Defining a system with adaptation that smoothly goes from Weber's law to no adaptation

Consider repertoire of ORN A_a responding to odor s and background b which are both vectors. We assume that background has been there for a while so if the system adapts it has adapted to b. $\epsilon_a[b]$ is the adaptation which is function of b.

$$A_a[s, b] = 1 \Big/ \left(1 + e^{\epsilon_a[b] - \mathrm{Log}[1 + K_{ai}\,(s_i + b_i)]}\right)$$

Here the K_{ai} is the matrix of inverse dissociation constants when the system is ACTIVE. (we neglected the other dissociation constants).

We assume that adaptation causes

$$A_a[0, b] = \frac{1}{1 + e^{\epsilon_{a0} - \beta\,\mathrm{Log}[1 + K_{ai}\,b_i]}} = \frac{1}{1 + \frac{1 - A_{a0}}{A_{a0}}\,e^{-\beta\,\mathrm{Log}[1 + K_{ai}\,b_i]}}$$

This second equation defines the adaptation function $\epsilon_a[b]$. We get:

$$\epsilon_a[b] = \epsilon_{a0} + (1 - \beta)\,\mathrm{Log}[1 + K_{ai}\,b_i]$$

Thus, if β=0 then the adaptation is perfect and we have Weber law. If β≠0 then the adaptation is imperfect and if β=1 then we have NO adaptation. Reintroducing into the first equation we have

$$\begin{aligned} A_a[s, b] &= 1 \Big/ \left(1 + e^{\epsilon_{a0} + (1-\beta)\,\mathrm{Log}[1 + K_{ai}\,b_i] - \mathrm{Log}[1 + K_{ai}\,(s_i + b_i)]}\right) \\ &= \frac{1}{1 + \frac{1 - A_{a0}}{A_{a0}}\,\frac{(1 + K_{ai}\,b_i)^{1-\beta}}{1 + K_{ai}\,(s_i + b_i)}} \end{aligned}$$

or in the free energy space defined as $F_a[s, b] = \mathrm{Log}\left[\frac{1 - A_a[s,b]}{A_a[s,b]}\right]$ we get:

$$\begin{aligned} F_a[s, b] &= \epsilon_{a0} - \mathrm{Log}\left[1 + K_{ai}\,\left(s_i + b_i\right)\right] + (1 - \beta)\,\mathrm{Log}[1 + K_{ai}\,b_i] \\ &= \epsilon_{a0} - \mathrm{Log}\left[\frac{1 + K_{ai}\,\left(s_i + b_i\right)}{\left(1 + K_{ai}\,b_i\right)^{1-\beta}}\right] \\ &= \epsilon_{a0} - \mathrm{Log}\left[\frac{K_{ai}\,s_i}{\left(1 + K_{ai}\,b_i\right)^{1-\beta}} + \left(1 + K_{ai}\,b_i\right)^{\beta}\right] \end{aligned}$$

For the intuition it is useful to define the following quantity because it grows with increasing A (so signs are not reversed):

$$\begin{aligned} U_a[s, b] = e^{-F_a[s,b]} &= U_{a0}\,\frac{1 + K_{ai}\,\left(s_i + b_i\right)}{\left(1 + K_{ai}\,b_i\right)^{1-\beta}} \\ &= U_{a0}\left(\frac{K_{ai}\,s_i}{\left(1 + K_{ai}\,b_i\right)^{1-\beta}} + \left(1 + K_{ai}\,b_i\right)^{\beta}\right) \end{aligned}$$

where

$$U_{a0} = \frac{A_{a0}}{1 - A_{a0}}$$

Why does Weber's law help coding?

we can define a distance between two odors with same background

$$\left(U_a[s_1, b] - U_a[s_2, b]\right)^2 = U_{a0}^2 \left(\frac{K_a\,(s_1 - s_2)}{\left(1 + K_a\, b\right)^{1-\beta}}\right)^2$$

to be compared to distance between responses to same odor but on different backgrounds

$$\left(U_a[s, b_1] - U_a[s, b_2]\right)^2 = U_{a0}^2 \left(\frac{1 + K_a\,\left(s + b_1\right)}{\left(1 + K_a\, b_1\right)^{1-\beta}} - \frac{1 + K_a\,\left(s + b_2\right)}{\left(1 + K_a\, b_2\right)^{1-\beta}}\right)^2$$

if β=0 we have Weber's law and we have

$$\left(U_a[s_1, b] - U_a[s_2, b]\right)^2 = U_{a0}^2 \left(K_a\,(s_1 - s_2)\right)^2 \left(\frac{1}{1 + K_a\, b}\right)^2$$

$$\left(U_a[s, b_1] - U_a[s, b_2]\right)^2 = U_{a0}^2 \left(K_a\,\left(b_2 - b_1\right)\right)^2 \left(\frac{K_a\, s}{\left(1 + K_a\, b_1\right)\,\left(1 + K_a\, b_2\right)}\right)^2$$

In terms of variances we should first write it as:

$$\left(U_a[s_1, b] - U_a[s_2, b]\right)^2 \left(1 + K_a\, b\right)^2 = U_{a0}^2 \left(K_a\,(s_1 - s_2)\right)^2$$
$$\left(U_a[s, b_1] - U_a[s, b_2]\right)^2 \left(\left(1 + K_a\, b_1\right)\,\left(1 + K_a\, b_2\right)\right)^2 = U_{a0}^2 \left(K_a\,\left(b_2 - b_1\right)\right)^2 \left(K_a\, s\right)^2$$

if β=1 we do not have adaptation and we have:

$$\left(U_a[s_1, b] - U_a[s_2, b]\right)^2 = U_{a0}^2 \left(K_a\,(s_1 - s_2)\right)^2$$

$$\left(U_a[s, b_1] - U_a[s, b_2]\right)^2 = U_{a0}^2 \left(K_a\,\left(b_1 - b_2\right)\right)^2$$