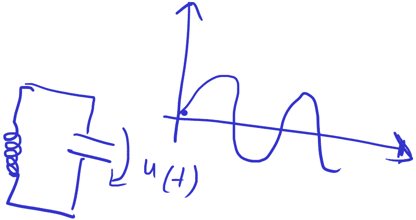
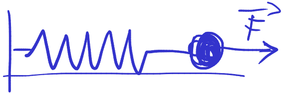
Embedded System Design and Modeling

II. Modeling of continuous systems

Continuous dynamics

- **Dynamic system** = system whose state evolves in time
- **Continuous dynamics** = the state is described by continuous functions, its evolution is governed by **differential equations**
- Example: mechanical, electrical physical processes
  - governed by mechanical / electrical differential equations
  - example: $m_1 x''(t) + K(x'(t) - x_0) = 0$
  - unknown $x(t)$+ its derivative + second derivative + ...
- Every electrical/mechanical component defines a certain relation between the unknowns



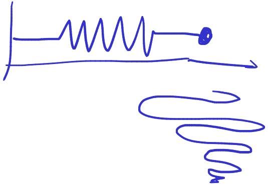


Electrical systems

Electrical systems:

- Unknown functions = voltage + current in all branches
- Electrical (ideal) elements:
  - resistance: $u(\overset{t}{\cancel{i}}) = R \cdot i(t)$
  - capacitance: $\underline{i(t)} = C \cdot \frac{d}{dt} u(t)$ ←
  - etc.
- One big system of linear differential equations (SCS course, basically)
  - Kirchhoff equations <=> equations between currents and voltages <=> linear differential equation system
- Example: an RC system (solve at blackboard)

Mechanical systems



Mechanical systems:

- Unknown functions = coordinates x(t), y(t), z(t)
  - speeds = derivatives of the positions
  - acceleration = derivative of speed = second derivative of positions
  - (forces: $\underline{F} = \underline{m \cdot a} = m \cdot \underline{\frac{d^2}{dt^2}x(t)}$)
- Mechanical (ideal) elements:
  - (Consider just a single dimension $x(t)$, is easier)
  - inertial force: $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$
  - friction force:
    - sliding friction: $\vec{F}_f = -\mu \vec{N} = -\mu \cdot m \cdot \frac{d^2}{dt^2}x(t)$
    - viscous friction: $\vec{F}_v = -C_v \cdot \vec{v} = -C_v \cdot \frac{d}{dt}x(t)$
  - etc. . .

Mechanical systems

- Mechanical elements are described by linear differential equations, just like electrical ones
  - they are just idealizations, physical processes can be highly nonlinear (more complex)
  - but wait, so are electrical devices actually, and this hasn't stopped us…
- Example: oscillations after releasing of a loaded spring
  - (solve at blackboard)

Equivalence spring = LC circuit

- A loaded spring oscillates (without any friction) according to the equation:

$$-\frac{1}{m}kx = \frac{d^2x}{dt^2}$$

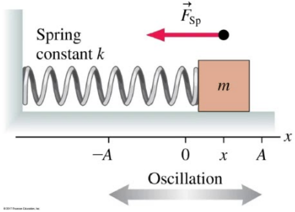


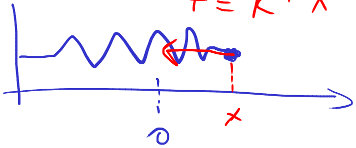
Figure 1: Spring oscillations

- image from *https://www.youtube.com/watch?v=M2m0ALqgcnQ*

$a = \frac{dv}{dt}$

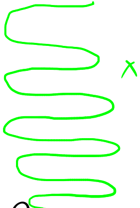
$v = \frac{dx}{dt}$

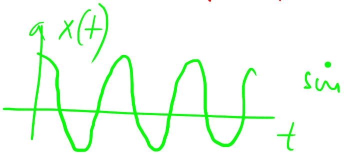
$F = K \cdot x = m \cdot a = m \cdot x''$



$m \cdot x''(t) - K \cdot x(t) = 0$

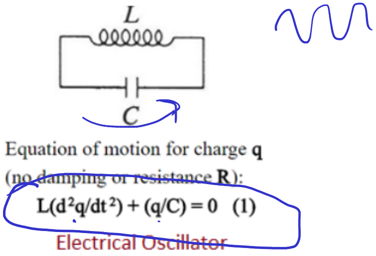
Ec. diferențială





Equivalence spring = LC circuit

- A LC circuit oscillates (without any resistance loss) according to the equation:



Equation of motion for charge **q**
(no damping or resistance **R**):

$$L(d^2q/dt^2) + (q/C) = 0 \quad (1)$$

Electrical Oscillator

q(t)

Figure 2: LC oscillations

- image from *https://www.rfwireless-world.com/Terminology/Mechanical-Oscillator-vs-Electrical-Oscillator.html*

Equivalence spring = LC circuit

$e^x = (e^x)' = (e^x)''$

$\sin' = \cos$

$\underline{\sin''} = \cos' = -\underline{\sin}$

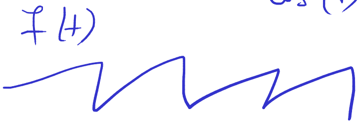
$\cos''(t) = -\cos(t)$

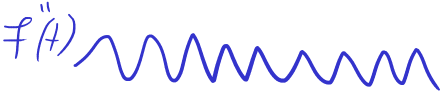
- Notice the similarities
- Same linear differential equation:

$$\boxed{\frac{d^2}{dt^2}\underline{f(t)} + A \cdot \underline{f(t)} = \underline{0}}$$

$f(t) = ?$

- Same solution
  - $f(t)$ = sinusoidal (why sinusoidal?)
- All kinds of continuous systems can be described in the same way: using linear differential equations





Electrical - mechanical analogies

- Multiple ways to define analogies between electrical and mechanical characteristics
- Here is the one we will use from now on:

Electr.	Mech. (linear)	Mech. (rotational)
Current [A] =	Force [N] =	Torque ("cuplu") [N.m]
Voltage [V] =	Speed [m/s] =	Angular speed [rad/s]

$$P = I \cdot V = F \cdot v$$

Mechanics: linear vs rotational

- Note: there are different quantities for **linear** vs **rotational** movements
  - **Force** in linear movement $\equiv$ **Torque** (cuplu) in rotational movement
  - Linear speed linear movement $\equiv$ Angular speed in rotational movement

Simple model of a DC motor

- Motor: gateway between the two electrical and mechanical domains
  - converts electric energy to mechanical energy, and vice-versa
- (Simple) model of a DC motor:

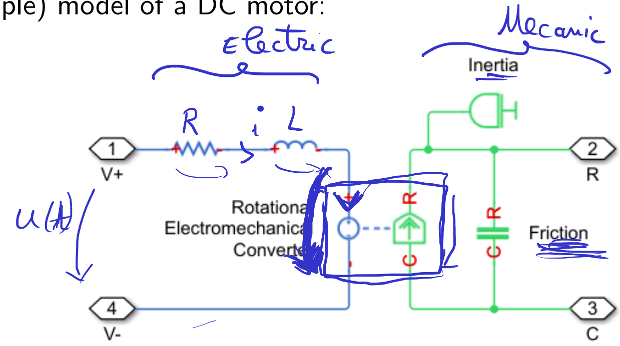


Figure 3: Simple model of a DC motor

Image from Mathworks Simulink (`ssc_dcmotor` example model)

DC motor model: electrical side

Electrical side of the DC motor model:

- Resistance: models the resistance of the windings

$$\underline{u(t)} = R \cdot \underline{i(t)} \qquad R$$

- Inductance: models the inductive behavior of the windings

$$\underline{u(t)} = L \cdot \frac{d}{dt} \underline{i(t)} \qquad L$$

- Controlled voltage source:
  - Voltage ("back electro-magnetic force voltage") is proportional to motor angular speed $S(t)$ on the mechanical side (think of a dynamo)

$$\underbrace{\boxed{\underline{u(t)}}}_{\text{tensiunea}} = K_e \cdot \underline{S(t)} \leftarrow \text{viteza} \qquad |$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- Controlled force/torque source
  - Generates force/torque proportional to the current $i(t)$ on the electrical side

$$\underline{T} = K_t \cdot \boxed{\underline{i(t)}}$$

- Inertia: models the inertial force of the moving part of the motor
  - Generates force/torque proportional to acceleration (derivative of speed)

$$T_i = -m \cdot acceleration = -m \cdot \frac{d}{dt} \underbrace{S(t)}_{\text{speed}}$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- Friction: models the (viscous) friction force of the moving part of the motor
    - Generates force/torque proportional to speed

$$\underline{T_f} = \underset{\substack{\uparrow \\ \text{minus}}}{-} C_v \cdot \underline{S(t)}$$

- Inertia and Friction forces/torques oppose the force/torque) of the motor, therefore they have minus sign

Laplace transform

$$u(t) = L \cdot \frac{di(t)}{dt}$$

$$U(s) = \boxed{L \cdot s}\, I(s)$$

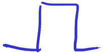
- Both electrical and mechanical sides are described by linear differential equations
- The Laplace transform is a useful tool (remember SCS)
  - derivation = multiplication by s
  - integration = multiplication by $1/s$
  - transform function H(s) = output(s)/input(s)
- Exercise: write the equations of all electrical and mechanical elements in Laplace transform

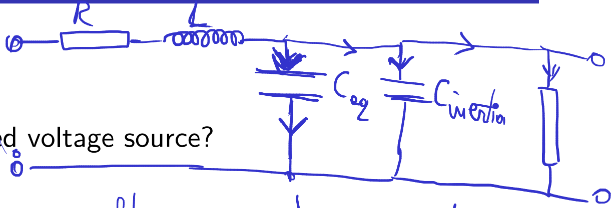
Full electrical model

- All the mechanical elements can be modeled in the electrical domain
  - since they are all just differential equations, basically
  - obtain a full model in the electrical domain only

- Next slides: find electrical correspondent to all mechanical elements

Model of the controlled voltage source





Electrical equivalent of DC motor

$a = \frac{dv}{dt}$

$v = \underline{v_0} + \int_0^t a$

$\underline{F} = m \cdot \underline{a}$

ct

- How to model the controlled voltage source?
- Like this:
  - voltage is proportional to speed: $\overset{\text{voltage}}{U(s)} = K_e \cdot \overset{\text{speed}}{S(s)}$
  - speed = integral of acceleration: $S(s) = S_0 + (1/s) \cdot A$
  - acceleration is proportional to force (force(torque) / mass) = $C_{const} \cdot T(s)$
  - force/torque = proportional to current: $\underline{T(s) = K_t \cdot I(s)}$
- Result:

$$\boxed{U(s) = K_e \cdot (\underbrace{S_0 + 1/s \cdot C_{const} \cdot K_t I(s)})}$$

$$U(s) = K_e \left(S_0 + \frac{1}{s} \cdot C_i \, C_2 \cdot I(s) \right) = C_1 + \frac{1}{s} \cdot C_2 \cdot I(s)$$

Model of the controlled voltage source

$$U(s) = \underbrace{\overset{\downarrow}{K_e} \cdot S_0}_{\mathit{Constant}} + \underbrace{\overset{\downarrow}{K_e} \overset{\downarrow}{C}_{const}}_{\mathit{Constant}} \cdot \frac{1}{s} \cdot I(s)$$

- Voltage proportional on integral of current, plus a constant initial value
  - what kind of electrical element acts like this?

- The controlled voltage source can be modeled as a **capacitance**
  - Voltage is proportional to integral of current
  - (Current is proportional to derivative of voltage)
  - The first constant term = the initial voltage on the capacity

- The equivalent capacitance value depends on the motor parameters

Model of the inertial force

- Inertia = a ~~force~~ Curent which opposes (i.e. reduces) the motor ~~force~~ curent, and is proportional to ~~acceleration~~ deriv. tensiunii
- Use the analogy listed before:
  - force = current
  - speed = voltage
  - acceleration = derivative of speed = derivative of voltage
- Inertia = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to derivative of *voltage*
  - what kind of electrical element acts like this?

Model of the inertial force

- Inertia model = a **capacity in parallel** with the controlled voltage source
  - current proportional to derivative voltage ⇔ a capacity
  - reduces the motor current ⇔ is in parallel with the controlled voltage source (steals some of its current)

Model of the friction force

- (Viscous) friction = a ~~force~~ current which opposes (i.e. reduces) the motor ~~force~~ current, and is proportional to ~~speed~~ voltage
- Use the same analogy:
  - force = current
  - speed = voltage
- (Viscous) friction = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to *voltage*
  - what kind of electrical element acts like this?

Model of the friction force

- (Viscous) friction model $=$ a **resistance in parallel** with the controlled voltage source
  - current proportional to voltage $\Leftrightarrow$ a resistance
  - reduces the motor current $\Leftrightarrow$ is in parallel with the controlled voltage source (steals some of its current)

Model of the sliding friction force

- There can also exist a sliding friction force = friction force which does not depend on speed, but is a constant
  - that's the friction force you likely encountered in high-school physics ("planul înclinat" etc.)
- Question: how is this force modeled in electrical domain?

Model of the sliding friction force

- Answer: a constant current source in parallel
  - constant current $\Leftrightarrow$ constant source
  - in parallel $\Leftrightarrow$ reduces the motor current

The full electrical model

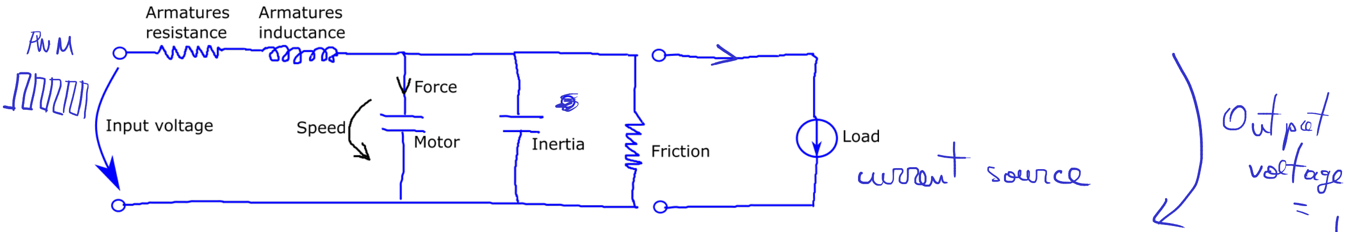
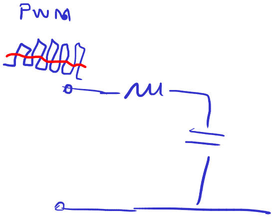


Figure 4: DC Motor electrical equivalent model

- This is **a second order model** (1L, 1C)
  - the two capacities are in parallel, so they can be added into a single one
- The L is the inductance of the armatures ⇒ small, often negligible
- Can be approximated by a **first order model** RC circuit



Transfer function of a DC motor

- We can derive a transfer function
  - input = voltage on motor input $U(s)$
  - output = motor speed $S(s)$ = voltage on equivalent motor capacity
- Transfer function (2^nd degree, approximately 1^{st} degree)

$$H(s) = \frac{S(s)}{U(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} \approx \frac{K}{\tau \cdot s + 1}$$

- Take home message:
  - Simple DC motor no-load model = a second order RLC model = approx a RC model
  - Behaves like a RC low-pass filter
- Note: This is a no-load model (motor doesn't move anything heavy)

Motor under load

- What happens if motor has a load?
  - e.g. the motor drags/lifts a constant weight
  - i.e. like a crane lifting a big weight from the ground

- How to model the load?

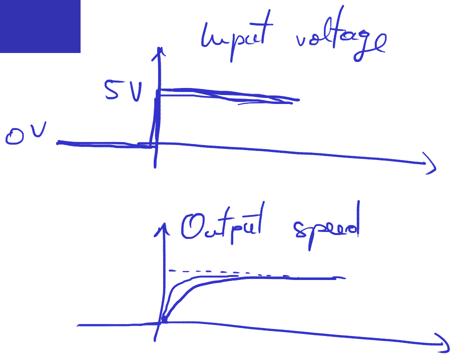
Motor under load

- How to model the load?
- Like a constant $\overset{\text{current}}{\text{force}}$/torque opposing the motor $\overset{\text{current}}{\text{force}}$/torque
  - i.e. like a sliding friction force
  - i.e. like a current source in parallel, stealing lots of current
- In practice, the load force/torque may not be constant
  - depends on mechanical properties
  - e.g. lifting the hatch/liftgate ("portbagaj") of a car: harder when lower, easier when higher

Simulink model

- Simulink has a DC motor model already integrated
- You will use it in the lab (maybe)

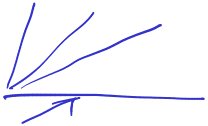
What to use the model for?



What to use the motor model for?

Simulate:

- how fast motor starts when supply is first applied
- what happens when supply fluctuates (e.g. PWM)
- what happens when motor parameters change (e.g. temperature rises, friction ~~slows~~) smaller
- what happens when load varies
- . . .



Motor speed controller

Basic problem: how to make sure motor speed stays **exactly** as desired:

- even if parameters vary
- even if load varies
- even if supply varies
- on power on, speed is reached as fast as possible

This is a job for a **motor controller**

- Today's special: the PID motor controller

need a speed sensor

Motor controllers

PID controller

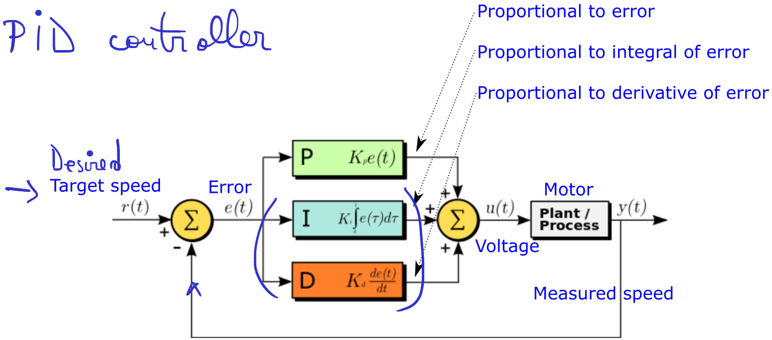


Figure 5: PID speed controller (image from Wikipedia)

- Negative feedback loop
- Can be used for any sort of process, not just motors
- Make output signal $y(t)$ follow the desired input $r(t)$

PID Controller

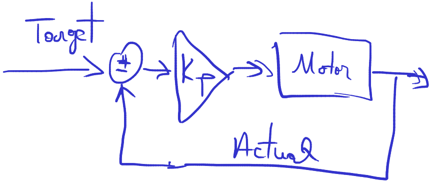
- PID controller = the simplest solution
- Input = error signal = target speed - actual measured speed
- Output = Sum of three components:
  - **P**roportional: P * input
  - **I**ntegral: I * integral of input
  - **D**erivative: D * derivative of input

PID Controller - P component

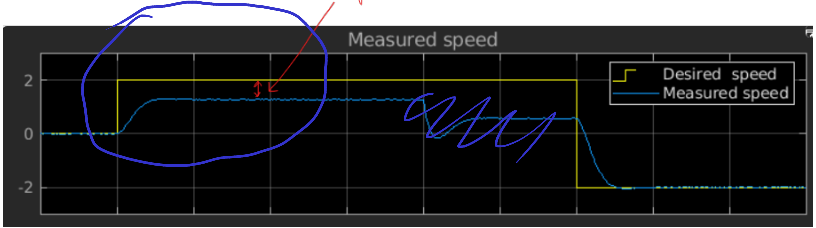
- Intuitive role of the *P* component:
  - If actual speed < target => increase motor voltage
  - If actual speed > target => decrease motor voltage

$$V = K_p \cdot \left(\underset{\text{speed}}{\text{target}} - \underset{\text{speed}}{\text{actual}} \right)$$

- This is not enough:
  - Non-zero motor voltage requires non-zero speed error => the motor never actually reaches the target speed
  - There is always a small systematic error ("**bias error**", "steady-state error")



PID controller - only P, systematic error



steady - state error

Figure 6: Systematic error for P-only controller

PID Controller - I component

- Intuitive role of the *I* component:
  - Eliminate the bias error of the *P* component, by slowly integrating the remaining error signal => integral slowly increases over time => motor voltage is pushed towards the correct value
  - Error signal cannot remain constant forever, because the integral would grow large => force changes to the motor input