Embedded System Design and Modeling

II. Modeling continuous systems

Continuous systems

- Physical processes are usually in continuous domain
  - e.g. electrical, mechanical
- Processes in continuous domain are described by **differential equations**
  - i.e. with unknown functions + their derivatives + their second derivatives + ...
- Simple (ideal) case: differential equations are linear:
  - only operations allowed: sum, multiplication by a constant
- Every electrical/mechanical part defines a certain relation between the unknowns

Electrical systems

Electrical systems:

- Unknown functions = voltage + current in all branches
- Electrical (ideal) elements:
  - resistance: $u(i) = R \cdot i(t)$
  - capacitance: $i(t) = C \cdot \frac{d}{dt} u(t)$
  - etc.
- One big system of linear differential equations (SCS course, basically)
  - Kirchhoff equations $<=>$ equations between currents and voltages $<=>$ linear differential equation system
- Example: an RC system (solve at blackboard)

Mechanical systems

Mechanical systems:

- Unknown functions = coordinates x(t), y(t), z(t)
  - speeds = derivatives of the positions
  - acceleration = derivative of speed = second derivative of positions
  - (forces: $F = m \cdot a = m \cdot \frac{d^2}{dt^2} x(t)$)
- Mechanical (ideal) elements:
  - (Consider just a single dimension $x(t)$, is easier)
  - inertial force: $F = m \cdot a = m \cdot \frac{d^2}{dt^2} x(t)$
  - friction force:
    - sliding friction: $\vec{F_f} = -\mu \vec{N} = -\mu \cdot m \cdot \frac{d^2}{dt^2} x(t)$
    - viscous friction: $\vec{F_v} = -C_v \cdot \vec{v} = -C_v \cdot \frac{d}{dt} x(t)$
  - etc...

Mechanical systems

- Mechanical elements are described by linear differential equations, just like electrical ones
  - they are just idealizations, physical processes can be highly nonlinear (more complex)
  - but wait, so are electrical devices actually, and this hasn't stopped us...
- Example: oscillations after releasing of a loaded spring
  - (solve at blackboard)

Electrical - mechanical analogies

- Multiple ways to define analogies between electrical and mechanical characteristics
- Here is the one we will use:

Electr.	Mech. (linear)	Mech. (rotational)
Current [A]	Force [N]	Torque (“cuplu”) [N.m]
Voltage [V]	Speed [m/s]	Angular speed [rad/s]

Simple model of a DC motor

- Motor: gateway between the two electrical and mechanical domains
  - converts electric energy to mechanical energy, and vice-versa
- (Simple) model of a DC motor:

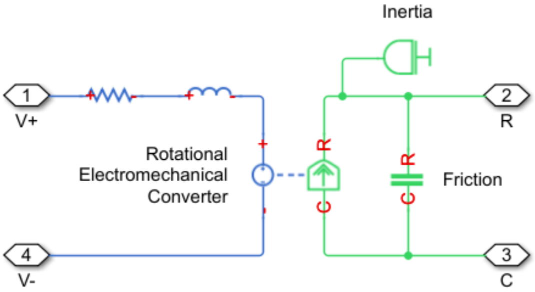

Figure 1: Simple model of a DC motor

Image from Mathworks Simulink (`ssc_dcmotor` example model)

DC motor model: electrical side

Electrical side of the DC motor model:

- Resistance: models the resistance of the windings

$$u(t) = R \cdot i(t)$$

- Inductance: models the inductive behavior of the windings

$$u(t) = L \cdot \frac{d}{dt} i(t)$$

- Controlled voltage source:
  - Voltage ("back electro-magnetic force voltage") is proportional to motor angular speed $S(t)$ on the mechanical side (think of a dynamo)

$$u(t) = K_e \cdot S(t)$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- Controlled force/torque source
  - Generates force/torque proportional to the current $i(t)$ on the electrical side

$$T = K_t \cdot i(t)$$

- Inertia: models the inertial force of the moving part of the motor
  - Generates force/torque proportional to acceleration (derivative of speed)

$$T_i = -m \cdot acceleration = -m \cdot \frac{d}{dt} S(t)$$

DC motor model: mechanical side

Mechanical circuit of the DC motor model (no load):

- Friction: models the (viscous) friction force of the moving part of the motor
  - Generates force/torque proportional to speed

$$T_f = -C_v \cdot S(t)$$

- Inertia and Friction forces/torques oppose the force/torque) of the motor, therefore they have minus sign

Laplace transform

- Both electrical and mechanical sides are described by linear differential equations
- The Laplace transform is a useful tool (remember SCS)
  - derivation = multiplication by s
  - integration = multiplication by $1/s$
  - transform function H(s) = output(s)/input(s)
- Exercise: write the equations of all electrical and mechanical elements in Laplace transform

Full electrical model

- All the mechanical elements can be modeled in the electrical domain
  - since they are all just differential equations, basically
  - obtain a full model in the electrical domain only
- Next slides: find electrical correspondent to all mechanical elements

Model of the controlled voltage source

- How to model the controlled voltage source?
- Like this:
  - voltage is proportional to speed: $U(s) = K_e \cdot S(s)$
  - speed = integral of acceleration: $S(s) = S_0 + 1/s \cdot A$
  - acceleration is proportional to force (force(torque) / mass) = $C_{const} \cdot T(s)$
  - force/torque = proportional to current: $T(s) = K_t \cdot I(s)$
- Result:

$$U(s) = K_e \cdot (S_0 + 1/s \cdot C_{const} \cdot K_t I(s))$$

Model of the controlled voltage source

$$U(s) = \underbrace{K_e \cdot S_0}_{Constant} + \underbrace{K_e C_{const}}_{Constant} \cdot \frac{1}{s} \cdot I(s)$$

- Voltage proportional on integral of current, plus a constant initial value
  - what kind of electrical element acts like this?
- The controlled voltage source can be modeled as a **capacitance**
  - Voltage is proportional to integral of current
  - (Current is proportional to derivative of voltage)
  - The first constant term = the initial voltage on the capacity
- The equivalent capacitance value depends on the motor parameters

Model of the inertial force

- Inertia = a force which opposes (i.e. reduces) the motor force, and is proportional to acceleration
- Use the analogy listed before:
  - force = current
  - speed = voltage
  - acceleration = derivative of speed = derivative of voltage
- Inertia = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to derivative of *voltage*
  - what kind of electrical element acts like this?

Model of the inertial force

- Inertia model = a **capacity in parallel** with the controlled voltage source
  - current proportional to derivative voltage $\Leftrightarrow$ a capacity
  - reduces the motor current $\Leftrightarrow$ is in parallel with the controlled voltage source (steals some of its current)

Model of the friction force

- (Viscous) friction = a force which opposes (i.e. reduces) the motor force, and is proportional to speed
- Use the same analogy:
  - force = current
  - speed = voltage
- (Viscous) friction = a *current* which opposes (i.e. reduces) the motor *current*, and is proportional to *voltage*
  - what kind of electrical element acts like this?

Model of the friction force

- (Viscous) friction model = a **resistance in parallel** with the controlled voltage source
  - current proportional to voltage $\Leftrightarrow$ a resistance
  - reduces the motor current $\Leftrightarrow$ is in parallel with the controlled voltage source (steals some of its current)

Model of the sliding friction force

- There can also exist a sliding friction force = friction force which does not depend on speed, but is a constant
  - that's the friction force you likely encountered in high-school physics ("planul înclinat" etc.)
- Question: how is this force modeled in electrical domain?

Model of the sliding friction force

- Answer: a constant current source in parallel
  - constant current $\Leftrightarrow$ constant source
  - in parallel $\Leftrightarrow$ reduces the motor current

The full electrical model

- Draw picture at blackboard: R in series with L in series with (R parallel with (C1 + C2))
- This is **a second order model** (1L, 1C)
  - the two capacities are in parallel, so they can be added into a single one
- The L is the inductance of the armatures $\Rightarrow$ small, often negligible
- Can be approximated by a **first order model**

Transfer function of a DC motor

- We can derive a transfer function
  - input = voltage on motor input $U(s)$
  - output = motor speed $S(s)$ = voltage on equivalent motor capacity
- Transfer function

$$H(s) = \frac{S(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_2} \approx \frac{\alpha}{s + \beta}$$

- Take home message: Simple DC motor no-load model = a second order RLC model = approx a RC model
- This is a no-load model (motor doesn't move anything heavy)

Motor under load

- What happens if motor has a load?
  - e.g. the motor drags/lifts a constant weight
  - i.e. like a crane lifting a big weight from the ground
- How to model the load?

Motor under load

- How to model the load?
- Like a constant force/torque opposing the motor force/torque
  - i.e. like a sliding friction force
  - i.e. like a current source in parallel, stealing lots of current
- In practice, the load force/torque may not be constant
  - depends on mechanical properties
  - e.g. lifting the hatch/liftgate ("portbagaj") of a car: harder when lower, easier when higher

Simulink model

- Simulink has a DC motor model already integrated
- You will use it in the lab