Markov Decision Processes

- MDPs are defined by:
  - Set of states S
  - Set of actions A
  - Transition Function $P(s'|s, a)$ defines all such transitions from all states s(row) to all successor states s'(column):
$$P = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}$$
  - Reward Function $R(s, a, s')$
  - Start State s_0 or distribution of start state probabilities
  - Discount Factor γ
  - Horizon H (how long we observe and act)
- MDPs describe a fully observable environment for RL
- Each state by itself can completely characterize the process
- Even partially observable problems can be converted to MDPs
- **Markov Property**
  - A state is Markov iff $P[S_{t+1}|S_t] = P[S_{t+1}|S+1, \cdots, S_t]$, i.e. the future is independent of the past given the present. The state is a sufficient stastistic of the future
  - For a Markov state s and successor state s', we can define a state transition probability $P_{ss'}$:
$$P_{ss'} = \mathbb{P}[S_{t+1} = s'|S_t = s]$$
- **Markov Process/Markov Chain:** A memory-less random process(sequence of random states $S_1, S_2, \cdots$), with the Markov Property
  - A tuple $\langle S, P \rangle$
    - S is a finite set of states
    - P is the state transition probability matrix

Markov Reward Process:

- A markov chain with values
- Formally, it is a tuple $\langle S, P, R, \gamma \rangle$
  - S is a finite set of states
  - P is the state transition probability matrix

- R is the reward function, $R_s = \mathbb{E}[R_{t+1}|S_t = s]$
- γ is the discount factor, $\gamma \in [0, 1]$. 0 is myopic evaluation, 1 is far-sighted evaluation

- We define the following things about MRPs
- **Return(G_t):** Total discounted reward from time step t
  - $G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$
  - γ acknowledges the value of future rewards at time step t
- **Value Function:** $v(s)$
  - The state value function of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}[G_t|S_t = s]$$

  - Given the current state, what is the expected return from all possible future time steps from this state
- **Bellman Equation**
  - The value function can be broken up into two parts:
    - immediate reward R_{t+1}
    - discounted value of successor state $\gamma v(S_{t+1})$

$$\begin{aligned} v(s) &= \mathbb{E}[G_t|S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots |S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \cdots)|S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1}|S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma v(s+1)|S_t = s] \end{aligned}$$

    - The value of the current state is equal to the sum of the expected immediate reward and the discounted value of the next state
    - $v(s) = R_s + \gamma \sum_{s' \in S} v(s')$
    - Vector Representation: $v = R + \gamma P v$
      - P is the state transition matrix
      - R is the reward matrix
      - v is the value vector
- Solving the Bellman Equation
  - Linear equation and can be solved directly

$$\begin{aligned} v &= R + \gamma P v \\ v(1 - \gamma P) &= R \\ v &= (1 - \gamma P)^{-1} R \end{aligned}$$

  - Complexity is $O(n^3)$. Not feasible for large MRPs.
  - There are other methods for solving MRPs:
    - Dynamic Programming
    - Monte-Carlo Evaluation
    - Temporal Difference Learning

Markov Decision Process(Formal Definition):

- A Markov Reward Process with decisions
- Formally, it is a tuple $\langle S, A, P, R, \gamma \rangle$
  - S is a finite set of states
  - A is a finite set of actions
  - P is the state transition matrix, redefined as $P^a_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$ to take into account A
  - R is the reward function, redefined as $R^a_s = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$ to take into account A
  - γ is the discount factor
- To maximize the goal of maximizing the sum or rewards, we need to define the behaviour of the agent. This is known as the policy.
- **Policy (π):** A distribution over actions given states

$$\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$$

  - If the agent is in a state s, the policy gives the probability of all the actions that can be taken from that state.
  - MDP Policies only depend on the current state and not the history. Policies are stationary(time independent): $A_t \sim \pi(\cdot|S_t), \forall t > 0$
  - If we have some MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and fix a policy π:
    - the state sequences $S_1, S_2, \cdots$ is a Markov Process $\langle \mathcal{S}, \mathcal{P}^\pi \rangle$ and
    - the state-reward sequence $S_1, R_1, S_2, R_2, \cdots$ is a Markov Reward Process $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^p i, \gamma \rangle$, defining the transition dynamics and reward function by averaging over the policy.

$$\mathcal{P}^\pi_{s,s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{s,s'}$$
$$\mathcal{R}^\pi_{s,s'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^a_{s,s'}$$

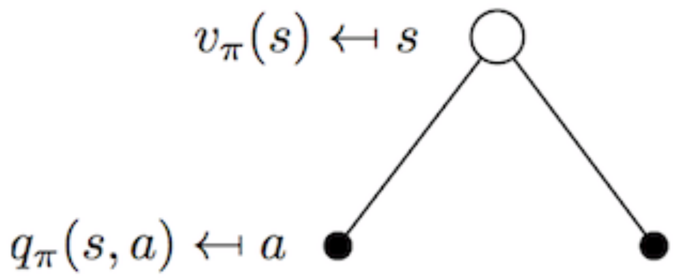
- **Value Function:**
  - state-value function $v_\pi(s)$ of a MDP is the expected return starting from s and following the policy π

$$v_\pi(s) = \mathbb{E}[G_t | S_t = s]$$

  - action-value function $q_\pi(s, a)$ of a MDP is the expected return starting from state s by taking an action a and then following a policy π.

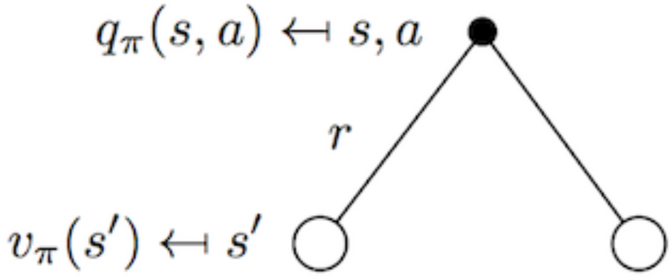
$$q_\pi(s, a) = \mathbb{E}[G_t | A_t = a, S_t = s]$$

- Relationship between state and action value functions:



$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

> An agent following a policy π is in state s and has choices for its actions, each of which have an associated q value. Hence the value of state s is the sum of q values of subsequent states multiplied by their probabilities.



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

> An agent following a policy π is in state s and does an action a. The value of this action is the sum of the reward for the action and the discounted value of the subsequent states given some transition matrix.

- **Bellman Expectation Equation**
  - Decomposing the state-value function — splitting expected reward from current state into sum of immediate reward and discounted reward of successor state — we can arrive at the **Bellman function**:

$$v_\pi(s) = \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$$

  - Similarly decomposing the action-value function, we get:

$$q_\pi(s,a) = \mathbb{E}[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | A_t = a, S_t = s]$$

- **Optimal Value Functions**
  - To define optimality, we define a partial ordering of policies: $\pi \geq \pi' \ \ if \ \ v_\pi(s) \geq v_{\pi'}(s) \forall s$
  - There always exists an optimal policy $\pi_* \geq \pi, \forall \pi$.
  - To find the highest reward, we want an optimal state-value function and optimal action-value function. This is usually defined as the best function over several alternate policies.

$$v_*(s) = \underset{\pi}{max} \ \ v_\pi(s)$$
$$q_*(s,a) = \underset{\pi}{max} \ \ q_\pi(s,a)$$

  - All optimal policies achieve the optimal state and action value functions $v_{\pi_*}(s) = v_*(s)$ and $q_{\pi_*}(s,a) = q_*(s,a)$
  - The optimal value function specify the best performance in an MDP and access to this function renders the MDP as solved.
  - We can find an optimal policy by maximizing over $q_*(s,a)$. If an agent is in a state s, it picks an action a with probability 1, that maximizes $q_*(s,a)$.

$$\pi_*(a|s) = \begin{cases} 1 & if \ \ a = \underset{a \in \mathcal{A}}{argmax} \ \ q_*(s,a) \\ 0 & otherwise \end{cases}$$

- **Bellman Optimality Equation for v_***

  $v_* (s) = \underset{a}{max}\ \ R_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_*(s')$

  > Max value from current state s is the sum of max reward from current state and expected value of next state s' according to optimal policy, discounted by γ.

- **Bellman Optimality Equation for q_***

  $q_*(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \underset{a'}{max}\ q_*(s', a')$

  > Max q value from current state s and given action a is the sum of immediate reward and expected q value for state s' and most optimal action a'.