Gravitational Waves: introduction to signals and data analysis

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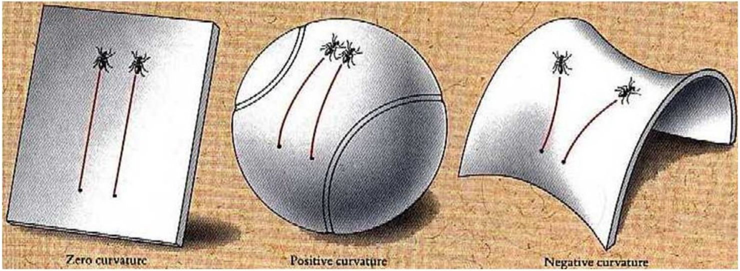
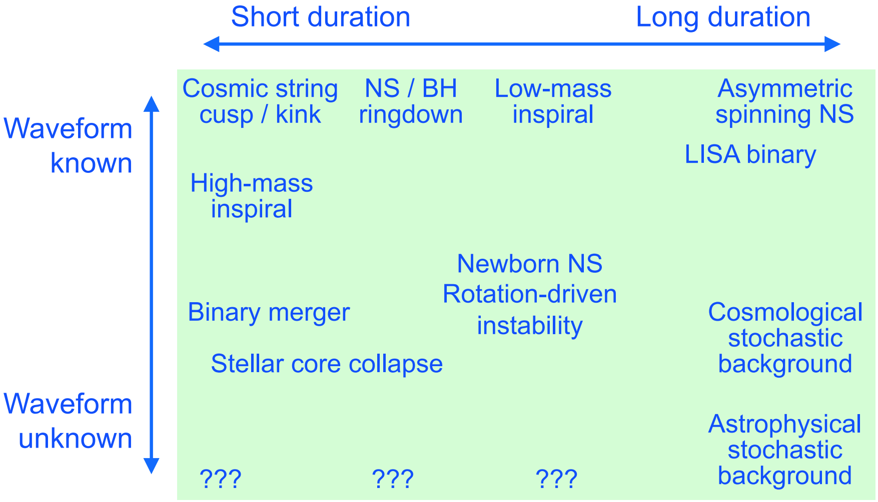
Based on ”Gravitational waves”: https://users.camk.edu.pl/bejger/lectures

Geodesic deviation in curved spacetime

- ⋆ In general relativity, trajectories of freely-falling particles are geodesics (the equivalent of straight lines in curved spacetime)
  - → Newton's 1st law: Unless acted upon by a non-gravitational force, a test mass will follow a geodesic.
- ⋆ The curvature of spacetime is revealed by the behavior of neighbouring geodesics
- ⋆ Non-zero curvature ↔ acceleration of geodesic deviation ↔ non-uniform gravitational field

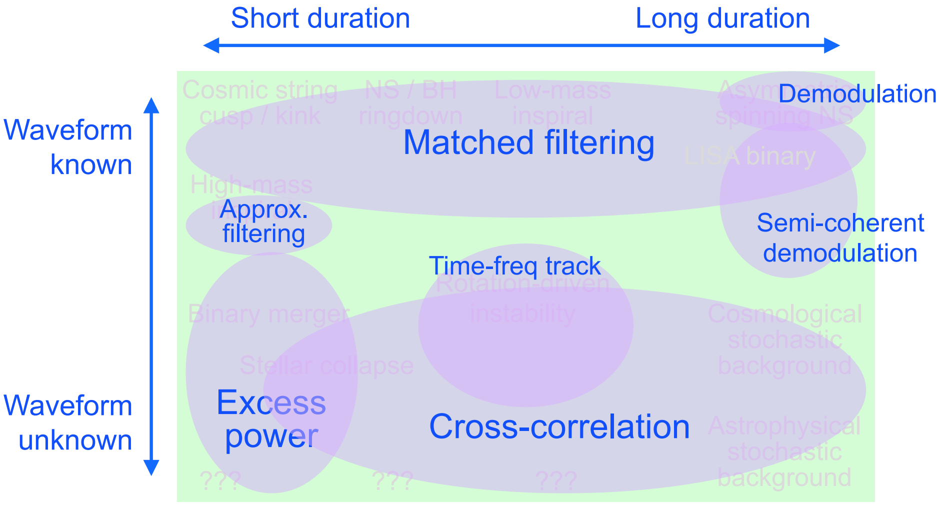


Taxonomy of signal and search types



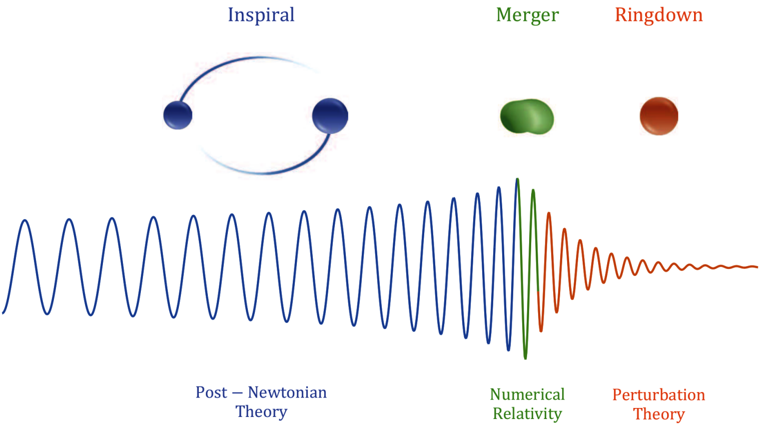
courtesy of Peter Shawhan

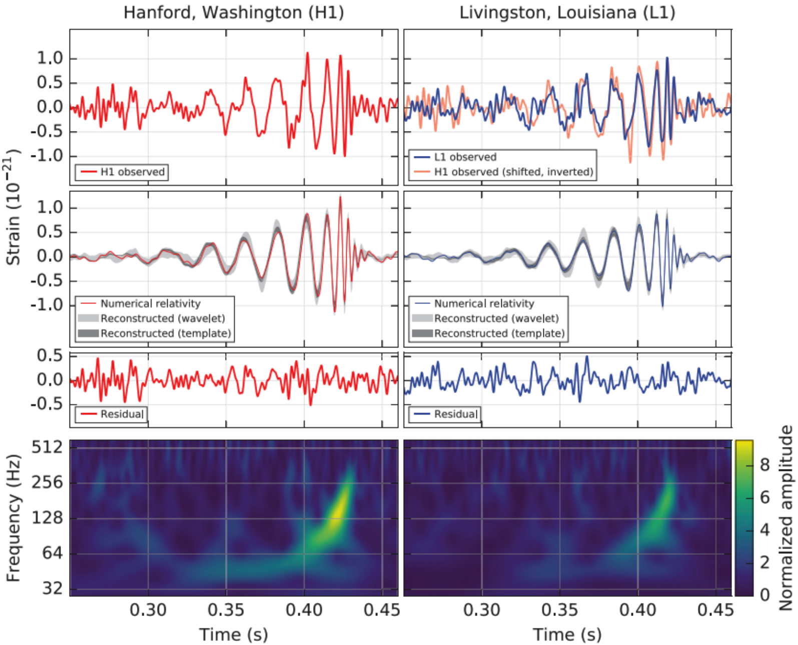
Taxonomy of signal and search types



courtesy of Peter Shawhan

Last orbits of a binary system





Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
Normalized amplitude
Time (s)

GW150914: parameters

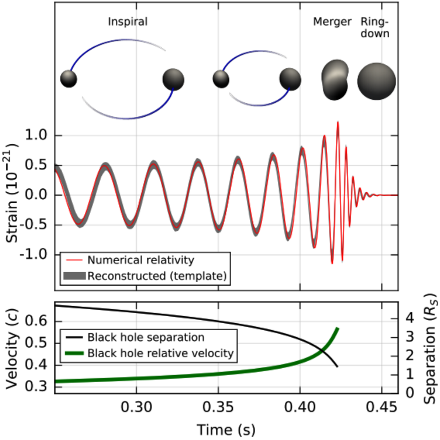
False alarm probability <1 in 5 million
False alarm rate <1 in 200000 years

- $M_1 = 36^{+5}_{-4}\ \mathrm{M}_\odot$, $M_2 = 29^{+4}_{-4}\ \mathrm{M}_\odot$,
- Final black hole parameters:
  - mass $M = 62^{+4}_{-4}\ \mathrm{M}_\odot$,
  - spin $a = 0.67^{+0.05}_{-0.07}$,
- Distance: $410^{+160}_{-180}\ \mathrm{Mpc}$
  i.e. 1 billion 300 million light years,
  redshift $z = 0.09^{+0.03}_{-0.04}$ (assuming standard cosmology).

(uncertainties define 90% credible intervals)

GW150914: parameters

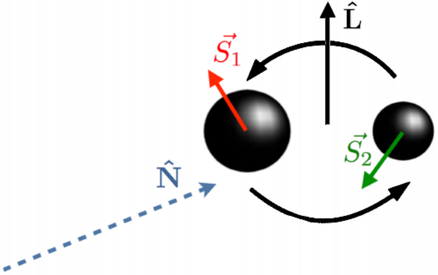
- ⋆ Duration: 0.2 s,
- ⋆ Final orbital velocity: $> 0.5\ \mathrm{c}$,
- ⋆ Total energy emitted in waves: $E = mc^2 = 3^{+0.5}_{-0.5}\ \mathrm{M_\odot c^2}$,
- ⋆ Peak "brightness": $3.6^{+0.5}_{-0.4} \times 10^{49}\ \mathrm{Joule/s}$ ($200^{+30}_{-30}\ \mathrm{M_\odot c^2/s}$), $\rightarrow$ much more than all the stars in the Universe radiate in EM!

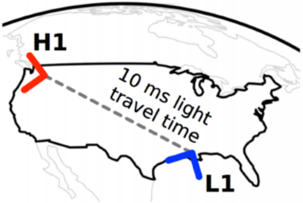


(uncertainties define 90% credible intervals)

Binary system: 15+ parameters

- Intrinsic:
  - masses
  - spins
  - tidal deformability





Credit: LIGO/Virgo

- Extrinsic:
  - Inclination, distance, polarisation
  - Sky location
  - Time, reference phase

Effects of various parameters on inspiral waveform

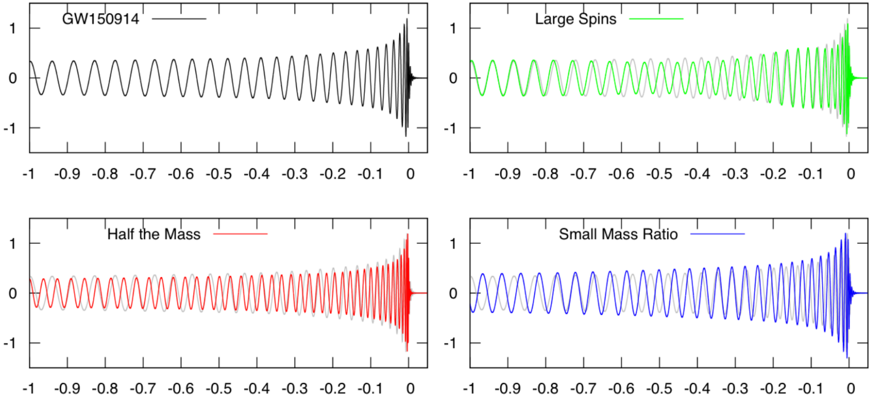


Illustration by N. Cornish and T. Littenberg

Initial estimates

For a spherical wave of amplitude $h(r)$,

- $\star$ flux of energy is $F(r) \propto h^2(r)$,
- $\star$ the luminosity $L(r) \propto 4\pi r^2 h^2(r)$.

Conservation of energy (flux through surface at r):

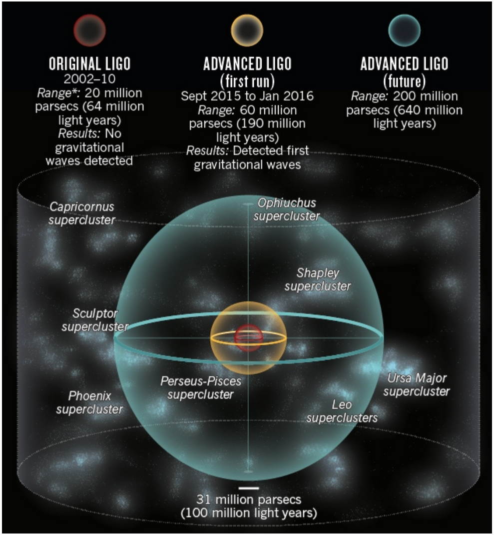
$$\Longrightarrow h(r) \propto 1/r.$$

Radiating modes: quadrupole and higher

For a mass distribution $\rho(r)$, conserved moments:

- $\star$ monopole $\int \rho(r) d^3r$ - total mass-energy (energy conservation),
- $\star$ dipole $\int \rho(r) r d^3r$ - center of mass-energy (momentum conservation).

Sensitivity → amplitude → volume



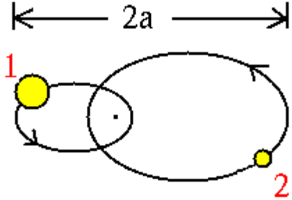
- ⋆ Detector's sensitivity (registering waves of amplitude h) is related to maximal range r, $h \propto 1/r$
- ⋆ Reachable cosmic volume $V \propto r^3$
- ⋆ Increase of sensitivity $h \to 0.1h$ gives $r \to 10r$, that is $V \to 1000V$.

More estimates

GWs correspond to accelerated movement of masses.

Consider a binary system of m_1 and m_2, semiaxis a with

- ⋆ total mass $M = m_1 + m_2$,
- ⋆ reduced mass $\mu = m_1 m_2 / M$,
- ⋆ mass quadrupole moment $Q \propto Ma^2$,
- ⋆ Kepler's third law $GM = a^3\omega^2$.



$$h(r) \propto \frac{1}{r}\frac{\partial^2 (Ma^2)}{\partial t^2} = \boxed{\frac{G^2}{c^4}\frac{1}{r}\frac{M\mu}{a} = \frac{G^{5/3}}{c^4}\frac{1}{r}M^{2/3}\mu\,\omega^{2/3}.}$$

Gravitational waves: quadrupole approximation

The quadrupole approximation (slowly-moving sources, Einstein 1918), wave amplitude is

$$h^{\mu\nu} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}^{\mu\nu}, \quad \text{or, in terms of kinetic energy,} \quad h \sim \frac{E^{nsph.}_{kin.}}{r}.$$

Resulting GW luminosity is

$$L_{GW} \equiv \frac{dE_{GW}}{dt} \approx \frac{1}{5}\frac{G}{c^5}\langle \dddot{Q}^{\mu\nu}\dddot{Q}_{\mu\nu}\rangle$$

$$\propto \frac{G}{c^5}Q^2\omega^6 \propto \frac{G^4}{c^5}\left(\frac{M}{a}\right)^5 \propto \frac{c^5}{G}\left(\frac{R_s}{a}\right)^2\left(\frac{v}{c}\right)^6.$$

($R_s = 2GM/c^2$, $c^5/G \simeq 3.6 \times 10^{52}$ Joule/s)

Binary system: evolution of the orbit

Waves are emitted at the expense of the orbital energy:

$$E_{orb} = -\frac{Gm_1 m_2}{2a}, \qquad \frac{dE_{orb}}{dt} \equiv \frac{Gm_1 m_2}{2a^2}\dot{a} = -\frac{dE_{GW}}{dt}.$$

Evolution of the semi-major axis:

$$\frac{da}{dt} = -\frac{dE_{GW}}{dt}\frac{2a^2}{G\underbrace{m_1 m_2}_{\mu M}} \rightarrow \frac{da}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^4}{a^3}.$$

The system will coalesce after a time τ,

$$\tau = \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^4},$$

where a_0 is the initial separation.

Binary system: chirp mass

Waves are emitted at the expense of the orbital energy:

$$E_{orb} = -\frac{Gm_1 m_2}{2a}, \qquad \frac{dE_{orb}}{dt} \equiv \frac{Gm_1 m_2}{2a^2}\dot{a} = -\frac{dE_{GW}}{dt}.$$

Resulting evolution of the orbital frequency ω:

$$\dot{\omega}^3 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mu^3 M^2 = \left(\frac{96}{5}\right)^3 \frac{\omega^{11}}{c^{15}} G^5 \mathcal{M}^5,$$

where $\mathcal{M} = \left(\mu^3 M^2\right)^{1/5} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$ is the chirp mass. GWs frequency from a binary system is primarily twice the orbital frequency ($2\pi f_{GW} = 2\omega$). $\mathcal{M}$ is a directly measured quantity:

$$\mathcal{M} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3} f_{GW}^{-11/3} \dot{f}_{GW}\right)^{3/5}.$$

Binary system: energy emitted in GWs

End of the chirp f^c_{GW} is related to critical distance between masses a_{fin}:

$$a_{fin} = R_{s1} + R_{s2} = \frac{2G}{c^2}\left(m_1 + m_2\right).$$

It can be used to estimate the total mass M:

$$M = m_1 + m_2 \approx \frac{c^3}{2\sqrt{2}G\pi}\frac{1}{f^c_{GW}}.$$

Energy emitted during the life of the binary system (rest-mass energy + orbital energy):

$$E = E_{rm} + E_{orb} = \left(m_1 + m_2\right)c^2 - \frac{Gm_1 m_2}{2a}.$$

(for $m_1 = m_2$, $a_{fin} = 2R_s = 4Gm_1/c^2$, $\Delta E \approx 6\%$).

Final black hole spin

Orbital angular momentum J (major semi-axis $a = a_1 + a_2$):

$$J = m_1 a_1^2 \omega + m_2 a_2^2 \omega = m_1 m_2 \sqrt{\frac{Ga}{m_1 + m_2}}.$$

Dimensionless spin magnitude χ of an object with J and M:

$$\chi = \frac{cJ}{GM^2}.$$

For $m_1 = m_2 = m$,

- $\star$ $a = a_{fin}$, $\chi \approx 0.35$,
- $\star$ $a = 2r_{isco}$ (innermost stable circular orbit around BH of mass m), $\chi \sim 0.7$.

Circularization of orbit by GW radiation

Peters (1964, p. 103):

The equation of the relative orbit of the motion is

$$r = \frac{a\,(1 - e^2)}{1 + e\cos(\Psi - \Psi_o)} \; . \qquad (5.36)$$

Equation 5.43 can be integrated to get a(e) during the collapse of a system. The integration is tedious but straight forward. a(e) is then found to be

$$a = \frac{c_o\, e^{12/19}}{(1 - e^2)} \left[1 + \frac{121}{304} e^2\right]^{\frac{870}{2299}} , \qquad (5.48)$$

where c_o is determined by the initial conditions $a = a_o$ when $e = e_o$. a is plotted against e in Figure 5.

For small e, this reduces to

$$a = c_o\, e^{12/19} \quad , \quad e^2 \ll 1 ,$$

For a - semi-major axis, e - eccentricity,

$$\frac{e}{e_0} \approx \left(\frac{a}{a_0}\right)^{19/12}$$

Reduction of a by ≈ 2 means reduction in eccentricity by ≈ 3
$\rightarrow$ radiation reaction quickly circularizes the orbit.

Parameter estimation basics (GW510914)

GW amplitude dependence for a binary system

$$h \propto \mathcal{M}^{5/3} \times f_{GW}^{2/3} \times r^{-1}$$

where $\mathcal{M}$ is the chirp mass, $\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$, known from the observations:

$$\mathcal{M} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f_{GW}^{-11/3} \dot{f}_{GW}\right]^{3/5}$$

From higher-order post-Newtonian corrections: $q = m_2/m_1$, spin components parallel to the orbital angular momentum...

$$\mathcal{M} \simeq 30 M_\odot \implies M = m_1 + m_2 \simeq 70 M_\odot \quad (\text{if } m_1 = m_2,\ M = 2^{6/5}\mathcal{M})$$

8 orbits observed until 150 Hz (orbital frequency 75 Hz):

- ⋆ Binary neutron star system is compact enough, but too light,
- ⋆ Neutron star-black hole system for a given total mass - black hole too big, would merge at lower frequency.

→ Black hole binary.

Binary system: source distance estimate

- $\star$ At cosmological distances, the observed frequency f_{GW} is redshifted by $(1+z)$:

$$f \to f/(1+z)$$

- $\star$ There is no mass scale in vacuum GR, so redshifting of f_{GW} cannot be distinguished from rescaling the masses

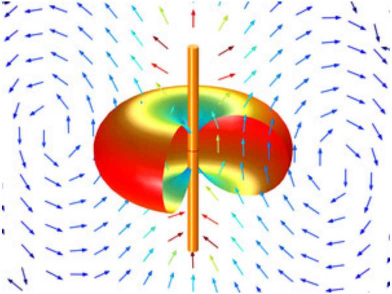
  because the waveform's phase is an expansion in powers of a small parameter, like the orbital frequency $v/c \propto (\pi M f_{GW})^{1/3}$

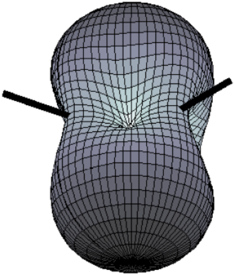
$\Longrightarrow$ inferred masses are $m = (1+z)m^{source}$

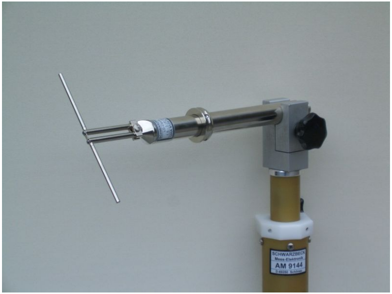
$\rightarrow$ Direct, independent **luminosity distance** measurement (but not z) from GW with f_{GW} and the strain h:

$$r = \frac{5}{96\pi^2}\frac{c}{h}\frac{\dot{f}_{GW}}{f_{GW}^3}.$$

Antenna directionality





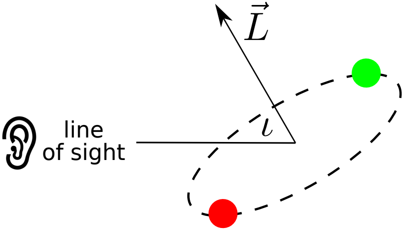


Binary system: distance-inclination degeneracy

Luminosity distance $\sim 1/h$, and

$$h = h_+ F_+ + h_\times F_\times$$

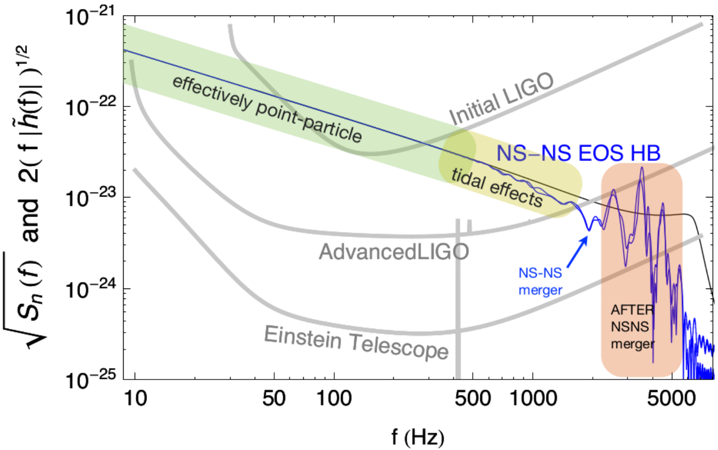
depends on the inclination of the binary with repect to the "line of sight".



Two independent polarizations h_+ and $h_\times$:

$$h_+ = \frac{2\mu}{r} (\pi M f_{GW})^{2/3} \left(1 + \cos^2 \iota\right) \cos(2\phi(t)),$$
$$h_\times = \frac{4\mu}{r} (\pi M f_{GW})^{2/3} \cos \iota \sin(2\phi(t)).$$

Binary inspiral vs the sensitivity curve



For extended-body interactions, phase evolution differs from point-particle description,

$$\Psi(f) = \Psi_{PP}(f) + \Psi_{tidal}(f)$$

Ψ_{tidal} breaks the v expansion degeneracy.

Astrophysically-interesting parameters

- ⋆ Chirp mass $\mathcal{M} = \left(\mu^3 M^2\right)^{1/5} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$,
- ⋆ Mass ratio $q = m_2/m_1$ (at 1PN), alternatively $\nu = m_1 m_2/(m_1 + m_2)^2$,
- ⋆ Spin-orbit and spin-spin coupling (at 2PN and 3PN, resp.) $\rightarrow$

$$\chi_{eff} = (m_1 \chi_{1z} + m_2 \chi_{2z})/(m_1 + m_2)$$

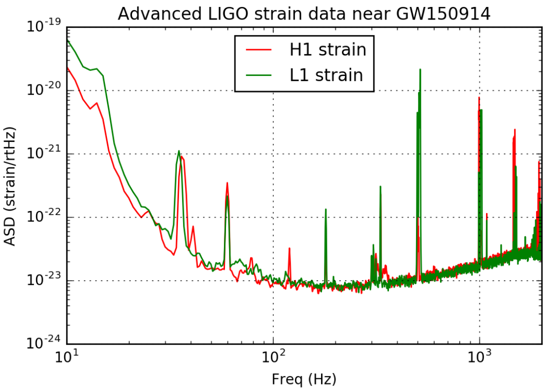
where χ_{iz} are spin components along system's total angular momentum,

- ⋆ Tidal deformability Λ (at 5PN) $\rightarrow$

$$\tilde{\Lambda} = \frac{16}{13} \frac{(m_1 + 12 m_2) m_1^4 \Lambda_1}{(m_1 + m_2)^5} + (1 \leftrightarrow 2)$$

- ⋆ Direct "luminosity" ("loudness") distance: binary systems are "standard sirens".

Sensitivity - amplitude spectral density of the noise



- ⋆ *Plot dominated by instrumental noise, lines*: mirror suspension resonances at 500 Hz and harmonics, calibration lines and power lines (60 Hz and harmonics) etc.,
- ⋆ Data sampled at 16384 Hz, so the Nyquist frequency is 8192 Hz,
- ⋆ Data stream: $\sim$50 MB/s (main GW + auxiliary ”witness” channels).

Sensitivity - amplitude spectral density of the noise

⋆ GW detectors register time series $x(t)$ (as light phase difference at the photodiode),

⋆ The average power P of $x(t)$ over T:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df$$

$$\text{Parseval's theorem:} \quad \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df,$$

with $\hat{x}(f)$ the Fourier transform of $x(t)$.

⋆ Power spectral density (PSD) is

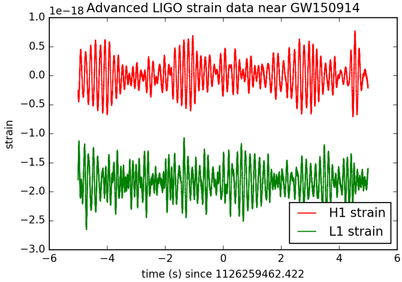
$$S(f) = \lim_{T\to\infty} \frac{1}{T} |\hat{x}(f)|^2, \quad \text{units: W/Hz.}$$

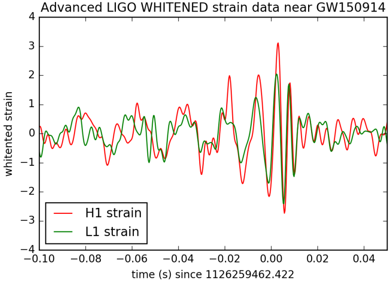
⋆ Amplitude spectral density (ASD) is

$$S_{ASD}(f) = \sqrt{S(f)}, \quad \text{units: } \sqrt{\text{W/Hz}}.$$

For a dimensionless amplitude $h = \Delta L/L$, the amplitude spectral density has units of $1/\sqrt{Hz}$.

How the data looks like



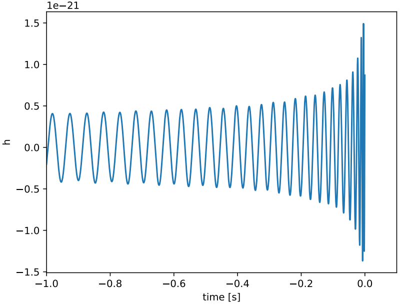


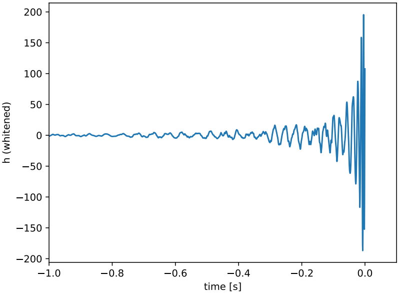
The data are dominated by the **low frequency noise** (L1 offset by -2×10^{-18} due to very low frequency oscillations).

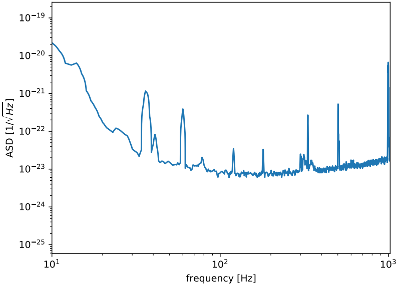
If one knows the signal is there:

- ⋆ filtering the frequencies outside the desired band with bandpass filter,
- ⋆ suppressing the instrumental lines,
- ⋆ whitening: dividing the data by the noise ASD in the Fourier domain to normalize the power for all frequencies for an easier comparison.

Raw vs whitened waveform

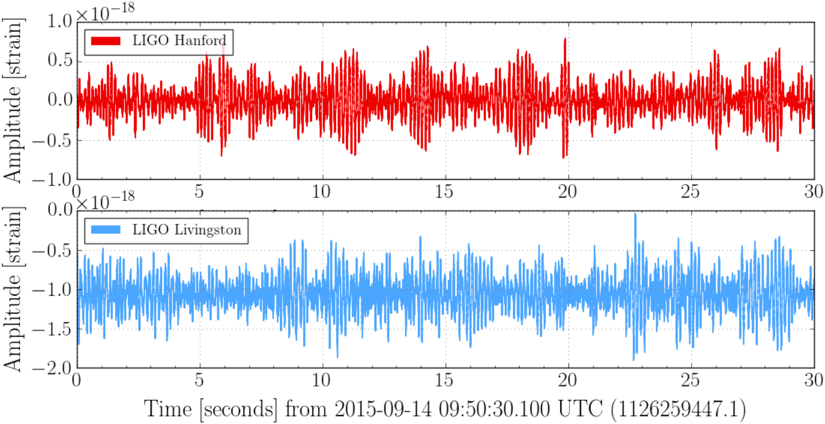






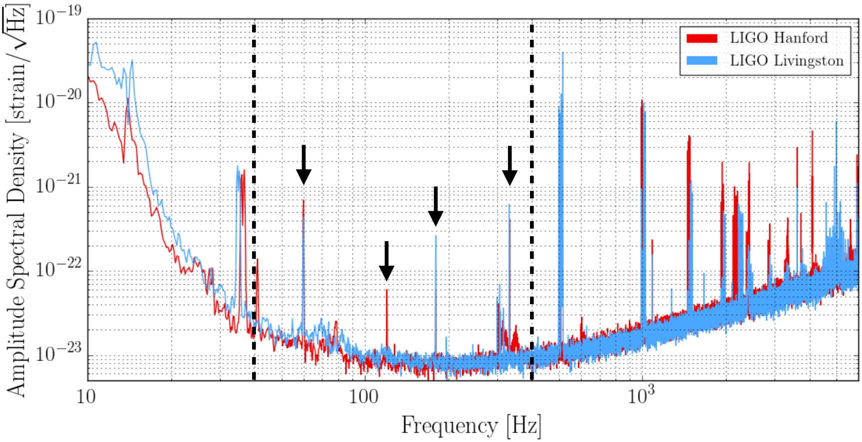
Whitening: $h_w(f) \to h(f)/ASD(f)$

Detecting GW150914: raw $h(t)$



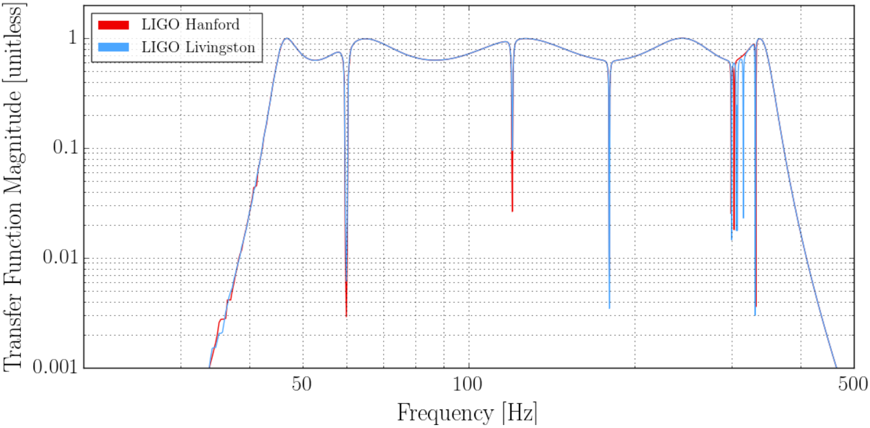
Raw data is dominated by the low-frequency content (mostly anthropogenic and seismic origin).

Detecting GW150914: ASD



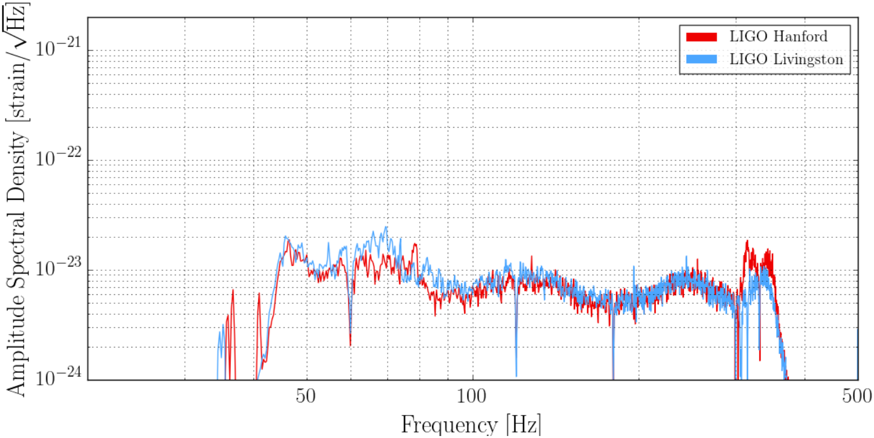
GW150914 signal is an inspiral and merger of two massive BH → restricted frequency range → bandpass filter (range indicated by vertical dashed lines). Additionally, applying the notch filter(s) to remove constant-frequency lines (calibration etc.), indicated by arrows.

Detecting GW150914: applying a transfer function

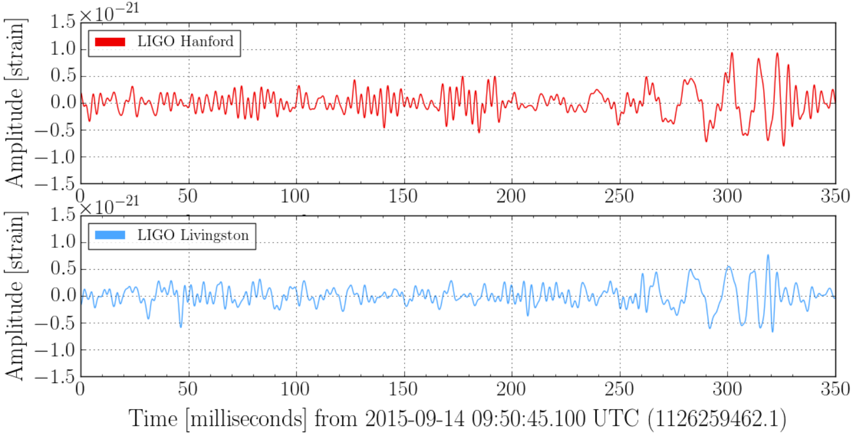


Removing specific frequencies using a sum of notch filters (transfer function).

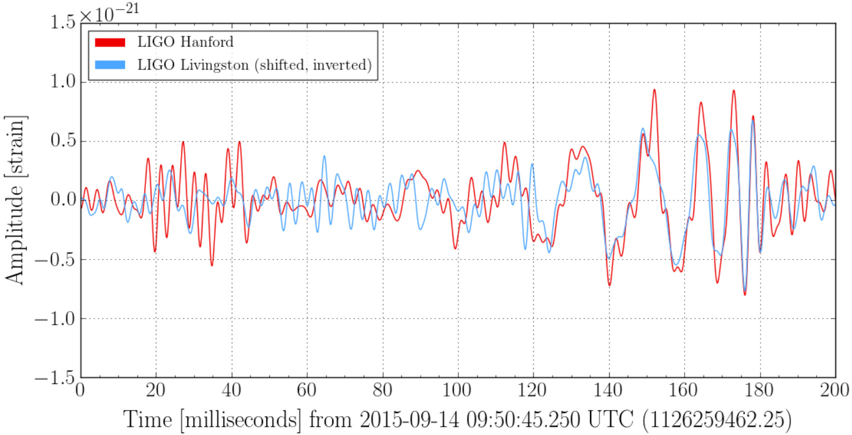
Detecting GW150914: bandpassed and notched data



Detecting GW150914: bandpassed, notched and whitened time domain data



Detecting GW150914: signal coherence



Comparison of two detectors' data: there is a coherent signal in both detector streams at the same time.

Matched filtering

Assuming a signal model h, looking for the "best match" correlation $C(t)$ in data stream x, for a given time offset t

$$C(t) = \int_{-\infty}^{\infty} \underbrace{x(t')}_{\text{Data}} \quad \times \quad \underbrace{h(t'-t)}_{\text{Template with time offset } t} \quad dt'$$

Rewrite correlation using Fourier transforms

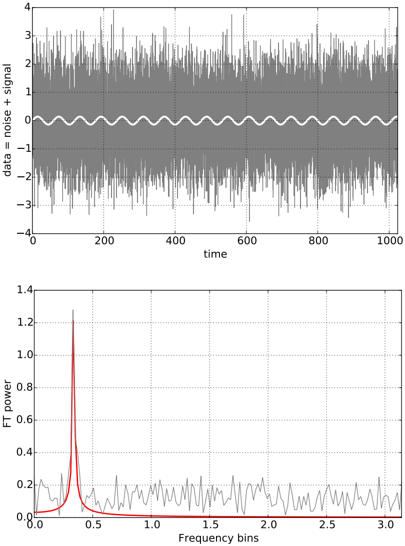
$$C(t) = 4\int_{0}^{\infty} \tilde{x}(f)\tilde{h}^{*}(f)e^{2\pi i f t}df$$

(an inverse FT of $\tilde{x}(f)\tilde{h}^{*}(f)$). In practice, optimal matched filtering with the frequency weighting

$$C(t) = (x|h) = 4\int_{0}^{\infty} \frac{\tilde{x}(f)\tilde{h}^{*}(f)}{S_n(f)}e^{2\pi i f t}df, \quad \text{with } S_n(f) \text{ (noise PSD).}$$

$$\boxed{\text{Matched filter SNR: } \rho = (x|h)/\sqrt{(h|h)}, \quad \text{Optimal SNR: } \rho_{opt} = \sqrt{(h|h)}}$$

Matched filtering: a monochromatic signal



In this case a Fourier transform is sufficient to detect the signal (simplest matched filter method):

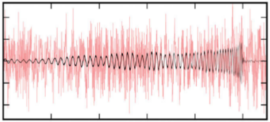
$$F = \left| \int_0^{T_0} x(t) \exp(-i\omega t) dt \right|^2$$

Signal-to-noise $SNR = h_0 \sqrt{\frac{T_0}{S_0}}$

T_0 - time series duration, S_0 - spectral density of the data.

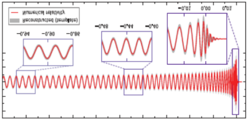
Matched filter in pictures

Data



X

Waveform



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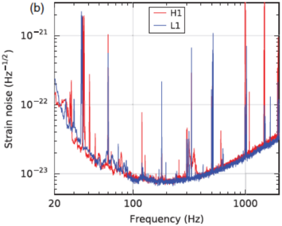
Sensitivity



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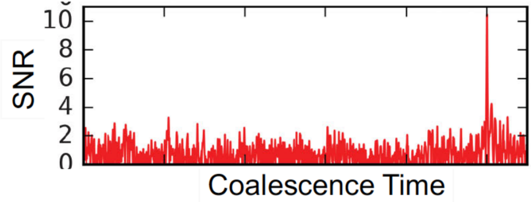
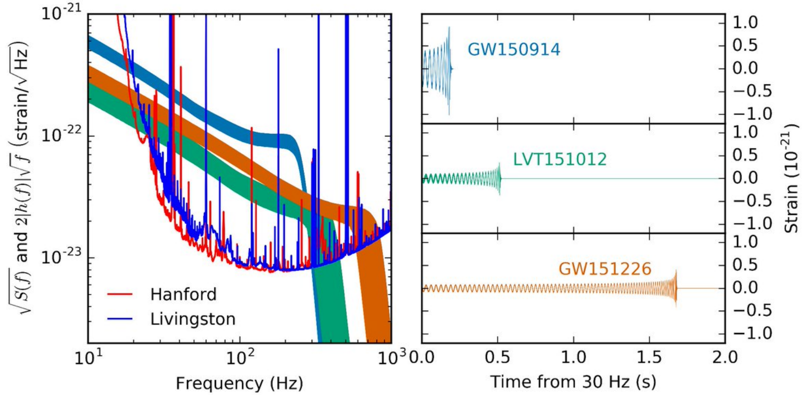


LIGO O1: 3 events

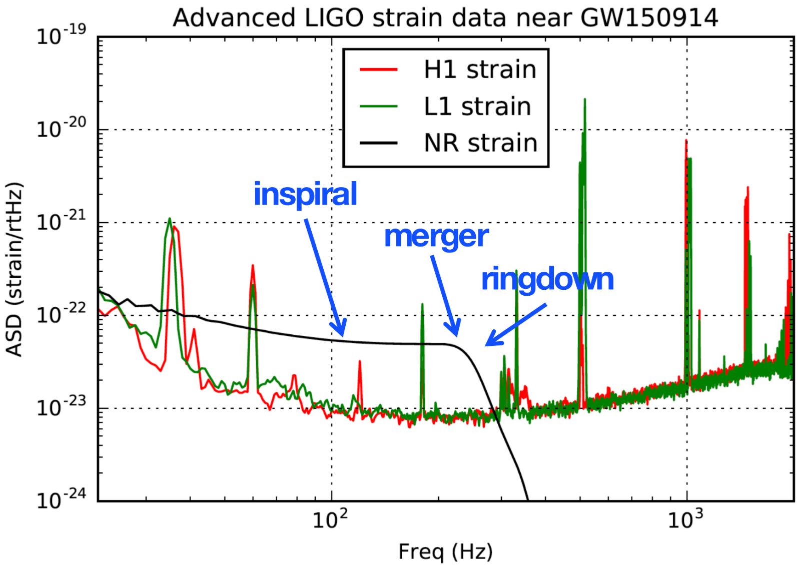


Optimal signal-to-noise ρ : $$\rho^2 = \int_0^\infty \left(\frac{2|\tilde{h}(f)|\sqrt{f}}{\sqrt{S_n(f)}} \right)^2 d\ln(f)$$

(GW150914: $\rho \simeq 24$, GW151226: $\rho \simeq 13$, GW151012: $\rho \simeq 10$)

+extra slides

Binary inspiral vs the sensitivity curve



Binary inspiral vs the sensitivity curve

The so-called *Newtonian* signal at instantaneous frequency f_{GW} is

$$h = Q(\text{angles}) \times \mathcal{M}^{5/3} \times f_{GW}^{2/3} \times r^{-1} \times e^{-i\Phi}.$$

where the signal's phase is

$$\Phi(t) = \int 2\pi f_{GW}(t')dt'.$$

The relation between f_{GW} and t

$$\pi \mathcal{M} f_{GW}(t) = \left(\frac{5\mathcal{M}}{256(t_c - t)} \right)^{3/8}$$

The orbital velocity is $v \propto (\pi M f_{GW})^{1/3}$ because from Kepler's 3rd law ($\omega^2 a^3 = GM$), one gets $\omega = 2\pi f = \pi f_{GW}$, $v = \omega a$ $\rightarrow v^3 = \pi G M f_{GW}$.

Binary inspiral vs the sensitivity curve

Matched filtering means that the signal is integrated with a proper phase as it sweeps through the range of frequencies.

Sensitivity curves most often show the effective (match-filtered) h_{eff}, and not the instantaneous h.

Dimensonal estimation of the frequency slope:

$$N_{cycles} \approx f_{GW}^2 \times \left(\frac{df_{GW}}{dt} \right)^{-1}$$

$$h_{eff} \propto \sqrt{N_{cycles}}\, h \propto \sqrt{f_{GW}\, t}\, h \propto \sqrt{f_{GW} \times f_{GW}^{-8/3}} \times f_{GW}^{2/3} \propto \boxed{f_{GW}^{-1/6}}.$$

Binary inspiral vs the sensitivity curve

Actually used in estimating the SNR is the frequency-domain match-filtering signal model $\tilde{h}(f) \simeq h_{eff}$ (Fourier transform of $h(t)$),

$$\tilde{h}(f) = Q(angles)\sqrt{\frac{5}{24}}\pi^{-2/3}\frac{\mathcal{M}^{5/6}}{r}f_{GW}^{-7/6}e^{-i\Psi(f)},$$

where the frequency domain phase Ψ is (in point-particle approximation):

$$\Psi(f) \equiv \Psi_{PP}(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3M}{128\mu v^{5/2}}\sum_{k=0}^{N}\alpha_k v^{k/2}.$$

Binaries are standard sirens

Binaries are *clean* systems: we have accurate models even in full general relativity.

Loss of energy to GWs causes orbit to decay, orbital frequency to go up. So the GWs will chirp up in frequency. Chirp time $t_{\text{chirp}} \sim f / [\mathrm{d}f / \mathrm{d}t]$.

Signal contains both apparent brightness (from h and f) and intrinsic luminosity (from t_{chirp}), from which we can compute the distance to the source:

$$\text{Distance} \propto c \frac{}{\text{frequency}^2 \times t_{\text{chirp}}}$$