

# Embedded System Design and Modeling

## II. Modeling of continuous systems

# Continuous systems

- ▶ Physical processes are usually in continuous domain
  - ▶ e.g. electrical, mechanical
- ▶ Processes in continuous domain are described by **differential equations**
  - ▶ i.e. with unknown functions + their derivatives + their second derivatives + ...
- ▶ Simple (ideal) case: differential equations are linear:
  - ▶ only operations allowed: sum, multiplication by a constant
- ▶ Every electrical/mechanical part defines a certain relation between the unknowns

# Electrical systems

## Electrical systems:

- ▶ Unknown functions = voltage + current in all branches
- ▶ Electrical (ideal) elements:
  - ▶ resistance:  $u(i) = R \cdot i(t)$
  - ▶ capacitance:  $i(t) = C \cdot \frac{d}{dt} u(t)$
  - ▶ etc.
- ▶ One big system of linear differential equations (SCS course, basically)
  - ▶ Kirchhoff equations  $\Leftrightarrow$  equations between currents and voltages  
 $\Leftrightarrow$  linear differential equation system
- ▶ Example: an RC system (solve at blackboard)

# Mechanical systems

## Mechanical systems:

- ▶ Unknown functions = coordinates  $x(t)$ ,  $y(t)$ ,  $z(t)$ 
  - ▶ speeds = derivatives of the positions
  - ▶ acceleration = derivative of speed = second derivative of positions
  - ▶ (forces:  $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$ )
- ▶ Mechanical (ideal) elements:
  - ▶ (Consider just a single dimension  $x(t)$ , is easier)
  - ▶ inertial force:  $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$
  - ▶ friction force:
    - ▶ sliding friction:  $\vec{F}_f = -\mu \vec{N} = -\mu \cdot m \cdot \frac{d^2}{dt^2}x(t)$
    - ▶ viscous friction:  $\vec{F}_v = -C_v \cdot \vec{v} = -C_v \cdot \frac{d}{dt}x(t)$
  - ▶ etc. . .

# Mechanical systems

- ▶ Mechanical elements are described by linear differential equations, just like electrical ones
  - ▶ they are just idealizations, physical processes can be highly nonlinear (more complex)
  - ▶ but wait, so are electrical devices actually, and this hasn't stopped us. . .
- ▶ Example: oscillations after releasing of a loaded spring
  - ▶ (solve at blackboard)

# Equivalence spring = LC circuit

- ▶ A loaded spring oscillates (without any friction) according to the equation:

$$-\frac{1}{m}kx = \frac{d^2x}{dt^2}$$

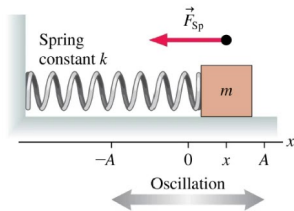
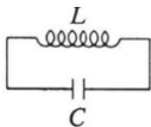


Figure 1: Spring oscillations

- ▶ image from <https://www.youtube.com/watch?v=M2m0ALqgcnQ>

## Equivalence spring = LC circuit

- ▶ A LC circuit oscillates (without any resistance loss) according to the equation:



Equation of motion for charge  $q$   
(no damping or resistance  $R$ ):

$$L(d^2q/dt^2) + (q/C) = 0 \quad (1)$$

Electrical Oscillator

Figure 2: LC oscillations

- ▶ image from  
<https://www.rfwireless-world.com/Terminology/Mechanical-Oscillator-vs-Electrical-Oscillator.html>



# Equivalence spring = LC circuit

- ▶ Notice the similarities
- ▶ Same linear differential equation:

$$\frac{d}{dt^2}f(t) + A \cdot f(t) = 0$$

- ▶ Same solution
  - ▶  $f(t) = \text{sinusoidal}$  (why sinusoidal?)
- ▶ All kinds of continuous systems can be described in the same way: using linear differential equations

# Electrical - mechanical analogies

- ▶ Multiple ways to define analogies between electrical and mechanical characteristics
- ▶ Here is the one we will use from now on:

Electr.	Mech. (linear)	Mech. (rotational)
Current [A]	Force [N]	Torque ("cuplu") [N.m]
Voltage [V]	Speed [m/s]	Angular speed [rad/s]

# Mechanics: linear vs rotational

- ▶ Note: there are different quantities for **linear** vs **rotational** movements
  - ▶ **Force** in linear movement  $\equiv$  **Torque** (cuplu) in rotational movement
  - ▶ Linear speed linear movement  $\equiv$  Angular speed in rotational movement

# Simple model of a DC motor

- ▶ Motor: gateway between the two electrical and mechanical domains
  - ▶ converts electric energy to mechanical energy, and vice-versa
- ▶ (Simple) model of a DC motor:

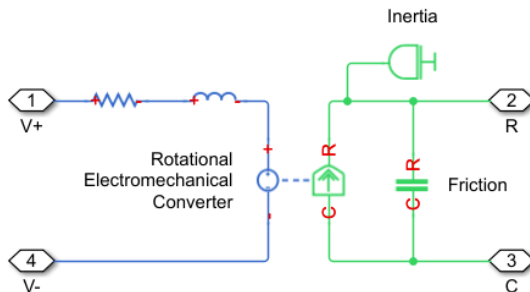


Figure 3: Simple model of a DC motor

Image from Mathworks Simulink (ssc\_dcmotor example model)

# DC motor model: electrical side

Electrical side of the DC motor model:

- ▶ Resistance: models the resistance of the windings

$$u(t) = R \cdot i(t)$$

- ▶ Inductance: models the inductive behavior of the windings

$$u(t) = L \cdot \frac{d}{dt} i(t)$$

- ▶ Controlled voltage source:

- ▶ Voltage (“back electro-magnetic force voltage”) is proportional to motor angular speed  $S(t)$  on the mechanical side (think of a dynamo)

$$u(t) = K_e \cdot S(t)$$

# DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- ▶ Controlled force/torque source
  - ▶ Generates force/torque proportional to the current  $i(t)$  on the electrical side

$$T = K_t \cdot i(t)$$

- ▶ Inertia: models the inertial force of the moving part of the motor
  - ▶ Generates force/torque proportional to acceleration (derivative of speed)

$$T_i = -m \cdot acceleration = -m \cdot \frac{d}{dt}S(t)$$

# DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- ▶ Friction: models the (viscous) friction force of the moving part of the motor
  - ▶ Generates force/torque proportional to speed

$$T_f = -C_v \cdot S(t)$$

- ▶ Inertia and Friction forces/torques oppose the force/torque) of the motor, therefore they have minus sign

# Laplace transform

- ▶ Both electrical and mechanical sides are described by linear differential equations
- ▶ The Laplace transform is a useful tool (remember SCS)
  - ▶ derivation = multiplication by  $s$
  - ▶ integration = multiplication by  $1/s$
  - ▶ transform function  $H(s) = \text{output}(s)/\text{input}(s)$
- ▶ Exercise: write the equations of all electrical and mechanical elements in Laplace transform



# Full electrical model

- ▶ All the mechanical elements can be modeled in the electrical domain
  - ▶ since they are all just differential equations, basically
  - ▶ obtain a full model in the electrical domain only
- ▶ Next slides: find electrical correspondent to all mechanical elements

# Model of the controlled voltage source

- ▶ How to model the controlled voltage source?
- ▶ Like this:
  - ▶ voltage is proportional to speed:  $U(s) = K_e \cdot S(s)$
  - ▶ speed = integral of acceleration:  $S(s) = S_0 + 1/s \cdot A$
  - ▶ acceleration is proportional to force (force(torque) / mass) =  $C_{const} \cdot T(s)$
  - ▶ force/torque = proportional to current:  $T(s) = K_t \cdot I(s)$
- ▶ Result:

$$U(s) = K_e \cdot (S_0 + 1/s \cdot C_{const} \cdot K_t I(s))$$

# Model of the controlled voltage source

$$U(s) = \underbrace{K_e \cdot S_0}_{Constant} + \underbrace{K_e C_{const}}_{Constant} \cdot \frac{1}{s} \cdot I(s)$$

- ▶ Voltage proportional on integral of current, plus a constant initial value
  - ▶ what kind of electrical element acts like this?
- ▶ The controlled voltage source can be modeled as a **capacitance**
  - ▶ Voltage is proportional to integral of current
  - ▶ (Current is proportional to derivative of voltage)
  - ▶ The first constant term = the initial voltage on the capacity
- ▶ The equivalent capacitance value depends on the motor parameters

# Model of the inertial force

- ▶ Inertia = a force which opposes (i.e. reduces) the motor force, and is proportional to acceleration
- ▶ Use the analogy listed before:
  - ▶ force = current
  - ▶ speed = voltage
  - ▶ acceleration = derivative of speed = derivative of voltage
- ▶ Inertia = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to derivative of *voltage*
  - ▶ what kind of electrical element acts like this?

# Model of the inertial force

- ▶ Inertia model = a **capacity in parallel** with the controlled voltage source
  - ▶ current proportional to derivative voltage  $\Leftrightarrow$  a capacity
  - ▶ reduces the motor current  $\Leftrightarrow$  is in parallel with the controlled voltage source (steals some of its current)

# Model of the friction force

- ▶ (Viscous) friction = a force which opposes (i.e. reduces) the motor force, and is proportional to speed
- ▶ Use the same analogy:
  - ▶ force = current
  - ▶ speed = voltage
- ▶ (Viscous) friction = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to *voltage*
  - ▶ what kind of electrical element acts like this?

# Model of the friction force

- ▶ (Viscous) friction model = a **resistance in parallel** with the controlled voltage source
  - ▶ current proportional to voltage  $\Leftrightarrow$  a resistance
  - ▶ reduces the motor current  $\Leftrightarrow$  is in parallel with the controlled voltage source (steals some of its current)

# Model of the sliding friction force

- ▶ There can also exist a sliding friction force = friction force which does not depend on speed, but is a constant
  - ▶ that's the friction force you likely encountered in high-school physics (“planul înclinat” etc.)
- ▶ Question: how is this force modeled in electrical domain?



# Model of the sliding friction force

- ▶ Answer: a constant current source in parallel
  - ▶ constant current  $\Leftrightarrow$  constant source
  - ▶ in parallel  $\Leftrightarrow$  reduces the motor current

# The full electrical model

- ▶ Draw picture at blackboard: R in series with L in series with (R parallel with  $(C1 + C2)$ )
- ▶ This is a **second order model** (1L, 1C)
  - ▶ the two capacities are in parallel, so they can be added into a single one
- ▶ The L is the inductance of the armatures  $\Rightarrow$  small, often negligible
- ▶ Can be approximated by a **first order model**

# Transfer function of a DC motor

- ▶ We can derive a transfer function
  - ▶ input = voltage on motor input  $U(s)$
  - ▶ output = motor speed  $S(s)$  = voltage on equivalent motor capacity
- ▶ Transfer function

$$H(s) = \frac{S(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_2} \approx \frac{K}{\tau \cdot s + 1}$$

- ▶ Take home message:
  - ▶ Simple DC motor no-load model = a second order RLC model = approx a RC model
  - ▶ Behaves like a RC low-pass filter
- ▶ Note: This is a no-load model (motor doesn't move anything heavy)

# Motor under load

- ▶ What happens if motor has a load?
  - ▶ e.g. the motor drags/lifts a constant weight
  - ▶ i.e. like a crane lifting a big weight from the ground
- ▶ How to model the load?

# Motor under load

- ▶ How to model the load?
- ▶ Like a constant force/torque opposing the motor force/torque
  - ▶ i.e. like a sliding friction force
  - ▶ i.e. like a current source in parallel, stealing lots of current
- ▶ In practice, the load force/torque may not be constant
  - ▶ depends on mechanical properties
  - ▶ e.g. lifting the hatch/liftgate (“portbagaj”) of a car: harder when lower, easier when higher

# Simulink model

- ▶ Simulink has a DC motor model already integrated
- ▶ You will use it in the lab

# Motor controllers

- ▶ DC motor behaves like a RC low-pass filter
  - ▶ input = voltage, output = speed
- ▶ Consequences:
  - ▶ Possible slow reaction time (exponential response to step function)
  - ▶ Little/None oscillations or overshoot
  - ▶ Final speed dependent on motor parameters
- ▶ How to improve behavior?

# Motor controllers

- ▶ Use a controller, in a negative feedback loop
- ▶ Draw at blackboard: schematic
- ▶ Role:
  - ▶ improve motor reaction speed (tradeoff: speed vs. overshoots)
  - ▶ robust against parameter or load variations



# PID Controller

- ▶ PID controller = the simplest solution
- ▶ Input = error signal = target speed - actual measured speed
- ▶ Output = Sum of three components:
  - ▶ **P**roportional:  $P * \text{input}$
  - ▶ **I**ntegral:  $I * \text{integral of input}$
  - ▶ **D**erivative:  $D * \text{derivative of input}$

# PID Controller

- ▶ Intuitive role of the  $P$  component:
  - ▶ If actual speed  $<$  target  $\Rightarrow$  increase motor voltage
  - ▶ If actual speed  $>$  target  $\Rightarrow$  decrease motor voltage
- ▶ This is not enough:
  - ▶ Non-zero motor voltage requires non-zero speed error  $\Rightarrow$  the motor never actually reaches the target speed
  - ▶ There is always a small systematic error (**bias error**)

- ▶ Intuitive role of the  $I$  component:
  - ▶ Eliminate the bias error of the  $P$  component, by slowly integrating the remaining error signal  $\Rightarrow$  integral slowly increases over time  $\Rightarrow$  motor voltage is pushed towards the correct value
  - ▶ Error signal cannot remain constant forever, because the integral would grow large  $\Rightarrow$  force changes to the motor input

- ▶ Intuitive role of the  $D$  component:
  - ▶ make the system react faster (jumpy) to fast input changes
  - ▶ improves system reaction time
- ▶ Problem:
  - ▶ fast reaction time = more oscillation behavior:
    - ▶ more overshoot
    - ▶ possibly unstable

- ▶ PID tuning: find P, I, D values for good behavior
  - ▶ Typical requirements:
    - ▶ stable system, overall
    - ▶ overshoot not larger than X%
    - ▶ fastest response in these conditions
- ▶ Find out more at the Vehicle Control Systems course