

Information Theory

Chapter I: Discrete information sources

Block diagram of a communication system

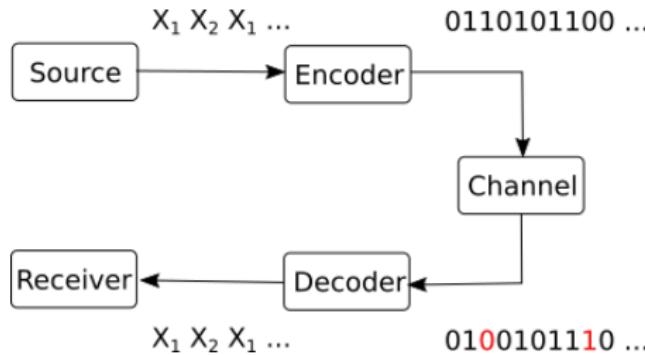


Figure 1: Block diagram of a communication system

- ▶ Source: creates information messages
- ▶ Encoder: converts messages into symbols for transmission (i.e bits)
- ▶ Channel: delivers the symbols, introduces errors
- ▶ Decoder: detects/corrects the errors, rebuilds the information messages

What is information?

Example:

- ▶ Consider the sentence: “your favorite football team lost the last match”
- ▶ Does this message carry information? How, why, how much?
- ▶ Consider the following facts:
 - ▶ the message carries information only when you don't already know the result
 - ▶ if you already known the result, the message is useless (brings no information)
 - ▶ if the result was to be expected, there is little information. If the result is highly unusual, there is more information in this message (think betting)

- ▶ We define the notion of **information** for a **probabilistic event**
 - ▶ the happening of a probabilistic event = creation of information
- ▶ Information brought by an event depends on the **probability** of the event
- ▶ Rule of thumb: if you can guess something most of the times, it has little information
- ▶ Questions:
 - ▶ does a sure event ($p = 1$) bring any information?
 - ▶ does an almost sure event (e.g. $P = 0.9999$) bring little or much information?
 - ▶ does a rare event (e.g. $P = 0.0001$) bring a little or much information?

- ▶ The information attached to a particular event (known as “message”)
 s_i is rigorously defined as:

$$i(s_i) = -\log_2(p(s_i))$$

- ▶ Properties:
 - ▶ $i(s_i) \geq 0$
 - ▶ lower probability (rare events) means higher information
 - ▶ higher probability (frequent events) means lower information
 - ▶ a certain event brings no information: $-\log(1) = 0$
 - ▶ an event with probability 0 brings infinite information (but it never happens...)
 - ▶ for two independent events, their information gets added

$$i(p(s_i) \cdot p(s_j)) = i(s_i) + i(s_j)$$

The choice of logarithm

- ▶ Any base of logarithm can be used in the definition.
- ▶ Usual convention: use binary logarithm $\log_2()$
- ▶ In this case, the information $i(s_i)$ is measured in **bits**
- ▶ If using natural logarithm $\ln()$, it is measured in *nats*.
- ▶ Logarithm bases can be converted to/from one another:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

- ▶ Information defined using different logarithms differ only in scaling:

$$i_b(s_i) = \frac{i_a(s_i)}{\log_a(b)}$$

- ▶ A probabilistic event is always part of a set of multiple events (options)
 - ▶ e.g: a football team can win/lose/draw a match (3 possible events)
 - ▶ each event has a certain probability. All probabilities are known beforehand
 - ▶ at a given time, only one of the events can happen
- ▶ An **information source** = the set of all events together with their probabilities
- ▶ One event is called a **message**
- ▶ Each message carries the information that **it** happened, the quantity of information is dependent on its probability

Sequence of messages

- ▶ An information source creates a **sequence of messages**
 - ▶ e.g. like throwing a coin or a dice several times in a row
- ▶ The probabilities of the messages are known and fixed
- ▶ Each time, a new message is randomly selected according to the probabilities

Discrete memoryless source

- ▶ A **discrete memoryless source** (DMS) is an information source which produces a sequence of **independent** messages
 - ▶ i.e. the choice of a message at one time does not depend on the previous messages
- ▶ Each message has a fixed probability. The set of probabilities is the **distribution** of the source:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Discrete memoryless source

$$S : \begin{pmatrix} s_1 & s_2 & s_3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

- ▶ Terminology:
 - ▶ Discrete: it can take a value from a discrete set ("alphabet")
 - ▶ Complete: $\sum p(s_i) = 1$
 - ▶ Memoryless: successive values are independent of previous values
(e.g. successive throws of a coin)
- ▶ A message from a DMS is also called a **random variable** in probabilistics.

Examples

- ▶ A coin is a discrete memoryless source (DMS) with two messages:

$$S : \begin{pmatrix} \text{heads} & \text{tails} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- ▶ A dice is a discrete memoryless source (DMS) with six messages:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

- ▶ Playing the lottery can be modeled as DMS:

$$S : \begin{pmatrix} s_1 & s_2 \\ 0.9999 & 0.0001 \end{pmatrix}$$

Examples

- ▶ An extreme type of DMS containing the certain event:

$$S : \begin{pmatrix} s_1 & s_2 \\ 1 & 0 \end{pmatrix}$$

- ▶ Receiving an unknown *bit* (0 or 1) with equal probabilities:

$$S : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Sequence of messages from DMS

- ▶ A DMS produces a sequence of messages by randomly selecting a message every time, with the same fixed probabilities
 - ▶ throwing a dice several times in a row you can get a sequence
4, 2, 3, 2, 1, 6, 1, 5, 4, 5
- ▶ If the sequence is very long (has N messages, N very large), each message s_i appears approximately $p(s_i) * N$ times in the sequence
 - ▶ gets more precise as $N \rightarrow \infty$

Entropy of a DMS

- ▶ We usually don't care about a single message. We are interested in long sequences of messages (think millions of bits of data)
- ▶ We are interested in the *average* information of a message from a DMS
- ▶ Definition: the **entropy** of a DMS source S is **the average information of a message**:

$$H(S) = \sum_k p(s_k) i(s_k) = - \sum_k p(s_k) \log_2(p_k)$$

where $p(s_k)$ is the probability of message k

Entropy of a DMS

- ▶ Since information of a message is measured in bits, entropy is measured in **bits** (or **bits / message**, to indicate it is an average value)
- ▶ Entropies using information defined with different logarithms differ only in scaling:

$$H_b(S) = \frac{H_a(S)}{\log_a(b)}$$

Examples

- ▶ Coin: $H(S) = 1\text{bit}/\text{message}$
- ▶ Dice: $H(S) = \log(6)\text{bits}/\text{message}$
- ▶ Lottery: $H(S) = -0.9999 \log(0.9999) - 0.0001 \log(0.0001)$
- ▶ Receiving 1 bit: $H(S) = 1\text{bit}/\text{message}$ (hence the name!)

Interpretation of the entropy

All the following interpretations of entropy are true:

- ▶ $H(S)$ is the *average uncertainty* of the source S
- ▶ $H(S)$ is the *average information* of the messages from source S
- ▶ A long sequence of N messages from S has total information
 $\approx N \cdot H(S)$
- ▶ $H(S)$ is the minimum number of bits (0,1) required to uniquely represent an average message from source S

Properties of entropy

We prove the following **properties of entropy**:

1. $H(S) \geq 0$ (non-negative)

Proof: via definition

2. $H(S)$ is maximum when all n messages have equal probability $\frac{1}{n}$. The maximum value is $\max H(S) = \log(n)$

Proof: only for the case of 2 messages, use derivative in definition

3. *Diversification* of the source always increases the entropy

Proof: compare entropies in both cases

The entropy of a binary source

- ▶ Consider a general DMS with two messages:

$$S : \begin{pmatrix} s_1 & s_2 \\ p & 1 - p \end{pmatrix}$$

- ▶ It's entropy is:

$$H(S) = -p \cdot \log(p) - (1 - p) \cdot \log(1 - p)$$

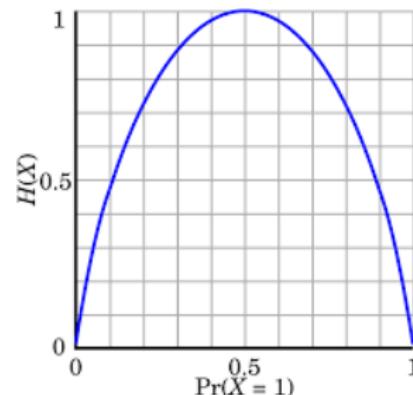


Figure 2: Entropy of a binary source

Example - Game

Game: I think of a number between 1 and 8. You have to guess it by asking yes/no questions.

- ▶ How much uncertainty does the problem have?
- ▶ How is the best way to ask questions? Why?
- ▶ What if the questions are not asked in the best way?
- ▶ On average, what is the number of questions required to find the number?

Example - Game v2

- ▶ Suppose I choose a number according to the following distribution:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

- ▶ On average, what is the number of questions required to find the number?
- ▶ What questions would you ask?
- ▶ What if the distribution is:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.14 & 0.29 & 0.4 & 0.17 \end{pmatrix}$$

- ▶ In general:
 - ▶ What distribution makes guessing the number the most difficult?
 - ▶ What distribution makes guessing the number the easiest?

Efficiency and redundancy

- ▶ Efficiency of a DMS:

$$\eta = \frac{H(S)}{H_{max}} = \frac{H(S)}{\log(n)}$$

- ▶ Absolute redundancy of a DMS:

$$R = H_{max} - H(S)$$

- ▶ Relative redundancy of a DMS:

$$\rho = \frac{H_{max} - H(S)}{H_{max}} = 1 - \eta$$

Information flow of a DMS

- ▶ Suppose that message s_i takes time t_i to be transmitted via some channel.
- ▶ Definition: the **information flow** of a DMS S is the average information transmitted per unit of time:

$$H_\tau(S) = \frac{H(S)}{\bar{t}}$$

where \bar{t} is the average duration of transmitting a message:

$$\bar{t} = \sum_i p_i t_i$$

- ▶ Measured in **bps** (bits per second)
- ▶ Important for data communication

Distance between distributions

- ▶ How to measure how similar / how different are two distributions?
 - ▶ must have the same number of messages
 - ▶ example: $p(s_1), \dots, p(s_n)$ and $q(s_1), \dots, q(s_n)$
- ▶ **Definition:** the **Kullback–Leibler distance** of two distributions P and Q is

$$D_{KL}(P||Q) = \sum_i p(s_i) \log\left(\frac{p(s_i)}{q(s_i)}\right)$$

- ▶ It is a way to measure the **distance (difference)** between two distributions
- ▶ Also known as *relative entropy*, or the Kullback-Leibler *divergence*

Properties of Kullback-Leibler distance

- ▶ Properties:
 - ▶ $D_{KL}(P||Q)$ is always ≥ 0 , and is equal to 0 only when P and Q are the same
 - ▶ the higher $D_{KL}(P||Q)$ is, the more different the distributions are
 - ▶ it is **not commutative**: $D_{KL}(P||Q) \neq D_{KL}(Q||P)$
- ▶ Example: at whiteboard
- ▶ Example usage: classification systems (cross-entropy loss)

- ▶ Definition: the **n-th order extension** of a DMS S , S^n is a source which has as messages all the combinations of n messages of S :

$$\sigma_i = \underbrace{s_j s_k \dots s_l}_n$$

- ▶ If S has k messages, S^n has k^n messages
- ▶ Since S is DMS, probabilities multiply:

$$p(\sigma_i) = p(s_j) \cdot p(s_k) \cdot \dots \cdot p(s_l)$$

Extended DMS - Example

► Examples:

$$S : \begin{pmatrix} s_1 & s_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$S^2 : \begin{pmatrix} \sigma_1 = s_1s_1 & \sigma_2 = s_1s_2 & \sigma_3 = s_2s_1 & \sigma_4 = s_2s_2 \\ \frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{9}{16} \end{pmatrix}$$

$$S^3 : \begin{pmatrix} s_1s_1s_1 & s_1s_1s_2 & s_1s_2s_1 & s_1s_2s_2 & s_2s_1s_1 & s_2s_1s_2 & s_2s_2s_1 & s_2s_2s_2 \\ \dots & \dots \end{pmatrix}$$

- ▶ Long sequence of binary messages:

010011001110010100...

- ▶ Can be grouped in bits, half-bytes, bytes, 16-bit words, 32-bit long words, and so on
- ▶ Can be considered:
 - ▶ N messages from a binary source (with 1 bit), or
 - ▶ $N/2$ messages from a source with 4 messages (with 2 bits)...
 - ▶ etc

- ▶ Theorem: The entropy of a n -th order extension is n times larger than the entropy of the original DMS

$$H(S^n) = nH(S)$$

- ▶ Interpretation: grouping messages from a long sequence in blocks of n does not change total information (e.g. groups of 8 bits = 1 byte)

An example [memoryless is not enough]

- ▶ The distribution (frequencies) of letters in English:

letter	probability	letter	probability
A	.082	N	.067
B	.015	O	.075
C	.028	P	.019
D	.043	Q	.001
E	.127	R	.060
F	.022	S	.063
G	.020	T	.091
H	.061	U	.028
I	.070	V	.010
J	.002	W	.023
K	.008	X	.001
L	.040	Y	.020
M	.024	Z	.001

- ▶ Text from a memoryless source with these probabilities:

OCRO HLI RGWR NMIELWIS EU LL NBNSEBYA TH EEI
ALHENHTPA OOBTTVA NAH BRL

(taken from *Elements of Information Theory*, Cover, Thomas)

- ▶ What's wrong? **Memoryless**

Sources with memory

- ▶ **Definition:** A source has **memory of order m** if the probability of a message depends on the last m messages.
- ▶ The last m messages = the **state** of the source (notation S_i).
- ▶ A source with n messages and memory $m \Rightarrow$ has n^m states in all.
- ▶ For every state, messages can have a different set of probabilities.
Notation: $p(s_i|S_k)$ = “*probability of s_i in state S_k* ”.
- ▶ Also known as *Markov sources*.

Example

- ▶ A source with $n = 4$ messages and memory $m = 1$

- ▶ if last message was s_1 , choose next message with distribution

$$S_1 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

- ▶ if last message was s_2 , choose next message with distribution

$$S_2 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.33 & 0.37 & 0.15 & 0.15 \end{pmatrix}$$

- ▶ if last message was s_3 , choose next message with distribution

$$S_3 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.2 & 0.35 & 0.41 & 0.04 \end{pmatrix}$$

- ▶ if last message was s_4 , choose next message with distribution

$$S_4 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$$

Transitions

- ▶ When a new message is provided, the source **transitions** to a new state:

... $\underbrace{s_i s_j s_k}_\text{old state} s_l$

... $s_i \underbrace{s_j s_k s_l}_\text{new state}$

- ▶ The message probabilities = the probabilities of transitions from some state S_u to another state S_v

Transition matrix

- ▶ The transition probabilities are organized in a **transition matrix** $[T]$

$$[T] = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \dots & \dots & \dots & \dots \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{bmatrix}$$

- ▶ p_{ij} is the transition probability from state S_i to state S_j
- ▶ N is the total number of states

Graphical representation

At whiteboard: draw states and transitions for previous example (source with $n = 4$ messages and memory $m = 1$)

Entropy of sources with memory

- ▶ What entropy does a source with memory have?
- ▶ Each state S_k has a different distribution \rightarrow each state has a different entropy $H(S_k)$

$$H(S_k) = - \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$

- ▶ Global entropy = average entropy

$$H(S) = \sum_k p_k H(S_k)$$

where p_k = probability that the source is in state S_k

- ▶ (i.e. after a very long sequence of messages, the fraction of time when the source was in state S_k)

- ▶ How to find out the weights p_k ?
- ▶ They are known as the **stationary probabilities**
- ▶ p_k = probability that the source is in state S_i , after running for a very long time
 - ▶ (i.e. after a very long sequence of messages, the fraction of time when the source was in state S_k)
- ▶ We need to answer the following question:

If we know the state S_k at time n , what will be the state at time $n + 1$?

Ergodic sources

- ▶ Let $p_i^{(n)}$ = the probability that source S is in state S_i at time n .
- ▶ In what state will it be at time $n + 1$? (after one more message)
 - ▶ i.e. what are the probabilities of the states at time $n + 1$?
- ▶ Just multiply with T

$$[p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}] \cdot [T] = [p_1^{(n+1)}, p_2^{(n+1)}, \dots, p_N^{(n+1)}]$$

- ▶ After one more message:

$$[p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}] \cdot [T] \cdot [T] = [p_1^{(n+2)}, p_2^{(n+2)}, \dots, p_N^{(n+2)}]$$

- ▶ For every new moment of time, one more multiplication with T

- ▶ In general, starting from time 0, after n messages the probabilities that the source is in a certain state are:

$$[p_1^{(0)}, p_2^{(0)}, \dots, p_N^{(0)}] \cdot [T]^n = [p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}]$$

- ▶ A source is called **ergodic** if every state can be reached from every state, in a finite number of steps.
- ▶ Property of ergodic sources:
 - ▶ After many messages, the probabilities of the states *become stationary* (converge to some fixed values), irrespective of the initial probabilities (no matter what state the source started from initially)

$$\lim_{n \rightarrow \infty} [p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}] = [p_1, p_2, \dots, p_N]$$

Finding the stationary probabilities

- ▶ How to find the value of the stationary probabilities?
- ▶ When n is very large, after n messages and after $n + 1$ messages the probabilities are the same:

$$[p_1, p_2, \dots, p_N] \cdot [T] = [p_1, p_2, \dots, p_N]$$

- ▶ This is an equation system in matrix form
- ▶ One line should be removed (linear combination), and replaced with:

$$p_1 + p_2 + \dots + p_N = 1$$

- ▶ Solve the resulting system of equations, find values of p_k

Entropy of ergodic sources with memory

- ▶ The entropy of an ergodic source with memory is

$$H(S) = \sum_k p_k H(S_k) = - \sum_k p_k \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$

Exercise

1. Consider a discrete source with memory, with the graphical representation given below. The states are defined as follows:
 $S_1 : s_1s_1, S_2 : s_1s_2, S_3 : s_2s_1, S_4 : s_2s_2.$

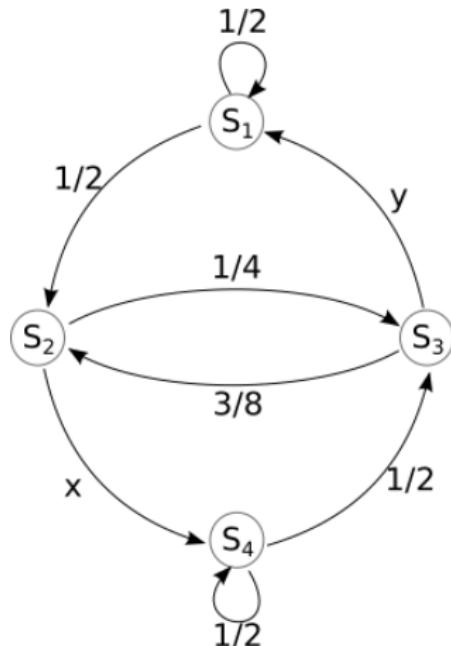


Figure 3: Graphical representation of the source

Exercise (continued)

Questions:

- a. What are the values of x and y ?
- b. Write the transition matrix $[T]$;
- c. Compute the entropy in state S_4 ;
- d. Compute the global entropy of the source;
- e. What are the memory order, m , and the number of messages of the source, n ?
- f. If the source is initially in state S_2 , in what states and with what probabilities will the source be after 2 messages?

Example English text as sources with memory

(taken from *Elements of Information Theory*, Cover, Thomas)

- ▶ Memoryless source, equal probabilities:

XFOML RXKHRJFFUJ ZLPWCFWKCYJ
FFJEYVKCQSGXYD QPAAMKBZAACIBZLHJQD

- ▶ Memoryless source, probabilities of each letter as in English:

OCRO HLI RGWR NMIELWIS EU LL NBNSEBYA TH EEI
ALHENHTTPA OOBTTVA NAH BRL

- ▶ Source with memory $m = 1$, frequency of pairs as in English:

ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY
ACHIN D ILONASIVE TUOOWE AT TEASONARE FUSO
TIZIN ANDY TOBE SEACE CTISBE

Example English text as sources with memory

- ▶ Source with memory $m = 2$, frequency of triplets as in English:

```
IN NO IST LAT WHEY CRATICT FROURE BERS GROCID  
PONDENOME OF DEMONSTURES OF THE REPTAGIN IS  
REGOACTIONA OF CRE
```

- ▶ Source with memory $m = 3$, frequency of 4-plets as in English:

```
THEGENERATEDJOBPROVIDUALBETTERTRANDTHEDISPLAYED  
CODE,ABOVERYUPOUNDLTSWELLTHECODERSTINTHESTICAL  
ITDOHOCKBOTHEMERG.(INSTATESCONSERTATION.NEVER  
ANYOPUBLEANDTOTHEORY.EVENTIALCALLEGANDTOELAST  
BENERATEDINWITHPIESASISWITHTHE)
```

Example application

- ▶ Suppose we receive a text with random missing letters
- ▶ We need to fill the blanks with the appropriate letters
- ▶ How?
 - ▶ build a model: source with memory of some order
 - ▶ fill the missing letter with the most likely letter given by the model

Chapter summary

- ▶ Information of a message: $i(s_k) = -\log_2(p(s_k))$
- ▶ Entropy of a memoryless source:
$$H(S) = \sum_k p_k i(s_k) = -\sum_k p_k \log_2(p_k)$$
- ▶ Properties of entropy:
 1. $H(S) \geq 0$
 2. Is maximum when all messages have equal probability
($H_{max}(S) = \log(n)$)
 3. *Diversification* of the source always increases the entropy
- ▶ Sources with memory: definition, transitions
- ▶ Stationary probabilities of ergodic sources with memory:
$$[p_1, p_2, \dots, p_N] \cdot [T] = [p_1, p_2, \dots, p_N], \sum_i p_i = 1.$$
- ▶ Entropy of sources with memory:

$$H(S) = \sum_k p_k H(S_k) = -\sum_k p_k \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$