Exercises 8: channels

1. Consider a communication process defined by the following **joint probability matrix**:

$$P(x_i \cap y_j) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

   a. compute the marginal probabilities and the marginal entropies $H(X)$ and $H(Y)$;
   b. Find the channel matrix $P(Y|X)$ and draw the graph of the channel;
   c. compute the mutual information $I(X,Y)$, and draw the geometrical representation.

1 a) $$P(X,Y) = P(x_i \cap y_j) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1/2 & 0 & 0 \\ x_2 & 0 & 1/4 & 1/4 \end{array} \begin{array}{l} \to p(x_1) = 1/2 \\ \to p(x_2) = 1/2 \end{array}$$

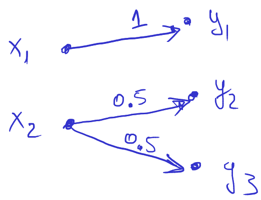
$$H(X) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = 1\,b$$

$$p(y_1) = \tfrac{1}{2} \quad p(y_2) = \tfrac{1}{4} \quad p(y_3) = \tfrac{1}{4}$$

$$H(Y) = -\tfrac{1}{2}\underbrace{\log\tfrac{1}{2}}_{-1} - 2\cdot\tfrac{1}{4}\underbrace{\log\tfrac{1}{4}}_{-2}$$

$$= \tfrac{1}{2} + 1 = 1.5\,b$$

b). $$P(Y|X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0 & 0 \\ x_2 & 0 & 0.5 & 0.5 \end{array}$$



c) $I(X,Y) = ?$

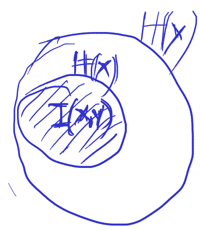
$H(X,Y)$ = entropy of all values in $P(X,Y)$

$$= -\tfrac{1}{2}\log\tfrac{1}{2} - 2\cdot\tfrac{1}{4}\log\tfrac{1}{4} - \underbrace{3\cdot 0\log 0}_{0} = 1.5\,b$$

$$I(X,Y) = H(X) + H(Y) - H(X,Y) =$$

$$= 1 + 1.5 - 1.5$$

$$= 1\,b$$



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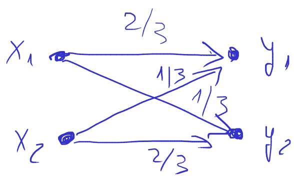
2. At the input of a binary symmetric channel with the following channel matrix

$$P(y_j|x_i) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

we apply two inputs x_1 and x_2 with probabilities $p(x_1) = \frac{3}{4}$ and $p(x_2) = \frac{1}{4}$.

a. Draw the graph of the channel
b. Find $H(X)$, $H(Y)$ and $I(X,Y)$
c. Compute the uncertainty remaining over the input X when output symbol y_2 is received
d. Compute the channel capacity, the redundancy and the efficiency of the channel.

a) $$P(Y|X) = P(y_j|x_i) = \begin{array}{c|cc|c} & y_1 & y_2 & \\ x_1 & 2/3 & 1/3 & \cdot\, p(x_1) \\ x_2 & 1/3 & 2/3 & \cdot\, p(x_2) \end{array}$$



$p(x_1) = \frac{3}{4}$

$p(x_2) = 1/4$

$\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$ $\quad \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

$1/3 \cdot \frac{1}{4} = 1/12$ $\quad \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

b). $H(Y) = ?$

$$P(x,y) = \begin{array}{c|cc|l} & y_1 & y_2 & \\ x_1 & 1/2 & 1/4 & \to p(x_1) = \frac{3}{4} \\ x_2 & \frac{1}{12} & 1/6 & \to p(x_2) = \frac{1}{4} \end{array}$$

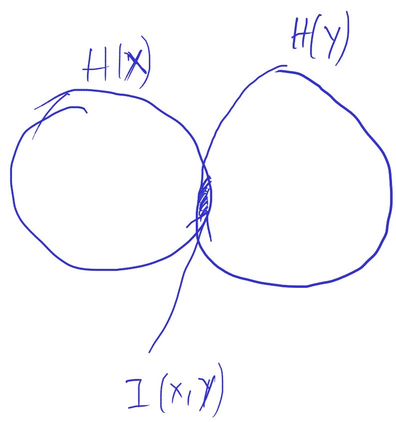
$p(y_1) = \frac{7}{12}$ $\quad p(y_2) = \frac{5}{12}$

$$H(X) = -\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4} = 0.81\ b$$

$$H(Y) = -\frac{7}{12}\log\frac{7}{12} - \frac{5}{12}\log\frac{5}{12} = 0.98\ b$$

$$H(X,Y) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{12}\log\frac{1}{12} - \frac{1}{6}\log\frac{1}{6} = 1.73\ b$$

$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$
$$= 0.81 + 0.98 - 1.73$$
$$= 1.79 - 1.73$$
$$= 0.06\ b$$



c). $\boxed{H(X|y_2) = ?}$

$$\frac{1}{2}\cdot\frac{12}{7} = \frac{6}{7} \qquad \frac{1}{4}\cdot\frac{12}{5} =$$

$$\frac{1}{2}\cdot\frac{12}{7} = \frac{1}{7}$$

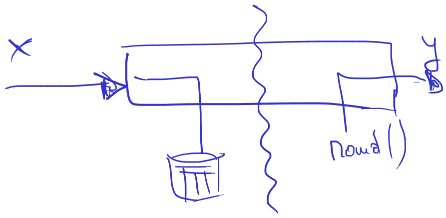
$$P(X|Y) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 6/7 & \frac{6}{10} \\ x_2 & 1/7 & \frac{4}{10} \end{array}$$

$$H(X|y_2) = -\frac{6}{10}\log\frac{6}{10} - \frac{4}{10}\log\frac{4}{10}$$

$$= 0.97\ b$$

d). We didn't do this in the lectures

③ $$P(X,Y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1/6 & 1/6 & 1/6 \\ x_2 & 1/6 & 1/6 & 1/6 \end{array}$$

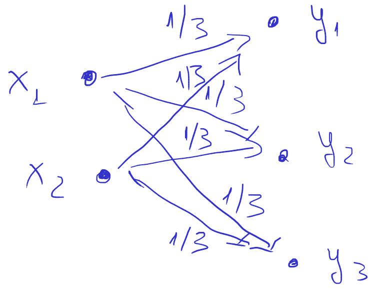


Coin: $p(x_1) = p(x_2) = \frac{1}{2}$

3-side dice: $p(y_1) = p(y_2) = p(y_3) = \frac{1}{3}$

independent: $\boxed{p(x_1 \cap y_1) = p(x_1)\cdot p(y_2)}$

b) $$P(Y|X) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$



c) $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \underline{1}$

$H(Y) = -3\cdot\frac{1}{3}\log\frac{1}{3} = \log 3$

$$H(X,Y) = -6\cdot\frac{1}{6}\log\frac{1}{6} = \log 6 = \log(2\cdot 3) = \underbrace{\log 2}_{\underbrace{1}_{H(X)}} + \underbrace{\log 3}_{H(Y)}$$

$$I(X,Y) = H(X) + H(Y) - H(X,Y) = 0$$

(4) $$P(Y|X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ x_1 & \multicolumn{3}{c}{\left[\begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]} \\ x_2 & & & \\ x_3 & & & \end{array} \qquad H(Y|X) = 0$$

(5) $$P(Y|X) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 \end{matrix} \\ \left[\begin{matrix} 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{matrix}\right] \end{array} \begin{array}{l} \\ \longrightarrow H(Y|x_1) \neq 0 = \log 3 \\ \longrightarrow H(Y|x_2) = 0 \end{array}$$