

Embedded System Design and Modeling

II. Modeling of continuous systems

Continuous dynamics

- ▶ **Dynamic system** = system whose state evolves in time
- ▶ **Continuous dynamics** = the state is described by continuous functions, its evolution is governed by **differential equations**
- ▶ Example: mechanical, electrical physical processes
 - ▶ governed by mechanical / electrical differential equations
 - ▶ example: $m_1 x''(t) + K(x'(t) - x_0) = 0$
 - ▶ unknown $x(t)$ + its derivative + second derivative + ...
- ▶ Every electrical/mechanical component defines a certain relation between the unknowns

Electrical systems

Electrical systems:

- ▶ Unknown functions = voltage + current in all branches
- ▶ Electrical (ideal) elements:
 - ▶ resistance: $u(i) = R \cdot i(t)$
 - ▶ capacitance: $i(t) = C \cdot \frac{d}{dt} u(t)$
 - ▶ etc.
- ▶ One big system of linear differential equations (SCS course, basically)
 - ▶ Kirchhoff equations \Leftrightarrow equations between currents and voltages
 \Leftrightarrow linear differential equation system
- ▶ Example: an RC system (solve at blackboard)

Mechanical systems

Mechanical systems:

- ▶ Unknown functions = coordinates $x(t)$, $y(t)$, $z(t)$
 - ▶ speeds = derivatives of the positions
 - ▶ acceleration = derivative of speed = second derivative of positions
 - ▶ (forces: $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$)
- ▶ Mechanical (ideal) elements:
 - ▶ (Consider just a single dimension $x(t)$, is easier)
 - ▶ inertial force: $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$
 - ▶ friction force:
 - ▶ sliding friction: $\vec{F}_f = -\mu \vec{N} = -\mu \cdot m \cdot \frac{d^2}{dt^2}x(t)$
 - ▶ viscous friction: $\vec{F}_v = -C_v \cdot \vec{v} = -C_v \cdot \frac{d}{dt}x(t)$
 - ▶ etc. . .

Mechanical systems

- ▶ Mechanical elements are described by linear differential equations, just like electrical ones
 - ▶ they are just idealizations, physical processes can be highly nonlinear (more complex)
 - ▶ but wait, so are electrical devices actually, and this hasn't stopped us. . .
- ▶ Example: oscillations after releasing of a loaded spring
 - ▶ (solve at blackboard)

Equivalence spring = LC circuit

- ▶ A loaded spring oscillates (without any friction) according to the equation:

$$-\frac{1}{m}kx = \frac{d^2x}{dt^2}$$

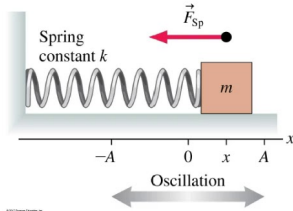
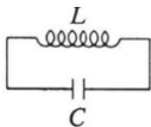


Figure 1: Spring oscillations

- ▶ image from <https://www.youtube.com/watch?v=M2m0ALqgcnQ>

Equivalence spring = LC circuit

- ▶ A LC circuit oscillates (without any resistance loss) according to the equation:



Equation of motion for charge q
(no damping or resistance R):

$$L(d^2q/dt^2) + (q/C) = 0 \quad (1)$$

Electrical Oscillator

Figure 2: LC oscillations

- ▶ image from
<https://www.rfwireless-world.com/Terminology/Mechanical-Oscillator-vs-Electrical-Oscillator.html>

Equivalence spring = LC circuit

- ▶ Notice the similarities
- ▶ Same linear differential equation:

$$\frac{d^2}{dt^2}f(t) + A \cdot f(t) = 0$$

- ▶ Same solution
 - ▶ $f(t) = \text{sinusoidal}$ (why sinusoidal?)
- ▶ All kinds of continuous systems can be described in the same way: using linear differential equations

Electrical - mechanical analogies

- ▶ Multiple ways to define analogies between electrical and mechanical characteristics
- ▶ Here is the one we will use from now on:

Electr.	Mech. (linear)	Mech. (rotational)
Current [A]	Force [N]	Torque ("cuplu") [N.m]
Voltage [V]	Speed [m/s]	Angular speed [rad/s]

Mechanics: linear vs rotational

- ▶ Note: there are different quantities for **linear** vs **rotational** movements
 - ▶ **Force** in linear movement \equiv **Torque** (cuplu) in rotational movement
 - ▶ Linear speed linear movement \equiv Angular speed in rotational movement

Simple model of a DC motor

- ▶ Motor: gateway between the two electrical and mechanical domains
 - ▶ converts electric energy to mechanical energy, and vice-versa
- ▶ (Simple) model of a DC motor:

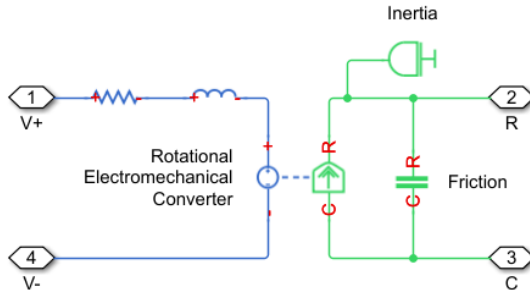


Figure 3: Simple model of a DC motor

Image from Mathworks Simulink (ssc_dcmotor example model)

DC motor model: electrical side

Electrical side of the DC motor model:

- ▶ Resistance: models the resistance of the windings

$$u(t) = R \cdot i(t)$$

- ▶ Inductance: models the inductive behavior of the windings

$$u(t) = L \cdot \frac{d}{dt} i(t)$$

- ▶ Controlled voltage source:

- ▶ Voltage (“back electro-magnetic force voltage”) is proportional to motor angular speed $S(t)$ on the mechanical side (think of a dynamo)

$$u(t) = K_e \cdot S(t)$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- ▶ Controlled force/torque source
 - ▶ Generates force/torque proportional to the current $i(t)$ on the electrical side

$$T = K_t \cdot i(t)$$

- ▶ Inertia: models the inertial force of the moving part of the motor
 - ▶ Generates force/torque proportional to acceleration (derivative of speed)

$$T_i = -m \cdot acceleration = -m \cdot \frac{d}{dt}S(t)$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- ▶ Friction: models the (viscous) friction force of the moving part of the motor
 - ▶ Generates force/torque proportional to speed

$$T_f = -C_v \cdot S(t)$$

- ▶ Inertia and Friction forces/torques oppose the force/torque) of the motor, therefore they have minus sign

Laplace transform

- ▶ Both electrical and mechanical sides are described by linear differential equations
- ▶ The Laplace transform is a useful tool (remember SCS)
 - ▶ derivation = multiplication by s
 - ▶ integration = multiplication by $1/s$
 - ▶ transform function $H(s) = \text{output}(s)/\text{input}(s)$
- ▶ Exercise: write the equations of all electrical and mechanical elements in Laplace transform

Full electrical model

- ▶ All the mechanical elements can be modeled in the electrical domain
 - ▶ since they are all just differential equations, basically
 - ▶ obtain a full model in the electrical domain only
- ▶ Next slides: find electrical correspondent to all mechanical elements

Model of the controlled voltage source

- ▶ How to model the controlled voltage source?
- ▶ Like this:
 - ▶ voltage is proportional to speed: $U(s) = K_e \cdot S(s)$
 - ▶ speed = integral of acceleration: $S(s) = S_0 + 1/s \cdot A$
 - ▶ acceleration is proportional to force (force(torque) / mass) = $C_{const} \cdot T(s)$
 - ▶ force/torque = proportional to current: $T(s) = K_t \cdot I(s)$
- ▶ Result:

$$U(s) = K_e \cdot (S_0 + 1/s \cdot C_{const} \cdot K_t I(s))$$

Model of the controlled voltage source

$$U(s) = \underbrace{K_e \cdot S_0}_{Constant} + \underbrace{K_e C_{const}}_{Constant} \cdot \frac{1}{s} \cdot I(s)$$

- ▶ Voltage proportional on integral of current, plus a constant initial value
 - ▶ what kind of electrical element acts like this?
- ▶ The controlled voltage source can be modeled as a **capacitance**
 - ▶ Voltage is proportional to integral of current
 - ▶ (Current is proportional to derivative of voltage)
 - ▶ The first constant term = the initial voltage on the capacity
- ▶ The equivalent capacitance value depends on the motor parameters

Model of the inertial force

- ▶ Inertia = a force which opposes (i.e. reduces) the motor force, and is proportional to acceleration
- ▶ Use the analogy listed before:
 - ▶ force = current
 - ▶ speed = voltage
 - ▶ acceleration = derivative of speed = derivative of voltage
- ▶ Inertia = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to derivative of *voltage*
 - ▶ what kind of electrical element acts like this?

Model of the inertial force

- ▶ Inertia model = a **capacity in parallel** with the controlled voltage source
 - ▶ current proportional to derivative voltage \Leftrightarrow a capacity
 - ▶ reduces the motor current \Leftrightarrow is in parallel with the controlled voltage source (steals some of its current)

Model of the friction force

- ▶ (Viscous) friction = a force which opposes (i.e. reduces) the motor force, and is proportional to speed
- ▶ Use the same analogy:
 - ▶ force = current
 - ▶ speed = voltage
- ▶ (Viscous) friction = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to *voltage*
 - ▶ what kind of electrical element acts like this?

Model of the friction force

- ▶ (Viscous) friction model = a **resistance in parallel** with the controlled voltage source
 - ▶ current proportional to voltage \Leftrightarrow a resistance
 - ▶ reduces the motor current \Leftrightarrow is in parallel with the controlled voltage source (steals some of its current)

Model of the sliding friction force

- ▶ There can also exist a sliding friction force = friction force which does not depend on speed, but is a constant
 - ▶ that's the friction force you likely encountered in high-school physics (“planul înclinat” etc.)
- ▶ Question: how is this force modeled in electrical domain?

Model of the sliding friction force

- ▶ Answer: a constant current source in parallel
 - ▶ constant current \Leftrightarrow constant source
 - ▶ in parallel \Leftrightarrow reduces the motor current

The full electrical model

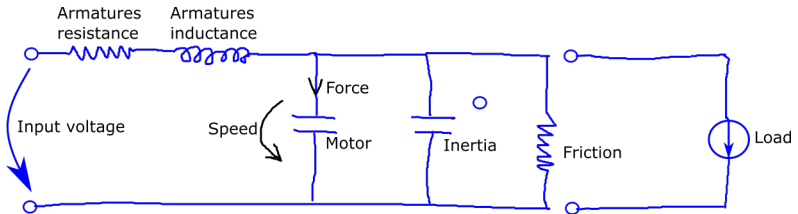


Figure 4: DC Motor electrical equivalent model

- ▶ This is a **second order model** (1L, 1C)
 - ▶ the two capacities are in parallel, so they can be added into a single one
- ▶ The L is the inductance of the armatures \Rightarrow small, often negligible
- ▶ Can be approximated by a **first order model**

Transfer function of a DC motor

- ▶ We can derive a transfer function
 - ▶ input = voltage on motor input $U(s)$
 - ▶ output = motor speed $S(s)$ = voltage on equivalent motor capacity
- ▶ Transfer function (2^{nd} degree, approximately 1^{st} degree)

$$H(s) = \frac{S(s)}{U(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} \approx \frac{K}{\tau \cdot s + 1}$$

- ▶ Take home message:
 - ▶ Simple DC motor no-load model = a second order RLC model = approx a RC model
 - ▶ Behaves like a RC low-pass filter
- ▶ Note: This is a no-load model (motor doesn't move anything heavy)

Motor under load

- ▶ What happens if motor has a load?
 - ▶ e.g. the motor drags/lifts a constant weight
 - ▶ i.e. like a crane lifting a big weight from the ground
- ▶ How to model the load?

Motor under load

- ▶ How to model the load?
- ▶ Like a constant force/torque opposing the motor force/torque
 - ▶ i.e. like a sliding friction force
 - ▶ i.e. like a current source in parallel, stealing lots of current
- ▶ In practice, the load force/torque may not be constant
 - ▶ depends on mechanical properties
 - ▶ e.g. lifting the hatch/liftgate (“portbagaj”) of a car: harder when lower, easier when higher

Simulink model

- ▶ Simulink has a DC motor model already integrated
- ▶ You will use it in the lab (maybe)

What to use the model for?

What to use the motor model for?

Simulate:

- ▶ how fast motor starts when supply is first applied
- ▶ what happens when supply fluctuates (e.g. PWM)
- ▶ what happens when motor parameters change (e.g. temperature rises, friction slows)
- ▶ what happens when load varies
- ▶ ...

Motor speed controller

Basic problem: how to make sure motor speed stays **exactly** as desired:

- ▶ even if parameters vary
- ▶ even if load varies
- ▶ even if supply varies
- ▶ on power on, speed is reached as fast as possible

This is a job for a **motor controller**

- ▶ Today's special: the PID motor controller

Motor controllers

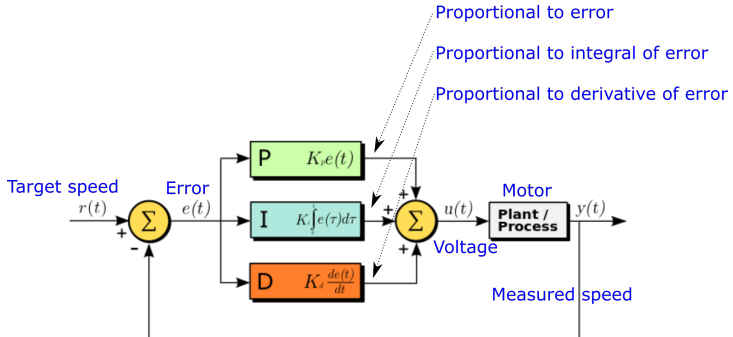


Figure 5: PID speed controller (image from Wikipedia)

- ▶ Negative feedback loop
- ▶ Can be used for any sort of process, not just motors
- ▶ Make output signal $y(t)$ follow the desired input $r(t)$

PID Controller

- ▶ PID controller = the simplest solution
- ▶ Input = error signal = target speed - actual measured speed
- ▶ Output = Sum of three components:
 - ▶ **Proportional:** $P * \text{input}$
 - ▶ **Integral:** $I * \text{integral of input}$
 - ▶ **Derivative:** $D * \text{derivative of input}$

PID Controller - P component

- ▶ Intuitive role of the P component:
 - ▶ If actual speed $<$ target \Rightarrow increase motor voltage
 - ▶ If actual speed $>$ target \Rightarrow decrease motor voltage
- ▶ This is not enough:
 - ▶ Non-zero motor voltage requires non-zero speed error \Rightarrow the motor never actually reaches the target speed
 - ▶ There is always a small systematic error (“**bias error**”, “steady-state error”)

PID controller - only P, systematic error

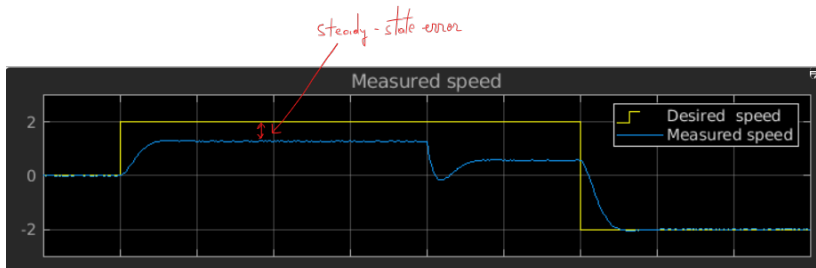


Figure 6: Systematic error for P-only controller

PID Controller - I component

- ▶ Intuitive role of the I component:
 - ▶ Eliminate the bias error of the P component, by slowly integrating the remaining error signal \Rightarrow integral slowly increases over time \Rightarrow motor voltage is pushed towards the correct value
 - ▶ Error signal cannot remain constant forever, because the integral would grow large \Rightarrow force changes to the motor input

PID controller - P and I

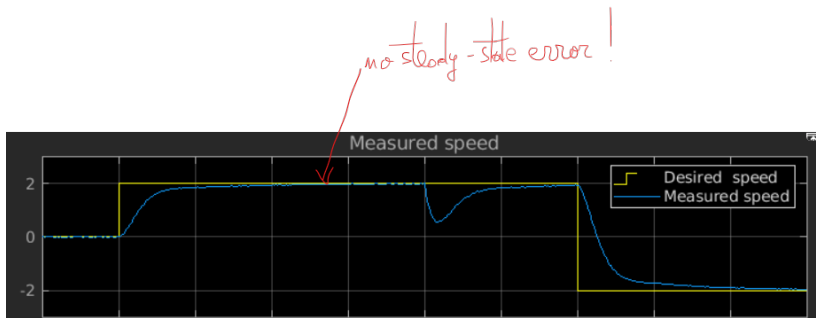


Figure 7: P and I components

PID Controller - D component

- ▶ Intuitive role of the D component:
 - ▶ make the system react faster (jumpy) to fast input changes
 - ▶ improves system reaction time
- ▶ Problem:
 - ▶ fast reaction time = more oscillation behavior:
 - ▶ more overshoot
 - ▶ possibly unstable

PID controller - P, I and D

faster response,
but more overshoot / oscillations!

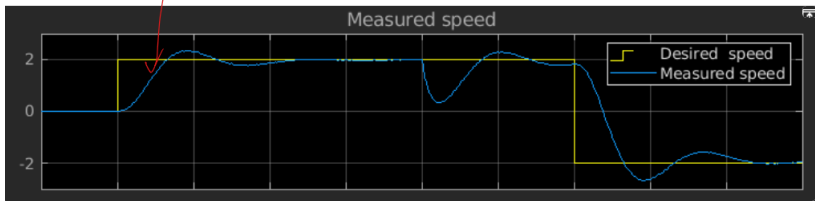


Figure 8: P, I and D components

PID tuning

- ▶ PID tuning: find P, I, D values for good behavior
 - ▶ Typical requirements:
 - ▶ stable system, overall
 - ▶ overshoot not larger than X%
 - ▶ fastest response in these conditions
- ▶ Find out more at the Vehicle Control Systems course (2nd semester, I think)