Embedded System Design and Modeling

II. Modeling of continuous systems

Actor model of systems

A system can be decomposed as inter-connected building blocks, called "actors"

- Each actor has:
  - 0, 1 or more input ports
  - 0, 1 or more output ports
  - an internal computation / function / what it does
- Connections = Signals

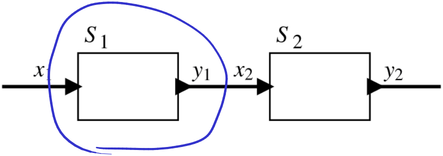


Figure 1: Actor model of systems[1]

[1] (Image from Lee&Seshia 2017)

Actor dynamics

How to describe what a component does?

- Continuous dynamics
- Discrete dynamics

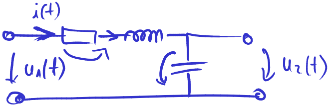
Continuous dynamics

$F = m \cdot a$ $\quad a = \frac{dv}{dt}(t)$

$v = \frac{dx}{dt}$

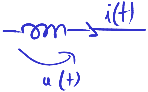
- **Dynamic system** = system whose state evolves in time
- **Continuous dynamics** = the state is described by continuous functions, its evolution is governed by **differential equations**
- Example: mechanical, electrical physical processes
  - governed by mechanical / electrical differential equations
  - example: $m_1 x''(t) + K(x'(t) - x_0) = 0$
  - unknown $x(t)$+ its derivative + second derivative + ...
- Every electrical/mechanical component defines a certain relation between the unknowns

Electrical systems



Electrical systems:

- Unknown functions = voltage + current in all branches
- Electrical (ideal) elements:
  - resistance: $u(i) = R \cdot i(t)$
  - capacitance: $i(t) = C \cdot \frac{d}{dt} u(t)$
  - etc.



$u(t) = L \cdot \frac{d}{dt} i(t)$

- One big system of linear differential equations (SCS course, basically)
  - Kirchhoff equations <=> equations between currents and voltages <=> linear differential equation system
- Example: an RC system (solve at blackboard)

Mechanical systems

Mechanical systems:

$F = m \cdot a = m \cdot \frac{d^2 x(t)}{dt^2}$

- Unknown functions = coordinates x(t), y(t), z(t)
  - speeds = derivatives of the positions
  - acceleration = derivative of speed = second derivative of positions
  - (forces: $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$)
- Mechanical (ideal) elements:
  - (Consider just a single dimension $x(t)$, is easier)
  - inertial force: $F = m \cdot a = m \cdot \frac{d^2}{dt^2}x(t)$
  - friction force:
    - sliding friction: $\vec{F_f} = -\mu \vec{N} = -\mu \cdot m \cdot \frac{d^2}{dt^2}x(t)$
    - viscous friction: $\vec{F_v} = -C_v \cdot \vec{v} = -C_v \cdot \frac{d}{dt}x(t)$
  - etc...

Mechanical systems

- Mechanical elements are described by linear differential equations, just like electrical ones
  - they are just idealizations, physical processes can be highly nonlinear (more complex)
  - but wait, so are electrical devices actually, and this hasn't stopped us...
- Example: oscillations after releasing of a loaded spring
  - (solve at blackboard)

Equivalence spring = LC circuit

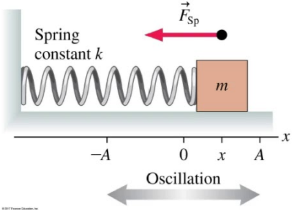
- A loaded spring oscillates (without any friction) according to the equation:

$$\boxed{\frac{d^2x(t)}{dt^2} + \frac{1}{m}\cdot k\cdot x = 0}$$

$$\boxed{-\frac{1}{m}kx = \frac{d^2x}{dt^2}}$$

$x(t)$

$$-\frac{1}{m}\cdot k\cdot x(t) = \frac{d^2x(t)}{dt^2}$$



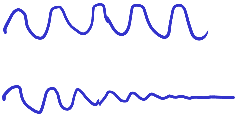


Figure 2: Spring oscillations

- image from *https://www.youtube.com/watch?v=M2m0ALqgcnQ*

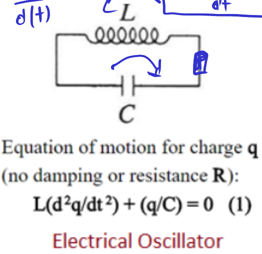
Equivalence spring = LC circuit

- A LC circuit oscillates (without any resistance loss) according to the equation:

$L \cdot \frac{di(t)}{d(t)}$

$i(t) = \frac{dq(t)}{dt}$

$$\boxed{L \cdot \frac{d^2 q(t)}{dt} + \frac{q}{C}} = 0 \Leftrightarrow \boxed{\frac{d^2 q(t)}{dt} + \frac{1}{LC} \cdot q(t) = 0}$$



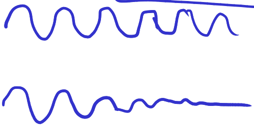


Figure 3: LC oscillations

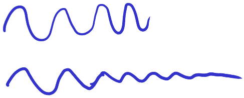
- image from *https://www.rfwireless-world.com/Terminology/Mechanical-Oscillator-vs-Electrical-Oscillator.html*

Equivalence spring = LC circuit

- Notice the similarities
- Same linear differential equation:

$f(t)$ = signal

$$\frac{d^2}{dt^2} f(t) + A \cdot f(t) = 0$$



- Same solution
  - $f(t)$ = sinusoidal (why sinusoidal?)
- All kinds of continuous systems can be described in the same way: using linear differential equations

Electrical - mechanical analogies

- Multiple ways to define analogies between electrical and mechanical characteristics
- Here is the one we will use from now on:

Electr.	Mech. (linear)	Mech. (rotational)
Current [A]	Force [N]	Torque ("cuplu") [N.m]
Voltage [V]	Speed [m/s]	Angular speed [rad/s]

Mechanics: linear vs rotational

- Note: there are different quantities for **linear** vs **rotational** movements
  - **Force** in linear movement $\equiv$ **Torque** (cuplu) in rotational movement
  - Linear speed linear movement $\equiv$ Angular speed in rotational movement

Simple model of a DC motor

- Example of continuous system modeling: model of a DC motor
- Motor: gateway between the two electrical and mechanical domains
  - converts electric energy to mechanical energy, and vice-versa
- (Simple) model of a DC motor:

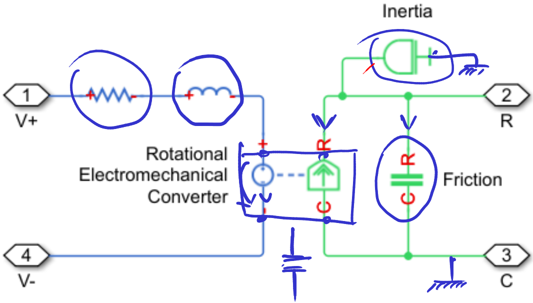


Figure 4: Simple model of a DC motor

DC motor model: electrical side

Electrical side of the DC motor model:

- Resistance: models the resistance of the windings

$$\boxed{u(t) = R \cdot i(t)}$$

- Inductance: models the inductive behavior of the windings

$$\boxed{u(t) = L \cdot \frac{d}{dt} i(t)}$$

- Controlled voltage source:
  - Voltage ("back electro-magnetic force voltage") is proportional to motor angular speed $S(t)$ on the mechanical side (think of a dynamo)

$$\boxed{u(t) = K_e \cdot S(t)}$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- Controlled force/torque source
  - Generates force/torque proportional to the current $i(t)$ on the electrical side

Torque $$\boxed{T = K_t \cdot i(t)}$$

- Inertia: models the inertial force of the moving part of the motor
  - Generates force/torque proportional to acceleration (derivative of speed)

$$T_i = -m \cdot acceleration = -m \cdot \frac{d}{dt} S(t)$$

DC motor model: mechanical side

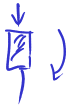
Mechanical circuit of the DC motor model (no load):

- Friction: models the (viscous) friction force of the moving part of the motor
  - Generates force/torque proportional to speed

$$T_f = -C_v \cdot S(t)$$

- Inertia and Friction forces/torques oppose the force/torque) of the motor, therefore they have minus sign

$$I = \underset{\frac{1}{R}}{Const} \cdot V$$



Laplace transform

$$U = L \cdot \frac{d\,i(t)}{dt}$$

$$U = s L \cdot i(t)$$

- Both electrical and mechanical sides are described by linear differential equations
- The Laplace transform is a useful tool (remember SCS)
  - derivation = multiplication by s
  - integration = multiplication by $1/s$
  - transform function H(s) = output(s)/input(s)

$$H(s) = \frac{s+1}{s+5}$$

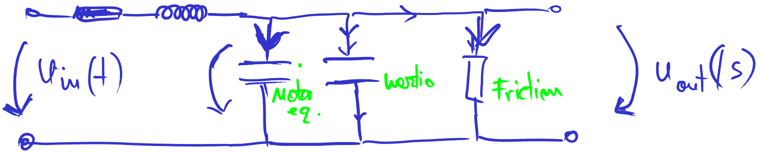
- Exercise: write the equations of all electrical and mechanical elements in Laplace transform

$$u''(t) + 2 \cdot u'(t) + u(t) = 0$$

$$\boxed{s^2 \cdot u(s) + 2 \cdot s \cdot u(s) + u(s) = 0}$$

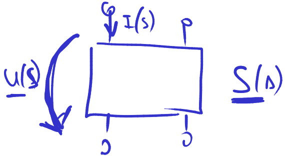
Full electrical model

- All the mechanical elements can be modeled in the electrical domain
  - since they are all just differential equations, basically
  - obtain a full model in the electrical domain only
- Next slides: find electrical correspondent to all mechanical elements



Electr. equivalent of a DC motor

Model of the controlled voltage source



- How to model the controlled voltage source?
- Like this:
  - voltage is proportional to speed: $U(s) = K_e \cdot S(s)$
  - speed = integral of acceleration: $S(s) = S_0 + 1/s \cdot A$
  - acceleration is proportional to force (force(torque) / mass) = $C_{const} \cdot T(s)$

$F = m \cdot a$

  - force/torque = proportional to current: $T(s) = K_t \cdot I(s)$
- Result:

$$\boxed{U(s) = K_e \cdot (S_0 + 1/s \cdot C_{const} \cdot K_t I(s))}$$

$$U(s) = K_e \cdot S_0 + K_e \cdot \underbrace{Const}_{K_t} \cdot \frac{1}{s} \cdot I(s)$$

$i(t) = C \cdot \frac{du(t)}{dt}$

$U(s) = U_0 + \frac{1}{s} I(s)$

Model of the controlled voltage source

$$U(s) = \underbrace{K_e \cdot S_0}_{Constant} + \underbrace{K_e C_{const}}_{Constant} \cdot \frac{1}{s} \cdot I(s)$$

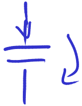
- Voltage proportional on integral of current, plus a constant initial value
  - what kind of electrical element acts like this?
- The controlled voltage source can be modeled as a **capacitance**
  - Voltage is proportional to integral of current
  - (Current is proportional to derivative of voltage)
  - The first constant term = the initial voltage on the capacity
- The equivalent capacitance value depends on the motor parameters

Model of the inertial force

- Inertia = a force which opposes (i.e. reduces) the motor force, and is proportional to acceleration

$A = \frac{dV}{dt}$

- Use the analogy listed before:
  - force = current
  - speed = voltage
  - acceleration = derivative of speed = derivative of voltage
- Inertia = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to derivative of *voltage*
  - what kind of electrical element acts like this?



$i(t) = C \cdot \frac{du(t)}{d(t)}$

Model of the inertial force

- Inertia model = a **capacity in parallel** with the controlled voltage source
  - current proportional to derivative voltage $\Leftrightarrow$ a capacity
  - reduces the motor current $\Leftrightarrow$ is in parallel with the controlled voltage source (steals some of its current)

Model of the friction force

- (Viscous) friction = a force which opposes (i.e. reduces) the motor force, and is proportional to speed
- Use the same analogy:
  - force = current
  - speed = voltage
- (Viscous) friction = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to *voltage*
  - what kind of electrical element acts like this?

Resistance

Model of the friction force

- (Viscous) friction model = a **resistance in parallel** with the controlled voltage source
  - current proportional to voltage ⇔ a resistance
  - reduces the motor current ⇔ is in parallel with the controlled voltage source (steals some of its current)

Model of the sliding friction force

- There can also exist a sliding friction force = friction force which does not depend on speed, but is a constant
  - that's the friction force you likely encountered in high-school physics ("planul înclinat" etc.)
- Question: how is this force modeled in electrical domain?

Model of the sliding friction force

- Answer: a constant current source in parallel
  - constant current $\Leftrightarrow$ constant source
  - in parallel $\Leftrightarrow$ reduces the motor current

The full electrical model

$$H(s) = \frac{\text{Speed}(s)}{\text{Input Voltage}(s)}$$

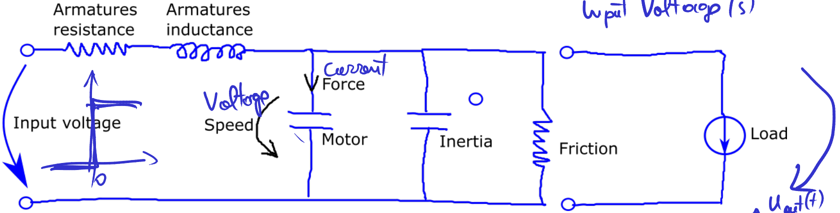


Figure 5: DC Motor electrical equivalent model

- This is **a second order model** (1L, 1C)
  - the two capacities are in parallel, so they can be added into a single one
- The L is the inductance of the armatures $\Rightarrow$ small, often negligible
- Can be approximated by a **first order model**

Transfer function of a DC motor

- We can derive a transfer function
  - input = voltage on motor input $U(s)$
  - output = motor speed $S(s)$ = voltage on equivalent motor capacity
- Transfer function ($2^n d$ degree, approximately 1^{st} degree)

$$
\begin{aligned}
H(s) = \frac{S(s)}{U(s)} &= \frac{R_{Fr}}{R_{Fr} + (R_{Arm} + sL_{Arm})(1 + sC_{M+I}R_{Fr})} \\
&= \frac{b_0}{s^2 + a_1 s + a_0} \\
&\approx \frac{K}{\tau \cdot s + 1}
\end{aligned}
$$

Transfer function of a DC motor

- Take home message:
  - Simple DC motor no-load model = a second order RLC model = approx a first-order RC model (ignoring L small)
  - Behaves like a RC low-pass filter
- Note: This is a no-load model (motor doesn't move anything heavy)
- What happens if motor has a load?
  - e.g. the motor drags/lifts a constant weight
  - i.e. like a crane lifting a big weight from the ground
- How to model the load?

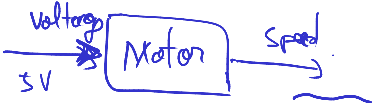
Motor under load

- How to model the load?
- Like a constant force/torque opposing the motor force/torque
  - i.e. like a sliding friction force
  - i.e. like a current source in parallel, stealing lots of current
- In practice, the load force/torque may not be constant
  - depends on mechanical properties
  - e.g. lifting the hatch/liftgate ("portbagaj") of a car: harder when lower, easier when higher

Simulink model

- Simulink has a DC motor model already integrated
- You will use it in the lab (maybe)

What to use the model for?



What to use the motor model for?

Simulate:

- how fast motor starts when supply is first applied
- what happens when supply fluctuates (e.g. PWM)
- what happens when motor parameters change (e.g. temperature rises, friction slows)
- what happens when load varies
- ...

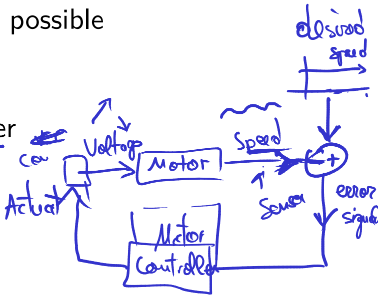
Motor speed controller

Basic problem: how to make sure motor speed stays **exactly** as desired:

- even if parameters vary
- even if load varies
- even if supply varies
- on power on, speed is reached as fast as possible

This is a job for a **motor controller**

- Today's special: the PID motor controller



Motor speed controller

This is a typical embedded system design problem:

- There is a physical process (the actual motor)
- We model it's behavior (use a motor model)
- We want to control it
- We design a controller system which steers the process as we want

Motor controllers

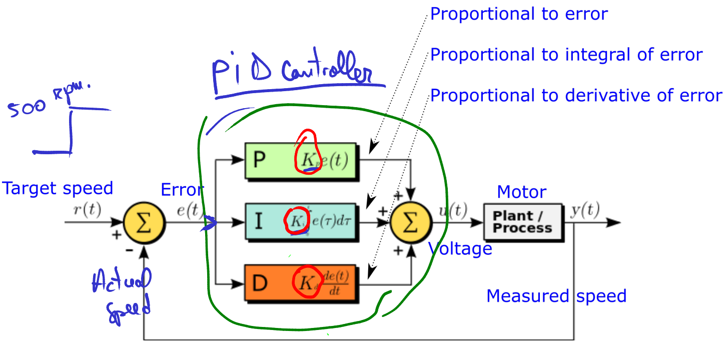


Figure 6: PID speed controller (image from Wikipedia)

- Negative feedback loop
- Can be used for any sort of process, not just motors
- Make output signal $y(t)$ follow the desired input $r(t)$

PID Controller

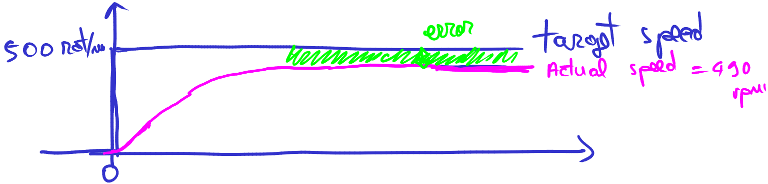
- PID controller = the simplest solution
- Input = error signal = target speed - actual measured speed
- Output = Sum of three components:
  - **P**roportional: P * input
  - **I**ntegral: I * integral of input
  - **D**erivative: D * derivative of input

PID Controller - P component

$$\boxed{V(t) = K_p \cdot e(t)}$$

- Intuitive role of the P component:
  - If actual speed < target => increase motor voltage
  - If actual speed > target => decrease motor voltage

- This is not enough:
  - Non-zero motor voltage requires non-zero speed error => the motor never actually reaches the target speed
  - There is always a small systematic error ("**bias error**", "steady-state error")



PID controller - only P, systematic error

steady-state error

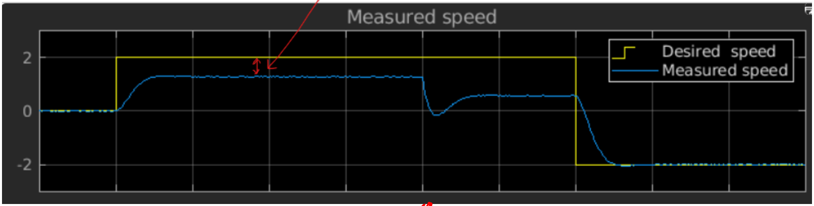
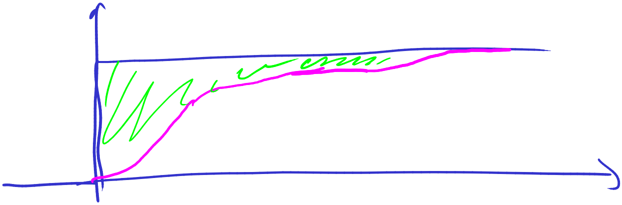


Figure 7: Systematic error for P-only controller

Add a load to the motor

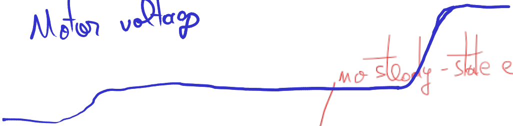
PID Controller - I component

- Intuitive role of the *I* component:
  - Eliminate the bias error of the *P* component, by slowly integrating the remaining error signal => integral slowly increases over time => motor voltage is pushed towards the correct value
  - Error signal cannot remain constant forever, because the integral would grow large => force changes to the motor input



PID controller - P and I

Motor voltage



no steady-state error !

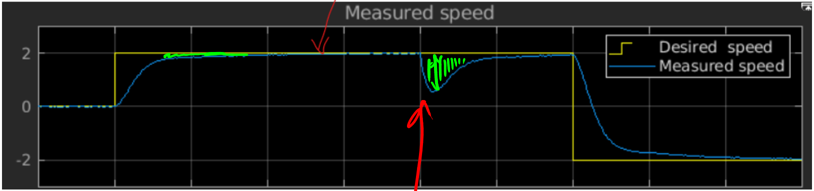


Figure 8: P and I components

I attach a weight to the motor

PID Controller - D component

- Intuitive role of the D component:
  - make the system react faster (jumpy) to fast input changes
  - improves system reaction time
- Problem:
  - fast reaction time = more oscillation behavior:
    - more overshoot
    - possibly unstable

PID controller - P, I and D

faster response,
but more overshoot / oscillations !

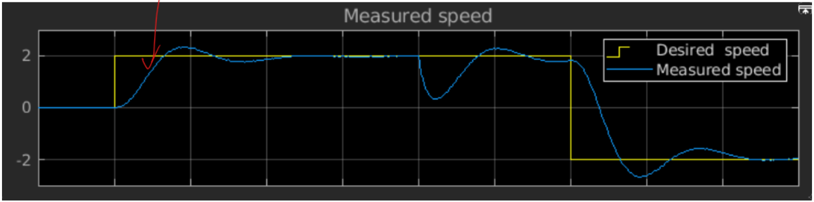


Figure 9: P, I and D components

PID tuning

- PID tuning: find P, I, D values for good behavior
  - Typical requirements:
    - stable system, overall
    - overshoot not larger than X%
    - fastest response in these conditions
- Find out more at the Vehicle Control Systems course (2nd semester, I think)