Exercises (Week 2)

Memoryless Sources

1) Done on the slides, 1st lecture

2)

3)

4) $S: \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$ $\qquad i(s_k) = -\log_2(p(s_k))$

a) $i(s_1) = -\log_2\left(\frac{1}{2}\right) = -\log_2 2^{-1} = -(-1) = 1$ bit

$i(s_2) = -\log_2(0) = \infty$ bits

$i(s_3) = -\log_2\left(\frac{1}{8}\right) = -\log_2 2^{-3} = 3$ bits

b). $\bar{i} = H(S) = -\sum_{i=1}^{5} p(s_i)\cdot\log(p(s_i)) = -\frac{1}{2}\log\frac{1}{2} - 0\cdot\log 0 - \frac{1}{8}\log\frac{1}{8} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{8}\log\frac{1}{8}$

$= \frac{1}{2} - 0\cdot\log 0 + \frac{3}{8} + \frac{2}{4} + \frac{3}{8} = \frac{7}{4} = 1.75$ bits

$-0\cdot\log_2 0 = ??$ $\quad (0\cdot\infty)$

l'Hopital

$\lim_{x\to 0} -x\cdot\log x = -\lim_{x\to 0} x\cdot\log_2 x = -\lim_{x\to 0}\frac{\log_2 x}{\frac{1}{x}} = -\lim_{x\to 0}\frac{(\log_2 x)'}{\left(\frac{1}{x}\right)'}$

$(\log_2 x)' = \frac{1}{x\cdot\ln 2}$

$= -\lim_{x\to 0}\frac{\frac{1}{x\cdot\ln 2}}{\frac{-1}{x^2}} = \lim_{x\to 0}\frac{1}{x\cdot\ln 2}\cdot\frac{x^2}{1} = \lim_{x\to 0}\frac{x}{\ln 2} = 0$

c). $\eta = \frac{H(S)}{H_{max}}$ $\qquad H_{max} = \log_2(M) = \log_2(5) = 2.32$

(M = # messages)

$\eta = \frac{1.75}{2.32} = 0.75 = 75\%$ $\qquad S_{max}: \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \Rightarrow H(S) = -\frac{1}{5}\log\frac{1}{5}\cdot 5 = \log_2 5$

$R = H_{max} - H(S) = 2.32 - 1.75 = 0.57$

$\rho = 1 - \eta = 0.25 = 25\% = \frac{0.57}{2.32}$

$\lim_{x\to\infty}\frac{2^x}{x^{10}} = \lim_{x\to\infty}\underbrace{2^x}_{\to\infty}\cdot\underbrace{\frac{1}{x^{10}}}_{\to 0} = \infty$

exponential >> polynomial >> log

a^x $\qquad x^b$ $\qquad \log(x\ldots)$

$s_1 \quad s_2 \quad s_3 \quad s_4$

5) $P = [0 \quad 0 \quad 0 \quad 1]$

$Q = [0.1 \quad 0.05 \quad 0.05 \quad 0.8]$

$$D_{KL}(P, Q) = \sum_i p_i \cdot \log \frac{p_i}{q_i}$$

$$D_{KL}(P, Q) = \underbrace{0 \cdot \log_2 \frac{0}{0.1}}_{0} + \underbrace{0 \cdot \log_2 \frac{0}{0.05}}_{0} + \underbrace{0 \cdot \log_2 \frac{0}{0.05}}_{0} + \underbrace{1 \cdot \log_2 \frac{1}{0.8}}_{0.32}$$

$0 \cdot \log 0 = 0$

$= 0.32$

$Q_2 : [0.5 \quad 0.2 \quad 0.1 \quad 0.2]$

$$D_{KL}(P, Q_2) = \underbrace{0 \cdot \log \frac{0}{0.5}}_{0} + \underbrace{0 \cdot \log \frac{0}{0.2}}_{0} + \underbrace{0 \cdot \log \frac{0}{0.1}}_{0} + \underbrace{1 \cdot \log \frac{1}{0.2}}_{2.32}$$

$= 2.32$