Information Theory

Chapter III: Error control coding

Chapter structure

What is error control coding?

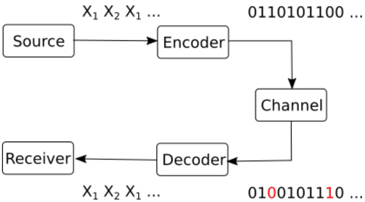

Figure 1: Communication system

- The second main task of coding: error control
- Protect information against channel errors

The need for error control coding

- In a transmission, the bits go through a **transmission channel**
  - The transmission channel is not ideal, it introduces some bit errors
  - Usually it is required that *all* bits are received correctly, no errors are allowed
- So what to do? **Error control coding**

Modelling the errors on the channel

- We consider only binary codes/ channels (symbols $= \{0, 1\}$)
- An **error** $=$ a bit that has changed from 0 to 1 or vice versa while going through channel
- Errors can appear:
  - **independently**: sporadic errors, each bit has a random chance of error, independent of all the others
  - in **packets of errors**: groups of consecutive errors

Modelling the errors on the channel

- Changing the value of a bit = modulo-2 sum with 1
- Value of a bit remains the same = modulo-2 sum with 0

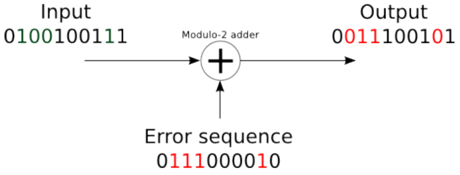

Figure 2: Channel error model

- Channel model we use (simple):
  - The transmitted sequence is summed modulo-2 with an **error sequence**

Modelling the errors on the channel

- Channel model we use (simple):
  - The transmitted sequence is summed modulo-2 with an **error sequence**
  - Error sequence has same length as the transmitted sequence
  - Where the error sequence is 1, there is a bit error
  - Where the error sequence is 0, there is no error

$$\mathbf{r} = \mathbf{c} \oplus \mathbf{e}$$

Mathematical properties of modulo-2 arithmetic

- Product is the same as for normal arithmetic
- Multiplication is distributive just like in normal case

$$a(b \oplus c) = ab \oplus ac$$

- Subtraction = addition. There is no negation. Each number is its own negative

$$a \oplus a = 0$$

Error detection vs correction

What can we do about errors?

- **Error detection**: find out if there is any error in the received sequence
  - don't know exactly where, so cannot correct the bits, but can discard whole sequence
  - perhaps ask the sender to retransmit (examples: TCP/IP, internet communication etc)
  - easier to do
- **Error correction**: find out exactly which bits have errors, if any
  - locating the error = correcting error (for binary channels)
  - can correct all errored bits by inverting them
  - useful when can't retransmit (data is stored: on HDD, AudioCD etc.)
  - harder to do than mere detection

Overview of error control coding process

The process of error control:

1. Want to send a sequence of k bits = **information word**

$$\mathbf{i} = i_1 i_2 \ldots i_k$$

2. For each possible information word, the coder assigns a **codeword** of length $n > k$:

$$\mathbf{c} = c_1 c_2 \ldots c_n$$

3. The codeword is sent on the channel instead of the original information word

4. The receiver receives a sequence $\hat{\mathbf{c}} \approx \mathbf{c}$, with possible errors:

$$\hat{\mathbf{c}} = \hat{c}_1 \hat{c}_2 \ldots \hat{c}_n$$

5. The decoding algorithm detects/corrects the errors in $\hat{\mathbf{c}}$

Definitions

- An **error correcting code** is an association between the set of all possible information words to a set of codewords
  - Each possible information word $\mathbf{i}$ has a certain codeword $\mathbf{c}$
- The association can be done:
  - randomly: codewords are selected and associated randomly to the information words
  - based on a certain rule: the codeword is computed with some algorithm from the information word
- A code is a **block code** if it operates with words of *fixed size*
  - Size of information word $\mathbf{i} = k$, size of codeword $\mathbf{c} = n$, $n > k$
  - Otherwise it is a *non-block code*
- A code is **linear** if any linear combination of codewords is also a codeword

Definitions

- A code is called **systematic** if the codeword contains all the information bits explicitly, unaltered
  - coding merely adds supplementary bits besides the information bits
  - codeword has two parts: the information bits and the parity bits
  - example: parity bit added after the information bits
- Otherwise the code is called **non-systematic**
  - the information bits are not explicitly visible in the codeword
- The **coding rate** of a code is:

$$R = k/n$$

Definitions

- A code C is an t-**error-detecting** code if it is able to **detect** t or less errors
- A code C is an t-**error-correcting** code if it is able to **correct** t or less errors
- Examples: at blackboard

A first example: parity bit

- Add parity bit to a 8-bit long information word, before sending on a channel
  - coding rate $R = 8/9$
  - can detect 1 error in a 9-bit codeword
  - detection algorithm: check if parity bit matches data
  - fails for 2 errors
  - cannot correct error (don't know where it is located)
- Add more parity bits to be able to locate the error
  - Example at blackboard
  - coding rate $R = 8/12$
  - can detect and correct 1 error in a 9-bit codeword

A second example: repetition code

- Repeat same block of data n times
  - want to send a k-bit information word
  - codeword to send = the information word repeated $n = 5$ times
  - coding rate $R = k/n = 1/5$
  - can detect and correct 2 errors, and maybe even more if they do not affect the same bit
  - error correcting algorithm = majority rule
  - not very efficient

Redundancy

- Merriam-Webster: **“redundant”** definition:
  a. *exceeding what is necessary or normal : superfluous*
  b. *characterized by or containing an excess; specifically : using more words than necessary*
- Because $k < n$, error control coding introduces **redundancy**
  - to transmit k bits of information we actually send more bits (n)

Redundancy

- Error control coding adds redundancy, while source coding aims to reduce redundancy. Contradiction?
- No:
  - Source coding reduces existing redundancy from the data, which served no purpose
  - Error control coding adds redundancy **in a controlled way**, with a purpose
- Source coding and error control coding in practice: do sequentially, independently
  1. First perform source coding, eliminating redundancy in representation of data
  2. Then perform error control coding, adding redundancy for protection

Transmission channels preview

- In Chapter IV we will study Transmission Channels = mathematical model of how information is handled from the sender to the receiver
- Each channel has a certain **capacity** value = the maximum amount of information that can be sent over the channel
  - e.g. a channel may have capacity $C = 0.8$ bits
- More about this in Chapter IV

Shannon's noisy channel theorem (second theorem, channel coding theorem)

- A coding rate is called **achievable** for a channel if, for that rate, there exists a coding and decoding algorithm guaranteed to correct all possible errors on the channel

Shannon's noisy channel coding theorem (second theorem)

For a given channel, all rates below capacity $R < C$ are achievable. All rates above capacity, $R > C$, are not achievable.

Channel coding theorem explained

In layman terms:

- For all coding rates $R < C$, **there is a way** to recover the transmitted data perfectly (decoding algorithm will detect and correct all errors)
- For all coding rates $R > C$, **there is no way** to recover the transmitted data perfectly

Channel coding theorem example

- We send bits on a channel with capacity 0.7 bits/message
- For any coding rate $R < 0.7$ there exists an error correction code that allows fixing of all errors
  - $R < 0.7$ means we send more than 10 bits for every 7 information bits, on average
- With less than 10 bits for every 7 information bits $=>$ no code exists that can fix all errors
- The theorem makes it clear when it is possible to fix all errors, and guarantees that a code exists in this case

Ideas behind channel coding theorem

- The rigorous proof of the theorem is too complex to present
- Key ideas of the proof:
  - Use very long information words, $k \to \infty$
  - Use random codes, compute the probability of having error after decoding
  - If $R < C$, *in average for all possible codes*, the probability of error after decoding goes to 0
  - If the average for all codes goes to 0, there exists at least on code better than the average
  - That is the code we should use

Ideas behind channel coding theorem

- **The theorem does not tell what code to use**, only that some code exists
  - There is no clue of how to actually find the code in practice
  - Only some general principles:
    - using longer information words is better
    - random codewords are generally good
- In practice, we cannot use infinitely long codewords, so we will only get a *good enough* code

Chapter structure

Chapter structure

1. General presentation
2. **Analyzing linear block codes with the Hamming distance**
3. Analyzing linear block codes with matrix algebra
4. Hamming codes
5. Cyclic codes

Distance between codewords

Practical ideas for error correcting codes:

- If a codeword $\mathbf{c_1}$ has errors and thus becomes identical to another codeword $\mathbf{c_2}$ ==> cannot detect any errors
  - Receiver will think it received a correct codeword $\mathbf{c_2}$, but actually it was $\mathbf{c_1}$
- We want codewords **as different as possible** from each other
- How to measure this difference? **Hamming distance**

Hamming distance

- The **Hamming distance** of two binary sequences **a**, **b** of length $n =$ the total number of bit differences between them

$$d_H(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} a_i \oplus b_i$$

- We need at least $d_H(a, b)$ bit changes to convert one sequence into another
- Example at blackboard

Hamming distance

- It satisfies the 3 properties of a metric function:
  1. $d_H(\mathbf{a}, \mathbf{b}) \geq 0 \quad \forall \mathbf{a}, \mathbf{b}$, with $d_H(\mathbf{a}, \mathbf{b}) = 0 \Leftrightarrow \mathbf{a} = \mathbf{b}$
  2. $d_H(\mathbf{a}, \mathbf{b}) = d_H(\mathbf{b}, \mathbf{a}), \forall \mathbf{a}, \mathbf{b}$
  3. $d_H(\mathbf{a}, \mathbf{c}) \leq d_H(\mathbf{a}, \mathbf{b}) + d_H(\mathbf{b}, \mathbf{c}), \forall \mathbf{a}, \mathbf{b}, \mathbf{c}$

- The **minimum Hamming distance of a code**, $d_{H\,min}$ = the minimum Hamming distance between any two codewords $\mathbf{c_1}$ and $\mathbf{c_2}$

Nearest-neighbor decoding scheme

Coding:

- Design a code with large $d_{H\,min}$
- Send a codeword $\mathbf{c}$ of the code

Decoding:

- Receive a word $\mathbf{r}$, that may have errors
- Error detecting:
  - check if r is part of the codewords of the code C:
  - if r is part of the code, decide that there have been no errors
  - if r is not a codeword, decide that there have been errors
- Error correcting:
  - if $\mathbf{r}$ is a codeword, decide there are no errors
  - else, choose codeword **nearest** to the received $\mathbf{r}$, in terms of Hamming distance
  - this is known as **nearest-neighbor decoding**

Performance of nearest neighbor decoding

Theorem:

- If the minimum Hamming distance of a code is d_{Hmin}, then:
  1. the code can *detect* up to $d_{Hmin} - 1$ errors
  2. the code can *correct* up to $\left\lfloor \frac{d_{Hmin}-1}{2} \right\rfloor$ errors using nearest-neighbor decoding

Consequence:

- It is good to have d_{Hmin} as large as possible
  - This implies longer codewords, i.e. smaller coding rate, i.e. more redundancy

Performance of nearest neighbor decoding

Proof:

1. at least d_{Hmin} binary changes are needed to change one codeword into another, $d_{Hmin} - 1$ is not enough $=>$ the errors are detected
2. the received word $\mathbf{r}$ is closer to the original codeword than to any other codeword $=>$ nearest-neighbor algorithm will find the correct one
    - because $\left\lfloor \frac{d_{Hmin}-1}{2} \right\rfloor$ = less than half the distance to another codeword

Note: if the number of errors is higher, can fail:

- Detection failure: decide that there were no errors, even if they were (more than $d_{Hmin} - 1$)
- Correction failure: choose a wrong codeword

Example: blackboard

Computational complexity

- **Computational complexity** = the amount of computational resources required by an algorithm
  - only refers to the **order of magnitude of the dominant term**
    - neglects the other terms
    - neglects actual coefficient values in front
- Computational complexity with respect to number of information bits k, of the search-based nearest neighbor decoding (as presented earlier), is

$$\mathcal{O}(2^k)$$

- Proof: Requires comparing with all codewords, and there are 2^k codewords in total

Computational complexity

- This implementation is **very inefficient**
  - k doubles => the amount of computations is squared
  - k increases 10 times => computations are raised to a power of 10
  - k increases 100 times => computations are raised to a power of 1000
  - for $k = 256$ you'd need all the energy of the Sun
- Need to find ways to make it simpler

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3. **Analyzing linear block codes with matrix algebra**
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Review of basic algebra

Informal definitions:

- **Vector space** = a set such that:
  a. one element + another element = still an element from the set
  b. one element × a constant = still an element from the set
  - Examples: Euclidian vector spaces: a line, points in 2D, 3D
  - Elements are called "vectors"
- **Basis** = a set of n independent vectors $\mathbf{e_1}, \ldots \mathbf{e_n}$
  - Any vector $\mathbf{v}$ can be expressed as a linear combination of the basis elements

$$\mathbf{v} = \mathbf{e_1} \cdot \alpha_1 + \ldots + \mathbf{e_n} \cdot \alpha_n$$

Review of basic algebra

- **Subspace** = a smaller dimensional vector space inside a larger vector space
  - Examples: a line in a plane
    - sum of two vectors on a line = still on the line
    - size of subspace = 1
    - size of larger space = 2
  - A plane in 3D space
    - sum of two vectors from the plane = still on the plane
    - size of subspace = 2
    - size of larger space = 3

Binary sequences form a vector space

- The set of all binary sequences of size n is a vector space of size n
  - sum of two sequences of size n is still a sequence of size n
- The sum operation = modulo-2 sum $\oplus$
- Multiplication with 0 and 1 = as in usual arithmetic

How to look at matrix-vector multiplications

- Matrix-vector multiplication
  - Output vector = linear combination of the matrix columns
- Vector-matrix multiplication
  - Output vector = linear combination of the matrix rows
- Explain at the blackboard, draw picture

How to look at matrix-vector multiplications

- Vector spaces can be perfectly described with matrix-vector multiplications
  - Matrix columns/rows = elements of the basis
  - The output vector = the vector
  - The multiplicated vector = the coefficients of the linear combination
- Any vector $\mathbf{v}$ can be expressed as a linear combination of the basis elements

$$\mathbf{v} = \mathbf{e_1} \cdot \alpha_1 + ... + \mathbf{e_n} \cdot \alpha_n$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & ... & \mathbf{e_n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_n \end{bmatrix}$$

- $\mathbf{e_1}, ...\mathbf{e_n}$ are **column vectors**
- Equation can be transposed => all vectors become row vectors

Codewords form a vector space

- The set of all binary codewords of a linear block code is a vector subspace of dimension k
- Proof:
  - code is linear => because sum (XOR) of two codewords is still a codeword
  - codeword $\times$ a constant (0 or 1) => still a codeword
  - total number of codewords is 2^k => dimension is k
- Length of codewords is n, but size of space is k => they form a **subspace** of the larger space of all binary sequences of length n

Codewords form a vector space

- Since all codewords form a (sub)space $=>$ all codewords can be expressed as matrix-vector multiplications
- Need to find a basis for the codewords

Generator matrix

- All codewords for a linear block code can be generated via a **matrix-vector multiplication**:

$$\mathbf{i} \cdot [G] = \mathbf{c}$$

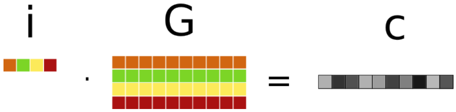

Figure 3: Codeword construction with generator matrix

- $[G]$ = **generator matrix** of size $k \times n$ ("fat" matrix, $k < n$)
  - it is fixed, it fully defines the whole code

Generator matrix

- Row-wise interpretation:
  - Any codeword $\mathbf{c}$ = a linear combination of rows in $[G]$
  - The rows of $[G]$ = a *basis* for the linear block code
  - Could also be transposed, i.e. use column vectors instead
- All operations are done in modulo-2 arithmetic
- There exists a separate codeword for all possible information words $\mathbf{i}$

Proof of linearity

- Prove that a codeword + another codeword = also codeword:

$$\mathbf{i_1} \cdot [G] = \mathbf{c_1}$$

$$\mathbf{i_2} \cdot [G] = \mathbf{c_2}$$

$$\mathbf{c_1} \oplus \mathbf{c_2} = (\mathbf{i_1} \oplus \mathbf{i_2}) \cdot [G] = \textit{codeword}$$

Parity check matrix

- Every generator matrix $[G]$ has a related **parity-check matrix** $[H]$ such that
$$\mathbf{0} = [H] \cdot [G]^T$$
  - also known as **control matrix**
  - size of $[H]$ is $(n-k) \times n$
  - $[G]$ and $[H]$ are related, one can be deduced from the other
- $[H]$ is very useful to check if a binary word is a codeword or not (i.e. for nearest neighbor error detection)

Using the parity check matrix

- Theorem: every codeword $\mathbf{c}$ generated with $[G]$ ($\mathbf{i} \cdot [G] = \mathbf{c}$) will produce a 0 vector when multiplied with the corresponding $[H]$ matrix:

$$\mathbf{0} = [H] \cdot \mathbf{c}^T$$

- Proof:

$$\mathbf{i} \cdot [G] = \mathbf{c}$$
$$[G]^T \cdot \mathbf{i}^T = \mathbf{c}^T$$
$$[H] \cdot \mathbf{c}^T = [H] \cdot [G]^T \cdot \mathbf{i}^T = \mathbf{0}$$

- All codewords generated with $[G]$ will produce 0 when multiplied with $[H]$
- All binary sequences that are not codewords will produce $\neq 0$ when multiplied with $[H]$

Relation between [G] and [H]

- [G] and [H] are related
  - The codewords form a k-dimensional subspace inside the larger n-dimensional vector space
  - The rows of $[H]$ are the "missing" dimensions of the subspace (the "orthogonal complement")
- Together $[G]$ and $[H]$ form a full square matrix $n \times n$, which is a basis for the full n-dimensional vector space
  - size of $[H]$ is $(n - k) \times n$
  - size of $[G]$ is $k \times n$
- Examples:
  - line in a 2D plane, has one orthogonal dimension
  - plane in 3D space, has one orthogonal dimension
  - line in 3D space, has 2 orthogonal dimension

Standard [G] and [H] for systematic codes

- For systematic codes, [G] and [H] have special forms (known as **"standard"** forms)
- Generator matrix
  - first part = identity matrix
  - second part = some matrix Q

$$[G]_{k \times n} = [I_{k \times k} \;\; Q_{k \times (n-k)}]$$

- Parity-check matrix
  - first part = same Q, but **transposed**
  - second part = identity matrix

$$[H]_{(n-k) \times n} = [Q^T_{(n-k) \times k} \;\; I_{(n-k) \times (n-k)}]$$

- Can easily compute one from the other
- Example at blackboard

Interpretation as parity bits

- Multiplication with G in standard form produces the codeword as
  - first part = information bits (since first part of $[G]$ is identity matrix)
  - additional bits = combinations of information bits = *parity bits*
- The additional bits added by coding are actually just parity bits
  - Proof: write the generation equations (example)
- Parity-check matrix in standard form $[H]$ checks if parity bits correspond to information bits
  - Proof: write down the parity check equation (see example)
- If all parity bits match the data, the result of multiplying with $[H]$ is 0
  - otherwise it is $\neq 0$

Interpretation as parity bits

- Generator & parity-check matrices are just mathematical tools for easy computation and checking of parity bits
- We're still just computing and checking parity bits, but we do it easier with matrices

Syndrome

- Nearest neighbor error detection = check if received word **r** is a codeword
- We do this easily by multiplying with $[H]$
- The resulting vector $z = [H] \cdot [r]^T$ is known as **syndrome**
- Column-wise interpretation of multiplication:

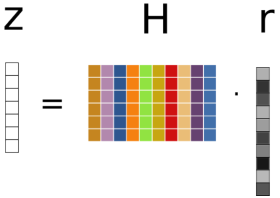

Figure 4: Codeword checking with parity-check matrix

Nearest neighbor error detection with matrices

Nearest neighbor error **detection** with matrices:

1. generate codewords with generator matrix:

$$\mathbf{i} \cdot [G] = \mathbf{c}$$

2. send codeword $\mathbf{c}$ on the channel
3. a random error word $\mathbf{e}$ is applied on the channel
4. receive word $\mathbf{r} = \mathbf{c} \oplus \mathbf{e}$
5. compute **syndrome** of $\mathbf{r}$:

$$\mathbf{z} = [H] \cdot \mathbf{r}^T$$

6. Decide:
   - If $\mathbf{z} = 0 => \mathbf{r}$ has no errors
   - If $\mathbf{z} \neq 0 => \mathbf{r}$ has errors

Nearest neighbor error correction with matrices

Nearest neighbor error **correction** with matrices:

- Syndrome $\mathbf{z} \neq 0 => \mathbf{r}$ has errors, we need to locate them
- The syndrome is the effect only of the error word:

$$\mathbf{z} = [H] \cdot \mathbf{r}^T = [H] \cdot (\mathbf{c}^T \oplus \mathbf{e}^T) = [H] \cdot \mathbf{e}^T$$

7. Create a **syndrome lookup table**:
    - for every possible error word $\mathbf{e}$, compute the syndrome $\mathbf{z} = [H] \cdot \mathbf{e}^T$
    - start with error words with 1 error (most likely), then with 2 errors (less likely), and so on
8. Locate the syndrome $\mathbf{z}$ in the table, read the corresponding error word $\widehat{\mathbf{e}}$
9. Find the correct word:
    - adding the error word again will invert the errored bits back to the originals

$$\widehat{\mathbf{c}} = \mathbf{r} \oplus \widehat{\mathbf{e}}$$

Example

Example: at blackboard

Computational complexity

- Computational complexity for error detection
  - Error detection = multiplication with $[H]$
  - Complexity is $\mathcal{O}(n^2)$ (size of $[H]$ is $(n-k) \times n$)
  - Much more efficient!
- Computational complexity for error correction
  - Need to check all possible error words => bad performance
  - In practice, other tricks are used to make it much faster (see Hamming codes for example)

Conditions on [H] for error detection and correction

- How to design a good matrix $[H]$?
- Conditions on [H] for successful error **detection**:
  - We can detect errors if the syndrome is **non-zero**
  - To detect a single error: every column of $[H]$ must be non-zero
  - To detect two errors: sum of any two columns of $[H]$ cannot be zero
    - that means all columns are different
  - To detect n errors: sum of any n or less columns of $[H]$ cannot be zero

Conditions on [H] for error detection and correction

- Conditions for syndrome-based error **correction**:
  - We can correct errors if the syndrome is **unique**
  - To correct a single error: all columns of $[H]$ are different
    - so the syndromes, for a single error, are all different
  - To correct n errors: sum of any n or less columns of $[H]$ are all different
    - much more difficult to obtain than for detection
- Conditions for error correction are more demanding than for detection
- Note: Rearranging the columns of $[H]$ (the order of bits in the codeword) does not affect performance

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1. General presentation
2. Analyzing linear block codes with the Hamming distance
3. Analyzing linear block codes with matrix algebra
4. **Hamming codes**
5. Cyclic codes

Hamming codes

- A particular class of linear error-correcting codes
- Definition: a **Hamming code** is a linear block code where the columns of $[H]$ are *the binary representation of all numbers from 1 to* $2^r - 1$, $\forall r \geq 2$
- Example (blackboard): (7,4) Hamming code
- Systematic: arrange the bits in the codeword, such that the control bits correspond to the columns having a single 1
  - no big difference from the usual systematic case, just a rearrangement of bits
  - makes implementation easier

Properties of Hamming codes

- From definition of $[H]$ it follows:
  1. Codeword has length $n = 2^r - 1$
  2. r bits are parity bits (also known as ***control bits***)
  3. $k = 2^r - r - 1$ bits are information bits

- Notation: **(n,k) Hamming code**
  - n = codeword length $= 2^r - 1$,
  - k = number of information bits $= 2^r - r - 1$
  - Example: (7,4) Hamming code, (15,11) Hamming code, $(127, 120)$ Hamming code

Structure of a Hamming codeword

- The codeword contains information bits and parity (control) bits
- The control bits correspond to the columns of the parity-check matrix $[H]$ which have a single 1 (i.e. columns which form the identity matrix)
- The information bits are placed in the remaining positions, where the columns of $[H]$ have two or more 1s
- Codeword for Hamming(7,4):

$$c_1 c_2 i_3 c_4 i_5 i_6 i_7$$

- Codeword for Hamming(15,11):

$$c_1 c_2 i_3 c_4 i_5 i_6 i_7 c_8 i_9 i_{10} i_{11} i_{12} i_{13} i_{14} i_{15}$$

Construction of Hamming codewords

- Every Hamming code has a generator matrix $[G]$, but we don't provide it explicitly, because it is hard to remember
- Instead, we deduce the values from the equation system of the parity-check matrix $[H]$, $\mathbf{0} = [H] \cdot \mathbf{c}^T$
- For example, for Hamming(7,4), we have:

$$\begin{cases} 0 = c_4 \oplus i_5 \oplus i_6 \oplus i_7 \\ 0 = c_2 \oplus i_3 \oplus i_5 \oplus i_6 \\ 0 = c_1 \oplus i_3 \oplus i_5 \oplus i_7 \end{cases}$$

which means:

$$\begin{cases} c_4 = i_5 \oplus i_6 \oplus i_7 \\ c_2 = i_3 \oplus i_5 \oplus i_6 \\ c_1 = i_3 \oplus i_5 \oplus i_7 \end{cases}$$

Properties of Hamming codes

- Can detect two errors
  - All columns are different => can detect 2 errors
  - Sum of two columns equal to a third => cannot correct 3

OR

- Can correct one error
  - All columns are different => can correct 1 error
  - Sum of two columns equal to a third => cannot correct 2
  - Non-systematic: syndrome = error position

BUT

- Not simultaneously!
  - same non-zero syndrome can be obtained with 1 or 2 errors, can't distinguish

Coding rate of Hamming codes

Coding rate of a Hamming code:

$$R = \frac{k}{n} = \frac{2^r - r - 1}{2^r - 1}$$

The Hamming codes can correct 1 OR detect 2 errors in a codeword of size n

- (7,4) Hamming code: $n = 7$
- (15,11) Hamming code: $n = 15$
- (31,26) Hamming code: $n = 31$

Longer Hamming codes are progressively weaker:

- weaker error correction capability
- better efficiency (higher coding rate)
- more appropriate for smaller error probabilities

Encoding & decoding example for Hamming(7,4)

See whiteboard.

In this example, encoding is done without the generator matrix G, directly with the matrix H, by finding the values of the parity bits c_1, c_2, c_4 such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [H] \begin{bmatrix} c_1 \\ c_2 \\ i_3 \\ c_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix}$$

For a single error, the syndrome **is the binary representation of the location of the error**.

Circuit for encoding Hamming(7,4)

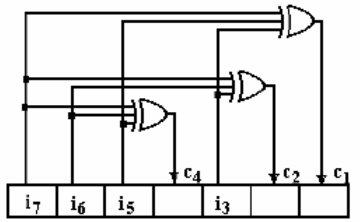

Figure 5: Hamming Encoder

- Components:
  - A **shift register** to hold the codeword
  - Logic XOR gates to compute the parity bits

Circuit for decoding Hamming(7,4)

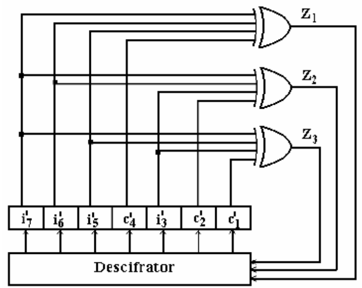

Figure 6: Hamming Decoder

- Components:
  - A **shift register** to hold the received word
  - Logic XOR gates to compute the bits of the syndrome (z_i)
  - **Binary decoder**: activates the output corresponding to the binary input value, fixing the error

SECDED Hamming codes

- Hamming codes can correct 1 error OR can detect 2 errors, but we cannot differentiate the two cases
- Example:
  - the syndrome $\mathbf{z} = [H] \cdot \mathbf{r}^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ can be caused by:
    - a single error in location 3 (bit i_3)
    - two errors in location 1 and 2 (bits c_1, bits c_2)
  - if we know it is a single error, we can go ahead and correct it, then use the corrected data
  - if we know there are two errors, we should NOT attempt to correct them, because we cannot locate the errors correctly
- Unfortunately, it is **not possible to differentiate** between the two cases.
- **Solution?** Add additional parity bit $\rightarrow$ SECDED Hamming codes

SECDED Hamming codes

- Add an additional parity bit to differentiate the two cases
  - c_0 = sum of all n bits of the codeword
- For (7,4) Hamming codes:

$$\mathbf{c_0} c_1 c_2 i_3 c_4 i_5 i_6 i_7$$

- The parity check matrix is extended by 1 row and 1 column

$$\tilde{H} = \begin{bmatrix} 1 & 1 \\ 0 & \mathbf{H} \end{bmatrix}$$

- Known as SECDED Hamming codes
  - **S**ingle **E**rror **C**orrection - **D**ouble **E**rror **D**etection

Encoding and decoding of SECDED Hamming codes

- Encoding:
  - compute codeword using $\tilde{H}$
  - alternatively, prepend $\mathbf{c_0}$ = sum of all other bits

Encoding and decoding of SECDED Hamming codes

- Decoding
  - Compute syndrome of the received word using $\tilde{H}$

$$\tilde{\mathbf{z}} = \begin{bmatrix} z_0 \\ \mathbf{z} \end{bmatrix} = [\tilde{H}] \cdot \mathbf{r}^T$$

  - z_0 is an additional bit in the syndrome corresponding to c_0
  - z_0 tells us whether the received c_0 matches the parity of the received word
    - $z_0 = 0$: the additional parity bit c_0 matches the parity of the received word
    - $z_0 = 1$: the additional parity bit c_0 does not match the parity of the received word

Encoding and decoding of SECDED Hamming codes

- Decoding (continued):
  - Decide which of the following cases happened:
    - If no error happened: $z_1 = z_2 = z_3 = 0, z_0 = \forall$
    - If 1 error happened: syndrome is non-zero, $z_0 = 1$ (does not match)
    - If 2 errors happened: syndrome is non-zero, $z_0 = 0$ (does match, because the two errors cancel each other out)
    - If 3 errors happened: same as 1, can't differentiate
- Now can simultaneously differentiate between:
  - 1 error: $\rightarrow$ perform correction
  - 2 errors: $\rightarrow$ detect, but do not perform correction
- Also, if correction is never attempted, can detect up to 3 errors
  - minimum Hamming distance $= 4$ (no proof given)
  - don't know if 1 error, 2 errors or 3 errors, so can't try correction

Summary until now

- Systematic codes: information bits + parity bits
- Generator matrix: use to generate codeword

$$\mathbf{i} \cdot [G] = \mathbf{c}$$

- Parity-check matrix: use to check if a codeword

$$0 = [H] \cdot \mathbf{c}^T$$

- Syndrome:

$$\mathbf{z} = [H] \cdot \mathbf{r}^T$$

- Syndrome-based error detection: syndrome non-zero
- Syndrome-based error correction: lookup table
- Hamming codes: $[H]$ contains all numbers $1 \ldots 2^r - 1$
- SECDED Hamming codes: add an extra parity bit

Chapter structure

Cyclic codes

Definition: **cyclic codes** are a particular class of linear block codes for which *every cyclic shift of a codeword is also a codeword*

- Cyclic shift: cyclic rotation of a sequence of bits (any direction)
- Are a particular class of linear block codes, so all the theory up to now still applies
  - they have a generator matrix, parity check matrix etc.
- But they can be implemented more efficient than general linear block codes (e.g. Hamming)
- Used **everywhere** under the common name **CRC** (**C**yclic **R**edundancy **C**heck)
  - Network communications (Ethernet), data storage in Flash memory

Usage example: Ethernet frame

- CRC codes are used in Ethernet frames:

802.3 Ethernet packet and frame structure

Layer	Preamble	Start of frame delimiter	MAC destination	MAC source	802.1Q tag (optional)	Ethertype (Ethernet II) or length (IEEE 802.3)	Payload	Frame check sequence (32-bit CRC)	Interpacket gap
	7 octets	1 octet	6 octets	6 octets	(4 octets)	2 octets	46–1500 octets	4 octets	12 octets
Layer 2 Ethernet frame			← 64–1522 octets →						
Layer 1 Ethernet packet & IPG	← 72–1530 octets →								← 12 oct. →

Figure 7: CRC value in an Ethernet frame

Binary polynomials

- Every binary sequence **a** corresponds to a polynomial $\mathbf{a}(\mathbf{x})$ with binary coefficients

$$a_0 a_1 \dots a_{n-1} \rightarrow \mathbf{a}(\mathbf{x}) = a_0 \oplus a_1 x \oplus \dots \oplus a_{n-1} x^{n-1}$$

- Example:

$$10010111 \rightarrow 1 \oplus x^3 \oplus x^5 \oplus x^6 \oplus x^7$$

- From now on, by "codeword" we also mean the corresponding polynomial.
- Can perform all mathematical operations with these polynomials:
  - addition, multiplication, division etc. (examples)
- There are efficient circuits for performing multiplications and divisions.

Generator polynomial

Theorem:

- All the codewords of a cyclic code are multiples of a certain polynomial $g(x)$, known as **generator polynomial**.

Properties of generator polynomial

The generator polynomial $g(x)$ must satisfy the following:

- $g(x)$ must have first and last coefficient equal to 1
- $g(x)$ must be a factor of $x^n \oplus 1$
- The *degree* of $g(x)$ is $n - k$, where:
  - n = the size of codeword (codeword polynomial has degree $n - 1$)
  - k = the size of the information word (information polynomial has degree $k - 1$)

$$(k-1) + (n-k) = n-1$$

- **The degree of $g(x)$ is the number of parity bits of the code.**

Example of generator polynomials

Example:

$$1 \oplus x^7 = (1 \oplus x)(1 \oplus x \oplus x^3)(1 \oplus x^2 \oplus x^3)$$

Each factor can generate a code:

- $1 \oplus x$ generates a (7,6) cyclic code
- $1 \oplus x \oplus x^3$ generates a (7,4) cyclic code
- $1 \oplus x^2 \oplus x^3$ generates a (7,4) cyclic code

Popular polynomials

```
Some popular polys are:
16 bits: (16,12,5,0)                                  [X25 standard]
         (16,15,2,0)                                  ["CRC-16"]
32 bits: (32,26,23,22,16,12,11,10,8,7,5,4,2,1,0)    [Ethernet]
```

Figure 8: Popular generator polynomials $g(x)$

- Image from *http://www.ross.net/crc/download/crc_v3.txt*
- Your turn: write the polynomials in mathematical form

Proving the cyclic property

Theorem:

- Any cyclic shift of a codeword is also a codeword.

Proof:

- It is enough to consider a cyclic shift by 1 position
- Original codeword

$$c_0 c_1 c_2 \ldots c_{n-1} \rightarrow \mathbf{c}(\mathbf{x}) = c_0 \oplus c_1 x \oplus \ldots \oplus c_{n-1} x^{n-1}$$

- Cyclic shift to the right by 1 position

$$c_{n-1} c_0 c_1 \ldots c_{n-2} \rightarrow \mathbf{c}'(\mathbf{x}) = c_{n-1} \oplus c_0 x \oplus \ldots \oplus c_{n-2} x^{n-1}$$

- We can rewrite:

$$\begin{aligned} \mathbf{c}'(\mathbf{x}) &= x \cdot \mathbf{c}(\mathbf{x}) \oplus c_{n-1} x^n \oplus c_{n-1} \\ &= x \cdot \mathbf{c}(\mathbf{x}) \oplus c_{n-1}(x^n \oplus 1) \end{aligned}$$

Proving the cyclic property

Proof (continued):

- Since $\mathbf{c}(\mathbf{x})$ is a multiple of $g(x)$, so is $x \cdot \mathbf{c}(\mathbf{x})$
- Also $(x^n \oplus 1)$ is always a multiple of $g(x)$
- $=>$ It follows that their sum $\mathbf{c}'(\mathbf{x})$ is a also a multiple of $g(x)$, which means it is a codeword.

QED

- Note that we relied on two properties mentioned before:
  - that a codeword $\mathbf{c}(\mathbf{x})$ is always a multiple of $g(x)$
  - that $g(x)$ is a factor of $(x^n \oplus 1)$

Coding and decoding of cyclic codes

- Cyclic codes can be used for detection or correction
- In practice, they are used mostly for **detection only** (e.g. in Ethernet)
  - because there are other codes with better performance for correction
- Can be systematic / non-systematic
  - In practice, the systematic variant is much preferred
- We study coding/decoding from 3 perspectives:
  - The mathematical way, with polynomials
  - The programming way, e.g. as a programming algorithm
  - The hardware way, via schematics

1. Coding and decoding - The mathematical way

Reminder: polynomial multiplication and division

- Two polynomials $a(x)$ and $b(x)$ can be multiplied
  - the result has degree = degree of $a(x)$ + degree of $b(x)$
- A polynomial $a(x)$ can be divided by another polynomial $b(x)$:

$$a(x) = b(x)q(x) \oplus r(x)$$

  - $q(x)$ = the quotient ("câtul")
  - $r(x)$ = the remainder ("restul")
  - the degree of $r(x)$ is strictly smaller than the degree of $b(x)$

1. Coding and decoding - The mathematical way

Coding

- We want to encode the **information word** with k bits

$$i_0 i_1 i_2 \dots i_{k-1} \rightarrow i(x) = i_0 \oplus i_1 x \oplus \dots \oplus i_{k-1} x^{k-1}$$

- **Non-systematic** codeword generation:

$$\boxed{c(x) = i(x) \cdot g(x)}$$

- The degrees match:
  - $i(x)$ has degree $k-1$ (k bits)
  - $g(x)$ has degree $n-k$ ($n-k+1$ bits)
  - $c(x)$ has degree $n-1 = (n-k)+(k-1)$ (n bits)

Systematic coding - The mathematical way

- **Systematic** codeword generation:

$$\boxed{c(x) = x^{n-k} \cdot i(x) \oplus b(x)}$$

- $b(x)$ is the remainder of dividing $x^{n-k}i(x)$ to $g(x)$:

$$x^{n-k}i(x) = a(x)g(x) \oplus b(x)$$

  - $b(x)$ is known as "the CRC value"
- Is this $c(x)$ really a multiple of $g(x)$? Yes, because:

$$c(x) = x^{n-k} \cdot i(x) \oplus b(x) = a(x)g(x) \oplus b(x) \oplus b(x) = a(x)g(x)$$

Interpretation

- Why is the code systematic?
- Let's analyze the systematic codeword generation step by step
- Consider the information word/polynomial

$$\mathbf{i} = [\underbrace{i_0 i_1 \ldots i_{k-1}}_{k}] \rightarrow i(x) = i_0 \oplus i_1 x \oplus \ldots \oplus i_{k-1} x^{k-1}$$

- Multiplying $x^{n-k} \cdot i(x)$ shifts all bits to the right with $(n-k)$ positions

$$[\underbrace{00\ldots0}_{n-k}\underbrace{i_0 i_1 \ldots i_{k-1}}_{k}] \rightarrow i(x) = i_0 x^{n-k} \oplus i_1 x^{n-k+1} \oplus \ldots \oplus i_{k-1} x^{n-1}$$

Interpretation (continued)

- The remainder $b(x)$ has degree strictly less than $n-k$ (degree of $g(x)$), so at most $n-k$ bits
- Therefore adding $b(x)$ will not overlap with $x^{n-k} \cdot i(x)$
  - the $(n-k)$ bits of $b(x)$ will fit in the first $n-k$ locations

$$\mathbf{c} = [\underbrace{b_0 b_1 \ldots b_{n-k}}_{n-k} \underbrace{i_0 i_1 \ldots i_{k-1}}] \rightarrow$$

$$\rightarrow c(x) = b_0 \oplus b_1 x \oplus \ldots \oplus b_{n-k-1} x^{n-k-1} \oplus i_0 x^{n-k} \oplus i_1 x^{n-k+1} \oplus \ldots \oplus i_k$$

- Hence the code is systematic: the information bits are in the codeword
- The code adds $b(x)$ (the remainder) = the **CRC value**

Interpretation

- Systematic cyclic codeword = compute a CRC value and append it to the data
- Different writing conventions:
  - when writing the codewords from LSB -> MSB (increasing order of degrees), the CRC appears in front
    - like in lecture slides
  - when writing the codewords from MSB -> LSB (decreasing order of degrees), the CRC appears at the end
    - like in laboratory
  - same thing, just bit ordering is reversed
  - (LSB = Least Significant Bit, MSB = Most Significant Bit)

Decoding - The mathematical way

Decoding

- We receive $\mathbf{r} = r_0 r_1 r_2 \ldots r_{n-1} \rightarrow \mathbf{r(x)} = r_0 \oplus r_1 x \oplus \ldots \oplus r_{n-1} x^{n-1}$\$
- Error **detection**: check if $r(x)$ is a codeword or not
- Check if the received $\mathbf{r(x)}$ still is a multiple of $g(x)$
  - Divide $\mathbf{r(x)}$ to $g(x)$:
    - If remainder of $r(x) : g(x)$ is 0 $=>$ it is a codeword, no errors present
    - If remainder is non-zero $=>$ it's not a true codeword, **errors detected**
- Computing the remainder = computing the CRC of the received data
  - Remember lab: decoding = compute CRC of all coded data, if 0 $=>$ OK, if non-zero $=>$ NOK

Decoding - The mathematical way

- Error **correction**: use a lookup table (just like with matrices)
  - build a lookup table for all possible error words (like with matrix codes)
  - for each error code, divide by $g(x)$ and compute the remainder
  - when the remainder is identical to the remainder obtained with $\mathbf{r(x)}$, we found the error word $=>$ correct errors
- Example: at blackboard

2. Coding and decoding - The programming way

- Only for systematic codes (mostly used)
- Steps:
  - 1. Compute the CRC $= b(x) =$ remainder of $x^{n-k} i(x)$ divided to $g(x)$
  - 2. Put the CRC in front of the information word, mirrored
- Good reference: *"A Painless Guide to CRC Error Detection Algorithms"*, Ross N. Williams
  - http://www.ross.net/crc/download/crc_v3.txt

Coding

- The mathematical polynomial division = just like XOR-ing successively with $g(x)$
  - align the binary sequence of $g(x)$ under the leftmost 1
  - XOR the sequences
  - repeat
  - just like in the lab
- See example at blackboard / lab

Example

```
______________
 11010110110000
 10011,,.,,....
 -----,,.,,....
  10011,.,,....
  10011,.,,....
  -----,.,,....
   00001.,,....
   00000.,,....
   -----.,,....
    00010,,....
    00000,,....
    -----,,....
     00101,....
     00000,....
     -----,....
      01011....
      00000....
      -----....
       10110...
       10011...
```

Figure 9: Polynomial division = XORing successively with $g(x)$

Decoding

- We receive $\mathbf{r} = r_0 r_1 r_2 \ldots r_{n-1} \rightarrow \mathbf{r(x)} = r_0 \oplus r_1 x \oplus \ldots \oplus r_{n-1} x^{n-1}$
- Step 1: Mirror the sequence **r** (CRC must be at the end!)
- Error detection:
  - compute the CRC of all sequence **r**
    - If the remainder is 0 => no errors
    - If the remainder is non-zero => errors detected!
- Error correction:
  - use a lookup table (just like with matrices)
    - build a lookup table for all possible error words (same as with matrix codes)
    - for each error word, compute the CRC
    - when the resulting remainder is identical to the remainder obtained with **r**, we found the error word => correct errors

Skip next slides for 2018-2019

The remaining slides in this file are skipped for the class of 2018-2019.

3. Coding and decoding - The hardware way

- Coding = based on polynomial multiplications and divisions
- Efficient circuits for multiplication / division exist, that can be used for systematic or non-systematic codeword generation (draw on blackboard)

Circuits for multiplication of binary polynomials

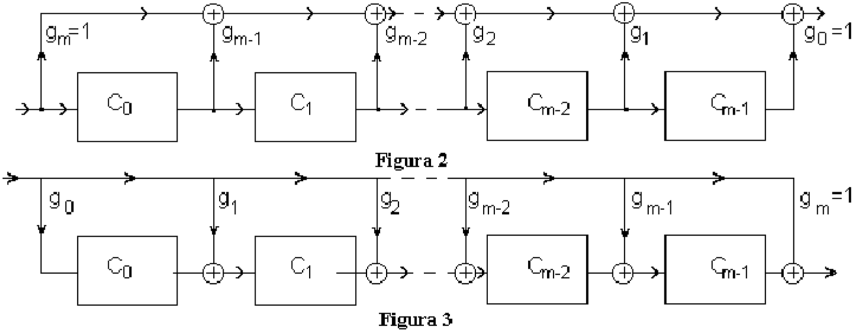

Figure 10: Circuits for polynomial multiplication

Operation of multiplication circuits

- The circuits multiply an input polynomial $a(x)$ with a polynomial $g(x)$ defined by their structure
- The input polynomial is applied at the input, 1 bit at a time, starting from highest degree
- The output polynomial is obtained at the output, 1 bit at a time, starting from highest degree
- Because output polynomial has larger degree, the circuit needs to operate a few more samples until the final result is obtained. During this time the input is 0.
- Examples: at the whiteboard

Circuits for division binary polynomials

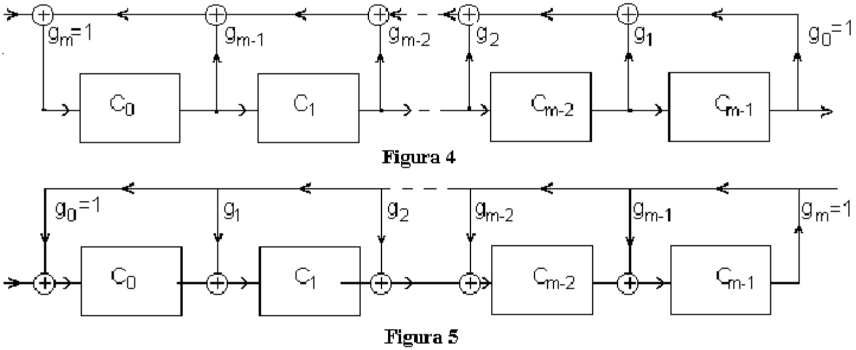

Figure 11: Circuits for polynomial division

Operation of division circuits

- The circuits divide an input polynomial $a(x)$ to a polynomial $g(x)$ defined by their structure
- The input polynomial is applied at the input, 1 bit at a time, starting from highest degree
- The output polynomial is obtained at the output, 1 bit at a time, starting from highest degree
- Because output polynomial has smaller degree, the circuit first outputs some zero values, until starting to output the result.
- If the remainder is 0, all the cells remain with 0 at the end
- Examples: at the whiteboard

Non-systematic cyclic encoder circuit

- Non-systematic cyclic encoder circuit:
  - simply a polynomial multiplication circuit
  - input is $i(x)$, output is $c(x)$

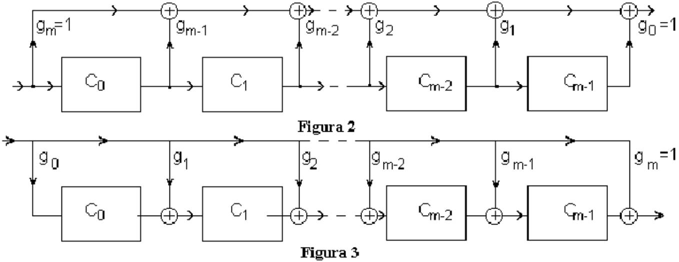

Figure 12: Circuits for polynomial multiplication

Systematic cyclic encoder circuit

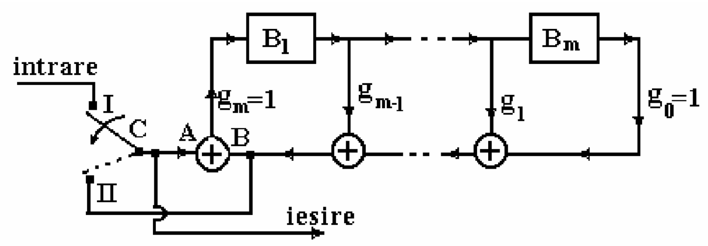

Figure 13: Systematic cyclic encoder circuit

- It contains inside a division circuit (upper right part)

Systematic cyclic encoder circuit

Operation of the cyclic encoder circuit:

- Initially all cells are 0
- Switch in position I:
  - information bits are applied to the output and to the division circuit
  - first bits of the output are the information bits => indeed systematic
  - the input bits are applied to the division circuit
- Switch in position II:
  - some output bits are put at the output
  - the same output bits are also applied to the input of the division circuit
- **In the end all cells end up with value 0**
  - because in phase II we add the input (A) with itself (B) at the input of the division circuit, so they cancel each other

Systematic cyclic encoder circuit

- Why is the output $c(x)$ the desired codeword? Because:
  1. has the information bits in the first part (systematic)
  2. is a multiple of $g(x)$
- Why is it a multiple of $g(x)$? Because:
  - the output $c(x)$ is always applied also to the input of the division circuit
    - in both phases of operation
  - after division, the cells end up in 0, which means there is no remainder of division
- Side note: we haven't really explained *why* the output $c(x)$ is a codeword, we just showed that it is so

The parity-check matrix for systematic cyclic codes

- Requires a more in-depth analysis of Linear Feedback Shift Registers (LFSR)

Linear-Feedback Shift Registers (LFSR)

- A **flip-flop** = a cell holding a bit value (0 or 1)
  - called *"bistabil"* in Romanian
  - operates on the edges of a clock signal
- A **register** = a group of flip-flops, holding multiple bits
  - example: an 8-bit register
- A **shift register** = a register where the output of a flip-flop is connected to the input of the next one
  - the bit sequence is shifted to the right
  - has an input (for the first cell)
- A **linear feedback shift register** (LFSR) = a shift register for which the input is a computed as a linear combination of the flip-flops values
  - input = usually a XOR of some cells from the register
  - like a division circuit without any input
  - feedback = all flip-flops, with coefficients g_i in general
  - example at whiteboard

States and transitions of LFSR

- **State** of the LFSR = the sequence of bit values it holds at a certain moment (in order: right to left)
- The state at the next moment, $S(k+1)$, can be computed by multiplication of the current state $S(k)$ with the **companion matrix** (or **transition matrix**) $[T]$:

$$S(k+1) = [T] \cdot S(k)$$

- The companion matrix is defined based on the feedback coefficients g_i:

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ g_0 & g_1 & g_2 & \dots & g_{m-1} \end{bmatrix}$$

- Note: reversing the order of bits in the state => transposed matrix
- Starting at time 0, then the state at time k is:

Period of LFSR

- The number of states is finite => they must repeat at some moment
- The state equal to 0 must not be encountered (in this case the LFSR will remain 0 forever)
- The **period** of the LFSR = number of time moments until the state repeats
- If period is N, then state at time N is same as state at time 0:

$$S(N) = [T]^N S(0) = S(0),$$

which means:

$$[T]^N = I_m$$

- Maximum period is $N_{max} = 2^m - 1$ (excluding state 0), in this case the polynomial $g(x)$ is called **primitive polynomial**

LFSR with inputs

- What if the LFSR has an input added to the feedback (XOR)?
  - exactly like a division circuit
  - assume the input is a sequence $a_{N-1}, ...a_0$
- Since a LFSR is a **linear circuit**, the effect is added:

$$S(1) = [T] \cdot S(0) \oplus \begin{bmatrix} 0 \\ 0 \\ ... \\ a_{N-1} \end{bmatrix}$$

- In general:

$$S(k_1) = [T] \cdot S(k) \oplus a_{N-k} \cdot [U],$$

where $[U]$ is:

$$[U] = \begin{bmatrix} 0 \\ 0 \\ ... \\ 1 \end{bmatrix}$$

The parity-check matrix for systematic cyclic codes

- Cyclic codes are linear block codes, so they have a parity-check and a generator matrix
  - but it is more efficient to implement them with polynomial multiplication / division circuits
- The parity-check matrix $[H]$ can be deduced by analyzing the states of the LFSR (divider) inside the encoder:
  - it is a LFSR with feedback and input
  - the input is the codeword $c(x)$
  - do computations at whiteboard ...
  - ... arrive at expression for matrix $[H]$

The parity-check matrix for systematic cyclic codes

- The parity check matrix $[H]$ has the form

$$[H] = [U, TU, T^2U, \dots T^{n-1}U]$$

- The cyclic codeword satisfies the usual relation

$$S(n) = 0 = [H]\mathbf{c}^{\mathbf{T}}$$

- In case of an error, the state at time n will be the syndrome (non-zero):

$$S(n) = [H]\mathbf{r}^{\mathbf{T}} \neq 0$$

Error detection and correction capability

Theorem:

Any (n,k) cyclic code with $g(x)$ being a primitive polynomial is capable of detecting 2 errors, or of correcting 1 error

- Proof:
  - $g(x)$ is primitive polynomial $=>$ the LSFR cycles through all possible states (non-zero)
  - therefore all the columns of [H] are distinct
  - Use the conditions based on the columns of [H] from first part of chapter
    - sum of any two columns is non-zero $=>$ can detect 2 errors
    - any two columns are distinct $=>$ can correct 1 error

Packets of errors

- Until now, we considered a single error (i.e errors appear independently)
- In real life, many times the errors appear in groups
- A **packet of errors** (*an error burst*) is a sequence of two or more **consecutive errors**
  - examples: *fading* in wireless channels
- The **length** of the packet = the number of consecutive errors

Condition on columns of [H] for packets of errors

- Conditions for packets of **e** errors are less restrictive than for **e** independent errors
- Error **detection** of e independent errors:
  - sum of **any** e or fewer columns is **non-zero**
- Error **detection** of a packet of e errors
  - sum of any **consecutive** e or fewer columns is **non-zero**
- Error **correction** of e independent errors
  - sum of **any** e or fewer columns is **unique**
- Error **correction** of a packet of e errors
  - sum of any **consecutive** e or fewer columns is **unique**

Detection of packets of errors

Theorem:

Any (n,k) cyclic code is capable of detecting any error packet of length $n - k$ or less

- A large fraction of longer bursts can also be detected (but not all)
- No proof (too complicated)

Cyclic decoder implemented with LFSR

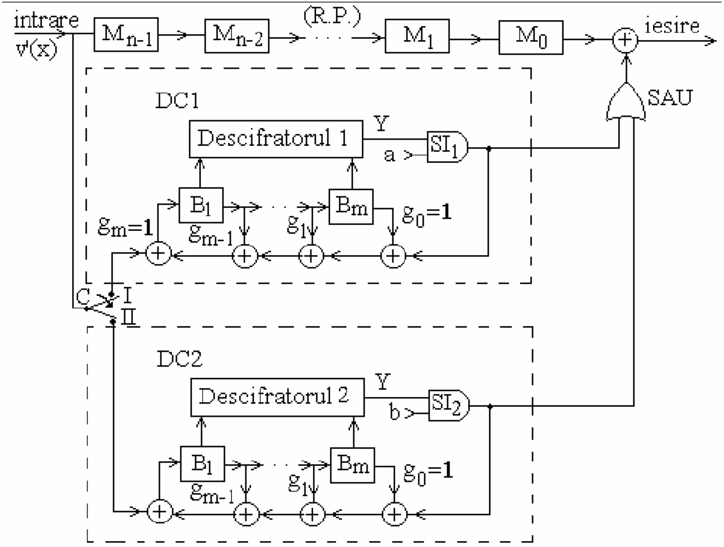

Figure 14: Cyclic decoder circuit

Cyclic decoder implemented with LFSR

- Consists of:
  - main shift register MSR
  - main switch SW
  - 2 LFSRs (divider circuits), built based on $g(x)$
  - 2 error locator blocks, one for each divider
  - 2 validation gates V1, V2, for each divider
  - output XOR gate for correcting errors

Cyclic decoder implemented with LFSR

- Operation phases:

1. Input phase: SW on position I, validation gate V1 blocked
    - The received codeword $r(x)$ is received one by one, starting with largest power of x^{n-1}
    - The received codeword enters the MSR and first LFSR (divider)
    - The first divider computes $r(x) : g(x)$
    - The validation gate V1 is blocked, no output

- Input phase ends after n moments, the switch SW goes into position II
- If the received word has no errors, all LFSR cells are 0 (no remainder), will remain 0, the error locator will always output 0,
    - the MSR will output the received bits unchanged

Cyclic decoder implemented with LFSR

2. Decoding phase: SW on position **II**, validation gate V1 open
   - LFSR keeps running with no input for n more moments
   - the MSR provides the received bits at the output, one by one
   - **exactly when the erroneous bit is at the main output of MSR, the error locator will output 1, and the output XOR gate will correct the bit (TO BE PROVEN)**
   - during this time the next codeword is loaded into MSR and into second LFSR (input phase for second LFSR)

- After n moments, the received word is fully decoded and corrected
- SW goes back into position **I**, the second LFSR starts decoding phase, while the first LFSR is loading the new receiver word, and so on
- **To prove:** error locator outputs 1 exactly when the erroneous bit is at the main output

Cyclic decoder implemented with LFSR

Theorem: if the k-th bit r_{n-k} from $r(x)$ has an error, the error locator will output 1 exactly after $k-1$ moments

- That's exactly when the erroneous k-th bit will be output from MSR => will be changed back to the good value
- **Proof:**
  1. assume error on position r_{n-k}
  2. the state of the LFSR at end of phase I = syndrome = column $(n-k)$ from $[H]$
  $$S(n) = [H]\mathbf{r}^T = [H]\mathbf{e}^T = T^{n-k}U$$
  3. after another $k-1$ moments, the state will be
  $$T^{k-1}T^{n-k}U = T^{n-1}U$$
  4. since $T^n = I_m$ -> $T^{n-1} = T^{-1}$
  5. $T^{-1}U$ is the state preceding state U, which is state
  $$\begin{bmatrix} 1 \\ 0 \\ \dots \end{bmatrix}$$

Cyclic decoder implemented with LFSR

- Step 5 above can be shown in two ways:
  - reasoning on the circuit
  - using the definition of T^{-1}

$$T = \begin{bmatrix} g_1 & g_2 & ...g_{m-1} & 1 & \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}$$

- The error locator is designed to detect this state $T^{-1}U$, i.e. it is designed as shown on blackboard
- Therefore, the error locator will correct an error
- This works only for 1 error, due to proof (1 column from [H])

Summary of cyclic codes

- Generated using a generator polynomial $g(x)$
- Non-systematic:

$$c(x) = i(x) \cdot g(x)$$

- Systematic:

$$c(x) = b(x) \oplus X^{n-k} i(x)$$

  - $b(x)$ is the remainder of dividing $X^{n-k} i(x)$ to $g(x)$
- A codeword is always a multiple of $g(x)$
- Error detection: divide by $g(x)$, look at remainder
- Schematics:
  - Cyclic encoder
  - Cyclic decoder with LFSR
  - Thresholding cyclic decoder
  - Encoder/decoder for packets of up to 2 errors