Digital Signal Processing

Chapter V. Digital filtering

Response of LTI systems to harmonic signals

$$x[n] = A\cdot e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \xrightarrow{\;y[n]\;}$$

- We consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$x[n-k] = A\cdot e^{j\omega_0 (n-k)} = A\cdot e^{j\omega_0 n}\cdot e^{-j\omega_0 k}$$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \boxed{\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}}\, \underbrace{Ae^{j\omega_0 n}}_{x[n]} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

$$h[n] \xleftrightarrow[DTFT]{\mathcal{F}} H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Transfer function

- $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

Response of LTI systems to harmonic signals

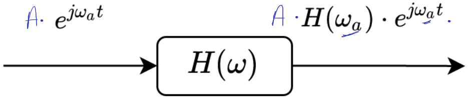


Figure 1: Output = a constant × Input

- $H(\omega)$ = Fourier transform of $h[n]$ evaluated for ω = **transfer function**

Eigen-function

$e^{j\omega_0 n}$

- Complex exponential signals are **eigen-functions** (funcții proprii) of LTI systems:
  - output signal = input signal × a (complex) constant
- $H(\omega_0)$ is a complex constant that multiplies the input signal
  - Amplitude of input gets multiplies by $|H(\omega_0)|$
  - Phase of input signal is added with $\angle H(\omega_0)$
- Why are sin/cos/exp functions important?
  - If input signal = sum of complex exponential (like coses + sinuses),
  - then output = same sum of complex exponentials, each scaled with some coefficients

$$H(\omega_0) = |H(\omega_0)| \cdot e^{j\angle H(\omega_0)}$$

$$y[n] = H(\omega_0) \cdot A e^{j\omega_0 n} = |H(\omega_0)| \cdot A \cdot e^{j(\omega_0 n + \angle H(\omega_0))}$$

Response to cosine and sine

- Cosine / sine = sum of two exponentials, via Euler

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}$$

$$\sin(\omega n) = \cos(\omega n - \frac{\pi}{2}) = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

- System is linear and real =>
  - amplitude is multiplied by $|H(\omega_0)|$
  - phase increases by $\angle H(\omega_0)$
- See proof at blackboard

$$\boxed{A \cdot \cos(\omega_0 n)} \longrightarrow \boxed{\begin{matrix} h[n] \\ H(\omega) \end{matrix}} \longrightarrow y[n]$$

$$\frac{A}{2} \cdot e^{j\omega_0 n} + \frac{A}{2} e^{-j\omega_0 n}$$

$$\frac{A}{2} \cdot H(\omega_0) \cdot e^{j\omega_0 n} + \frac{A}{2} \cdot H(-\omega_0) \cdot e^{-j\omega_0 n}$$

$$= \frac{A}{2} \cdot |H(\omega_0)| \cdot e^{j(\omega_0 n + \angle H(\omega_0))} + \frac{A}{2} \cdot |H(\omega_0)| \cdot e^{-j(\omega_0 n + \angle H(\omega_0))} =$$

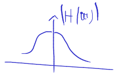
$$= \boxed{A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \angle H(\omega_0))}$$

Same for sin()

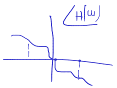
H(ω) is an even function:

$$H(\omega) = H(-\omega)^*$$

$$|H(\omega)| = |H(-\omega)|$$



$$-\angle H(\omega) = \angle H(-\omega)$$



Frequency response

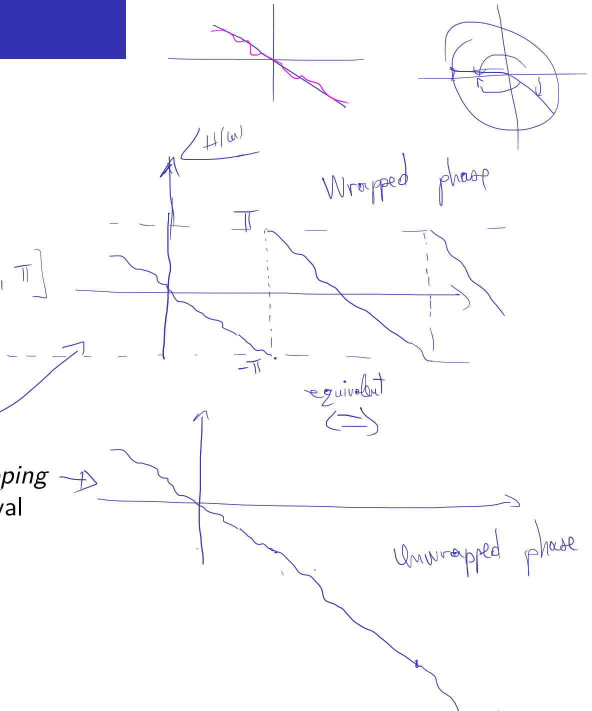
- Naming:

  or transfer function

  - $H(\omega)$ = **frequency response** of the system
  - $|H(\omega)|$ = **amplitude response** (or magnitude response)
  - $\angle H(\omega)$ = **phase response**

  $\angle H(\omega) \in [-\pi, \pi]$

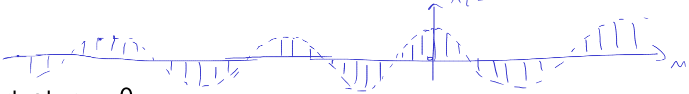
- Magnitude response is non-negative: $|H(\omega)| \geq 0$
- Phase response is an angle: $\angle H(\omega) \in (-\pi, pi]$
  - Phase response may have jumps of 2π (wrapped phase)
  - Stitching the pieces in a continuous function = phase *unwrapping*
  - Unwrapped phase: continuous function, may go outside interval $(-\pi, pi]$
  - Example: at blackboard



Permanent and transient response

Permanent response
$$x[n] = A \cdot \cos(\omega_0 n + \varphi), \quad n \in \mathbb{Z}$$
$$y[n] = A \cdot |H(\omega_0)| \cdot \cos(\omega_0 n + \varphi + \angle H(\omega_0))$$

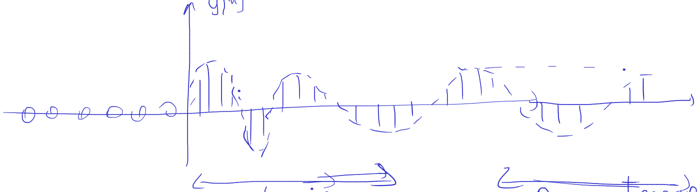
$x[n] = A\cos(\omega_0 n + \varphi)$



- Warning: $\cos(\omega n)$ does not start at $n = 0$
- The above harmonic signals start at $n = -\infty$.
- What's wrong if the signal starts at some time n?

$x[n] = A \cdot \cos(\omega_0 n + \varphi) \cdot u[n]$

$y[n]$



Permanent and transient response

- What if the signal starts at some time n?
- Total response $=$ transient response $+$ permanent response
  - transient response goes towards 0 as n increases
  - permanent response $=$ what remains
- So the above relations are valid only in **permanent regime**
  - i.e. after the transient regime has passed
  - i.e. after the transient response has practically vanished
  - i.e. when the signal started very long ago (from $n = -\infty$)
  - i.e. when only the permanent response remains in the output signal
- Example at blackboard

Permanent response of LTI systems to periodic inputs

- Consider an input $x[n]$ which is periodic with period N

Inverse DFT

- Then it can be represented as a Fourier series with coefficients c_k:

ω_0

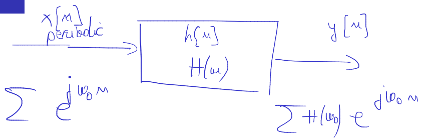
$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{j 2\pi k n / N}}$$

- Since the system is linear, each component k gets multiplied with $H\left(\frac{2\pi}{N} k\right)$
- So the total output is:

X_k of $y[n]$

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N} k\right) e^{j 2\pi k n / N}$$

- The output is still periodic, same period, same frequencies



Response of LTI systems to non-periodic signals

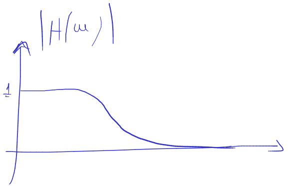
- Consider a general input $x[n]$ (not periodic)
- The output $=$ input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- Output spectrum $=$ Input spectrum $\times$ Transfer function

Response of LTI systems to non-periodic signals



- The transfer function $H(\omega)$ "shapes" the spectrum

$$\boxed{Y(\omega) = X(\omega) \cdot H(\omega)}$$

- In polar form:
  - modulus is multiplied

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$

  - phases is added:

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

Response of LTI systems to non-periodic signals

- The system **attenuates/amplifies** the input frequencies and **changes their phases**
- $H(\omega)$ = the **transfer function**
- $H(z)$ = the **system function**
- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

Roc

Power spectral density

$x\{n\} \longleftrightarrow X(\omega)$

$|X(\omega)|^2 = S_{xx}(\omega) =$ Power Spectral Density

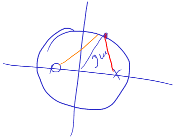
$|Y(\omega)|^2 = |X(\omega)|^2 \cdot |H(\omega)|^2$

$S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$

- $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\cancel{\Omega}\,\omega)$
- The poles and zeros of $S(\omega)$ come in pairs $(z, 1/z$ and $p, 1/p)$

Digital filters

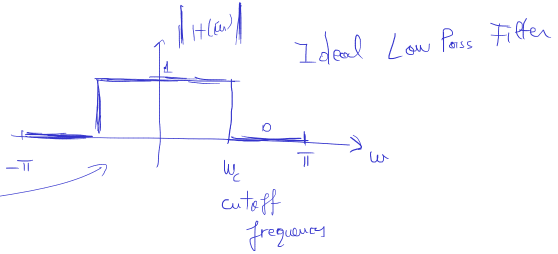
- LTI systems are also known as **filters** because their transfer function shapes ("filters") the frequencies of the input signals
- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

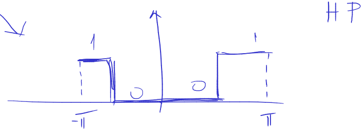


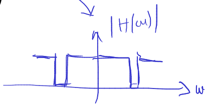
$\frac{\quad}{\quad} = |H(\omega)|$

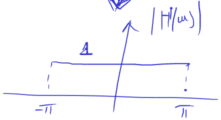
Ideal filters

- Draw at whiteboard the ideal transfer function of a:
  - low-pass filter
  - high-pass filter
  - band-pass filter
  - band-stop filter
  - all-pass filter (*changes the phase*)



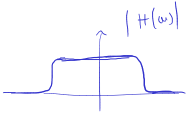




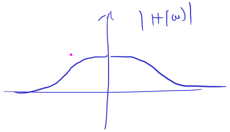


Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
  - i.e. largest power of z or z^{-1}
- Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1}) 
- Higher order -> better filter transfer function
  - closer to ideal filter
  - more complex to implement
  - more delays (bad)



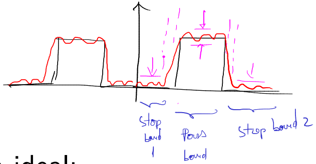
- Lower order
  - worse transfer function (not close to ideal)
  - simpler, cheaper
  - faster response

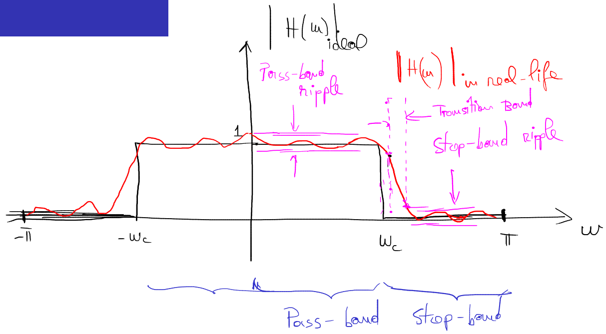


Filter design by pole and zero placements

- Based on geometric method
- The gain coefficient must be found by separate condition
  - i.e. specify the desired magnitude response at one frequency
- Examples at blackboard

Filter distortions





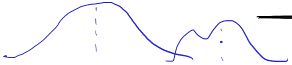
- When a filter is non-ideal:
  - non-constant amplitude –> amplitude distortions
  - non-linear phase –> phase distortions
- Phase distortions may be tolerated by certain applications
  - e.g. human auditory system is largely insensitive to phase distortions of sounds

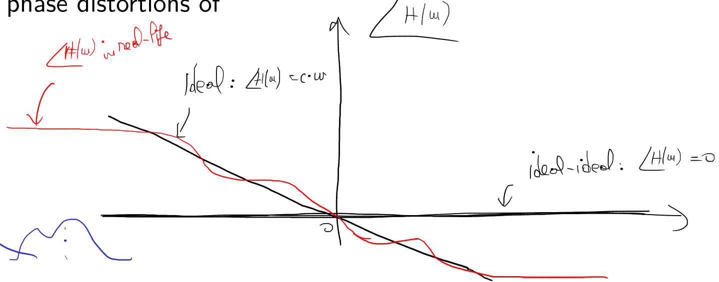
How should $\angle H(\omega)$ be in the ideal case:

1) Ideal - ideal : $\angle H(\omega) = 0 = 0 \cdot \omega$

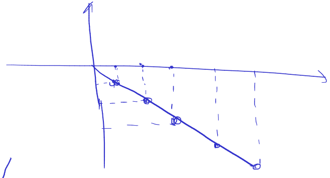
2) Ideal : $\angle H(\omega) = \text{linear} = c \cdot \omega$

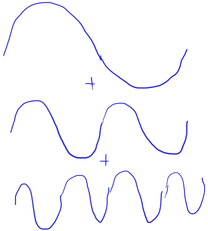
$\boxed{\text{Linear-phase}}$



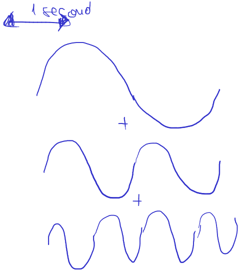


Effect of system's phase





- What is the effect of system's phase response $\angle H(\omega)$?
- Extra phase = delay
  - different frequencies are delayed differently
  - phase
- **Linear-phase** filter: delays all frequencies with the same amount of time
  - i.e. the whole signal is delayed, but otherwise not distorted
  - otherwise, we get distortions



Linear-phase filters

- For a sinusoidal signal, extra phase of 2π = delay of a period $N = \frac{1}{f}$
- To ensure same delay for all frequencies (in time), the phase $\angle H(\omega)$ must be proportional to the frequency
  - draw at blackboard
  - hence the name **linear**

Linear-phase filters

- Example: consider the following filter with **linear phase** function:

$$H(\omega) = \underset{=1}{C} \cdot e^{-j\omega n_0}$$

$|H(\omega)| = 1 \Rightarrow H(\omega) = 1 \cdot e^{-j\omega n_0}$

$\angle H(\omega) = \text{linear} = c \cdot \omega = (-n_0) \cdot \omega$

- The output signal is

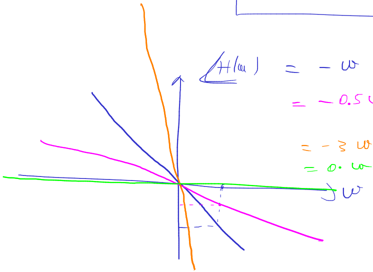
$$Y(\omega) = X(\omega) \cdot \overbrace{\underset{=1}{C} \cdot e^{-j\omega n_0}}^{H(\omega)}$$

Fourier Transf. property:

$$\boxed{\begin{aligned} x[n] &\longleftrightarrow X(\omega) \\ x[n-k] &\longleftrightarrow e^{-j\omega k} \cdot X(\omega) \end{aligned}}$$

$$y[n] = \underset{=1}{C} \cdot x[n - n_0]$$

- Linear phase means **just a delaying** of the input signal
  - Fourier property: $x[n - n_0] <\!\!-\!\!-\!\!> X(\omega) e^{-j\omega n_0}$



$\angle H(\omega) = -\omega \Rightarrow y[n] = x[n-1]$

$= -0.5\,\omega \quad y[n] = x[n-0.5]$

$= -3\,\omega \quad y[n] = x[n-3]$

$= 0 \cdot \omega \quad y[n] = x[n]$

Group delay

- Group delay = The time delay experienced by a component of frequency ω when passing through the filter
  - as opposed to "phase delay" = the phase added by the filter
- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

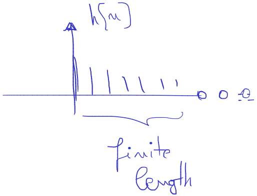
- Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Linear-phase FIR filters

IIR : $H(z) =$

FIR : $H(z) = \frac{\quad}{1}$

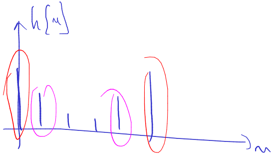
$h[n] =$



What type of filters can have linear phase?

- IIR filters cannot have linear phase (no proof provided)
- Only FIR filters can have linear phase, and only if they satisfy some symmetry conditions

Symmetry conditions for linear-phase FIR



- Let filter have an impulse response of length M (order is $M-1$)
- The filter coefficients are $h[0], \ldots \; h[M-1]$
- Linear-phase is guaranteed in two cases :
  - **Positive symmetry**

    $$\boxed{h[n] = h[M-1-n]}$$

  - **Negative symmetry (anti-symmetry)**

    $$\boxed{h[n] = -h[M-1-n]}$$

- The delay = the delay of the middle point of the symmetry

$$\frac{M}{2}$$

$h[0] = h[M-1]$
$h[1] = h[M-2]$
$h[2] = h[M-3]$
$\vdots$

$h[0] \quad h[1] \quad h[2] \quad h[3] \quad h[4] \quad h[5] \quad \ldots \quad h[M-1]$

$h[n] = \ldots 0\,0\,[\,3, 2, 1, 0, 4, 0, 1, 2, 3,\,]\,0.0\,0 \ldots$

h[4] must be 0

$h[0]$

$h[n] = [\,3 \;\; -5 \;\; 1 \;\; 4 \;\; 0 \;\; -4 \;\; -1 \;\; 5 \;\; -3\,]$

Cases of linear-phase FIR

- Proofs at blackboard

1. Positive symmetry, M $=$ odd
2. Positive symmetry, M $=$ even
3. Negative symmetry, M $=$ odd
4. Negative symmetry, M $=$ even

- Check constraints for $H(0)$ and $H(\pi)$
- For what types of filters is each case appropriate?

Zero-phase FIR filters

- Can we avoid delay altogether?
- **Zero-phase** filter = a particular type of linear-phase filter with zero delay
- For a zero-phase filter, the phase response $\angle H(\omega) = 0$
  - (Group) delay = derivative of $\angle H(\omega)$
  - delay 0 $\Leftrightarrow$ flat $\angle H(\omega)$ $\Leftrightarrow$ $\angle H(\omega) = 0$
- Delay is 0 $\Leftrightarrow$ symmetry with respect to $h[0]$
  - the system cannot be causal

Zero-phase FIR filters

- Zero-phase filters must be non-causal
  - left side of $h[n]$ symmetrical to right side of $h[n]$
- For causal, we need to delay $h[n]$ to be wholly on the right side $=>$ delay

Example

- Linear-phase filter (low-pass):

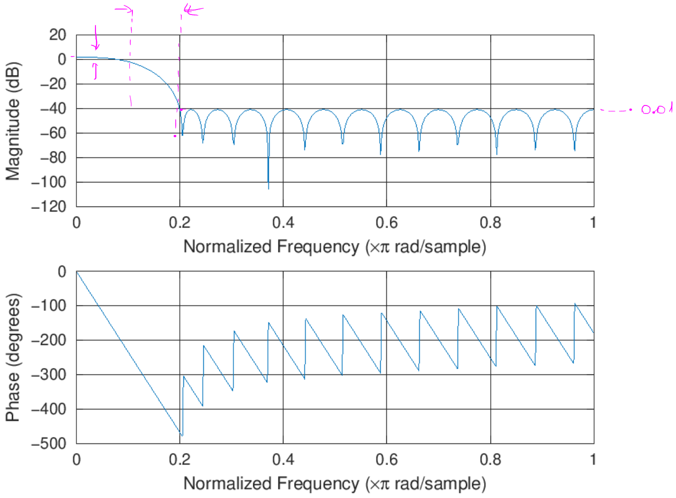


Figure 2: Transfer function of linear-phase filter

Example

- The impulse response (positive symmetry):

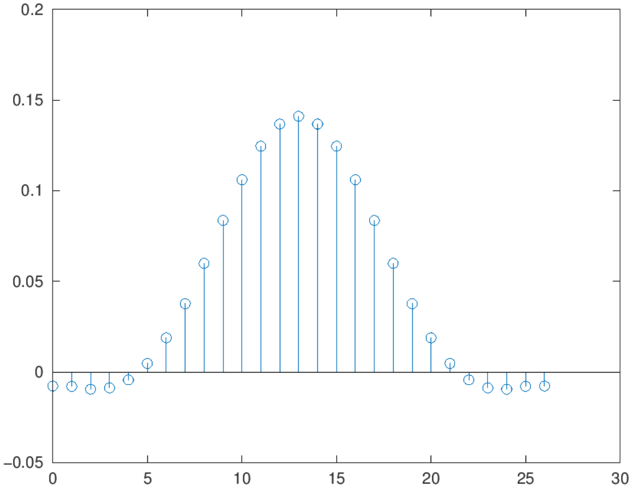


Figure 3: Impulse response of linear-phase filter

Example

- ECG signal: original and filtered. Filtering introduces **delay**

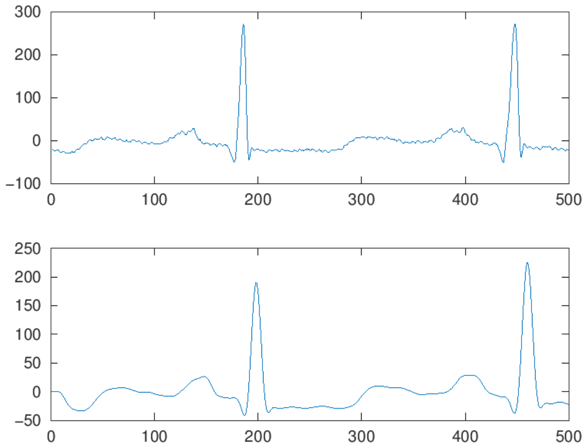


Figure 4: Delay introduced by filtering

Example

- Solution: zero-phase filter (positive symmetry, and centered in 0):
- But filter is **not causal** anymore

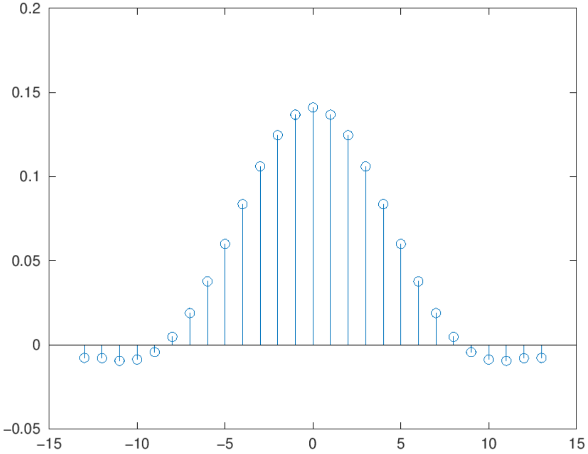


Figure 5: Impulse response of zero-phase filter

Example

- Filtering with zero-phase filter introduces **no delay**

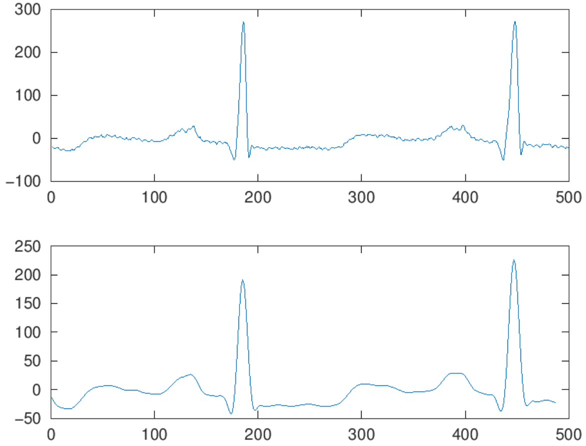


Figure 6: Zero-phase filter introduces no delay

Particular classes of filters

- **Digital resonators** = very selective band pass filters
  - poles very close to unit circle
  - may have zeros in 0 or at 1/-1
- **Notch filters**
  - have zeros exactly on unit circle
  - will completely reject certain frequencies
  - has additional poles to make the rejection band very narrow
- **Comb filters**
  - = periodic notch filters

Digital oscillators

- **Oscillator** = a system which produces an output signal even in absence of input
- Has a pair of complex conjugate poles **exactly on unit circle**
- Example at blackboard

Inverse filters

- Sometimes is necessary to **undo** a filtering
  - e.g. undo attenuation of a signal passed through a channel
- Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle –> unstable
- Examples at blackboard