

Digital Signal Processing

## II. Discrete signals and systems

## II.1 Discrete signals

A discrete signal can be represented:

- ▶ graphically
- ▶ in table form
- ▶ as a vector:  $x[n] = [..., 0, 0, 1, 3, 4, 5, 0, ...]$ 
  - ▶ an **arrow** indicates the origin of time ( $n = 0$ ).
  - ▶ if the arrow is missing, the origin of time is at the first element
  - ▶ the dots ... indicate that the value remains the same from that point onwards

Examples: at blackboard

Notation:  $x[4]$  represents the value of the fourth sample in the signal  $x[n]$

# Basic signals

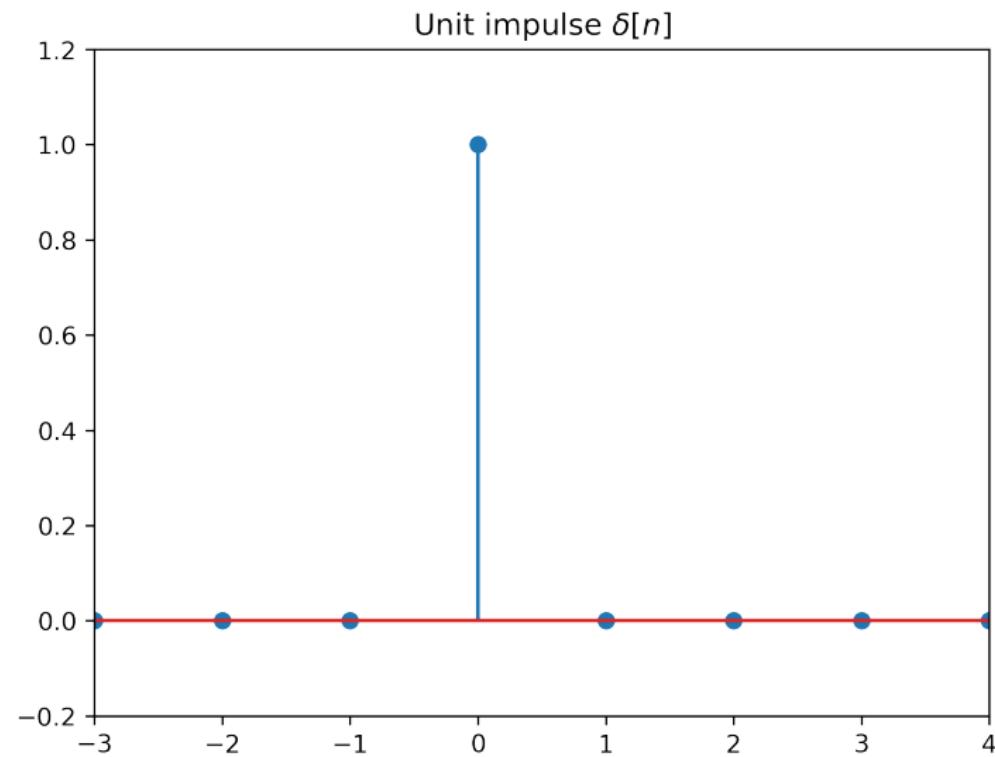
Some elementary signals are presented below.

## Unit impulse

Contains a single non-zero value of 1 located at time 0. It is denoted with  $\delta[n]$ .

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

# Representation



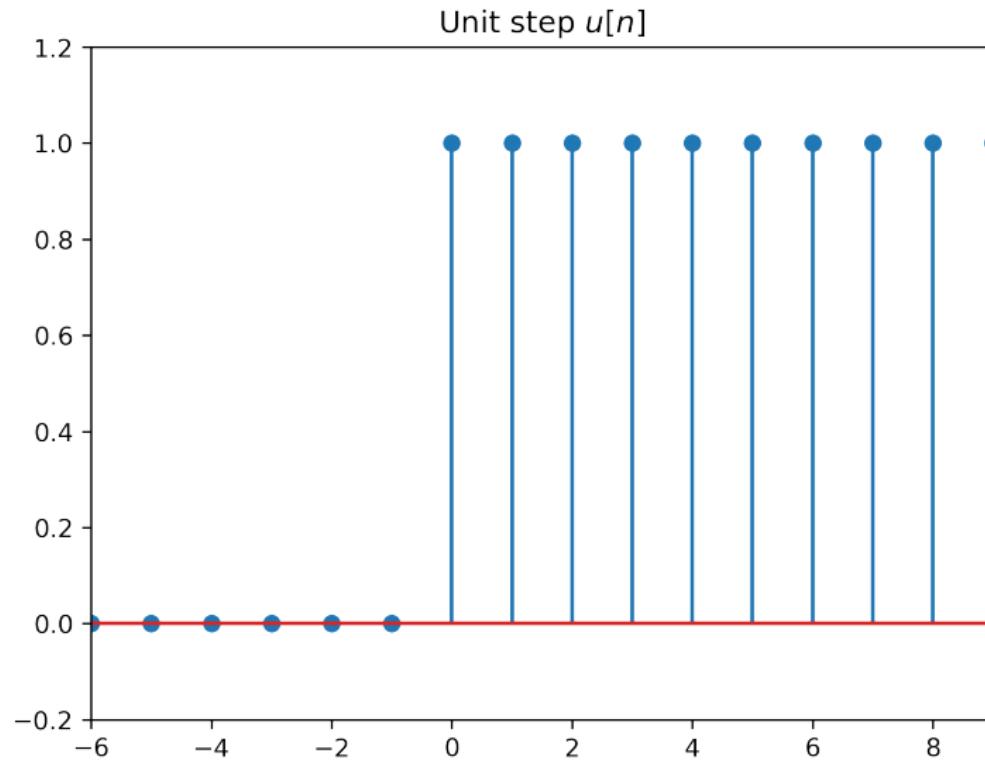
## Unit step

### Unit step

It is denoted with  $u[n]$ .

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

# Representation



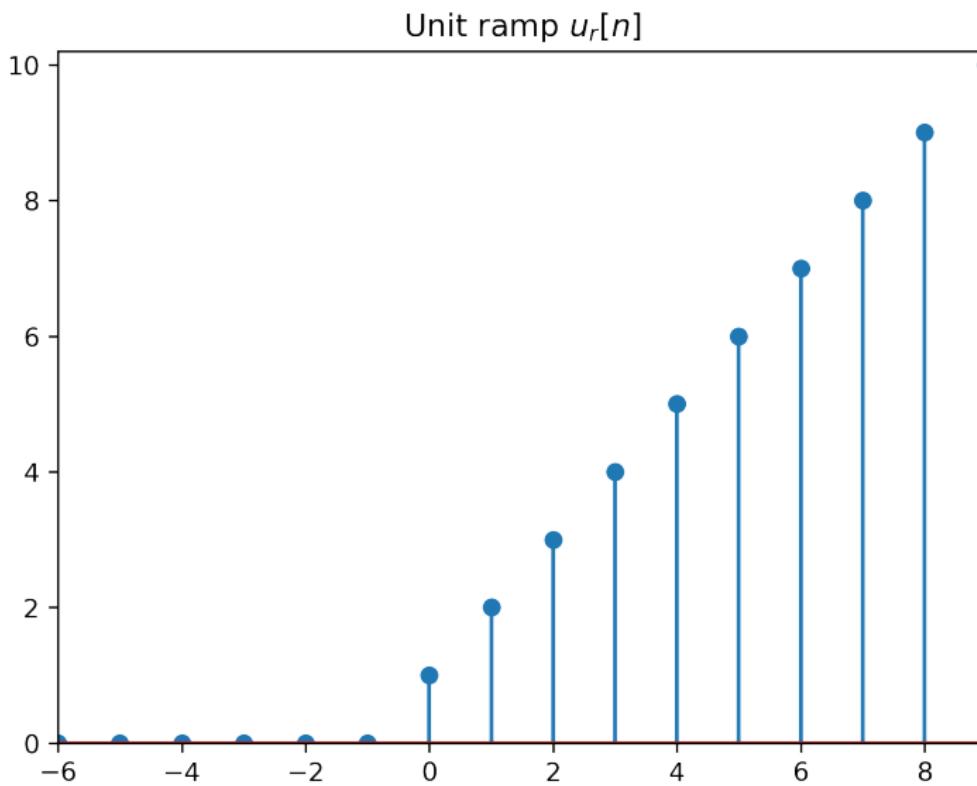
## Unit ramp

### Unit ramp

It is denoted with  $u_r[n]$ .

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

## Representation



# Exponential signal

## Exponential signal

It does not have a special notation. It is defined by:

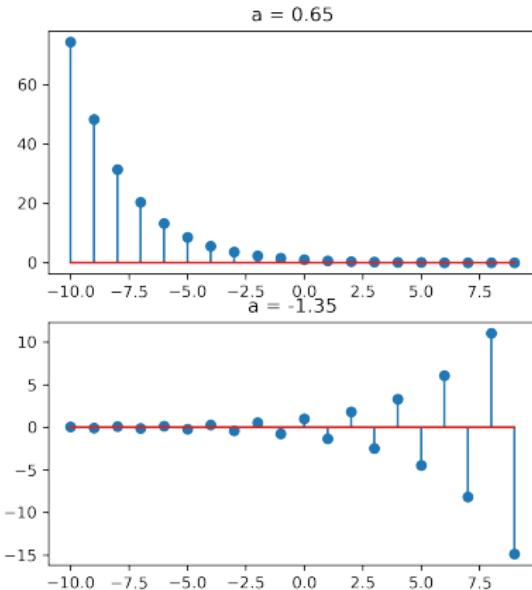
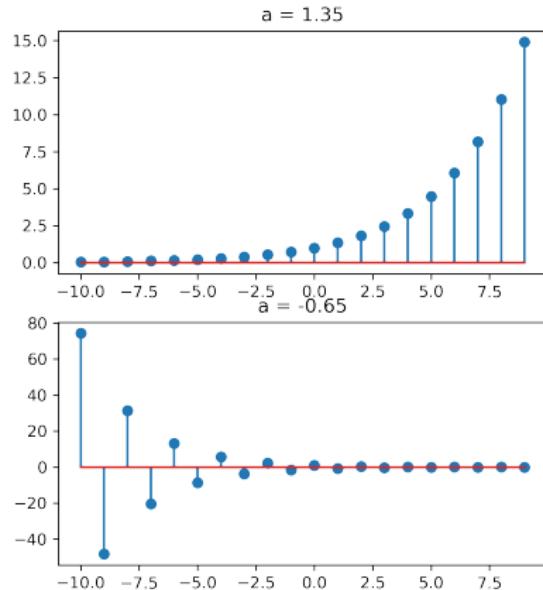
$$x[n] = a^n.$$

$a$  can be a real or a complex number. Here we consider only the case when  $a$  is real.

Depending on the value of  $a$ , we have four possible cases:

1.  $a \geq 1$
2.  $0 \leq a < 1$
3.  $-1 < a < 0$
4.  $a \leq -1$

# Representation



## II.2 Types of discrete signals

- ▶ The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} (x[n])^2.$$

- ▶ If  $E$  is finite, the signal is said to have finite energy.
- ▶ Examples:
  - ▶ unit impulse has finite energy
  - ▶ unit step does not

- ▶ Cross-link with DEDP course:

$$E = \|\mathbf{x} - \mathbf{0}\|^2 = \|\mathbf{x}\|^2$$

- ▶ Energy of a signal = **squared Euclidean distance to 0**

- ▶ geometric interpretation: squared length of the segment from 0 to the point  $\mathbf{x}$
- ▶ holds for continuous signals as well:

$$E = \|\mathbf{x}\|^2 = \int_{-\infty}^{\infty} x^2(t)dt$$

- ▶ The **average power of a discrete signal** is defined as

$$P = \lim_{N \rightarrow \infty} \frac{\sum_{n=-N}^N (x[n])^2}{2N + 1}.$$

- ▶ In other words, the average power is the average energy per sample.
- ▶ If  $P$  is finite, the signal is said to have finite power.
- ▶ A signal with finite energy has finite power ( $P = 0$  if the signal has infinite length). A signal with infinite energy can have finite or infinite power.
- ▶ Example: unit step has finite power  $P = \frac{1}{2}$  (proof at blackboard).

## Connection with DEDP class

- ▶ Average power = temporal average squared value  $\overline{X^2}$ 
  - ▶ i.e. average value of the square of samples

## Periodic and non-periodic signals

- ▶ A signal is called **periodic** if its values repeat themselves after a certain time (known as **period**).

$$x[n] = x[n + N], \forall t$$

- ▶ The **fundamental period** of a signal is the minimum value of  $N$ .
- ▶ Periodic signals have infinite energy, and finite power equal to the power of a single period.

## Even and odd signals

- ▶ A real signal is **even** if it satisfies the following symmetry:

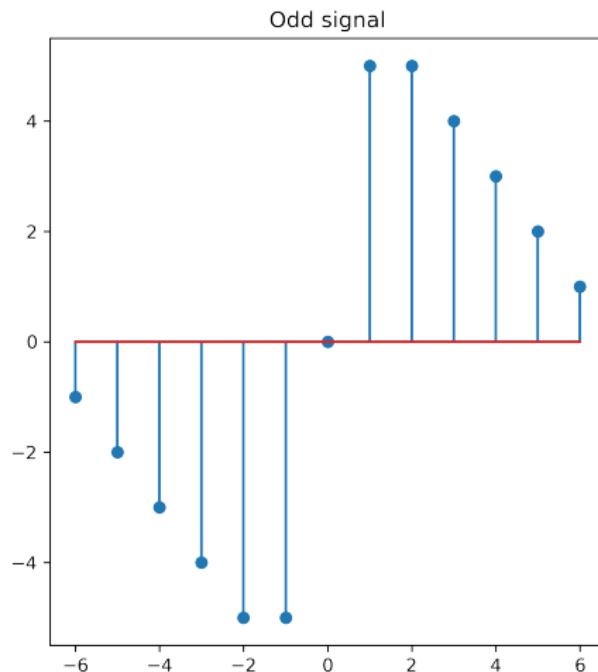
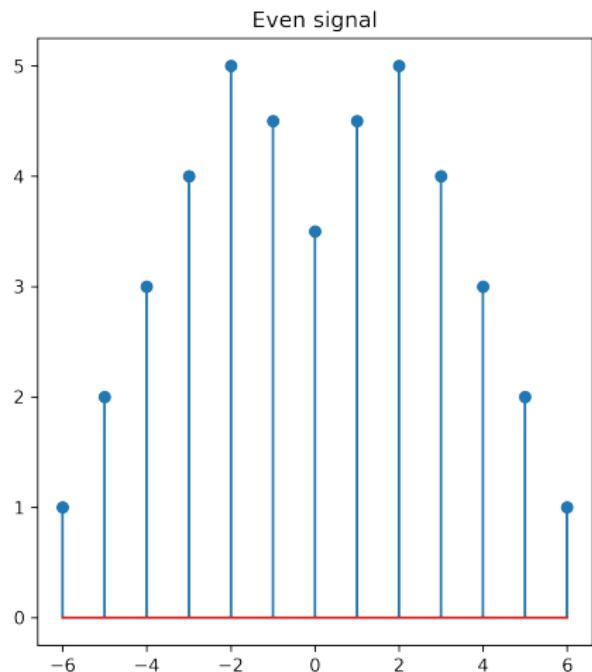
$$x[n] = x[-n], \forall n.$$

- ▶ A real signal is **odd** if it satisfies the following anti-symmetry:

$$-x[n] = x[-n], \forall n.$$

- ▶ There exist signals which are neither even nor odd.

## Even and odd signals: example



## Even and odd parts of a signal

- ▶ Every signal can be written as the sum of an even signal and an odd signal:

$$x[n] = x_e[n] + x_o[n]$$

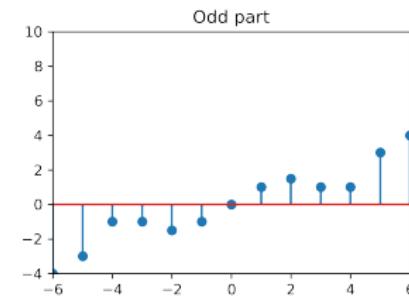
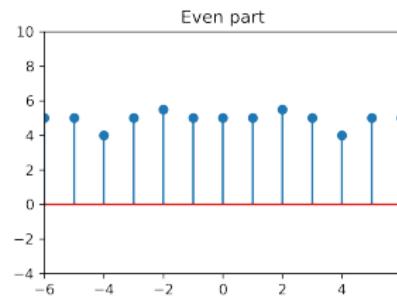
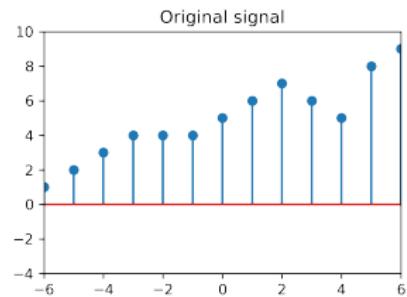
- ▶ The even and the odd parts of the signal can be found as follows:

$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

$$x_o[n] = \frac{x[n] - x[-n]}{2}.$$

- ▶ Proof: check that  $x_e[n]$  is even,  $x_o[n]$  is odd, and their sum is  $x[n]$

## Even and odd parts: example



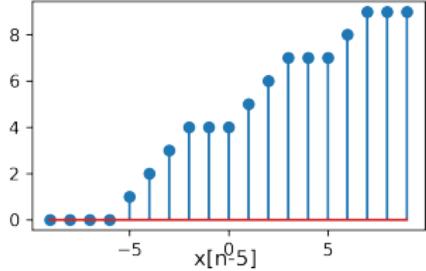
## II.3 Basic operations with discrete signals

## Time shifting

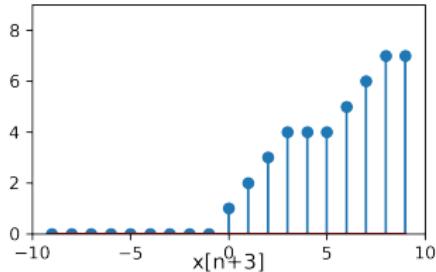
- ▶ The signal  $x[n - k]$  is  $x[n]$  **delayed with  $k$  time units**
  - ▶ Graphically,  $x[n - k]$  is shifted  $k$  units to the **right** compared to the original
- ▶ The signal  $x[n + k]$  is  $x[n]$  **anticipated with  $k$  time units**
  - ▶ Graphically,  $x[n + k]$  is shifted  $k$  units to the **left** compared to the original signal.

# Time shifting: representation

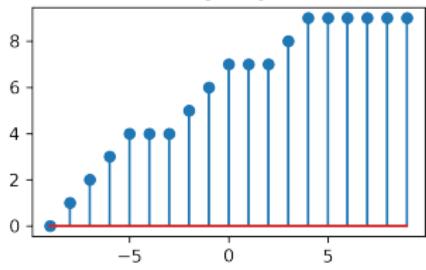
Original signal  $x[n]$



$x[n^0-5]$

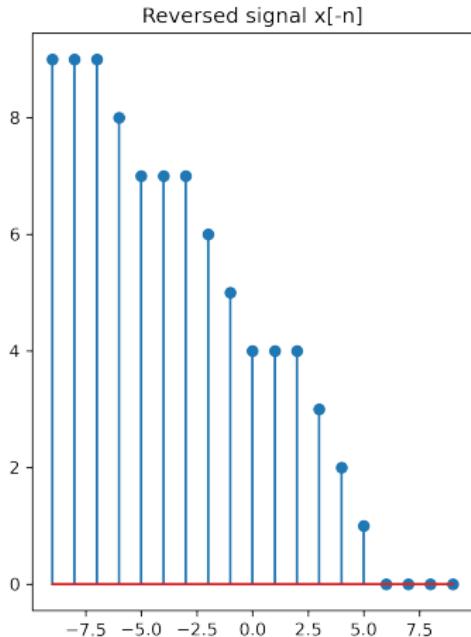
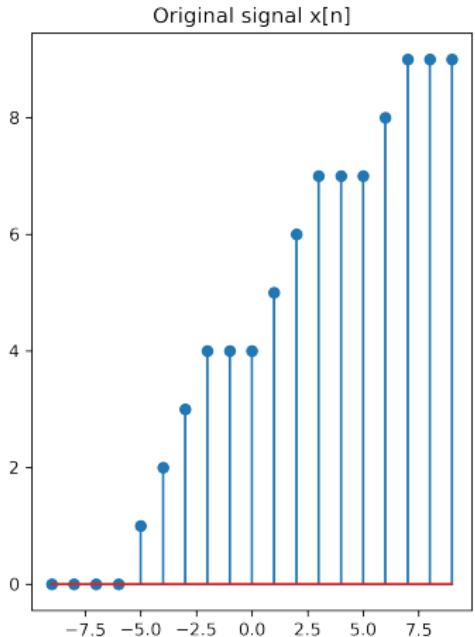


$x[n^0+3]$



# Time reversal

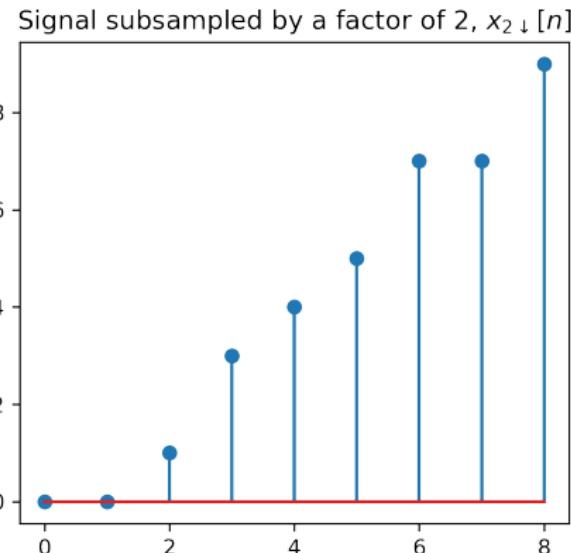
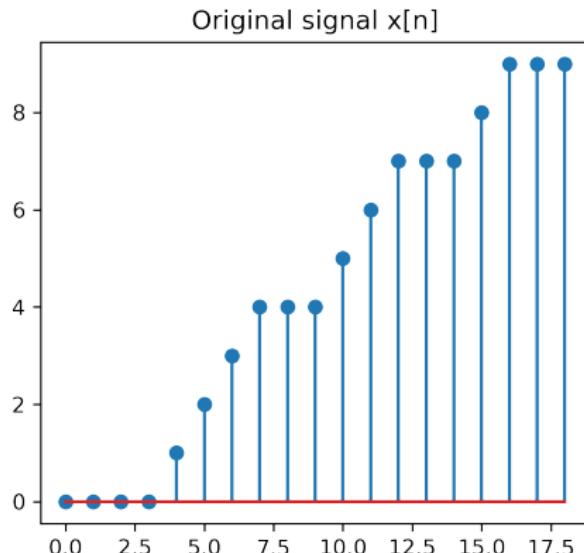
- ▶ Changing the variable  $n$  to  $-n$  produces a signal  $x[-n]$  which mirrors  $x[n]$ .



# Subsampling

- ▶ **Subsampling** by a factor of  $M =$  keep only 1 sample from every  $M$  of the original signal
  - ▶ Total number of samples is reduced  $M$  times

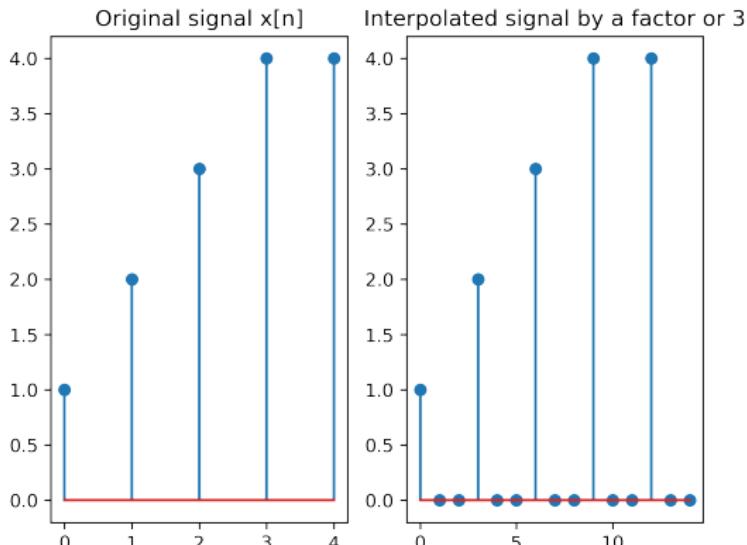
$$x_{M\downarrow}[n] = x[Mn]$$



# Interpolation

- ▶ **Interpolation** by a factor of  $L$  adds  $(L - 1)$  zeros between two samples in the original signal
  - ▶ Total number of samples increases  $L$  times

$$x_{L\uparrow} = \begin{cases} x\left[\frac{n}{L}\right] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$



## Mathematical operations

- ▶ A signal  $x[n]$  can be **scaled** by a constant  $A$ , i.e. each sample is multiplied by  $A$ :

$$y[n] = Ax[n].$$

- ▶ Two signals  $x_1[n]$  and  $x_2[n]$  can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- ▶ Two signals  $x_1[n]$  and  $x_2[n]$  can be **multiplied** by multiplying the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

## II.4 Discrete systems

## Definition

- ▶ **System** = a device or algorithm which produces an **output signal** based on an **input signal**
- ▶ We will only consider systems with a single input and a single output
- ▶ Figure here: blackboard.
- ▶ Common notation:
  - ▶  $x[n]$  is the input
  - ▶  $y[n]$  is the output
  - ▶  $H$  is the system.

## Notations

► Notations:

$$y[n] = H[x[n]]$$

("the system  $H$  applied to the input  $x[n]$  produces the output  $y[n]$ ")

$$x[n] \xrightarrow{H} y[n]$$

("the input  $x[n]$  is transformed by the system  $H$  into  $y[n]$ ")

# Equations

- ▶ Usually, a system is described by the **input-output equation** (or **difference equation**) which explains how  $y[n]$  is defined in terms of  $x[n]$ .

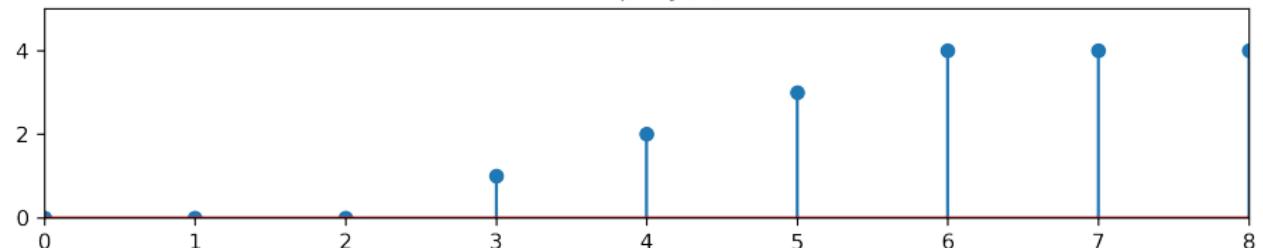
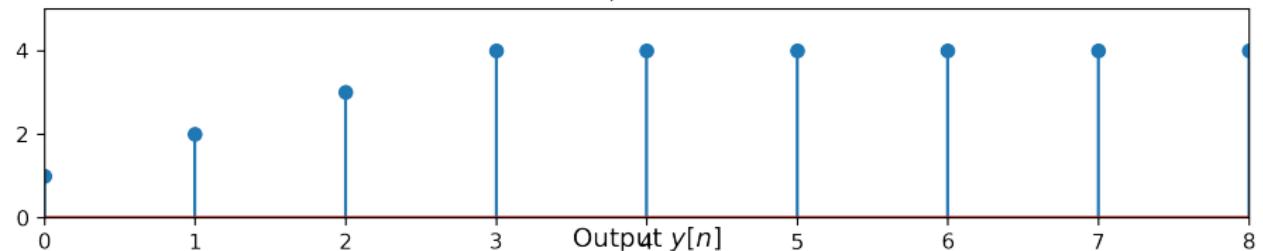
Examples:

1.  $y[n] = x[n]$  (the identity system)
2.  $y[n] = x[n - 3]$
3.  $y[n] = x[n + 1]$
4.  $y[n] = \frac{1}{3}(x[n + 1] + x[n] + x[n - 1])$
5.  $y[n] = \max(x[n + 1], x[n], x[n - 1])$
6.  $y[n] = (x[n])^2 + \log_{10} x[n - 1]$
7.  $y[n] = \sum_{k=-\infty}^n x[k] = x[n] + x[n - 1] + x[n - 2] + \dots$

## Example

$$y[n] = x[n - 3]$$

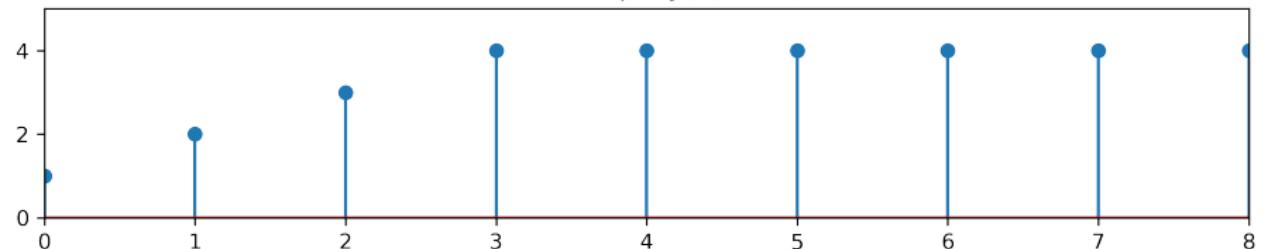
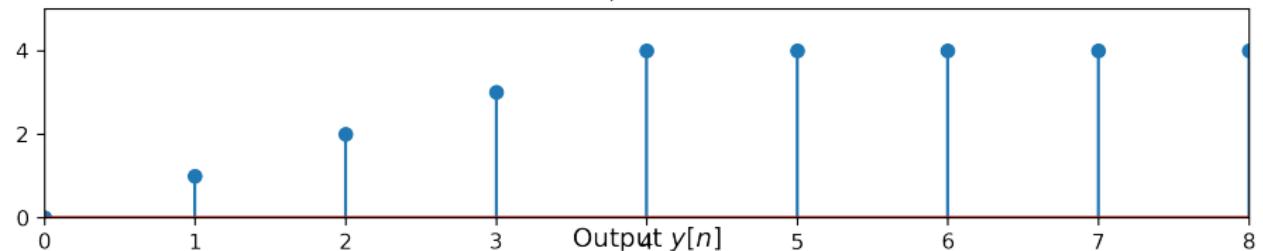
Input  $x[n]$



## Example

$$y[n] = x[n + 1]$$

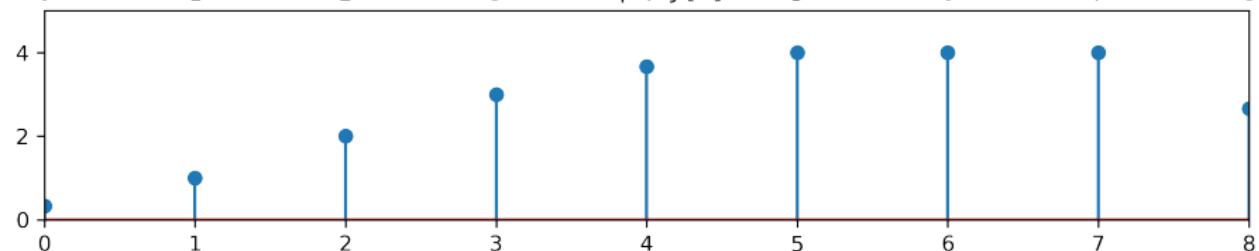
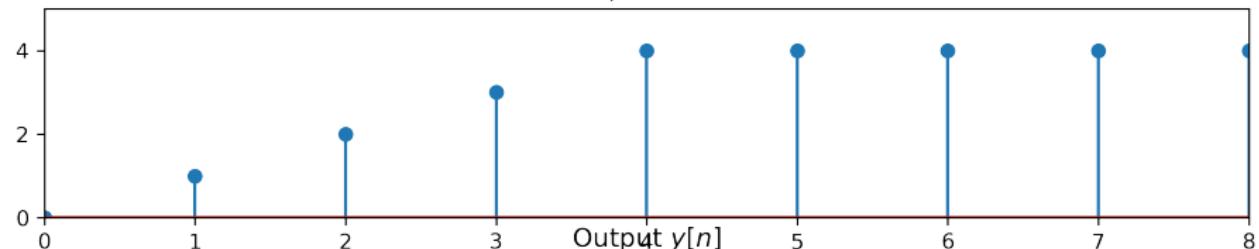
Input  $x[n]$



## Example

$$y[n] = (x[n+1] + x[n] + x[n-1])/3$$

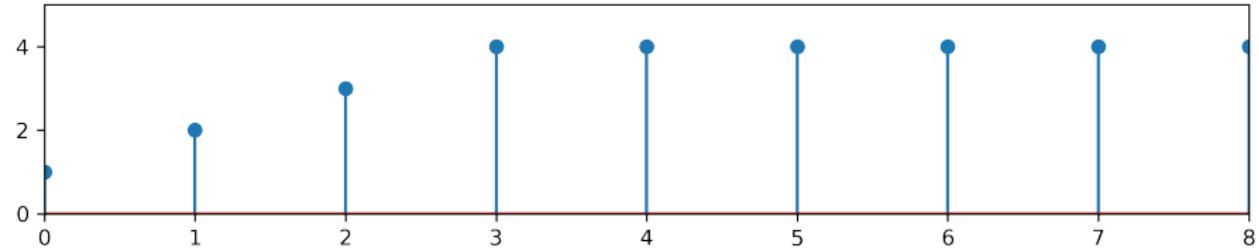
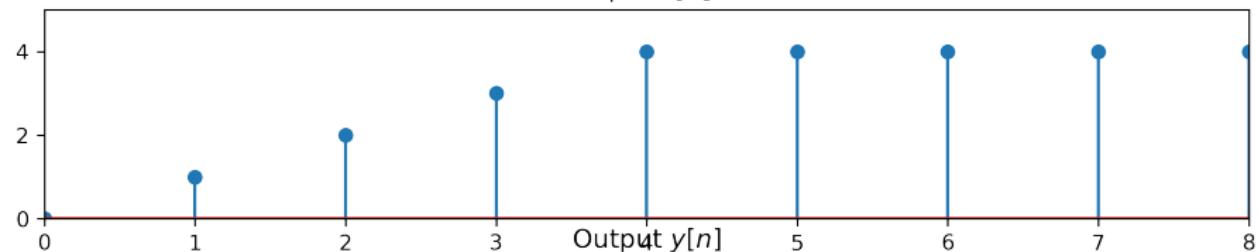
Input  $x[n]$



## Example

$$y[n] = \max(x[n+1], x[n], x[n-1])$$

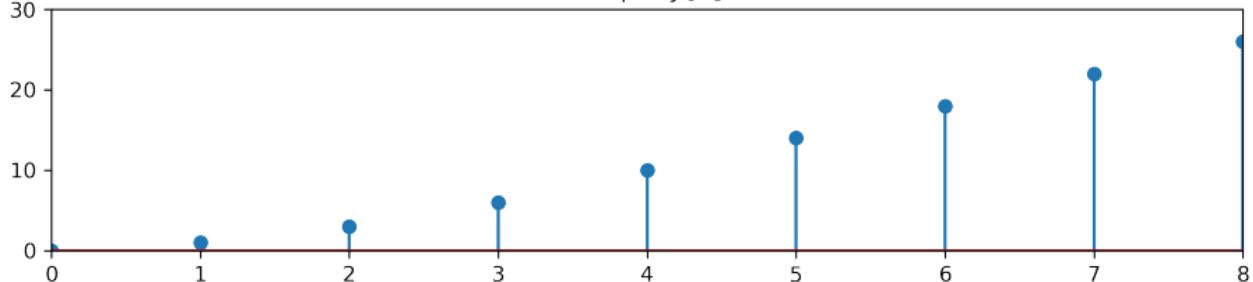
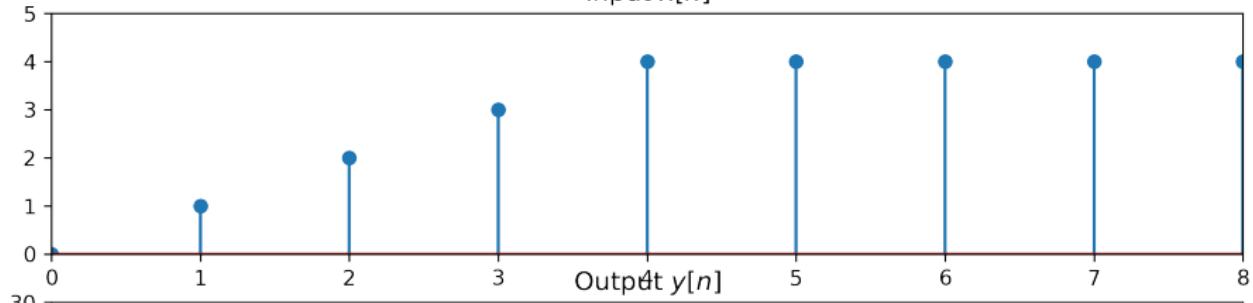
Input  $x[n]$



## Example

$$y[n] = \sum_{k=-\infty}^n x[k] = x[n] + x[n-1] + x[n-2] + \dots$$

Input  $x[n]$



- ▶ Some systems can/must be written in **recursive form**

$$y[n] = y[n - 1] + x[n],$$

- ▶ Must always specify **initial conditions**
  - ▶ i.e. initial value (e.g.  $y[-1] = 2.5$ )
  - ▶ if not mentioned, assume they are 0 ("relaxed system")
  - ▶ they represent the internal state of the system at the starting moment
- ▶ For recursive systems, the output signal depends on both the input signal **and** on the initial conditions
  - ▶ different initial conditions lead to different outputs, even if input signal is the same
  - ▶ a recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ( $x[n] = 0$ )

- ▶ The operation of a system can be described graphically (see examples on blackboard):
  - ▶ summation of two signals
  - ▶ scaling of a signal with a constant
  - ▶ multiplication of two signals
  - ▶ delay element
  - ▶ anticipation element
  - ▶ other blocks for more complicated math operations

## II.4 Classification of discrete systems

- ▶ **Memoryless (or static)**: output at time  $n$  depends only on the input from the same moment  $n$
- ▶ Otherwise, the system **has memory (dynamic)**
- ▶ Examples:
  - ▶ memoryless:  $y[n] = (x[n])^3 + 5$
  - ▶ with memory:  $y[n] = (x[n])^3 + x[n - 1]$

- ▶ Memory of size  $N$ :
  - ▶ output at time  $n$   $y[n]$  depends only up to the last  $N$  inputs,  
 $x[n - N], x[n - (N - 1)], \dots, x[n]$ ,
  - ▶ if  $N$  is finite: the system has **finite memory**
  - ▶ if  $N = \infty$ , the system has **infinite memory**
- ▶ Examples:
  - ▶ finite memory of order 4:  $y[n] = x[n] + x[n - 2] + x[n - 4]$
  - ▶ infinite memory:  $y[n] = 0.5y[n - 1] + 0.8x[n]$ 
    - ▶ recursive systems usually have infinite memory

## Memoryless / systems with memory

- ▶ An input sample has an effect on the output only for the next  $N$  time moments
- ▶ For systems infinite memory, any sample influences **all** the following samples, forever
  - ▶ but, if system is stable, the influence gets smaller and smaller

## Time-Invariant and Time-Variant systems

- ▶ A relaxed system  $H$  is **time-invariant** if and only if:

$$x[n] \xrightarrow{H} y[n]$$

implies,  $\forall x[n], \forall k$ , that

$$x[n - k] \xrightarrow{H} y[n - k]$$

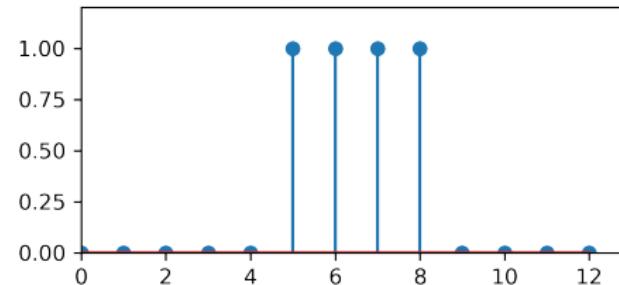
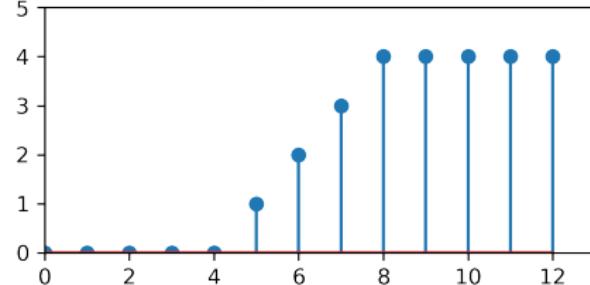
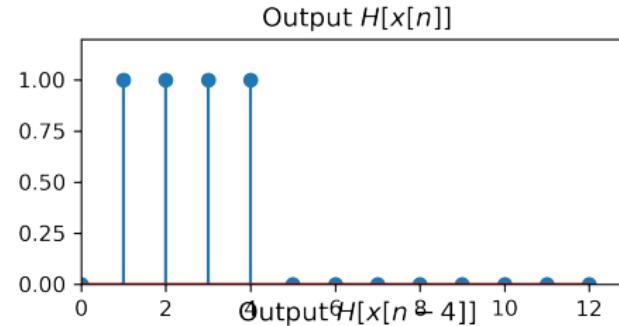
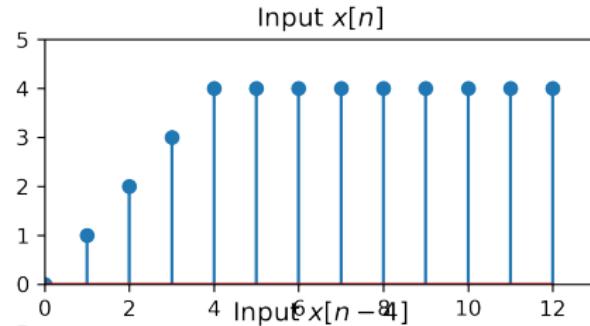
- ▶ Delaying the input signal with  $k$  will only delay the output with the same amount, otherwise the output is not affected
  - ▶ Must be true for all input signals, for all possible delays (positive or negative)
- ▶ Otherwise, the system is said to be **time-variant**

## Time-Invariant and Time-Variant systems

- ▶ Examples:
  - ▶  $y[n] = x[n] - x[n - 1]$  is time-invariant
  - ▶  $y[n] = n \cdot x[n]$  is not time-invariant
- ▶ A system is time-invariant if it depends on  $n$  only through the input signal  $x[n]$

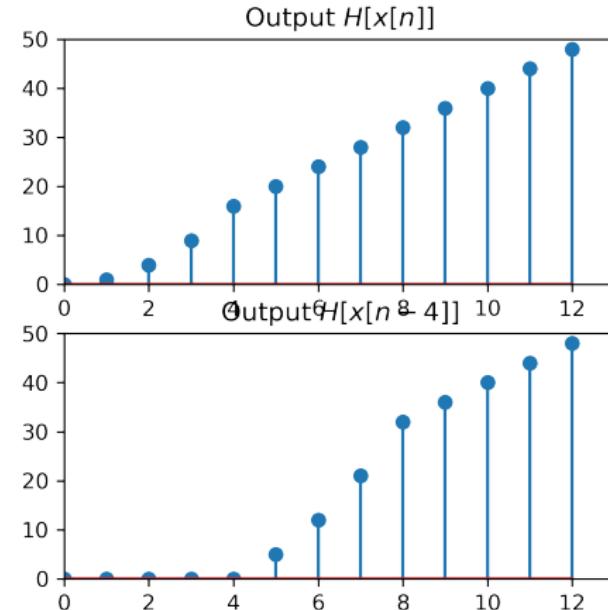
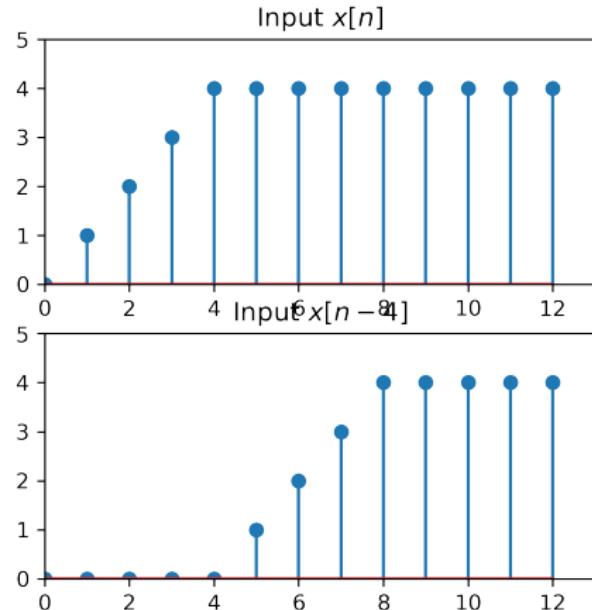
# Example

Time-invariant system  $y[n] = x[n] - x[n - 1]$



## Another example

Time-variant system  $y[n] = n \cdot x[n]$



- ▶ A system  $H$  is **linear** if:

$$H [ax_1[n] + bx_2[n]] = aH [x_1[n]] + bH [x_2[n]].$$

- ▶ Composed of two parts:
  - ▶ Applying the system to a sum of two signals = applying the system to each signal, and adding the results
  - ▶ Scaling the input signal with a constant  $a$  is the same as scaling the output signal with  $a$
- ▶ The same relation will be true for a sum of many signals, not just two

- ▶ Advantage of linear systems
  - ▶ Complicated input signals can be decomposed into a sum of smaller parts
  - ▶ The system can be applied to each part independently
  - ▶ Then the results are added back
- ▶ Examples:
  - ▶ linear system:  $y[n] = 3x[n] + 5x[n - 2]$
  - ▶ nonlinear system:  $y[n] = 3(x[n])^2 + 5x[n - 2]$

- ▶ For a system to be linear, the input samples  $x[n]$  must not undergo non-linear transformations.
- ▶ **The only transformations** of the input  $x[n]$  allowed to take place in a linear system are:
  - ▶ scaling (multiplication) with a constant
  - ▶ delaying
  - ▶ summing different delayed versions of the signal (not summing with a constant)

## Causal and non-causal systems

- ▶ **Causal:** the output  $y[n]$  depends only on the current input  $x[n]$  and the past values  $x[n - 1], x[n - 2] \dots$ , but not on the future samples  $x[n + 1], x[n + 2] \dots$
- ▶ Otherwise the system is **non-causal**.
- ▶ A causal system can operate in real-time
  - ▶ we need only the input samples from the past
  - ▶ non-causal systems need samples from the future
- ▶ Examples:
  - ▶  $y[n] = x[n] - x[n - 1]$  is causal
  - ▶  $y[n] = x[n + 1] - x[n - 1]$  is non-causal
  - ▶  $y[n] = x[-n]$  is non-causal

- ▶ **Bounded** signal: if there exists a value  $M$  such that all the samples of the signal or smaller than  $M$ , in absolute values

$$x[n] \in [-M, M]$$

$$|x[n]| \leq M$$

- ▶ **Stable** system: if for any bounded input signal it produces a bounded output signal
  - ▶ not necessarily with the same  $M$
  - ▶ known as BIBO (Bounded Input  $\rightarrow$  Bounded Output)
- ▶ In other words: when the input signal has bounded values, the output signal does not go towards  $\infty$  or  $-\infty$ .

► Examples:

- ▶  $y[n] = (x[n])^3 - x[n + 4]$  is stable
- ▶  $y[n] = \frac{1}{x[n] - x[n-1]}$  is unstable
- ▶  $y[n] = \sum_{k=-\infty}^n x[k] = x[n] + x[n - 1] + x[n - 2] + \dots$  is unstable

Impulse response of Linear Time-Invariant (LTI) systems

# Linear Time-Invariant (LTI) systems

- ▶ Notation: An **LTI** system (**Linear Time-Invariant**) is a system which is simultaneously **linear** and **time-invariant**.
- ▶ LTI systems have an equation like this:

$$\begin{aligned}y[n] &= -a_1y[n-1] - a_2y[n-2] - \dots - a_Ny[n-N] + \\&\quad + b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M] \\&= -\sum_{k=1}^N a_k y[n-k] + \sum_{k=1}^M b_k x[n-k]\end{aligned}$$

- ▶ the above is for causal systems; non-causal can also have [n+k]

## The impulse response

- ▶ **Impulse response** of a system = output (response) of when the input signal is the impulse  $\delta[n]$ :

$$h[n] = H(\delta[n])$$

- ▶ The impulse response of a LTI system **fully characterizes the system**:
  - ▶ based on  $h[n]$  we can compute the response of the system to **any** input signal
  - ▶ all the properties of LTI systems can be described via characteristics of the impulse response

## Signals are a sum of impulses

- ▶ Any signal  $x[n]$  can be composed as a **sum of scaled and delayed impulses**  $\delta[n]$ .

- ▶ Example:

$$x[n] = \{3, 1, -5, 0, 2\} = 3\delta[n] + \delta[n - 1] - 5\delta[n - 2] + 2\delta[n - 2]$$

- ▶ In general

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$$

i.e. a sum of impulses  $\delta[n]$ , each one delayed with  $k$  and scaled with the corresponding value  $x[k]$

# Convolution

- ▶ The response of a LTI system to a sum of impulses, delayed with  $k$  and scaled with  $x[k]$ , **is a sum of impulse responses, delayed with  $k$  and scaled with  $x[k]$ .**

$$\begin{aligned}y[n] &= H(x[n]) \\&= H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) \\&= \sum_{k=-\infty}^{\infty} x[k]H(\delta[n-k]) \\&= \sum_{k=-\infty}^{\infty} x[k]h[n-k].\end{aligned}$$

- ▶ Convolution in short:
  - ▶ The input signal is composed of separate impulses
  - ▶ Each impulse will generate its own response (LTI)
  - ▶ Output signal is the sum of impulse responses, delayed and scaled
- ▶ Convolution only applies for LTI systems

# Convolution

- ▶ This operation = the **convolution** of two signals  $x[n]$  and  $h[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- ▶ The response of a LTI system to an input signal  $x[n]$  is **the convolution of  $x[n]$  with the system's impulse response  $h[n]$**

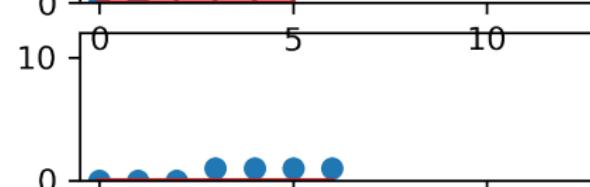
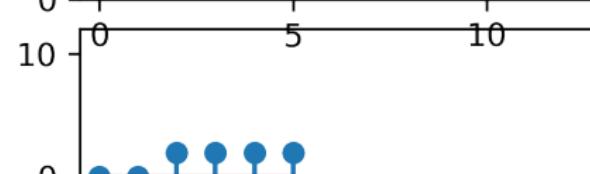
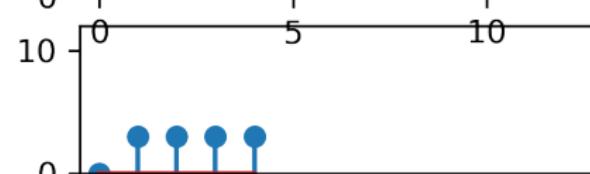
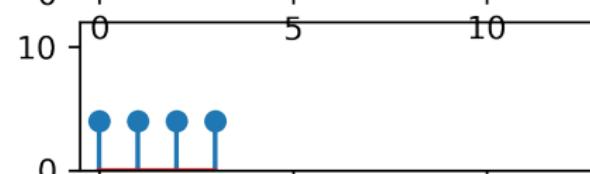
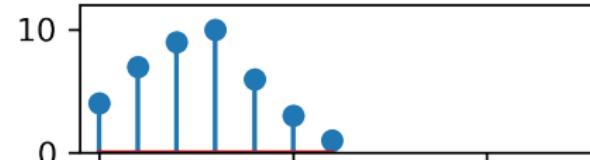
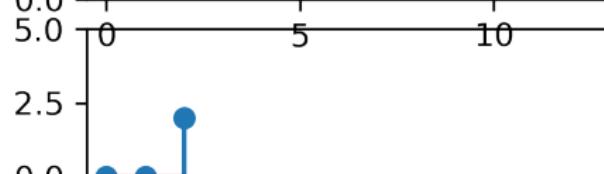
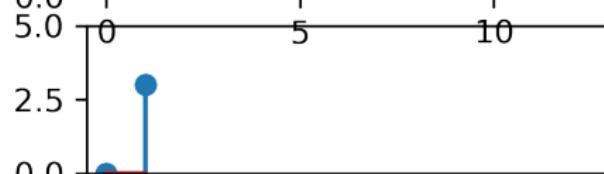
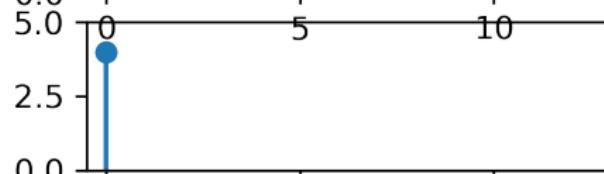
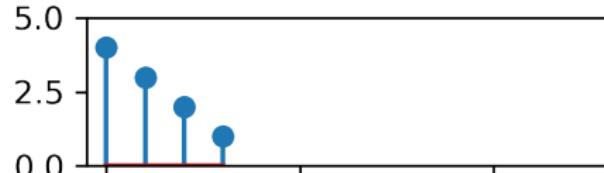
$$y[n] = x[n] * h[n]$$

# Convolution

- ▶ Convolution is commutative:  $x[n] * h[n] = h[n] * x[n]$ 
  - ▶ in equation it doesn't matter which signal has  $[k]$  and which with  $[n - k]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

## Example



## Interpretation of the convolution equation

The convolution equation can be interpreted in two ways:

1. The output signal  $y[n] =$  a sum of a lot of impulse responses  $h[n]$ , each one delayed by  $k$  (hence  $[n - k]$ ) and scaled by  $x[k]$

- ▶ one for each sample in the input signal
- ▶ explain at blackboard

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

## Interpretation of the convolution equation

2. Each output sample  $y[n] =$  a **weighted sum** of the input samples around it

$$y[n] = \dots + h[2] \cdot x[n-2] + h[1] \cdot x[n-1] + h[0] \cdot x[n] + h[-1] \cdot x[n+1] + \dots$$

- ▶ If  $h[n]$  has finite length (e.g. non-zero only between  $h[-2] \dots h[2]$ ), then there are only a few terms in the sum
  - ▶ Example at blackboard

$$x[n] * h[n] = \mathbf{y}[\mathbf{n}] = \sum_{k=-\infty}^{\infty} h[k] x[\mathbf{n} - k]$$

# Interpretation of the convolution equation

$$f \quad \boxed{1 \ 4 \ 2 \ 5} \quad g \quad \boxed{3 \ 4 \ 1} \quad c = f * g$$

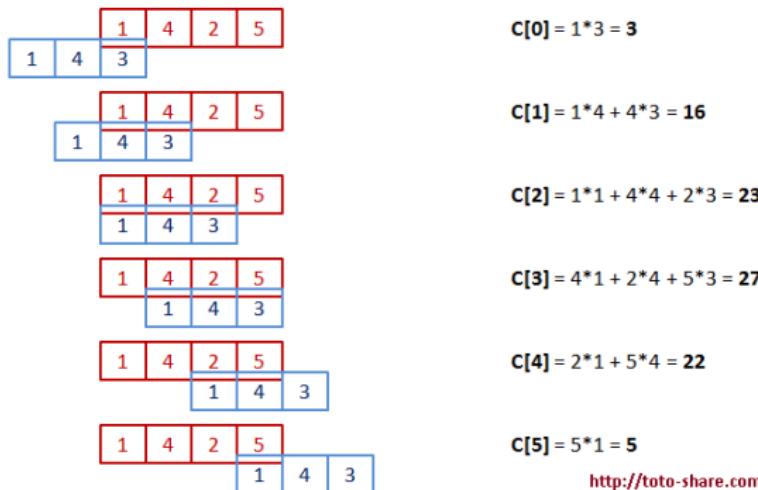


Figure 1: Convolution as weighted sum

- ▶ image from <http://www.stokastik.in>

## Interpretation of the convolution equation

- ▶ Watch the following:

<https://www.youtube.com/watch?v=uIKbLD6BRJA>

## Example

The impulse response can be read directly from the system equation (for non-recursive systems):

- ▶ Suppose we have the system:

$$y[n] = 3x[n+1] + 5x[n] - 2x[n-1] + 4x[n-2]$$

- ▶ What is the impulse response of the system?
- ▶ Answer:  $h[n] = \{ \dots, 0, 3, 5, \underset{\uparrow}{-2}, 4, 0, \dots \}$

## Convolution as matrix multiplication

- ▶ Convolution can we written as multiplication with a **circulant** (or "Toeplitz") matrix
  - ▶ in this example, assuming  $h[n]$  is non-zero only from  $h[-1]$  to  $h[3]$

$$\begin{bmatrix} \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \end{bmatrix} = \begin{bmatrix} \dots & \vdots & \dots \\ \dots & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & 0 & \dots \\ \dots & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & h_{-1} & \dots \\ \dots & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & \dots \\ \dots & \vdots & \dots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ x_{n-4} \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \\ x_{n+1} \\ x_{n+2} \\ x_{n+3} \\ \vdots \end{bmatrix}$$

## 2D convolution

- ▶ Convolution can be applied in 2D (for images)
- ▶ The input signal = an image  $I[x, y]$
- ▶ The impulse response (the ***kernel***) = a matrix  $H[x, y]$
- ▶ The convolution result:

$$Y[x, y] = I * H = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} I[x - i, y - j] \cdot H[i, j]$$

## 2D convolution

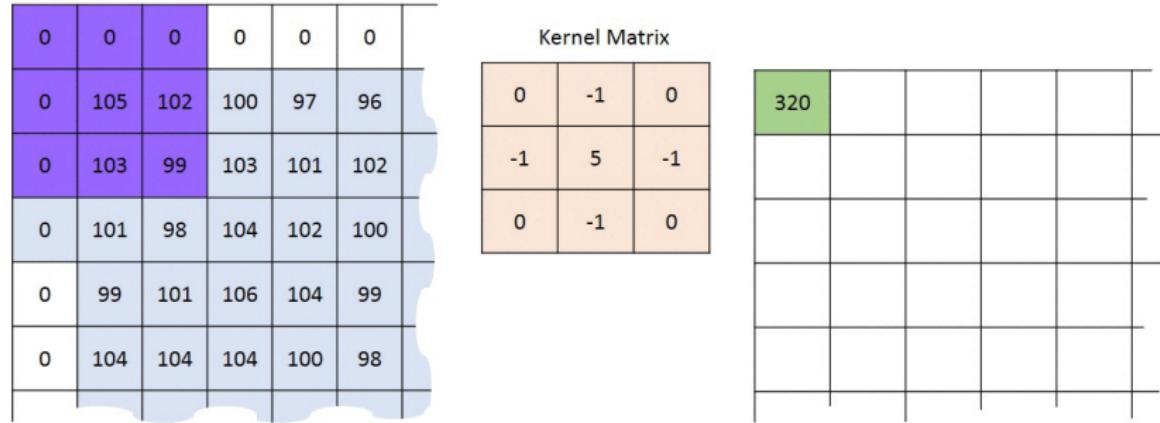


Image Matrix

$$\begin{aligned} & 0 * 0 + 0 * -1 + 0 * 0 \\ & + 0 * -1 + 105 * 5 + 102 * -1 \\ & + 0 * 0 + 103 * -1 + 99 * 0 = 320 \end{aligned}$$

Output Matrix

**Convolution with horizontal and vertical strides = 1**

Figure 2: 2D Convolution as weighted sum

- ▶ image from <http://machinelearningguru.com>

## 2D Convolution

- ▶ Watch this:

[http://machinelearningguru.com/computer\\_vision/basics/convolution/cor](http://machinelearningguru.com/computer_vision/basics/convolution/cor)

## 2D Convolution

- ▶ Simple image effects with 2D convolutions:
  - ▶ the "kernel" = the impulse response  $H[x, y]$
- ▶ See here: [https://en.wikipedia.org/wiki/Kernel\\_\(image\\_processing\)](https://en.wikipedia.org/wiki/Kernel_(image_processing))
- ▶ What are their 1D counterparts?

## Properties of convolution

### Basic properties of convolution

- ▶ Convolution is **commutative** (the order of the signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- ▶ Proof: make variable change  $(n - k) \rightarrow l$ , change all in equation
- ▶ Convolution is **associative**:

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

- ▶ (No proof)

## Properties of convolution

- The unit impulse is **neutral element** for convolution:

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

- Proof: equation
- Convolution is a **linear operation** (or **distributive**):

$$(\alpha \cdot a[n] + \beta \cdot b[n]) * c[n] = \alpha(\cdot a[n] * c[n]) + \beta \cdot (b[n] * c[n])$$

- Proof: by linearity of the corresponding system

## 1. Identity system

- ▶ A system with  $h[n] = \delta[n]$  produces an response equal to the input,  
 $y[n] = x[n], \forall x[n]$ .
- ▶ Proof:  $\delta[n]$  is neutral element for convolution.

## 2. Series connection is commutative

- ▶ LTI systems connected in series can be interchanged in any order
- ▶ Proof: by commutativity of convolution.
- ▶ LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * \dots * h_N[n]$$

## 3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + \dots + h_N[n]$$

### 4. Response of LTI systems to unit step

- If the input signal is  $u[n]$ , the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^n h[k]$$

## Properties of LTI systems expressed with $h[n]$

### ► Proof:

- The signal  $\sum_{k=-\infty}^n h[k]$  is a *discrete-time integration* of  $h[n]$
- The unit step  $u[n]$  itself is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

$$\delta[n] = u[n] - u[n - 1]$$

- Therefore the system response to the integral of the impulse = the integral of the system response to the impulse
- The interchanging of the integration with the system is due to the linearity of the system and is valid for all signals:

$$H \left( \sum_{k=-\infty}^n x[k] \right) = \sum_{k=-\infty}^n H(x[k])$$

Relation between LTI system properties and  $h[n]$

## Relation between LTI system properties and $h[n]$

- ▶ For an LTI system, if we know  $h[n]$ , we know **everything** about the system
- ▶ Therefore, the properties (causal, memory, stability) must be reflected somehow in  $h[n]$ 
  - ▶ Not linearity and time-invariance, they must be true, otherwise we wouldn't talk about  $h[n]$

# 1. Causal LTI systems and their $h[n]$

If a LTI system is causal, then

$$h[n] = 0, \forall n < 0$$

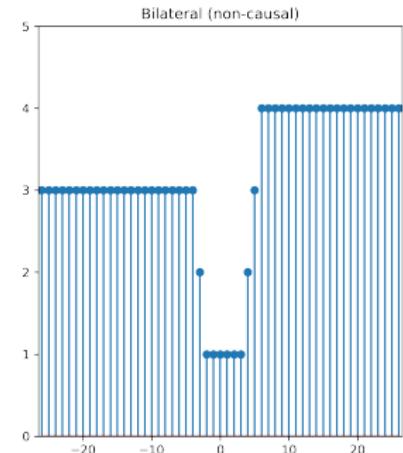
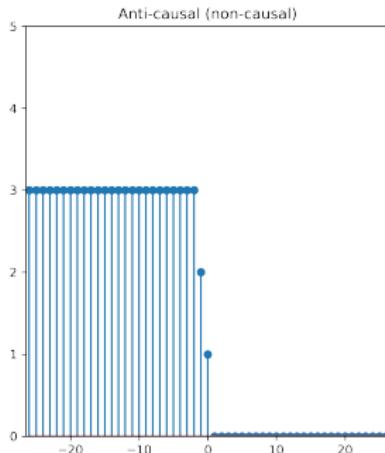
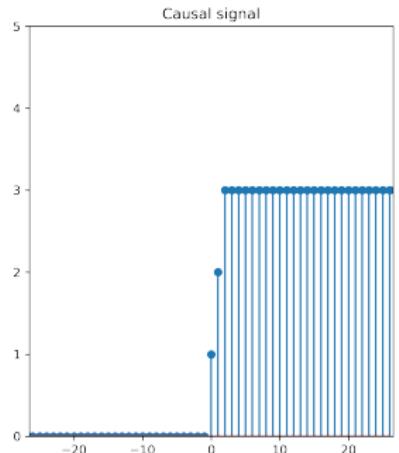
► Proof:

- ▶  $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$ ,
- ▶  $y[n]$  does not depend on  $x[n + 1], x[n + 2], \dots$
- ▶ it means that these terms are multiplied with 0
- ▶ the value  $x[n + 1]$  is multiplied with  $h[n - (n + 1)] = h[-1]$ ,  $x[n + 2]$  is multiplied with  $h[n - (n + 2)] = h[-2]$ , and so on
- ▶ Therefore:

$$h[n] = 0, \forall n < 0$$

- ▶ A **signal** which is 0 for  $n < 0$  is called a **causal** signal
- ▶ Otherwise the signal is **non-causal**
- ▶ We can say that a **system** is causal if and only if it has a causal **impulse response**
- ▶ Further definitions:
  - ▶ a signal which 0 for  $n > 0$  is called an **anti-causal** signal
  - ▶ a signal which has non-zero values both for some  $n > 0$  and for some  $n < 0$  (and thus is neither causal nor non-causal) is called **bilateral**

# Example



## 2. Stable systems and their $h[n]$

- ▶ Considering a bounded input signal,  $|x[n]| \leq A$ , the absolute value of the output is:

$$\begin{aligned}|y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \\&\leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| \\&= \sum_{k=-\infty}^{\infty} |x[k]| |h[n-k]| \\&\leq A \sum_{k=-\infty}^{\infty} |h[n-k]|\end{aligned}$$

- ▶ Therefore a **LTI system is stable if**

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

### 3. Memoryless systems and their $h[n]$ (Exercise)

#### Exercises:

- ▶ What can we say about the impulse response  $h[n]$  of a memoryless system?
- ▶ What about a system with finite memory  $M$ ?

FIR and IIR systems

- ▶ The **support** of a discrete signal = the smallest interval of  $n$  such that the signal is 0 everywhere outside the interval.
- ▶ Examples: at whiteboard
- ▶ Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.

- ▶ A **Finite Impulse Response (FIR)** system has an impulse response with **finite support**
  - ▶ i.e. the impulse response is 0 outside a certain interval.
  - ▶ i.e.  $h[n]$  is zero beyond some element  $h[M]$
- ▶ The system equation for a FIR system:

$$y[n] = \sum_{k=0}^M h[k]x[n-k] = h[0] \cdot x[n] + h[1] \cdot x[n-1] + \dots + h[M] \cdot x[n-M]$$

- ▶ is non-recursive (depends only on  $x$ )
- ▶ goes only up to some term  $h[M]x[n-M]$
- ▶ for causal system, starts from  $h[0]x[n]$ ; for non-causal, can start from  $h[-k]x[n+k]$

- ▶ For a causal FIR system, the output is a **linear combination of the last M+1 input samples**
- ▶ For non-causal FIR system, some future input samples enter the combination

- ▶ An **Infinite Impulse Response (FIR)** system has an impulse response with **infinite support**
  - ▶ i.e. the impulse response never becomes completely 0 forever.
- ▶ The output  $y[n]$  potentially depends on all the preceding input samples
  - ▶ from the convolution equation:

$$\begin{aligned}y[n] &= \sum_{k=0}^{\infty} h[k]x[n - k] \\&= h[0] \cdot x[n] + h[1] \cdot x[n - 1] + \dots h[M] \cdot x[n - M] + \dots \text{goes on} + \dots\end{aligned}$$

- ▶ An IIR system has infinite memory

- ▶ IIR systems must have **recursive** equations:
  - ▶ they depend on previous outputs  $y[n - 1]$  up to  $y[n - N]$
  - ▶ they also depend on input, going back up to  $x[n - k]$
- ▶ General equation of an IIR system:

$$\begin{aligned}y[n] &= -a_1y[n - 1] - a_2y[n - 2] - \dots - a_Ny[n - N] + \\&\quad + b_0x[n] + b_1x[n - 1] + \dots + b_Mx[n - M] \\&= -\sum_{k=1}^N a_k y[n - k] + \sum_{k=1}^M b_k x[n - k]\end{aligned}$$

- ▶ the impulse response cannot be read explicitly from the equation
- ▶ IIR equations are more general than FIR

## General equation of an LTI system

Recap:

- ▶ The general equation of an LTI system is:

$$\begin{aligned}y[n] &= -a_1y[n-1] - a_2y[n-2] - \dots - a_Ny[n-N] + \\&\quad + b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M] \\&= -\sum_{k=1}^N a_k y[n-k] + \sum_{k=1}^M b_k x[n-k]\end{aligned}$$

- ▶ If all  $a_i = 0$ , it is a FIR system, no  $y[n-k]$  term
  - ▶ in this case the coefficients  $b_k = h[k]$  (impulse response)
- ▶ If some  $a_i \neq 0$ , it is an IIR system
  - ▶ impulse response  $h[n]$  is infinitely long, is more complicated to find
- ▶ Note: if system is non-causal, can start from  $x[n+k]$

- ▶ Recursive systems need **initial conditions** (starting values)
  - ▶ since they rely on previous outputs
- ▶ If initial conditions are all 0, the system is **relaxed**
  - ▶ the output depends only on the input signal
- ▶ If initial conditions are not zero, the output depends on the input signal **and** the initial conditions

## Initial conditions for recursive systems

- ▶ The effect of the input signal and the effect of initial conditions are **independent**
  - ▶ the system behaves **linear** with respect to them
  - ▶ total output = output due to input + output due to initial conditions

| Input  | Init.Cond. | O | utput                          |
|--------|------------|---|--------------------------------|
| $x[n]$ | 0          |   | $y[n] = y_{zs}[n]$             |
| 0      | non-zero   |   | $y[n] = y_{zi}[n]$             |
| $x[n]$ | non-zero   |   | $y[n] = y_{zs}[n] + y_{zi}[n]$ |