Due Tuesday, April 2

1 Introduction

In the previous homework guidelines were given for selecting the weights (the matrices $\mathbf{Q}$ and $\mathbf{R}$) in an LQR design for a plant in which the state variables had physical significance. It is common for the state-space model of a plant to be obtained from physical measurements of inputs and outputs. In such cases, only the input and output variables have physical significance. As shown below, the output variables are easy to incorporate into the LQR performance index. If it is desired to add some damping to lightly damped plant poles, the state-space model may be transformed to modal canonical form to determine the appropriate contribution to the $\mathbf{Q}$ matrix.

Suppose we want to regulate a plant whose state-space model is $(\mathbf{A}, \mathbf{B})$. Although we are considering the design of a state-feedback regulator, it may be that an output equation $\mathbf{y} = \mathbf{Cx}$ defines some variables (those in the vector $\mathbf{y}$) that are important to keep small by putting $\mathbf{y}^T\mathbf{y}$ into the performance index. Because $\mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{C}^T\mathbf{Cx}$, we can put a penalty on the sum of squares of outputs using a $\mathbf{Q}$ matrix equal to $\mathbf{C}^T\mathbf{C}$. It is also possible to add other $\mathbf{Q}$ matrices to the performance index to penalize other combinations of state variables. That is, the overall $\mathbf{Q}$ matrix may be given by $\mathbf{Q}_1 + \mathbf{Q}_2 + \cdots$, where each individual matrix $\mathbf{Q}_i$ is chosen for a specific purpose.

1. On page 222 the book considers the design of a regulator to suppress the structural resonances of an aircraft due to wind-gust turbulence. Specifically, the goal is to suppress horizontal vibrations at the flight deck and in the tail section by means of LQG control. In this problem we will design only the LQR part. The important signals to regulate are the forward and aft acceleration measurements, which are modeled as $\mathbf{y} = \mathbf{Cx}$. The first element of $\mathbf{y}$ is the forward acceleration (y_f) and the second element of $\mathbf{y}$ is the aft acceleration y_a. The single control input is the rudder position u.

   The book gives a special type of state-space model for the plant but we will use the following $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ state-space model that describes the same aircraft (note that the web site provides an m-file `prob1_model.m` containing these matrices):

$$\mathbf{A} = \begin{bmatrix} -6.9022e-03 & -1.3638e-04 & 3.7363e-02 & 4.7472e-02 & 2.3868e-03 & -3.9850e-03 \\ -1.3674e-04 & -2.7308e-06 & 8.1129e-04 & 1.0894e-03 & 6.1301e-05 & -9.1141e-05 \\ 4.3923e-02 & 9.4691e-04 & -7.1408e-01 & -2.1274e+01 & -5.1138e-01 & 1.4589e-01 \\ -5.2285e-02 & -1.1836e-03 & 2.1274e+01 & -7.2245e-01 & -6.2964e-03 & 8.4503e-01 \\ -1.0534e-02 & -2.3345e-04 & 5.8809e-01 & -3.1519e-01 & -5.9742e-02 & 1.5971e+00 \\ -1.6102e-02 & -3.5323e-04 & 4.1146e-01 & -9.4134e-01 & -1.5981e+00 & -1.6758e-01 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2.2226e-01 \\ 3.9932e-03 \\ -7.7855e-01 \\ 7.5711e-01 \\ 1.6710e-01 \\ 2.6297e-01 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1.5986e-01 & 2.8072e-03 & -2.3653e-01 & -2.3903e-01 & 9.4852e-02 & -1.1733e-01 \\ 1.5442e-01 & 2.8399e-03 & -7.4175e-01 & -7.1839e-01 & -1.3757e-01 & 2.3534e-01 \end{bmatrix}$$

   The wind-gust turbulence is modeled as white process noise $\mathbf{w}$ in the state-space model:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u + \mathbf{w}$$

and the spectral density matrix of $\mathbf{w}$ is given in the book.

The following information is useful for this problem. A plant containing lightly-damped complex poles located at $-\alpha + j\beta$ will, when subject to a white process noise disturbance, will have a resonance in the output signals at frequency of about β radians per second. The amplitude of the resonance peak is related to $1/\alpha^2$. Thus, a regulator that results in closed-loop poles having a larger value of α will attenuate the resonance.

(a) Compute the eigenvalues of $\mathbf{A}$ and identify the frequencies (in radians per second) of the two resonances. Which of the two is expected to have the higher peak value? This will be called the "bad" resonance.

(b) The book recommends using as performance index the integral of

$$V = \mathbf{y}^T\mathbf{y} + 0.2u^2.$$

Choose the corresponding LQR weighting matrices $\mathbf{Q}_1$ and $\mathbf{R}_1$ and calculate `K=lqr(A,B,Q1,R1)`.

(c) Compute the closed-loop poles. About how much would you expect the bad resonance to be attenuated by this regulator?

(d) The book suggests adding another $\mathbf{Q}$ matrix to penalize the state variables associated with the bad mode. Unfortunately, this cannot be done by simple inspection of the state-space model given above. The way to proceed is to transform the plant into modal canonical form $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ using the following code:

```
plant=ss(A,B,C,zeros(2,1));
[plant_mode,Tc] = canon(plant,'modal')
[Abar,Bbar,Cbar,Dbar]=ssdata(plant_mode);
```

The new state-space model (modal canonical form) is obtained from the original state-space model by transforming the state vector $\mathbf{x}$ of the original plant into the state vector $\bar{\mathbf{x}}$ of modal canonical form. The transformation matrix $\mathbf{T}_c$ is computed by the code given above, and

$$\bar{\mathbf{x}} = \mathbf{T}_c\mathbf{x}. \tag{1}$$

The matrix $\bar{\mathbf{A}}$ is block diagonal; its eigenvalues are the eigenvalues of its diagonal blocks. A 2×2 matrix of the form

$$\begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix}$$

has eigenvalues $-\alpha \pm j\beta$. By looking at the matrix $\bar{\mathbf{A}}$ it is easy to identify the two state variables in $\bar{\mathbf{x}}$ that are associated with the bad resonance. The book suggests putting a weight of 4 on each of these state variables. So choose

$$\bar{\mathbf{Q}}_2 = \mathrm{diag}([* \quad * \quad * \quad * \quad * \quad *])$$

where all but two of the entries on the diagonal of $\bar{\mathbf{Q}}_2$ are zero and the other two (those associated with the bad resonance) are equal to 4. The term $\bar{\mathbf{x}}^T\bar{\mathbf{Q}}_2\bar{\mathbf{x}}$ can be written in terms of the original plant state variables using (1) as follows:

$$\bar{\mathbf{x}}^T\bar{\mathbf{Q}}_2\bar{\mathbf{x}} = \mathbf{x}^T\mathbf{T}_c^T\bar{\mathbf{Q}}_2\mathbf{T}_c\mathbf{x} = \mathbf{x}^T\mathbf{Q}_2\mathbf{x}$$

where $\mathbf{Q}_2 = \mathbf{T}_c^T\bar{\mathbf{Q}}_2\mathbf{T}_c$. The $\mathbf{Q}_2$ matrix can be added to $\mathbf{Q}_1$ from part (a) to achieve further damping of the bad mode. The book also suggests changing $\mathbf{R}_1$ to $\mathbf{R}_2 = 1$.

Use $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2$ and $\mathbf{R}_2$ to compute a new feedback gain vector. Calculate the closed-loop poles. About how much would you expect the bad resonance to be attenuated by this regulator? [Note: by increasing the nonzero entries in $\mathbf{Q}_2$ it would be possible to attenuate this resonance even further with full-state feedback. However, in the realistic case of having only noisy measurements available, the observer-based LQG regulator will not show further improvement in attenuation.]

2. Consider the following plant:

$$\mathbf{A} = \begin{bmatrix} -5 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [\,0 \quad 0 \quad 2\,].$$

Check that this plant has no zeros (you can use `tzero(A,B,C,D)`). Thus an observer-based regulator (also called an LQG regulator) can be designed simply on the basis of settling time. The desired settling time for the regulator is $T_s = 1$ second. Recall the following information from class: if $\mathbf{R} = \mathbf{I}$ and $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$ then the feedback gain matrix `K=lqr(A,B,rho*Q,R)` has the property that, as `rho` goes to infinity, the closed-loop poles (eigenvalues of $\mathbf{A} - \mathbf{BK}$) go to the zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ or minus infinity (this is true in the special case the the number of rows of $\mathbf{C}$ is equal to the number of columns of $\mathbf{B}$.

(a) Because this plant has no zeros, the choice $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$ will allow all of the closed-loop poles to go to minus infinity as ρ goes to infinity. Find the smallest value of ρ that gives a set of closed-loop poles in which the slowest poles have a real part (just) to the left of -4.5. Compute the stability robustness bounds δ_1 and δ_2 using `rb_regsf`.

(b) Design an observer using the Kalman filter formula for observer gain: `L=lqr(A',C',Qo,1)'`. Because the transposed system also has no zeros you can use `Qo=rho*B*B'`. Vary the value of ρ until the following two conditions are satisfied: the real parts of the observer poles are to the left of -15 and the the stability robustness bound δ_2 is greater than 0.6. Use `rb_regob` to compute δ_1 and δ_2 for observer-based regulators.

3. Consider the following plant:

$$\mathbf{A} = \begin{bmatrix} -5 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [\,0 \quad 0.25 \quad 1\,].$$

Verify that this plant has a zero at -2. The desired settling time is 1 second. The plant zero at -2 will be canceled (or nearly so) by an observer pole when we get to the observer design. However, the presence of a zero at -2 will complicate the calculation of $\mathbf{K}$. We can no longer use $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$ because one of the closed-loop poles will be drawn to the zero at -2 and this would give a settling time of about 2.3 seconds, which is too slow. One way to proceed is to calculate a vector $\mathbf{C}_1$ such that the zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{C}_1)$ are in desired locations. I recommend choosing the desired zero locations to be 50% faster than the desired closed-loop poles. The reason for this is that the closed-loop poles approach these zero locations as $\rho \to \infty$ and we do not want to have to use an infinite value of ρ.

(a) If the settling time of the desired zero locations is chosen to be $T_{ss} = T_s/1.5$, then we can use scaled Bessel poles as the desired zero locations, `dzeros=s2/Tss`, where

$$\texttt{s2} = [\,(-4.053 + j2.34) \quad (-4.053 - j2.34)\,].$$

In order to proceed we need calculate the coefficients of a polynomial $d_1 s^2 + d_2 s + d_3$ whose roots are the desired zero locations. Let the vector $\mathbf{d} = [\,d_1 \quad d_2 \quad d_3\,]^T$. Then the calculation is done as follows:

```
d=poly(dzeros)'
```

Note that the `poly` command produces a vector of polynomial coefficients corresponding to a specified set of roots. The last step is to compute the row vector $\mathbf{C}_1$ with the property that the zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{C}_1)$ equal `dzeros`. It turns out that, for SISO systems, $\mathbf{C}_1$ can be calculated from the vector $\mathbf{d}$ by solving a system of linear equations. That is,

$$\mathbf{C}_1^T = \mathbf{M}^{-1}\mathbf{d}. \tag{2}$$

The matrix $\mathbf{M}$ is computed from $(\mathbf{A}^T, \mathbf{C}^T)$ (the method to do this will be discussed later). For this problem the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Calculate $\mathbf{C}_1$ using (2) and verify that the zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{C}_1)$ are equal to `dzeros`.

(b) Choose $\mathbf{Q} = \rho \mathbf{C}_1^T \mathbf{C}_1$, $\mathbf{R} = 1$, and vary ρ to achieve closed-loop poles whose real parts are to the left of -4.5. Compute the stability robustness bounds δ_1 and δ_2 using `rb_regsf`.

(c) To design the observer, choose $\mathbf{Q}_o = \rho \mathbf{B}\mathbf{B}^T$. For various values of ρ calculate $\mathbf{L}$, the observer poles, as well as δ_1 and δ_2 for the observer-based regulator. Find the value of ρ for which δ_2 is about 0.7. Are the resulting observer poles fast enough to preserve the 1-second settling time of the regulator?

4. Consider the following plant:

$$\mathbf{A} = \begin{bmatrix} -5 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [\,0 \quad 0.25 \quad -1\,].$$

This plant has a zero at 2. Because of this right half-plane zero we cannot ask for a settling time of 1 second because it will lead to unacceptable stability margins. (Using the procedure of Problem 3 I designed feedback gains for a settling time of 1 second and and observer with a settling time of 0.2 seconds. The resulting stability margins were both about 0.05! I am not asking you to do this part as you would need the $\mathbf{M}$ matrices for both $\mathbf{K}$ and $\mathbf{L}$ design). Asking for a settling time of 5 seconds is possible because the necessary poles need a real part around -1. Thus, the desired settling time for this problem is 5 seconds, which means the closed-loop poles should have real parts to the left of -1.

(a) Choose $\mathbf{Q} = \rho \mathbf{C}^T \mathbf{C}$, calculate $\mathbf{K}$, and find the value of ρ that makes the slowest poles have a real part (just) to the left of -1. Calculate δ_1 and δ_2.

(b) To design the observe chose $\mathbf{Q}_o = \rho \mathbf{B}\mathbf{B}^T$ and calculate $\mathbf{L}$. Calculate the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ as well as δ_1 and δ_2. Find the value of ρ for which $\delta_2 \geq 0.6$.