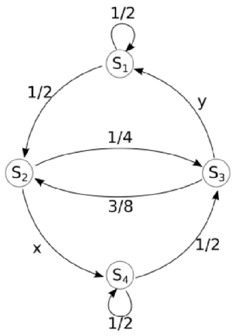
Sources With Memory

1).



S_1 : $s_1 s_1$

S_2 : $s_1 s_2$

S_3 : $s_2 s_1$

S_4 : $s_2 s_2$

a) $x = ? = \frac{3}{4}$ (sum of all output prob = 1)

$y = ? = \frac{5}{8}$

b) $$[T] = \begin{array}{c|cccc} & S_1 & S_2 & S_3 & S_4 \\ \hline S_1 & 1/2 & 1/2 & 0 & 0 \\ S_2 & 0 & 0 & 1/4 & 3/4 \\ S_3 & 5/8 & 3/8 & 0 & 0 \\ S_4 & 0 & 0 & 1/2 & 1/2 \end{array}$$

c). $p(s_1)$ if the current state is S_3?

$p(s_1 | S_3) = ?$

$$\underbrace{s_2 s_1 . s_1}_{S_3 \to S_1}$$

$p(s_1 | S_3) = p(\text{going to } S_1 \text{ from } S_3) = \frac{5}{8}$

d). Start at S_4

$p(s_1 s_2 s_2 s_1)$?

$$s_2 s_2 \; s_1 \; s_2 \; s_2 \; s_1$$
$$S_4 \to S_3 \to S_2 \to S_4 \to S_3$$
$$\frac{1}{2} \cdot \frac{3}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} \qquad = \frac{1}{2} \cdot \frac{3}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{9}{2^7} = \frac{9}{128}$$

Similar question:

if $S_4 \to S_3 \to S_2 \to S_3 \to S_1 \to S_1 \to S_1 \to S_2$

~~the~~ what are the messages generated?

$$-\log \frac{a}{b} = \log \frac{b}{a}$$
$$\log \frac{a}{b} = \log a - \log b$$

e). $H(S_4) = \underbrace{-0 \cdot \log 0}_{0} - \underbrace{0 \log 0}_{0} - \frac{1}{2} \cdot \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = 1$ bit

f). $H(S_1) = 1$ bit as in S_4

$H(S_2) = -\frac{1}{4} \log \frac{1}{4} - \frac{3}{4} \log \frac{3}{4} = \frac{1}{4} \cdot 2 + \frac{3}{4} (\underbrace{\log 4}_{2} - \log 3) = \frac{1}{2} + \frac{3}{2} - \frac{3}{4} \log 3 = 2 - \frac{3}{4} \log 3 = 0.81$ bit

$H(S_3) = -\frac{5}{8} \log \frac{5}{8} - \frac{3}{8} \log \frac{3}{8} = \frac{5}{8} (\underbrace{\log 8}_{3} - \log 5) + \frac{3}{8} (\underbrace{\log 8}_{3} - \log 3) = \frac{24 \; \cancel{15}}{8} - \frac{5}{8} \log 5 - \frac{3}{8} \log 3 = 0.95$ bit

System:

$$[p_1 \; p_2 \; p_3 \; p_4] \cdot \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} = [p_1 \; p_2 \; p_3 \; p_4]$$

$$\begin{cases} \frac{1}{2} p_1 + \frac{5}{8} p_3 = p_1 & (1) \\ \cancel{\frac{1}{2} p_1 + \frac{3}{8} p_3 = p_2} & \\ \frac{1}{4} p_2 + \frac{1}{2} p_4 = p_3 & (3) \\ \frac{3}{4} p_2 + \frac{1}{2} p_4 = p_4 & (2) \end{cases}$$

$p_1 + p_2 + p_3 + p_4 = 1$ (4)

(1) $\frac{5}{8} p_3 = \frac{1}{2} p_1 \Rightarrow \boxed{p_1 = \frac{5}{4} p_3}$

(3) $\frac{1}{4} p_2 + \frac{1}{2} \cdot \frac{3}{2} p_2 = p_3 \iff \boxed{p_2 = p_3}$

(2) $\frac{3}{4} p_2 = \frac{1}{2} p_4 \Rightarrow \boxed{p_4 = \frac{3}{2} p_2}$

(4) $\frac{5}{4} p_3 + p_3 + p_3 + \frac{3}{2} p_3 = 1 \Rightarrow \frac{19}{4} p_3 = 1 \Rightarrow \boxed{p_3 = \frac{4}{19} = p_2}$

$p_1 = \frac{5}{4} \cdot \frac{4}{19} = \frac{5}{19}$

$p_2 = p_3 = \frac{4}{19}$

$p_4 = \frac{3}{2} \cdot \frac{4}{19} = \frac{6}{19}$

$$H(S) = \sum_k P_k \cdot H(S_k) = \frac{5}{19} \cdot 1 + \frac{4}{19} \cdot 0.81 + \frac{4}{19} \cdot 0.95 + \frac{6}{19} \cdot 1 = \dots$$

g). $n = 2$ Number of states n^m $\underbrace{\_\ \_\ \_}_{m}$

$m = 2$

h).

$$\underbrace{\begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \hline 0 & 1 & 0 & 0 \end{array} \cdot \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}} \cdot \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} = \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \underbrace{\frac{1}{4} \cdot \frac{5}{8}}_{\frac{5}{32}} & \underbrace{\frac{1}{4} \cdot \frac{3}{8}}_{\frac{3}{32}} & \underbrace{\frac{3}{4} \cdot \frac{1}{2}}_{\frac{3}{8}} & \underbrace{\frac{3}{4} \cdot \frac{1}{2}}_{\frac{3}{8}} \end{array}$$