

Digital Signal Processing

Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

Recap: Complex numbers

- ▶ real and imaginary part
- ▶ **modulus and phase**
- ▶ graphical interpretation
- ▶ Euler formula
- ▶ modulus and phase of e^{jx}

Definition of Z transform

- ▶ The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$$

- ▶ Notation:

$$\mathcal{Z}(x[n]) = X(Z)$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(Z)$$

Definition of Z transform

- ▶ Similar to the Laplace transform for continuous signals
- ▶ The Z transform associates a **polynomial** to a signal (think Information Theory class)
- ▶ Why?
 - ▶ Easy representation of convolution
 - ▶ Convolution of two signals = multiplication of polynomials
 - ▶ Efficient descriptions of complicated systems with poles and zeros

Region of convergence

- ▶ $X(Z)$ is a sum dependent on some variable z (complex number)
- ▶ The **Region Of Convergence (ROC)** = the values of z for which the sum is convergent (does not go to $\pm\infty$)

Examples

Exercises:

- Compute Z transform for the following signals:

$$x[n] = 1, 2, 5, 7, 0, \text{ (with time origin in 1 or in 5)}$$

$$\delta[n], \delta[n - k], \delta[n + k]$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Region of convergence

- ▶ z is a complex number
- ▶ Region of convergence (ROC) is displayed as an area in the complex plane (also known as the Z plane)

Region of convergence

- ▶ For **finite-support** signals, the ROC is the **whole** Z plane, possibly except 0 or ∞
- ▶ For **causal** signals, the ROC is the **outside** of a circle:

$$|z| > r_1$$

- ▶ e.g. if $|z|$ is big enough, the sum is convergent
- ▶ For **anti-causal** signals, the ROC is the **inside** of a circle:

$$|z| < r_2$$

- ▶ e.g. if $|z|$ is small enough, the sum is convergent
- ▶ Why circles? Because only modulus of z matters
 - ▶ complex numbers on a circle have the same modulus

Region of convergence

- ▶ For **bilateral** signals, the ROC is the area **between** two circles:

$$r_1 < |z| < r_2$$

- ▶ bilateral signals have a causal part and an anti-causal part
- ▶ For finite-support signals, the two “circles” are of “radius” 0 and ∞
- ▶ Two different signals can have the same expression of $X(z)$, but with different ROC!
 - ▶ ROC is an essential part in specifying a Z transform
 - ▶ it should never be omitted

The Inverse Z Transform

- ▶ From a purely mathematical point of view, $X(z)$ is a complex function

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- ▶ Proper definition of inverse transform is based on the theory of complex functions
- ▶ Multiply with z^{n-1} and integrate along a contour C inside the ROC:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{k=-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{k=-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

The Inverse Z Transform

- ▶ The Cauchy integral theorem says that:

$$\frac{1}{2\pi j} \oint_C z^{n-k-1} dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- ▶ And therefore:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- ▶ We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

1. Linearity

If $x_1[n] \xrightarrow{Z} X_1(z)$ with ROC1, and $x_2[n] \xrightarrow{Z} X_2(z)$ with ROC2, then:

$$ax_1[n] + bx_2[n] \xrightarrow{Z} aX_1(z) + bX_2(z)$$

and ROC is at least the intersection of ROC1 and ROC2.

Proof: use definition

Properties of Z transform

2. Shifting in time

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, then:

$$x[n - k] \xleftrightarrow{Z} z^{-k} X(z)$$

with same ROC, possibly except 0 and ∞ .

Proof: by definition

- ▶ valid for all k , also for $k < 0$
- ▶ delay of 1 sample $= z^{-1}$

Properties of Z transform

3. Modulation in time

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, then:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(e^{-j\omega_0} z)$$

with same ROC

Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \xrightarrow{Z} X(z)$ with ROC $r_1 < |z| < r_2$, then:

$$x[-n] \xrightarrow{Z} X(z^{-1})$$

with ROC $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, then:

$$nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$$

with same ROC

Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, then:

$$x[n] - x[n-1] \xleftrightarrow{Z} (1 - z^{-1})X(z)$$

with same ROC except $z = 0$.

Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, then:

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{Z} \frac{X(z)}{(1 - z^{-1})}$$

with same ROC except $z = 1$.

Proof: $x[n] = y[n] - y[n - 1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \xleftrightarrow{Z} X(z)$ with ROC, and $x[n]$ is a complex signal, then:

$$x^*[n] \xleftrightarrow{Z} X^*(z^*)$$

with same ROC except $z = 0$.

Proof: apply definition

Consequence

If $x[n]$ is a real signal, the poles / zeroes are either real or in complex pairs

Properties of Z transform

9. Convolution in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with ROC1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with ROC2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

and ROC the intersection of ROC1 and ROC2.

Proof: use definition

- ▶ **Very important property!**
- ▶ Can compute the convolution of two signals via the Z transform

Properties of Z transform

10. Correlation in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with ROC1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with ROC2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \xleftrightarrow{Z} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and ROC the intersection of ROC1 and with the ROC of $X_2(z^{-1})$ (see reflection property)

Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

When $z \rightarrow \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z > a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z < a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$ z > r$

III.2. Z transforms which are Rational Functions

Rational functions

- ▶ Many Z transforms are in the form of a **rational function**, i.e. a **fraction** where
 - ▶ numerator = **polynomial** in z^{-1} or z
 - ▶ denominator = **polynomial** in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

- ▶ Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

Poles and zeros

- ▶ Any polynomial is completely determined by its **roots** and a **scaling** factor

$$\text{Any polynomial}(X) = G \cdot (X - x_1) \dots (X - x_n)$$

- ▶ The **zeros** of $X(z)$ are the **roots of the numerator** $B(z)$
- ▶ The **poles** of $X(z)$ are the **roots of the denominator** $A(z)$
- ▶ The zeros are usually named z_1, z_2, \dots, z_M , and the poles p_1, p_2, \dots, p_N .

Poles and zeros

- ▶ The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1) \dots (z - z_M)}{(z - p_1) \dots (z - p_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1}) \dots (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \dots (1 - p_N z^{-1})}$$

- ▶ It has:
 - ▶ M zeros with finite values
 - ▶ N poles with finite values
 - ▶ and either $N-M$ zeros in 0, if $N > M$, or $N-M$ poles in 0, if $N < M$ (trivial poles/zeros)

► Example:

$$\begin{aligned}X(z) &= \frac{2z^2 + 0.4z - 1}{3z^3 + 2.4z^2 - 3z - 2.4} \\&= \frac{2}{3} \cdot \frac{(z - 0.3)(z + 0.5)}{(z - 1)(z + 1)(z + 0.8)} \\&= z^{-1} \cdot \frac{(2 + 0.4z^{-1} - 1z^{-2})}{3 + 2.4z^{-1} - 3z^{-2} - 2.4z^{-3}} \\&= z^{-1} \cdot \frac{2}{3} \cdot \frac{(1 - 0.3z^{-1})(1 + 0.5z^{-1})}{(1 - z^{-1})(1 + z^{-1})(1 + 0.8z^{-1})}\end{aligned}$$

► Multiple ways of writing same expression

Graphical representation

- ▶ The graphical representation of poles and zeros in the complex plane is called **the pole-zero plot**
- ▶ Graphical: poles = “x”, zeros = “0”
- ▶ ROC cannot contain poles
- ▶ Example: at whiteboard

III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

Inverse Z Transform:

- ▶ We have $X(z)$ and the ROC, what is the signal $x[n]$ =?

Methods:

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition** (the one we'll actually use)

Partial fraction decomposition

Any rational function

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$X(z) = c_0 + c_1 z^{-1} + \dots c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + \dots \frac{A_N}{z - p_N}$$

- ▶ Each pole p_i has a corresponding partial fraction $\frac{A_i}{z - p_i}$
- ▶ First terms appear if $M \leq N$
- ▶ Based on linearity, we invert each term individually (simple)

Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, **divide numerator to denominator** to obtain the first terms.
 - ▶ The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller than denominator
2. In the remaining fraction, **eliminate the negative powers** of z by multiplying with z^N . We want all powers like z^N , not z^{-N}
3. **Divide by z :**

$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$

Procedure for Inverse Z Transform

4. Compute the poles of $\frac{X_1(z)}{z}$ and **decompose in partial fractions**:

$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + \dots$$

5. **Multiply back with z :**

$$X_1(z) = A_1 \frac{z}{z - p_1} + \dots$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- ▶ If all poles are distinct:

$$A_k = (z - p_k) \frac{X(z)}{z} \Big|_{z=p_k}$$

- ▶ If poles are in complex conjugate pairs
 - ▶ group the two fractions into a single fraction of degree 2
- ▶ If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \dots + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!} \frac{d^{m-i}}{dz^{m-i}} \left[(z - p_k)^m \cdot \frac{X(z)}{z} \right] \Big|_{z=p_k}$$

- ▶ example for $m = 2$

Real signals and complex poles/zeros

- ▶ Consequence of the complex-conjugate property of Z transform:
- ▶ A signal $x[n]$ with **real values** can have only:
 - ▶ **real-valued** poles or zeroes
 - ▶ complex poles and zeroes in **conjugate pairs**, which can be grouped into a single term of degree 2, with real coefficients
- ▶ If a Z transform has a complex pole or zero **without** its conjugate pair, then the corresponding signal $x[n]$ is **complex**

Position of poles and signal behavior

- ▶ A rational Z transform $X(z)$ = sum of partial fractions, as we just saw
 - ▶ and some initial terms z^k in front
 - ▶ Each **partial fraction** (pole) generates an **exponential signal**:
 - ▶ $a^n u[n]$,
- OR
- ▶ $-a^n u[-n - 1]$
 - ▶ For a single partial fraction (one pole only), we will analyze the relation between the position of the pole and the signal in time

Position of poles and signal behaviour - 1 pole

- ▶ Consider a single partial fraction with **1 pole** $p_1 = a$:

$$X(z) = C \cdot \frac{z}{z - a}, \quad \text{ROC} : |z| > |a|$$

- ▶ Consider only real signals $x[n] \in \mathbb{R} \rightarrow a$ is real
- ▶ Consider only causal signals $x[n] \rightarrow \text{ROC is } |z| > |a|$
- ▶ Let's analyze how the corresponding signal looks like
 - ▶ use the formulas:

$$x[n] = a^n u[n]$$

Position of poles and signal behavior - 1 pole

How does the signal look like, depending on the pole value $p_1 = a$:

- ▶ Pole **inside** the unit circle ($|a| < 1$) = **exponentially decreasing** signal
- ▶ Pole **outside** the unit circle ($|a| > 1$) = **exponentially increasing** signal
- ▶ Pole **exactly on** unit circle ($|a| = 1$) = not increasing, not decreasing, but **constant** signal
- ▶ **Negative** pole ($a < 0$) \rightarrow **alternating** signal
- ▶ **Positive** value ($a > 0$) \rightarrow **non-alternating** signal

Position of poles and signal behavior - 1 pole

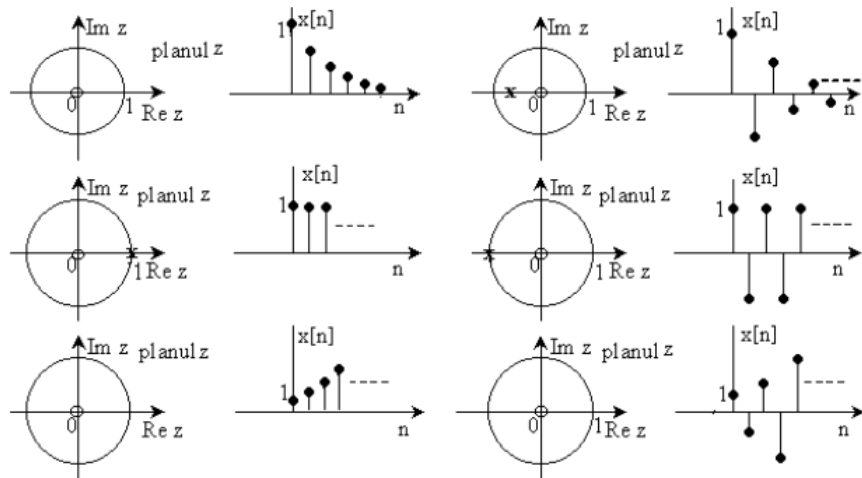


Figure 2: Signal behavior for 1 pole

Position of poles and signal behavior - 1 double pole

- ▶ Consider a **double pole** ($p_1 = a, p_2 = a$):

$$X(z) = C \frac{az}{(z-a)^2} = C \frac{az^{-1}}{(1-az^{-1})^2}, \text{ROC} : |z| > |a|$$

- ▶ The corresponding signal is:

$$x[n] = na^n u[n]$$

Effect of double pole in $p_1 = p_2 = a$:

- ▶ Pole inside the unit circle ($|a| < 1$) = decreasing signal
- ▶ Pole outside the unit circle ($|a| > 1$) = increasing signal
- ▶ Pole **exactly on unit** circle ($|a| = 1$) = **increasing signal**
- ▶ Negative pole ($a < 0$) = alternating signal
- ▶ Positive value ($a > 0$) = non-alternating signal

Position of poles and signal behavior - 1 double pole

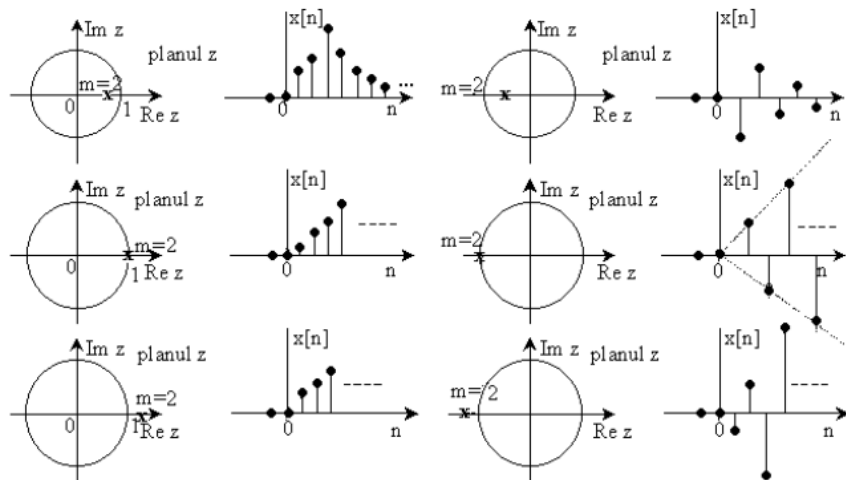


Figure 3: Signal behavior for 1 double pole

Position of poles and signal behavior - conjugate poles

- ▶ Consider a **pair of complex conjugate** poles ($p_1 = a$, $p_2 = a^*$):

$$X(z) = \frac{1 - a \cos \omega_0 z^{-1}}{1 - 2a \cos \omega_0 z^{-1} + a^2 z^{-2}}, \text{ ROC : } |z| > |a|$$

- ▶ The corresponding signal is:

$$x[n] = a^n \cos(\omega_0 n) u[n]$$

- ▶ Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**

Position of poles and signal behavior - conjugate poles

Effect of a pair of complex conjugate poles = **sinusoidal with exponential envelope**

- ▶ **phase** of poles gives the **frequency** of the sinusoidal
- ▶ **modulus** of poles gives the **exponential envelope**
 - ▶ poles **inside** unit circle = **decreasing** signal
 - ▶ poles **outside** unit circle \rightarrow **increasing** signal
 - ▶ poles **on** unit circle \rightarrow **oscillating signal**, constant amplitude, neither increasing nor decreasing

What if the poles are double?

- ▶ poles **on** unit circle \rightarrow **increasing** signal
- ▶ otherwise, similar to above

Position of poles and signal behavior - conjugate poles

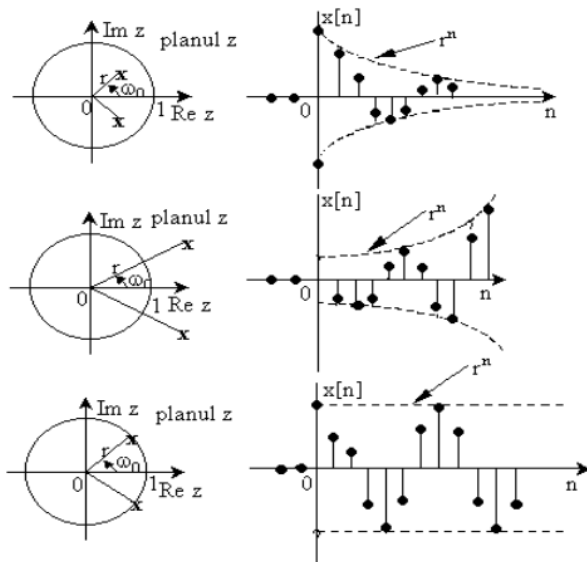


Figure 4: Signal behavior for 1 double pole

Position of poles and signal behavior

Summary: position of poles and behavior of signal

- ▶ A Z transform can be decomposed into **partial fractions**, i.e. separate poles
- ▶ Each pole means a separate fraction, means a separate component within the signal
- ▶ Conclusions (for real signals, causal):
 - ▶ **all poles inside** unit circle = **bounded** signal
 - ▶ because all components are exponentially decreasing
 - ▶ **simple poles on** unit circle = **bounded** signal
 - ▶ not increasing to infinity, but also not decreasing
 - ▶ **otherwise** = **unbounded** signal
 - ▶ poles **closer to 0** = **faster decreasing** signal
 - ▶ poles **farther from 0** = **slower decrease** of signal

III.4 LTI systems and the Z Transform

System function of a LTI system

- ▶ Consider a LTI system with impulse response $h[n]$
- ▶ If we apply an input signal $x[n]$, the output is (convolution):

$$y[n] = x[n] * h[n]$$

- ▶ In Z transform, **convolution = product** of transforms

$$Y(z) = X(z) \cdot H(z)$$

- ▶ **The system function $H(z)$ of a LTI system = the Z transform of the impulse response $h[n]$**
- ▶ The system function of a LTI system is (you know this from SCS):

$$H(z) = \frac{Y(z)}{X(z)}$$

System function and the system equation

- ▶ Reminder: any LTI system has an equation:

$$\begin{aligned}y[n] &= - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k] \\&= - a_1 y[n-1] - a_2 y[n-2] - \dots - a_N y[n-N] + \\&\quad + b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]\end{aligned}$$

- ▶ which can be rewritten as:

$$y[n] + \sum_{k=1}^n a_k y[n-k] = \sum_{k=0}^m b_k x[n-k]$$

$$\begin{aligned}y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N] &= \\&= b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]\end{aligned}$$

System function and the system equation

- ▶ The system function $H(z)$ can be written **directly from the equation**
- ▶ We apply the Z transform to the whole equation
 - ▶ every $y[n - k]$ becomes $z^{-k}Y(z)$
 - ▶ every $x[n - k]$ becomes $z^{-k}X(z)$
 - ▶ $Y(z)$, $X(z)$ are pulled in front as common factors
- ▶ We obtain:

$$Y(z) \left(1 + \sum_{k=1}^N a_k z^{-k} \right) = X(z) \left(\sum_{k=0}^M b_k z^{-k} \right)$$

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \\ &= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \end{aligned}$$

System function and the system equation

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \\ &= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \end{aligned}$$

- ▶ coefficients b_k of $x[n]$, $x[n-1]$... appear at **numerator**
- ▶ coefficients a_k of $y[n-1]$, $y[n-2]$... appear at **denominator**
 - ▶ beware of the sign change of a_k
 - ▶ the coefficient of $y[n]$ itself is always $a_0 = 1$

System function of FIR systems

Particular cases:

- ▶ **FIR systems:** when all $a_k = 0$
 - ▶ only zeros, no poles (“all-zero system”), no denominator in $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^M b_k z^{-k}$$

- ▶ the coefficients b_k are really the impulse response $h[n]$

System function of IIR systems

Particular cases:

- ▶ If some $a_k \neq 0$ we have an **IIR system**
 - ▶ $H(z)$ has some polynomial at the denominator
 - ▶ If denominator is just b_0 : **all-pole system**
 - ▶ has only poles, no zeros

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

- ▶ Or it can be a general IIR system with both poles and zeros (i.e. polynomials both at numerator or denominator)

Stability of a system and $H(z)$

Reminders from chapter 2:

- ▶ **Stable** system = a **bounded input** implies a **bounded output** (BIBO)
- ▶ A system is stable if:

$$\sum |h[n]| < \infty \quad (\text{is convergent})$$

Stability of a system and $H(z)$

- ▶ For a system with system function $H(z)$ we have:

$$|H(z)| = \left| \sum h[n]z^{-n} \right| \leq \sum |h[n]| \cdot |z^{-n}|$$

- ▶ Now let's consider z **on the unit circle**, i.e. $|z| = |z^{-n}| = 1$:

$$|H(z)|_{|z|=1} \leq \sum |h[n]|$$

- ▶ If the system is **stable**, $\sum |h[n]| < \infty$ (**convergent**), so $|H(z)|_{|z|=1} < \infty$
 - ▶ i.e. the unit circle $|z| = 1$ **is in the ROC**

Stability of a system and $H(z)$

- ▶ A LTI system is stable if the **unit circle is inside the Region of Convergence** of $H(z)$
 - ▶ one can also prove the reciprocal, so there is equivalence
- ▶ When the system is also **causal**:
 - ▶ ROC of causal system = exterior of a circle given by the largest pole
 - ▶ stable = unit circle inside the ROC
 - ▶ therefore stable = all poles **inside** unit circle
- ▶ A **causal** LTI system is stable if **all the poles are inside the unit circle**

Stability of a system and $H(z)$

- ▶ Alternative explanation:
 - ▶ If one pole is **outside** unit circle, the signal component for that partial fraction will be exponentially **increasing** \rightarrow whole signal is **unbounded**

Natural and forced response

- ▶ Consider a causal LTI system with initial conditions = 0
 - ▶ I.C. are relevant for recursive implementations (IIR)
- ▶ Consider an input signal:

$$x[n] \xleftrightarrow{Z} X(z) = \frac{N(z)}{Q(z)}$$

- ▶ Consider an impulse response (system function):

$$h[n] \xleftrightarrow{Z} H(z) = \frac{B(z)}{A(z)}$$

- ▶ Then the output signal is:

$$y[n] = x[n] * h[n] \xleftrightarrow{Z} Y(z) = X(z)H(z) = \frac{N(z)B(z)}{Q(z)A(z)}$$

- ▶ (Some poles and zeros might simplify, if exactly identical)

Natural and forced response

- ▶ Denote the poles of $X(z)$ as q_i and the poles of $H(z)$ as p_i
 - ▶ Assume all poles are *simple* (i.e. no multiplicity)
 - ▶ Assume all poles \neq all zeros, so no simplification
- ▶ The output signal has components dependent on the **input signal** and also of the **system itself**

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

- ▶ and $y[n]$ is

$$y[n] = \underbrace{\sum_{k=1}^N A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^L Q_k (q_k)^n u[n]}_{\text{forced response}}$$

Natural and forced response

Any output $y[n]$ is the **sum of two signals**:

- ▶ **Natural response** $y_{nr}[n]$ = the part given by the poles of **the system** itself
- ▶ **Forced response** $y_{fr}[n]$ = given by the poles of **the input signal**
- ▶ Together they form the **zero-state response** of the system = the output signal when initial conditions are 0

Zero-input response

If there are **non-zero** initial conditions, there is a **third component** as well:

- ▶ **Zero-input response** $y_{zi}[n]$ = given by the initial conditions of the system
 - ▶ It behaves similarly to the natural response, i.e. depends on the system's poles

Transient and permanent response

- ▶ For a **stable** system, all system poles $|p_k| < 1$
 - ▶ therefore, both natural response and zero-input response are made of decreasing exponentials
- ▶ For a stable system, the natural response and the zero-input response **die out exponentially**
- ▶ Thus, the natural response and the zero-input response are called **transient** response
 - ▶ they fade away, usually quickly
- ▶ Input signals last indefinitely \Rightarrow the forced response is a **permanent response**

Transient and permanent regime

Operating regimes:

- ▶ When the input signal is first applied, and the transient response is present, the system is in **transient regime**
- ▶ When the transient response has died out, the system remains in **permanent regime**, where only the input signal determines the output

Example: apply a infinitely long sinusoidal, starting from $n = 0$

- ▶ the output has some irregularities at the beginning, due to the natural responses
- ▶ afterwards, it becomes perfectly regular