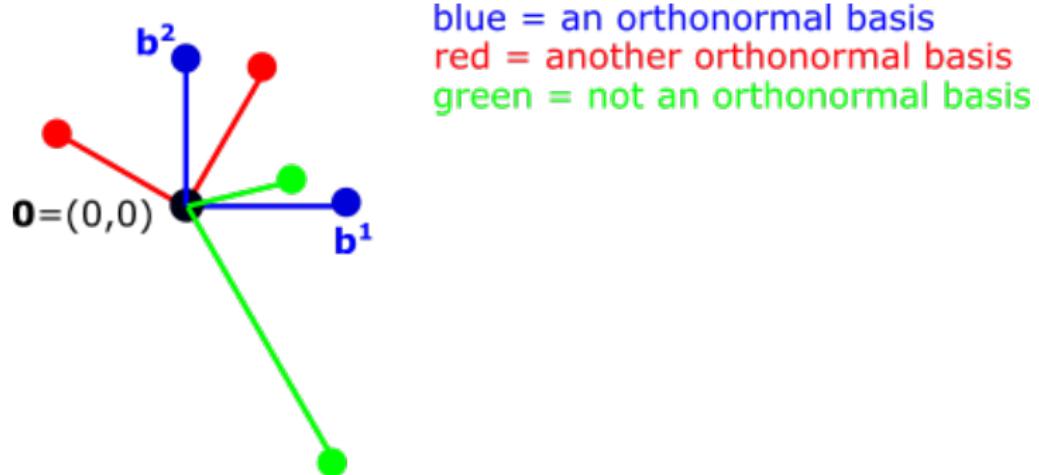


Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- ▶ Suppose we have an **orthonormal basis** $B = \{\mathbf{b}^i\}$
- ▶ Suppose we have a vector \mathbf{x}
- ▶ We can write (expand) \mathbf{x} as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ Question: how to **find** the coefficients α_i ?

Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{b}^i \rangle &= \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\&= \alpha_i\end{aligned}$$

Basis expansion of a vector

- ▶ Any vector \mathbf{x} can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- ▶ For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle \mathbf{x}, \mathbf{b}^i \rangle$$

Why bases

- ▶ How does all this talk about bases help us?
- ▶ To better understand the Fourier transform
- ▶ The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- ▶ The Fourier Transform of a signal x = finding the coefficients of x in this basis
- ▶ The Inverse Fourier Transform = expanding x with the elements of this basis
- ▶ Same **generic** thing every time, only the type of signals differ

IV.2 Introducing the Fourier Transforms

Reminder

► Reminder:

$$\boxed{e^{jx}} = \cos(x) + j \sin(x)$$

$$e^{jx} = \cos(x) + j \sin(x)$$

$$\Rightarrow \cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\Rightarrow \sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\Rightarrow \sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

Why sinusoidal signals

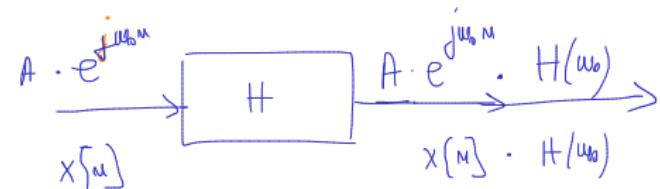
- ▶ Why are sinusoidal signals sin() and cos() **so prevalent** in signal processing?
- ▶ Answer: because they are combinations of an e^{jx} and an e^{-jx}
- ▶ Why are these e^{jx} so special?
- ▶ Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

- ▶ Consider an LTI system with $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} \underbrace{Ae^{j\omega_0 n}}_{x[n]} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

$$\begin{aligned}x[n] &= A \cdot e^{j\omega_0 n} \\x[n-k] &= A \cdot e^{j\omega_0(n-k)} = \underbrace{A \cdot e^{j\omega_0 n}}_{x[n]} \cdot e^{-j\omega_0 k}\end{aligned}$$



- ▶ Output signal = input signal \times a (complex) constant ($H(\omega_0)$)

Eigen-function

- **Eigen-function** of a system ("funcție proprie") = a function f which, if input in a system, produces an output proportional to it

$$H\{f\} = \lambda \cdot f, \lambda \in \mathbb{C}$$

$$\boxed{A \cdot v = \lambda \cdot v}$$

???

$$\lambda \cdot v = A \cdot v$$

$$\lambda \cdot \boxed{v} = \boxed{A} \cdot \boxed{v}$$

- just like **eigen-vectors** of a matrix (remember algebra): $\boxed{A \tilde{v} = \lambda \tilde{v}}$
- we call the "functions" to allow for continuous signals as well

- Complex exponential signals $e^{j\omega t}$ (or $e^{j\omega n}$) are eigen-functions of Linear and Time Invariant (LTI) systems:

- output signal = input signal \times a (complex) constant



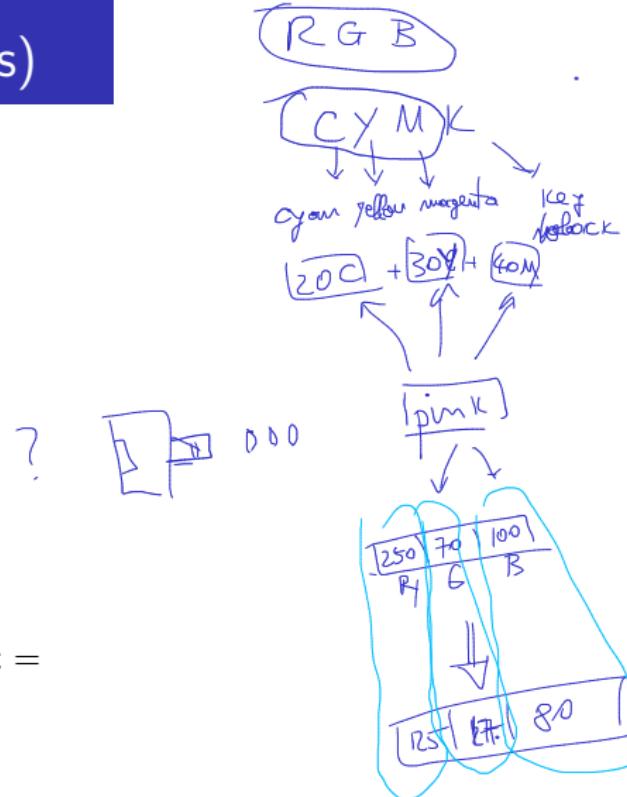
$$e^{j\omega_0 \cdot n} \xrightarrow{H_1} e^{j\omega_1 \cdot n}$$
$$e^{j\omega_1 \cdot n} \xrightarrow{H_2} e^{j\omega_2 \cdot n}$$
$$e^{j\omega_2 \cdot n} \xrightarrow{H_3} \dots$$

Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system very easily if we decompose all signals $x[n]$ as a combination of $\{e^{j\omega n}\}$

- Example: RGB color filter

- suppose we have some photographic filters (lenses):
 - one reduces red to 50% $\Rightarrow R \rightarrow 0.5R$
 - one reduces green to 25% $\Rightarrow G \rightarrow 0.25G$
 - one reduces blue to 80% $\Rightarrow B \rightarrow 0.8B$
 - RGB are eigen-functions of the system: input = 200 Blue, output = $0.8 * 200$ Blue
 - what is the output color if input is "pink"?
 - Answer is easy if we represent all colors in RGB



Representation with respect to eigen-functions (-vectors)

- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$

- All vector space theory becomes useful now:

- $\{e^{j\omega n}\}$ is an **orthonormal basis**
- decomposing signals = finding coefficients α_i
- we know how to do this, just like for any orthonormal basis

$$x[n] = \sum \alpha_\omega \cdot e^{j\omega n}$$

$$\alpha_\omega = \langle x, e^{j\omega n} \rangle$$

$$\begin{aligned} x[n] &= \sum_{\omega} e^{j\omega n} \\ &= A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n} + \dots \end{aligned}$$



$$y[n] = H_0 A_1 e^{j\omega_0 n} + H_1 A_2 e^{j\omega_1 n} + \dots$$

Discrete-Time Fourier Transform (DTFT)

- ▶ Consider the vector space of **non-periodic infinitely-long signals**
- ▶ This vector space is **infinite-dimensional**
- ▶ The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis**
- ▶ We can expand (almost) any x in this basis:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_\omega} e^{j2\pi fn} df$$

- ▶ The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_\omega = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

Discrete-Time Fourier Transform (DTFT)

$x[n]$

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n} df$$

- A signal $x[n]$ can be written as a linear combination of $\{e^{j2\pi f n}\}$, $\forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi f n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi f n}\}$ is found using the inner product $\langle x, e^{j2\pi f n} \rangle$

$$x[n] \longleftrightarrow X_f(\omega)$$

$$\left(\frac{1}{2\pi} \text{d}f = df \right) \quad (df = \frac{1}{2\pi} d\omega) \quad \boxed{\frac{1}{2}}$$

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) \cdot e^{j2\pi f n} df$$

! $\rightarrow \frac{1}{2}$

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f n}$$

SCS:

$$X_f(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Discrete-Time Fourier Transform (DTFT)

- ▶ Alternative form with ω
- ▶ We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

$$x[n] = \underbrace{\left(\frac{1}{2\pi}\right)}_{\text{circled}} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \langle x[n], e^{j\omega n} \rangle = \sum_n x[n] e^{-j\omega n}$$

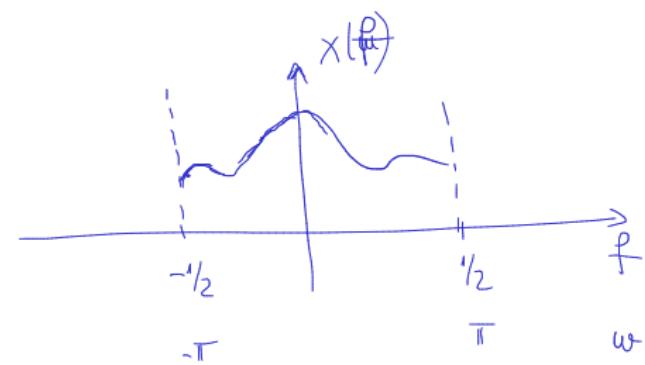
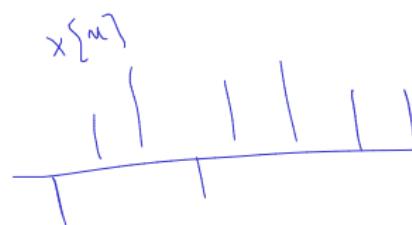
Discrete-Time Fourier Transform (DTFT)

- A non-periodic signal $x[n]$ has a continuous spectrum $X(\omega)$, with

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- e.g. $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$-\frac{\pi}{T} \quad \frac{\pi}{T}$$



Discrete Fourier Transform (DFT)

$$x[n] = \text{periodic}, \text{per} = N$$

$$f = \left\{ 0, \frac{1}{N}, \dots, \frac{N-1}{N} \right\}$$

- Consider the vector space of periodic signals with period N

~~signals~~

- for some fixed $N = 2, 3 \dots$ etc

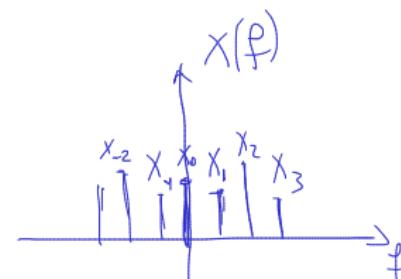
- This is a vector space of **dimension N**

- we need N numbers to identify a signal (specify its period)

- We can consider $x[n]$ only for **one period**, i.e. $n = 0, \dots, N - 1$

- The signals $\{e^{j2\pi f n}\}, \forall f \in \{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}$ form an **orthonormal basis** with N elements

- It is a **discrete** set of frequencies: $f = \frac{k}{N}, \forall k \in \{0, 1, \dots, N - 1\}$



Discrete Fourier Transform (DFT)

Inverse Discrete Fourier Transform

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- A periodic signal $x[n]$ can be written as a linear combination of k signals $\{e^{j2\pi kn/N}\}$, with some coefficients X_k

Discrete Fourier Transform

$$X_k = \frac{1}{N} \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle x, e^{j2\pi fn} \rangle$

$$x[n] = \sum_{k=0}^{N-1} X_k \cdot e^{j2\pi \left(\frac{k}{N}\right) n}$$
$$x[n] = \left(\sum_{k=0}^{N-1} X_k \cdot e^{j2\pi \left(\frac{k}{N}\right) n} \right)$$

$$X_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \left(\frac{k}{N}\right) n}$$

Discrete Fourier Transform (DFT)

- ▶ A periodic signal $x[n]$ with period N has a discrete spectrum $X(\omega)$ composed of only N frequencies $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}\}$
- ▶ Each frequency $\frac{k}{N}$ has a coefficient X_k
 - ▶ also written as c_k
 - ▶ The N coefficients X_k are the equivalent of $X(\omega)$
- ▶ It is also known as the "Fourier Series for Discrete Signals"

IV.3 The Discrete-Time Fourier Transform (DTFT)

Definition

Definitions (again):

Inverse Discrete-Time Fourier Transform (DTFT)

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi f n} df = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform (DTFT)

$$X(f) = \langle x[n], e^{j2\pi f n} \rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

Basic properties of DTFT

- $X(\omega)$ is defined only for $\omega \in [-\pi, \pi]$

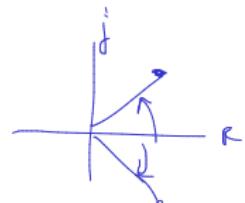
- or $f \in [-\frac{1}{2}, \frac{1}{2}]$

- $X(\omega)$ is complex (has $|X(\omega)|$, $\angle X(\omega)$)

- If the signal $x[n]$ is real, $X(\omega)$ is even

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

$$\alpha = |\alpha| \cdot e^{j\angle \alpha}$$

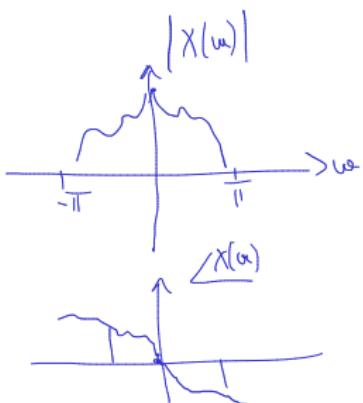


$$X(\omega) \in \mathbb{C} \Rightarrow |X(\omega)|, \angle X(\omega)$$

$$f(x)^* = f(-x)$$

- This means:

- modulus is even: $|X(\omega)| = |X(-\omega)|$
- phase is odd: $\angle X(\omega) = -\angle X(-\omega)$



Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi} \int_{-\pi}^0 X(\omega) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^\pi X(\omega) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_0^\pi (X(\omega) e^{j\omega n} + X(-\omega) e^{j(-\omega)n}) d\omega \\
 &= \frac{1}{2\pi} \int_0^\pi |X(\omega)| (e^{j\omega n} \angle X(\omega) + e^{-j\omega n} \angle X(\omega)) d\omega \\
 &= \frac{1}{2\pi} \int_0^\pi 2|X(\omega)| \cos(\omega n + \angle X(\omega)) d\omega
 \end{aligned}$$

$X(\omega)$

A $f = \text{or } \frac{1}{2}$ MORE

- Any signal $x[n]$ is a sum of sinusoids with all frequencies
 $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

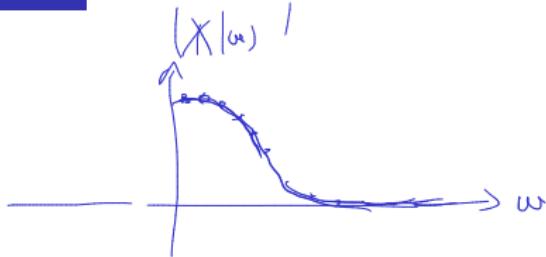
$$X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(f) e^{j(2\pi f)n} df = \int_{-2\pi}^{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$\left. \begin{array}{l} X(\omega) = |X(\omega)| \cdot e^{j \angle X(\omega)} \\ X(-\omega) = |X(\omega)| \cdot e^{-j \angle X(\omega)} \end{array} \right\}$$

$$\frac{e^{j\omega n} + e^{-j\omega n}}{2} = \cos(\omega n)$$

Expressing as sum of sinusoids

- ▶ Any signal $x[n]$ is a sum of sinusoids with all frequencies
 $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
 - ▶ this is the fundamental practical interpretation of the Fourier transform
- ▶ The modulus $|X(\omega)|$ is the amplitude of the sinusoids ($\times 2$)
 - ▶ for $\omega = 0$, $|X(\omega = 0)|$ = the DC component
- ▶ The phase $\angle X(\omega)$ gives the initial phase



Properties of DTFT

$$x[n] \longleftrightarrow X(\omega)$$

$$\begin{array}{ccc} x_1[n] & \longleftrightarrow & X_1(\omega) \\ x_2[n] & \longleftrightarrow & X_2(\omega) \end{array}$$

$$x[n] \longleftrightarrow X(\omega)$$

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$\underbrace{x_1 + x_2}_{x_1 + x_2}$

Properties of DTFT

$$\begin{aligned}x[n] &\xrightarrow{z} X(z) \\x[n+k] &\xrightarrow{z^{-k}} z^{-k} \cdot X(z) \\z = e^{j\omega}\end{aligned}$$

$\xleftrightarrow{\mathcal{Z}}$

$$\begin{aligned}x[n] &\xleftrightarrow{\mathcal{Z}} X(\omega) \\x[n-k] &\xleftrightarrow{\mathcal{Z}} e^{-j\omega k} \cdot X(n)\end{aligned}$$

$$\begin{aligned}\alpha &= |a| \cdot e^{j\angle a} \\&\stackrel{j\text{ something}}{=} |a| \cdot e^{j(\angle a + \text{something})} \\a \cdot e^{j\text{ something}} &= |a| \cdot e^{j\angle a}\end{aligned}$$

2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n-k] \cdot e^{-j\omega n} &= \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega(m+k)} \\&= e^{-j\omega k} \cdot \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega m} \\&= X(\omega)\end{aligned}$$

Properties of DTFT

3. Modulation in time

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of DTFT

$$x_1[n] * x_2[n] \leftrightarrow X_1(z) \cdot X_2(z)$$

$$\Downarrow z = e^{j\omega}$$

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- ▶ Not circular convolution, this is the normal convolution

6. Product in time

Product in time \leftrightarrow convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$$

Properties of DTFT

Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

Wiener-Khinchin theorem

Autocorrelation of a signal \leftrightarrow Power spectral density

$$r_{xx}[l] \xrightarrow{\mathcal{F}} S_{xx}(\omega) = |X(\omega)|^2$$

Parseval theorem

- **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = \int_{-\pi/2}^{\pi/2} |X(f)|^2 df$$

(► Is true for all orthonormal bases)

IV.4 The Discrete Fourier Transform (DFT)

Definitions

Definitions (again)

Inverse Discrete Fourier Transform (DFT)

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

$\begin{matrix} k=0 & \dots & 5 \\ \downarrow & \dots & \downarrow \\ 1 & 2 & \dots & 3 \end{matrix}$

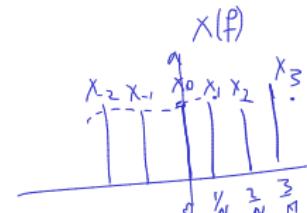
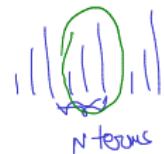
$$\frac{k}{N} = f$$

$f = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \dots, \frac{N-1}{N}$

$e^{-j \cdot 2\pi \cdot \frac{k}{N} \cdot n}$

Discrete Fourier Transform (DFT)

$$X_k = \frac{1}{N} \langle x[n], e^{j2\pi fn} \rangle = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$



$$f = \frac{6}{7} = -\frac{1}{7}$$

$$f = \frac{5}{7} = -\frac{2}{7}$$

$$f = \frac{4}{7} = -\frac{3}{7}$$

Periodicity and notation

- In discrete domain, $f = \frac{N-k}{N} = \frac{-k}{N}$ (aliasing, we can subtract 1 from f)

- We can consider X_{N-k} as X_{-k} , due to periodicity

- Example: a signal with period $N = 6$ has 6 DFT coefficients

- we can call them $X_0, X_1, X_2, X_3, X_4, X_5 \rightarrow \frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$

- we have $X_5 = X_{-1}, X_4 = X_{-2}$

- we can also call them $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$

$$\begin{matrix} z \\ X_4 & X_5 \end{matrix} =$$

$$f = \frac{5}{6} = \frac{-1}{6}$$

$$f = \frac{4}{6} = \frac{-2}{6}$$

$$X_5 = X_{-1}$$

Basic Properties of the DFT

$$X_k = |X_k| \cdot e^{j\angle X_k}$$

- X_k is complex (has $|X_k|$, $\angle X_k$)
- If the signal $x[n]$ is real, the coefficients are **even**

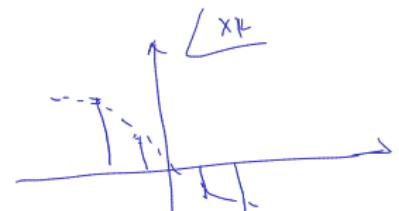
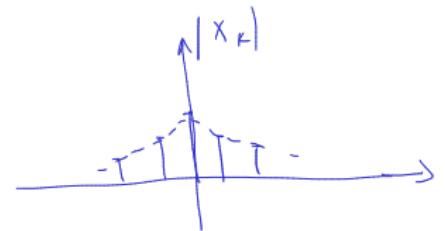
$$x[n] \in \mathbb{R} \rightarrow X_{-k} = X_k^*$$

- This means:

- modulus is even: $|X_k| = |X_{-k}|$
- phase is odd: $\angle X_{-k} = -\angle X_k$

$$X_{-k} = X_k^* \Rightarrow |X_{-k}| = |X_k|$$

$$\angle X_{-k} = -\angle X_k$$



Expressing as sum of sinusoids

- Grouping terms with k and $-k$:
- If N is odd, we have X_0 and pairs (X_k, X_{-k}) :

$$x[n] = X_0 e^{j0n} + \frac{1}{N} \sum_{k=-(N-1)/2}^{N-1} X_k e^{j2\pi kn/N} + \frac{1}{N} \sum_{k=0}^{(N-1)/2} X_k e^{j2\pi kn/N}$$

$$= X_0 + \frac{1}{N} \sum_{k=0}^{(N-1)/2} (X_k e^{j2\pi kn/N} + X_{-k} e^{-j2\pi kn/N})$$

$$= X_0 + \frac{1}{N} \sum_{k=0}^{(N-1)/2} |X_k| (e^{j2\pi kn/N + \angle X(k)} + e^{-j2\pi kn/N + \angle X(k)})$$

$$= X_0 + \frac{1}{N} \sum_{k=0}^{(N-1)/2} 2|X_k| \cos(2\pi k/N n + \angle X_k)$$

$\cancel{k = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{2}/N}$

$$X[m] = \frac{1}{N} \cdot \sum_{k=0}^{\frac{N-1}{2}} X_k e^{j2\pi \frac{k}{N} m}$$

$\cancel{k=0}$
 $\cancel{-\frac{N-1}{2}}$

$$X_{N-1} = X_{-1}$$

$$X_{N-2} = X_{-2}$$

Example:

$$N = 5$$

$$x[n] = \frac{1}{5} \sum_{k=0}^4 X_k e^{j2\pi \frac{k}{5} m}$$

$$= X_0 e^{j2\pi \frac{0}{5} m} + X_1 e^{j2\pi \frac{1}{5} m} + X_2 e^{j2\pi \frac{2}{5} m} + X_{-2} e^{j2\pi \frac{(-2)}{5} m} + X_{-1} e^{j2\pi \frac{(-1)}{5} m}$$

$$X_1 = |X_1| \cdot e^{j\angle X_1}$$

$$X_{-1} = |X_1| \cdot e^{-j\angle X_1}$$

$$|X_1| \cdot \left(e^{j(\angle X_1 + 2\pi \frac{1}{5} m)} + e^{-j(\angle X_1 + 2\pi \frac{1}{5} m)} \right)$$

Expressing as sum of sinusoids

- If N is even, we have X_0 and pairs (X_k, X_{-k}) , with an extra term $X_{N/2}$ which has no pair

e.g. $N = 6$: $X_{-2}, X_{-1}, X_0, X_1, X_2, X_3$

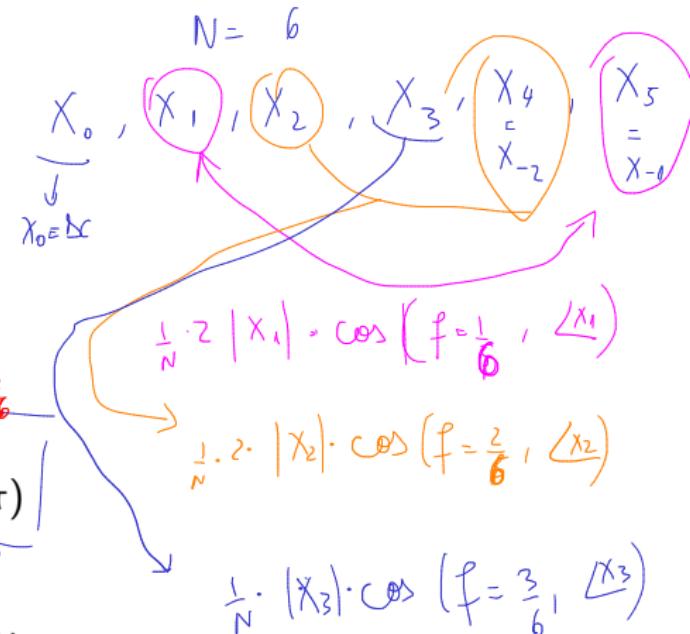
- $X_{N/2}$ must be a real number

- The extra term will be $\frac{1}{N}X_{N/2}e^{j2\pi N/2n/N} = X_{N/2} \cos(n\pi)$

- Overall:

$$x[n] = \underbrace{\frac{1}{N}X_0}_{\text{DC.}} + \frac{1}{N} \sum_{k=0}^{(N-2)/2} 2|X_k| \cos(2\pi k/Nn + \angle X_k) + \underbrace{\frac{1}{N}X_{N/2} \cos(n\pi)}_{\text{surplus}}$$

- Any signal $x[n]$ is a sum of sinusoids with frequencies $f = 0, 1/N, 2/N, \dots, (N-1)/2$ or $N/2$



Expressing as sum of sinusoids

$$f = \frac{k}{N}$$

- ▶ Any periodic signal $x[n]$ with period N is a sum of ~~N~~ sinusoids with frequencies $f = 0, 1/N, 2/N, \dots, (N-1)/2$ or $N/2$
- ▶ The modulus $|X_k|$ gives the **amplitude** of the sinusoids (sometimes $\times 2$)
 - ▶ for $\omega = 0$, $|X_0|$ = the DC component
 - ▶ when modulus = 0, that frequency has amplitude 0
- ▶ The phase $\angle X_k$ gives the initial phase

$$N = 6 \rightarrow$$

$$f = 0$$

$$f = \frac{1}{6}$$

$$f = \frac{2}{6}$$

$$f = \frac{3}{6}$$

$$N = 5 \rightarrow$$

$$f = 0$$

$$f = \frac{1}{5}$$

$$f = \frac{2}{5}$$

Example

- Consider a periodic signal $x[n]$ with period $N = \cancel{6}^5$ and the DFT coefficients:

$$X_k = [15.0000 + 0.0000i, \cancel{-2.5000 + 3.4410i}, \cancel{-2.5000 + 0.8123i}, \\ \cancel{-2.5000 - 0.8123i}, \cancel{-2.5000 - 3.4410i}]$$

Write $x[n]$ as a sum of sinusoids.

- Do the same for a periodic signal $x[n]$ with period $N = \cancel{5}^3$ and the DFT coefficients:

$$X_k = [21.0000 + 0.0000i, -3.0000 + 5.1962i, -3.0000 + 1.7321i, \\ -3.0000 + 0.0000i, -3.0000 - 1.7321i, -3.0000 - 5.1962i]$$

Write $x[n]$ as a sum of sinusoids.

$$X_1 = \underbrace{X_4^*}_{= X_{-1}} .$$

Properties of the DFT

$$\begin{aligned}x_1[n] &\leftrightarrow \left\{ X_k^{(1)} \right\} \\x_2[n] &\leftrightarrow \left\{ X_k^{(2)} \right\}\end{aligned}$$

1. Linearity

If the signal $x_1[n]$ has the DFT coefficients $\{X_k^{(1)}\}$, and $x_2[n]$ has $\{X_k^{(2)}\}$, then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\}$$

Proof: via definition

$$\underbrace{a \cdot x_1[n] + b \cdot x_2[n]}_{X_s[n]} \leftrightarrow \left\{ a \cdot X_k^{(1)} + b \cdot X_k^{(2)} \right\}$$

$$\begin{aligned}X_k &= \sum_{n=0}^{N-1} \underbrace{x_s[n]}_{ax_1[n] + bx_2[n]} \cdot e^{-j \frac{2\pi}{N} kn} \\&= a \cdot \sum_{n=0}^{N-1} x_1[n] e^{-j \frac{2\pi}{N} kn} + b \cdot \sum_{n=0}^{N-1} x_2[n] e^{-j \frac{2\pi}{N} kn} = a \cdot X_k^{(1)} + b \cdot X_k^{(2)}\end{aligned}$$

Properties of the DFT

$$\underbrace{|a| \cdot e^{j\angle a}} \cdot e^{j\text{const.}} = \underbrace{|a| \cdot e^{j(\angle a + \text{const.})}}$$

$$x[n] \leftrightarrow \{X_k\}$$

$$x[m-n_0] \leftrightarrow \left\{ e^{-j\frac{2\pi k m_0}{N}} \cdot X_k \right\}$$

2. Shifting in time

If $x[n] \leftrightarrow \{X_k\}$, then

$$x[n - n_0] \leftrightarrow \underbrace{\left\{ e^{(-j2\pi k n_0/N)} X_k \right\}}$$

Proof: via definition

- The amplitudes $|X_k|$ are not affected, shifting in time affects only the phase

$$\sum x[m] \cdot e^{-j\frac{2\pi k m}{N}} = \sum_m x[m-n_0] \cdot e^{-j\frac{2\pi k m}{N}} = \underbrace{\left(\sum_m x[m] \cdot e^{-j\frac{2\pi k m}{N}} \right)}_{X_k} \cdot e^{-j\frac{2\pi k m_0}{N}} = e^{-j\frac{2\pi k m_0}{N}} \cdot X_k$$

$\underbrace{1}_{(-1)} \cdot e^{-j\frac{2\pi k m_0}{N}} =$

$$z : \rightarrow z^{-m_0}$$

$$z = e^{j\omega} = e^{j2\pi f} = e^{j\frac{2\pi k}{N}}$$

$$x[m-n_0] \leftrightarrow \underbrace{e^{-j\frac{2\pi k m_0}{N}}}_{z} \cdot X_k$$

Properties of the DFT

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{X_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{X_{-k}^*\}$$

Properties of the DFT

5. Circular convolution

Circular convolution of two signals \leftrightarrow product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot X_k^{(1)} \cdot X_k^{(2)}\}$$

Circular convolution definition:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- ▶ takes two periodic signals of period N, result is also periodic with period N
- ▶ Example at the whiteboard: how it is computed

Example

Linear convolution

$$x_1 = [1, 2, 3, 0, \dots]$$

$$x_2 = [\dots, 4, 4, 4, 4, 0, \dots]$$

*

$$x_1 * x_2 =$$

$$\begin{array}{r} 4 \quad 4 \quad 4 \quad 4 \\ 8 \quad 8 \quad 8 \quad 8 \\ \hline 12 \quad 12 \quad 12 \quad 12 \\ \hline \dots, 0, 4 \quad 12 \quad 24 \quad 24 \quad 20 \quad 12, \dots \end{array}$$

Example (write on slides)

$$x_1 = \text{per...}, \underbrace{1, 2, 3, 4}_{N}, \dots \text{- periodic}$$

$$x_2 = \dots, \underbrace{5, 4, 3, 2}_N, \dots \text{- periodic..}$$

(*)

$$\begin{array}{c} x_1 \circledast x_2 : \\ \begin{array}{ccccccccc} 5, 4, 3, 2 & & 5, 4, 3, 2 & & \dots, 5, 4, 3, 2 \\ | & & | & & | & & | \\ 10, 8, 6 & & 4, 10, 8, 6 & & 4 & & 4 \\ | & & | & & | & & | \\ 15, 9 & & 12, 1, 6, 15, 9, 1 & & 12, 1, 6 \\ | & & | & & | & & | \\ 20 & & 16, 1, 12, 8, 20 & & 16, 1, 12, 8 \\ | & & | & & | & & | \\ \dots, 37 & & 32, 34, 37 & & \dots, 37 & & \dots, 37 \end{array} \end{array}$$

Circular convolution

- ▶ We are in the vector space of periodic signals with period N
- ▶ Linear (e.g. normal) convolution produces a result which is longer periodic with period N
- ▶ Circular convolution takes two sequences of length N and produces another sequence of length N
 - ▶ each sequence is a period of a periodic signal
 - ▶ circular convolution = like a convolution of periodic signals

6. Product in time

Product in time \leftrightarrow circular convolution of DFT coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} X_m^{(1)} X_{(k-m)_N}^{(2)} = X_k^{(1)} \otimes X_k^{(2)}$$

Properties of the DFT

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{2\pi} \sum_{k=0}^N |X_k|^2$$

- ▶ Is true for all orthonormal bases

Relationship between DTFT and DFT

- ▶ How are DTFT and DFT related?

- ▶ Discrete Time Fourier Transform:

- ▶ for non-periodical signals
- ▶ spectrum is continuous

\neq

- ▶ Discrete Fourier Transform

- ▶ for periodical signals
- ▶ spectrum is discrete

$$f = \frac{k}{N}$$

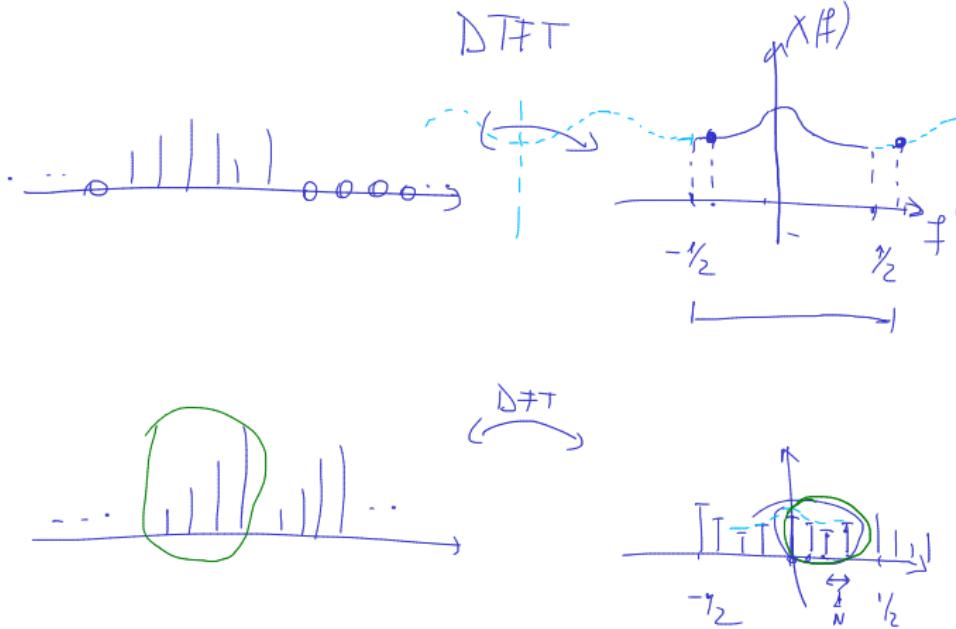
- ▶ Duality: periodic in time \leftrightarrow discrete in frequency

discrete in time \leftrightarrow

Both

periodic in frequency

Both

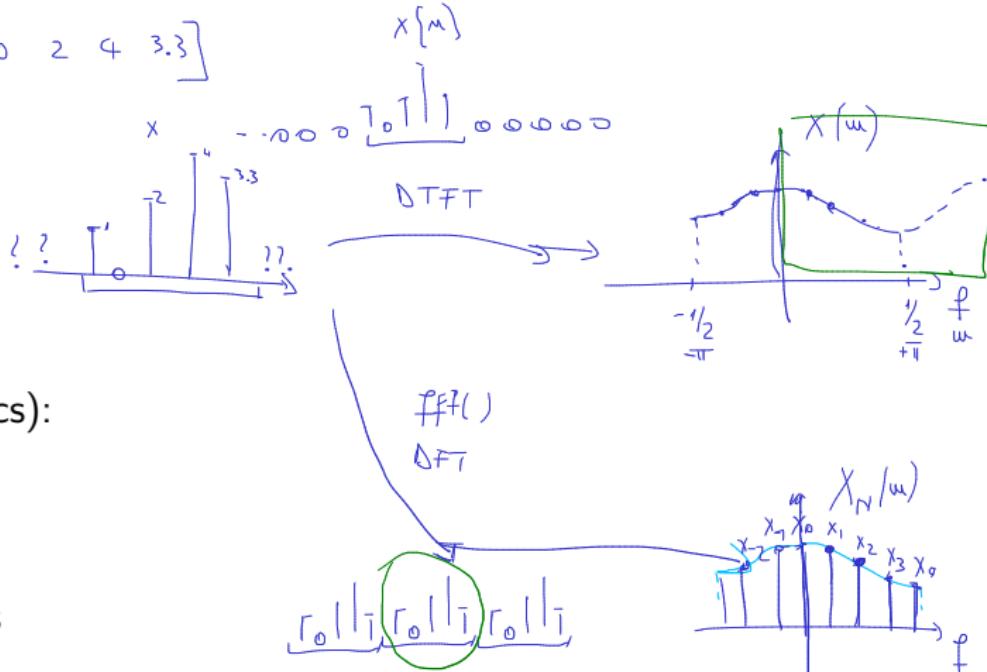


Relationship between DTFT and DFT

- ▶ Consider a non-periodic signal $x[n]$
- ▶ It has a continuous spectrum $X(\omega)$
- ▶ If we periodize it by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

$$x[n] = [1 \ 0 \ 2 \ 4 \ 3.3]$$



- ▶ then the Fourier transform is discrete (made of Diracs):

$$X_N(\omega) = 2\pi X_k \delta(\omega - k \frac{2\pi}{N})$$

- ▶ The coefficients of the Diracs = the DFT coefficients

$$\boxed{X_k = X(2\pi k / N)}$$

- ▶ They are samples from the continuous $X(\omega)$ of the non-periodized signal

Relationship between DTFT and DFT

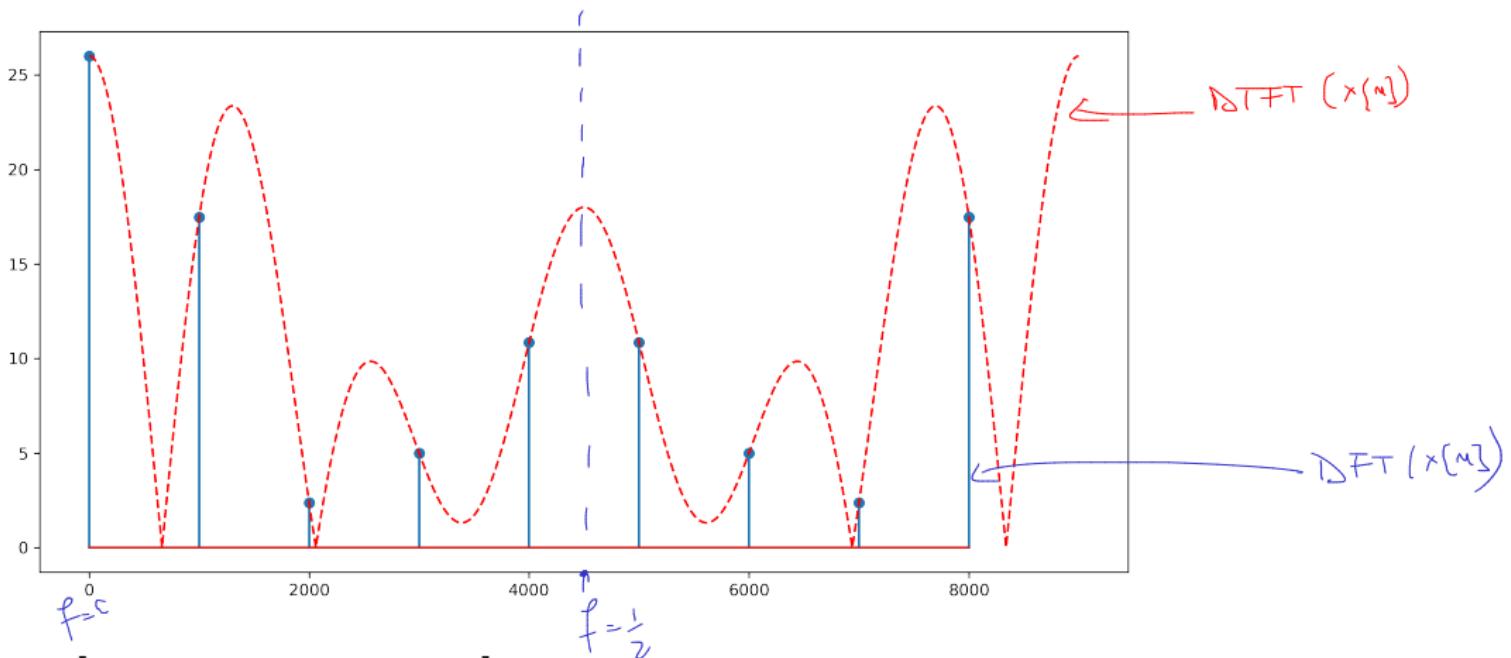
- ▶ Example: consider a sequence of 7 values

$$x = [6, 3, -4, 2, 0, 1, 2] \quad \begin{matrix} \dots & \dots & \dots & \dots \\ - & - & \dots & \dots & 6 & 3 & -4 & \dots & 12 & 6 & 3 & -4 & \dots \end{matrix}$$

- ▶ If we consider a non-periodic $x[n]$ with infinitely long zeros on either side, we have a continuous spectrum $X(\omega)$ (DTFT)
- ▶ If we consider that x is just a period of a periodic signal, we have a discrete spectrum X_k (DFT)
- ▶ Moreover, the discrete X_k are just samples from $X(\omega)$:

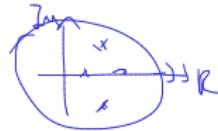
$$X_k = X(2\pi k / Nn)$$

Relationship between DTFT and DFT



- ▶ red line = $\text{DFT}(x)$ if x not periodical
 - ▶ actually run as $\text{fft}(x, 10000)$, x is extended with 9991 zeros
- ▶ blue = $\text{fft}(x)$

Relation between DTFT and Z transform



- Z transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(z) = \frac{1 + 0.5 z^{-1}}{1 + 0.2 z^{-1}}, \quad z \in \mathbb{C}$$

$\downarrow z = e^{j\omega}$

- DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(\omega) = \frac{1 + 0.5 e^{-j\omega}}{1 + 0.2 e^{-j\omega}}, \quad \omega \in \mathbb{R}$$

- DTFT can be obtained from Z transform with

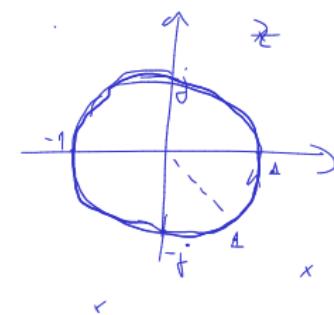
$$z = e^{j\omega}$$

- These $z = e^{j\omega}$ are points on the unit circle

- $|z| = |e^{j\omega}| = 1$ (*modulus*)
- $\angle z = \angle e^{j\omega} = \omega$ (*phase*)

$$\begin{aligned} e^{j\omega} &\Rightarrow |e^{j\omega}| = 1 \\ &= \cos(\omega) + j \sin(\omega) \\ e^{j\omega} &= \omega \end{aligned}$$

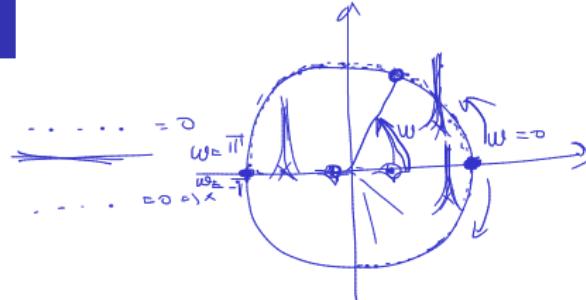
$$\left| e^{jx} = \cos(x) + j \sin(x) \right|$$



Relation between DTFT and Z transform

$$x[n] =$$

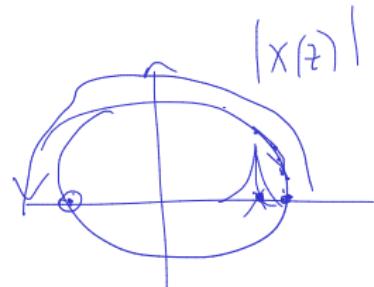
$$\left| X(z) \right| = \dots = \dots = \dots = \dots = \dots = \dots$$



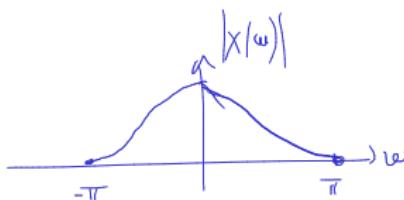
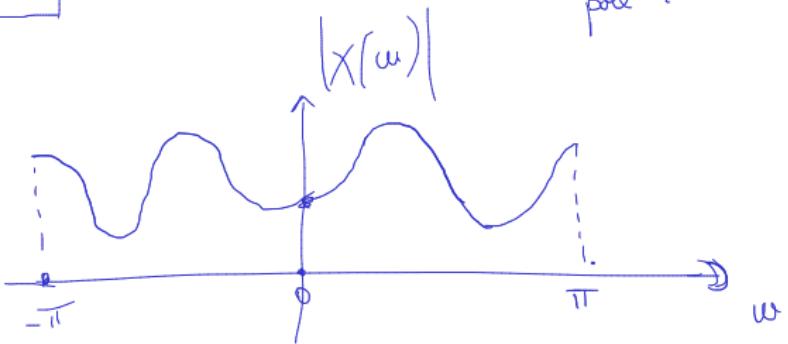
pole-zero diagram

► Fourier transform = Z transform evaluated **on the unit circle**

- if the unit circle is in the convergence region of Z transform
 - otherwise, equivalence does not hold
- This is true for most usual signals we work with
- some details and discussions are skipped



\Rightarrow



Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$\downarrow z = e^{j\omega}$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

zeros poles

► Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

► Phase:

$$\angle X(\omega) = \angle C + \underbrace{\angle(e^{j\omega} - z_1)}_{+ \cdots +} + \cdots + \underbrace{\angle(e^{j\omega} - z_M)}_{- \cdots -} - \underbrace{\angle(e^{j\omega} - p_1)}_{- \cdots -} - \cdots - \underbrace{\angle(e^{j\omega} - p_N)}_{- \cdots -}$$

$$|a \cdot b| = |a| \cdot |b|$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

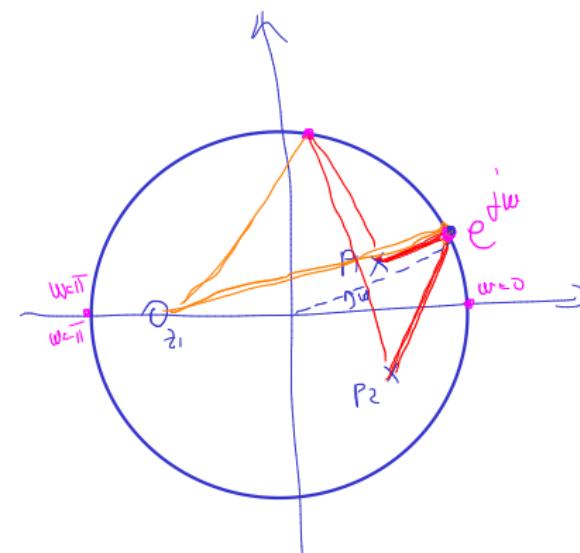
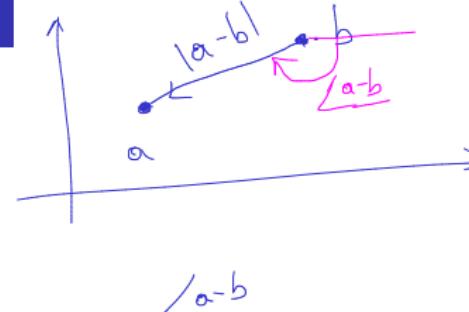
$$\begin{aligned}\angle a \cdot b &= \angle a + \angle b \\ \angle \frac{a}{b} &= \angle a - \angle b\end{aligned}$$

Geometric interpretation of Fourier transform

- ▶ For complex numbers:

- ▶ modulus of $|a - b|$ = the length of the segment between a and b
- ▶ phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- ▶ So, for a point on the unit circle $z = e^{j\omega}$
 - ▶ modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
 - ▶ phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

$$|X(\omega)| = \frac{c}{|z_1|} \cdot |z_2|$$



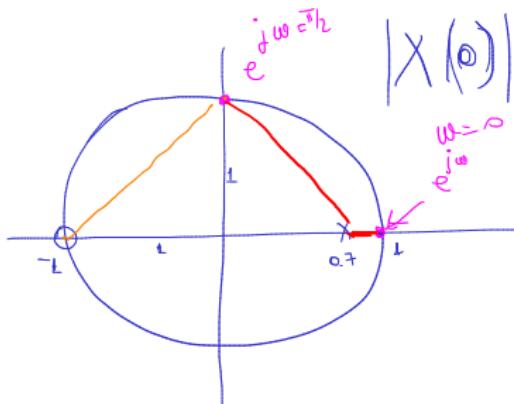
Geometric interpretation of Fourier transform

► Consequences:

- when a pole is very close to unit circle \rightarrow Fourier transform is large at this point
- when a zero is very close to unit circle \rightarrow Fourier transform is small at this point

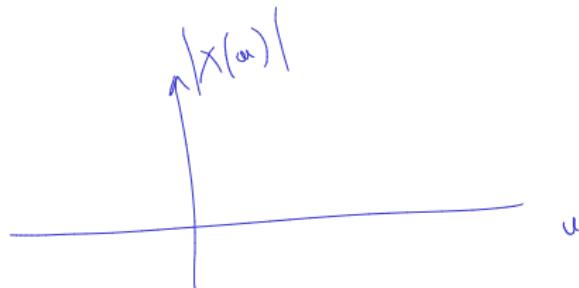
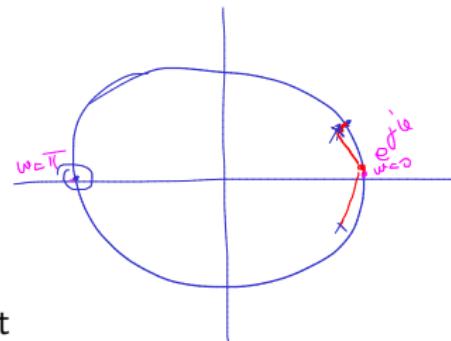
► Examples: ...

$$p_1 = 0.7 \\ z_1 = -1$$



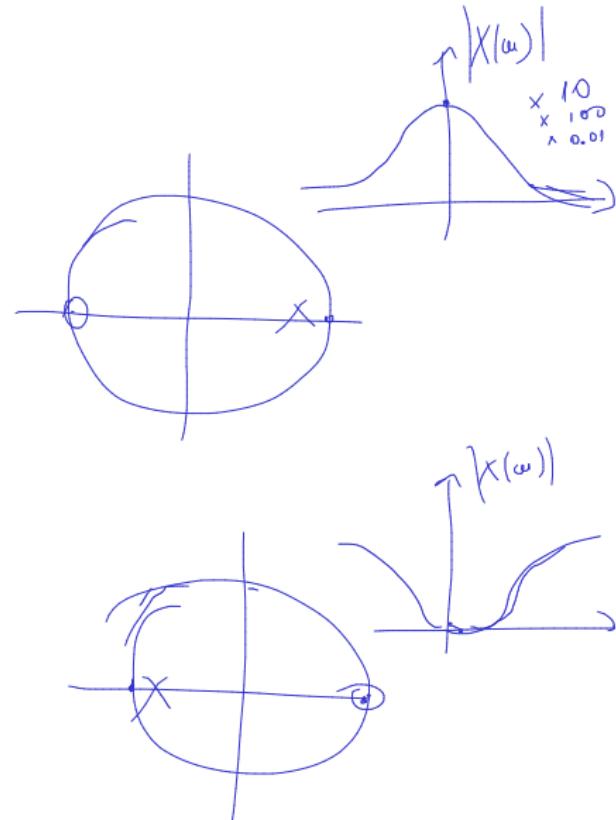
$$|X(0)| = C \cdot \frac{2}{0.3}$$

$$|X(\frac{\pi}{2})| = C \cdot \frac{\sqrt{2}}{\sqrt{1+0.7^2}}$$



Geometric interpretation of Fourier transform

- ▶ Simple interpretation for modulus $|X(\omega)|$:
 - ▶ Z transform $X(z)$ is like a landscape
 - ▶ poles = **mountains** of infinite height
 - ▶ zeros = **valleys** of zero height
 - ▶ Fourier transform $X(\omega) = \text{"Walking over this landscape along the unit circle"}$
 - ▶ The height profile of the walk gives the amplitude of the Fourier transform
 - ▶ When close to a mountain \rightarrow road is high \rightarrow Fourier transform has large amplitude
 - ▶ When close to a valley \rightarrow road is low \rightarrow Fourier transform has small amplitude



Geometric interpretation of Fourier transform

- ▶ Note: $X(z)$ might also have a constant C in front!
 - ▶ It does not appear in pole-zero plot
 - ▶ The value of $|C|$ and $\angle C$ must be determined separately
- ▶ This “geometric method” can be applied for phase as well

Time-frequency duality

- ▶ Duality properties related to all Fourier transforms
- ▶ Discrete \leftrightarrow Periodic
 - ▶ discrete in time \rightarrow periodic in frequency
 - ▶ periodic in time $\overset{\text{both}}{\underset{\text{both}}{\xrightarrow{}}}$ discrete in frequency
- ▶ Continuous \leftrightarrow Non-periodic
 - ▶ continuous in time \rightarrow non-periodic in frequency
 - ▶ non-periodic in time $\overset{\text{both}}{\underset{\text{both}}{\xrightarrow{}}}$ continuous in frequency

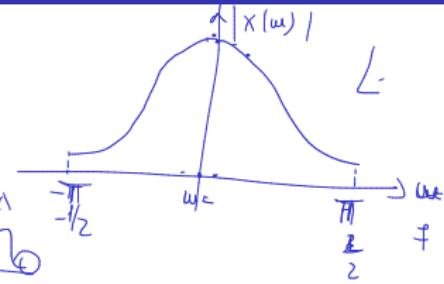
Terminology

- ▶ Based on frequency content:

- ▶ **low-frequency** signals

- ▶ **mid-frequency** signals (band-pass)

- ▶ **high-frequency** signals



- ▶ **Band-limited** signals: spectrum is 0 beyond some frequency f_{max}

- ▶ **Bandwidth B** : frequency interval $[F_1, F_2]$ which contains 95% of energy

- ▶ $B = F_2 - F_1$

- ▶ Based on bandwidth B :

- ▶ **Narrow-band** signals: $B \ll$ central frequency $\frac{F_1+F_2}{2}$

- ▶ **Wide-band** signals: not narrow-band

