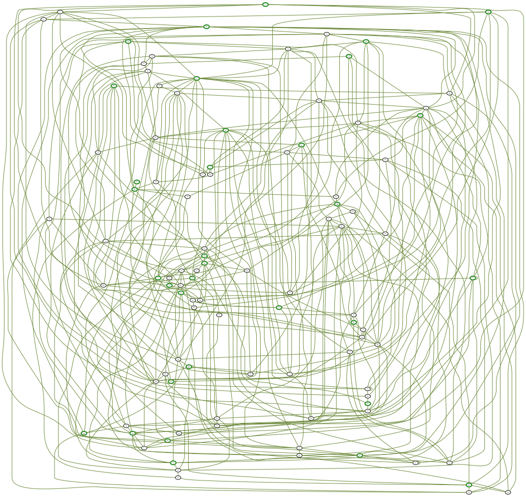
Algorithmic complexity and graphs: graph problems

28 septembre 2023

Graph problems



Graph problems

We will look at famous graph problems, typically of the form :

- "what is the largest subset of nodes of the graph, verifying some property ?"
- "what is the largest subset of edges of the graph, such that some property is verified ?"

networx

We will use **networx** to visualize graphs.

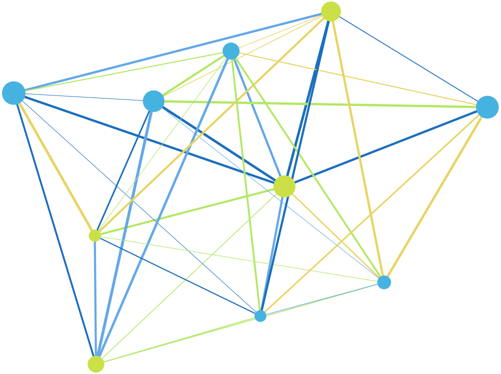


Figure – Undirected random graph generated with python

Warm up question

Given an **unoriented** graph with n nodes, how many edges can we build ?

Notation of a graph : $G(V, E)$

- V : set of n vertices
- E : set of edges

Warm up question

Given an **unoriented** graph with n nodes, how many edges can we build ?

Notation of a graph : $G(V, E)$

- V : set of n vertices
- E : set of edges, maximum size : $\frac{n(n-1)}{2} = \binom{n}{2} = \frac{n!}{2!(n-2)!}$

Networkx

- In order to do the following exercises, you will need **networx**

Exercice 1 : Please **cd ./graphs/random_graphs** and use the notebook **Random_undirected_graph.ipynb** or **random_undirected_graph.py** to generate a random undirected graph with a chosen number of nodes and edges.

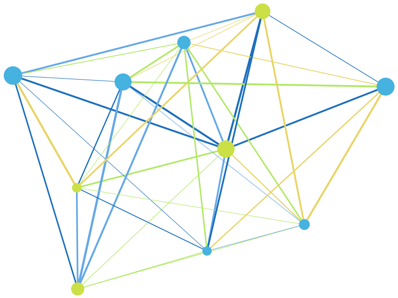


Figure – Random undirected graph with 10 nodes, 40 edges

Exercice 2 : Please use **random_directed_graph.py** to generate a random directed graph with a chosen number of nodes and edges.

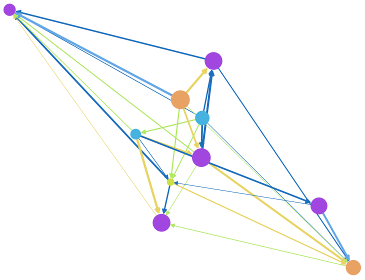
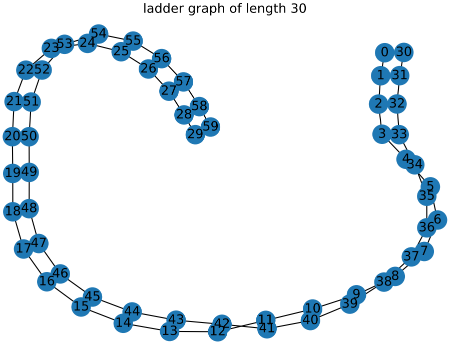


Figure – Random directed graph with 10 nodes, 30 edges

Networkx lib

We can also generate graphs with **networkx**. `https://networkx.org/documentation/stable/reference/generators.html`



Networkx

Networkx can be used to convert to or from common data structures (see **conversion_nx.py**)

```
G edges
[(0, 19), (0, 17), (0, 1), (0, 5), (0, 8), (0, 15), (0, 4), (1, 4), (1, 12),
 19), (6, 15), (6, 10), (7, 9), (7, 16), (8, 11), (8, 9), (9, 17), (9, 14), (
7), (15, 16), (17, 18)]

G as dictionary of lists
{0: [19, 17, 1, 5, 8, 15, 4], 1: [0, 4, 12, 13], 2: [10, 3, 12], 3: [16, 11,
, 14, 11, 8, 16, 12], 10: [2, 4, 6, 12], 11: [9, 8, 3, 17], 12: [4, 19, 5, 2
 [9, 0, 18, 13, 11], 18: [17], 19: [0, 12, 6, 5, 13]}

G as a numpy array
[[0. 1. 0. 0. 1. 1. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 1. 0. 1.]
 [1. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 1. 0. 0. 0.]
 [1. 1. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 1. 0. 0. 0. 0. 0. 0. 0.]
 [1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 1. 0. 0. 0. 1.]
 [0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 0.]
 [1. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1. 1. 0. 0. 1. 1. 0. 1. 0. 1. 1. 0. 0.]
 [0. 0. 1. 0. 1. 0. 1. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0.]
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 [0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 0. 1.]
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 [1. 0. 0. 0. 0. 0. 1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0. 0. 0.]
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 [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 1. 0. 0.]
 [1. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 1. 1. 0. 0. 0. 0. 0. 0.]]
➜  random_graphs git:(correction) ✗ pp conversion_nx.py
```
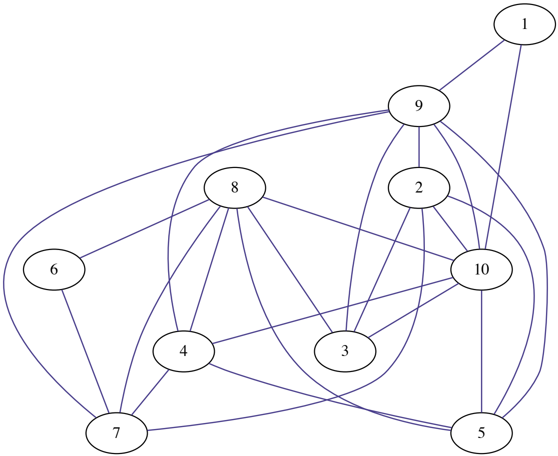

The dominating set problem



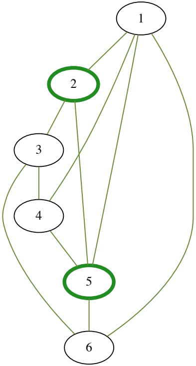
Dominating set

Say you want to cover a internet network. Some nodes (the emitters) are able to transmit information in the network, but not to all nodes : only to the nodes that are close enough.

Dominating set

Say you want to cover a internet network. Some nodes (the emitters) are able to transmit information in the network, but not to all nodes : only to the nodes that are close enough.
Optimization problem : You need to cover the network, but with the smallest possible number of emitters (in order to save money, infrastructure, material, etc.).
Exercice 3 : How would you formalize this problem with a **graph** ?

The dominating set problem



Mathematically speaking : if $G(V, E)$ is the graph. We look for a **subset of nodes** D such that **all nodes in the graph** are the neighbor of **at least one node** in D.

The dominating set problem

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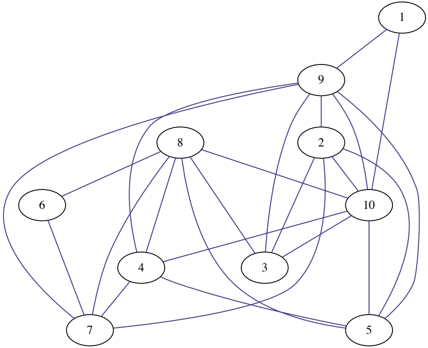
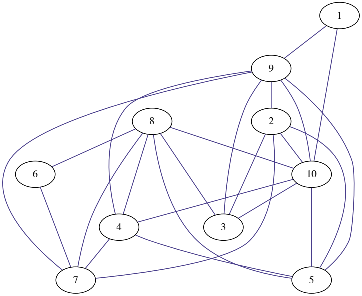


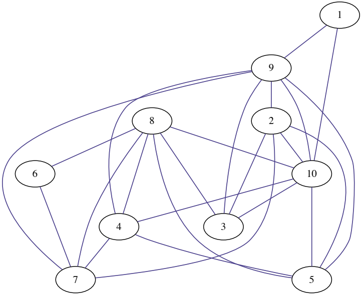
The dominating set problem

Mathematically speaking : if $G(V, E)$ is the graph. We look for a **subset of nodes** D such that **all nodes in the graph** are the neighbor of **at least one node** in D. And we want to pick the **smallest D** that "dominates" the network.



The dominating set problem

What is the most trivial dominating subset ?



Dominating set : example 1

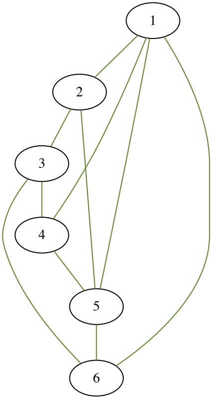


Figure – Some simple graph

Dominating set : example 1

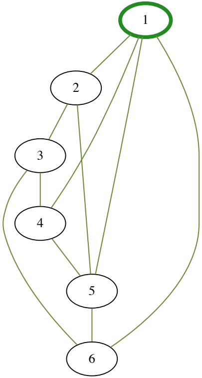


Figure – Is this a dominating subset ?

Dominating set : example 1

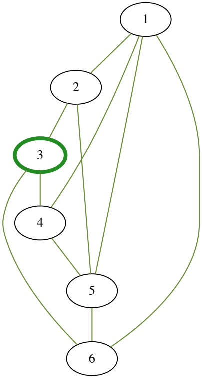


Figure – Is this a dominating subset ?

Dominating set : example 1

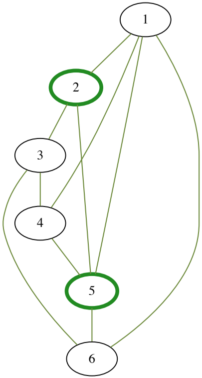


Figure – Is this a dominating subset ?

Dominating set : example 1

A **minimal dominating set** is a dominating set D such that removing any node from D prevents it from still being dominating.

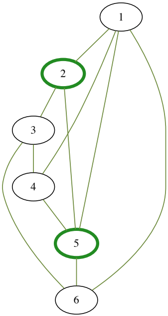


Figure – Concept of minimal dominating set.

Dominating set : example 1

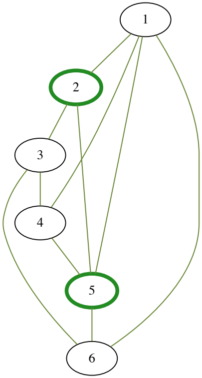
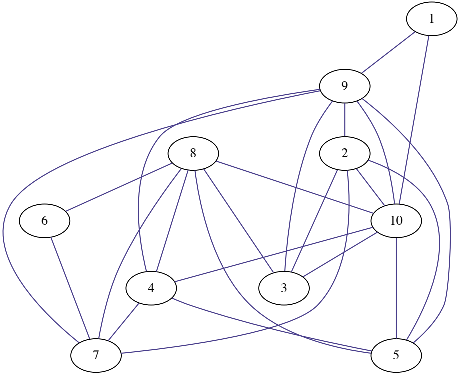


Figure – Is this a dominating subset ? Yes. Is it minimal ?

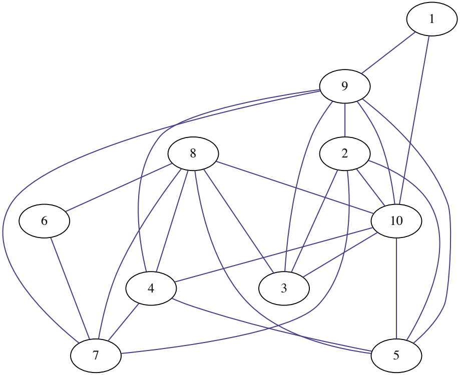
Dominating set : example 2

Please find a dominating set in this graph.



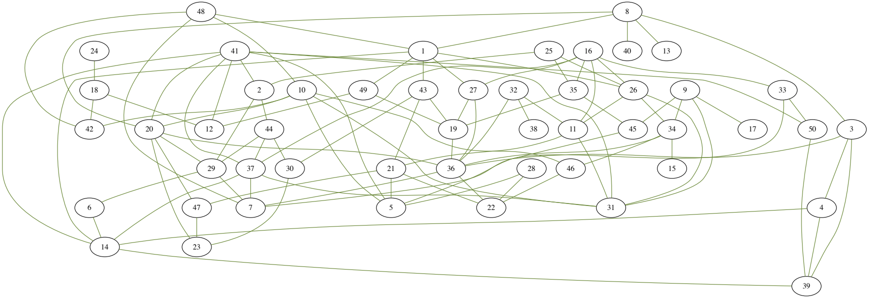
Dominating set : example 2

Please find a **minimal** dominating set in this graph.



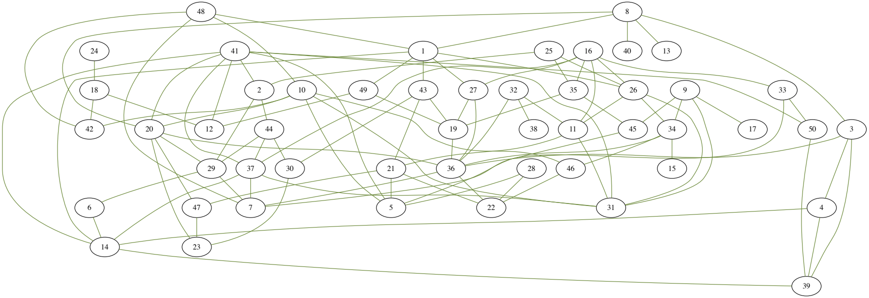
Dominating set : example 3

Please find a **minimal** dominating set in this graph.



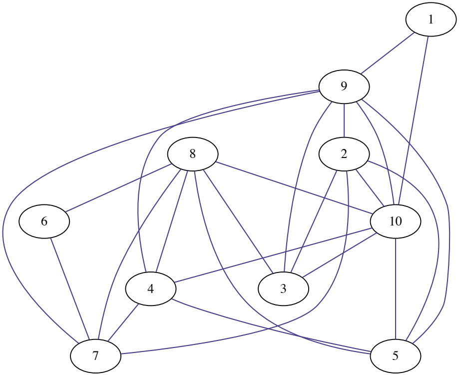
Dominating set : example 3

Is **minimal** the same thing as minimum ?



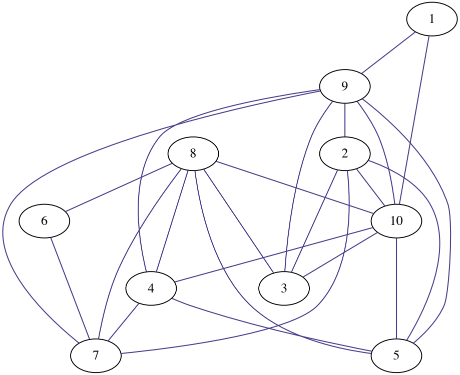
Dominating set : exhaustive search

What would be the **exhaustive search** in the case of the Dominating set problem ?



Dominating set : exhaustive search

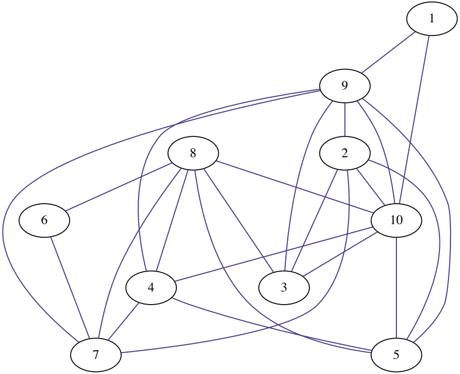
How many possibilities do have to try as a function of n ?



Dominating set : exhaustive search

How many possibilities do have to try as a function of n ?

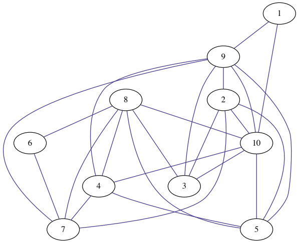
The number of subsets in $[1 : n]$ is :



Dominating set : exhaustive search

How many possibilities do have to try as a function of n ?
The number of subsets in $[1 : n]$ is :

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{1}$$



Heuristic

The exhaustive search is no possible. So what method should we use ?

Heuristic

Ok so the exhaustive search is no possible. So what method should we use ?

Let's build a **greedy algorithm** (heuristic).

Greedy algorithm

In a graph (unweighted), the **degree of a node** is its number of neighbors.

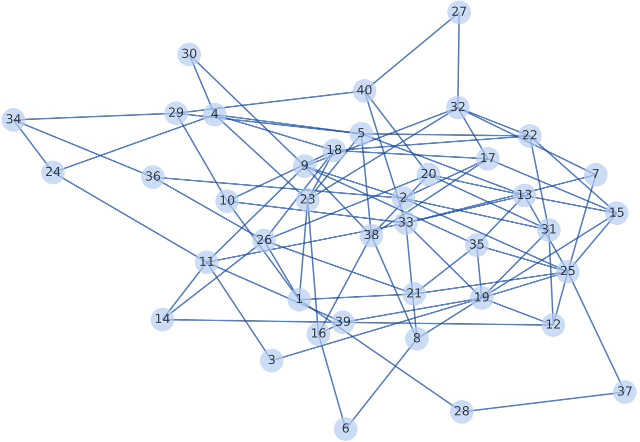
dominating set

Exercice 4 : Greedy algorithm implementation

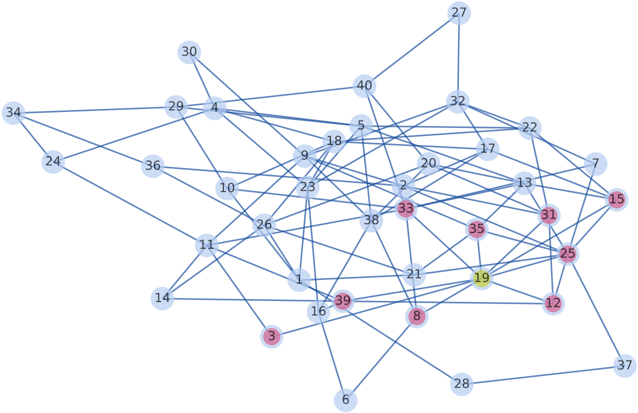
cd graphs/dominating_set and modify **greedy_standard.py** in order to apply the greedy algorithm :

- sort nodes by degree
- progressively add the to the set until it's dominating

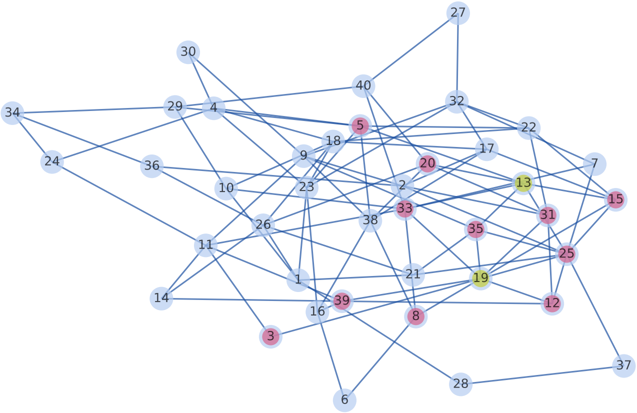
Initial graph



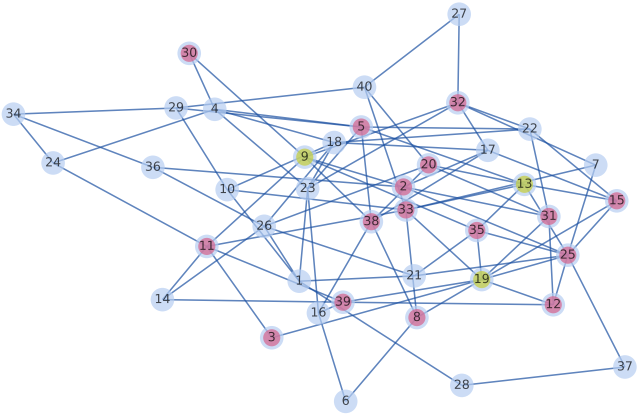
Subset size: 1
Algo step: 1



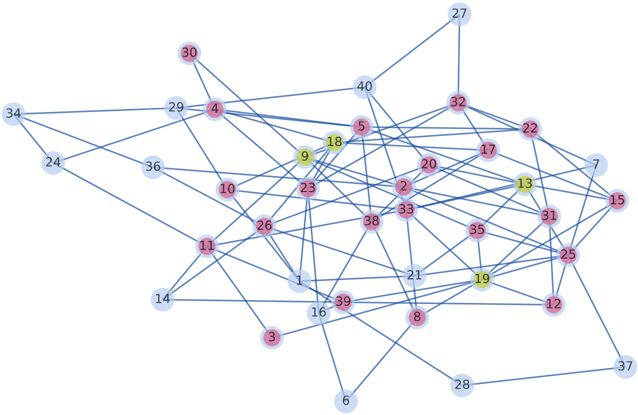
Subset size: 2
Algo step: 2



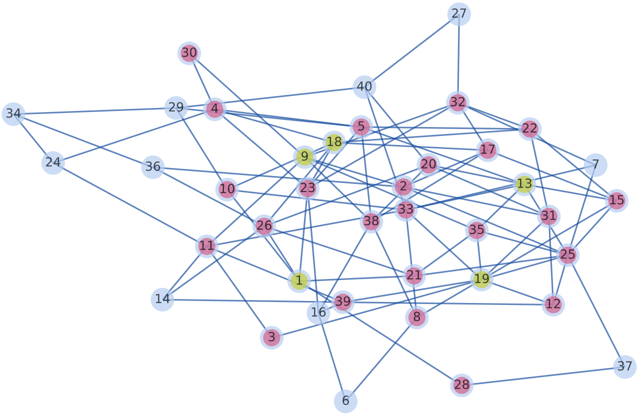
Subset size: 3
Algo step: 3



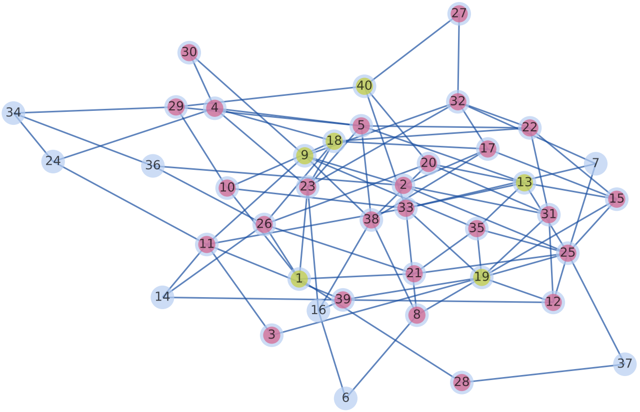
Subset size: 4
Algo step: 4



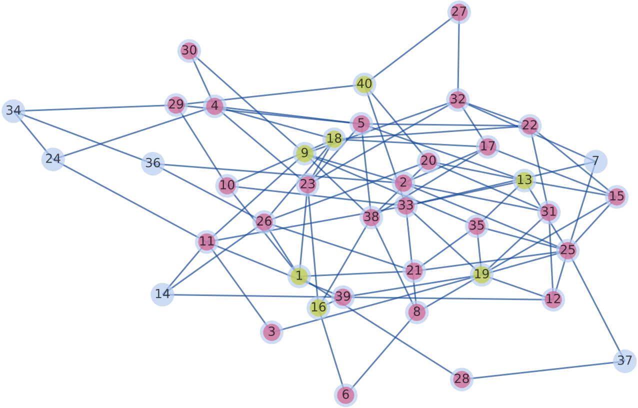
Subset size: 5
Algo step: 5



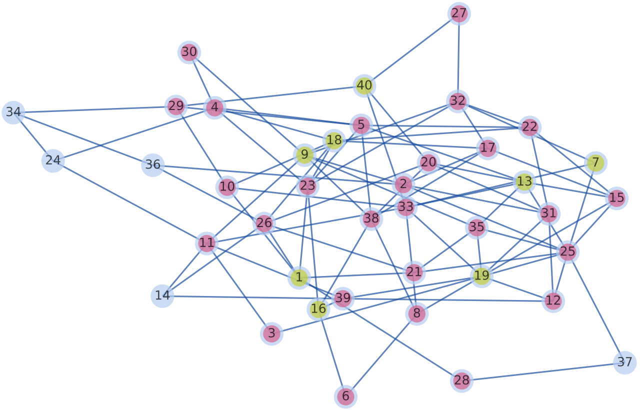
Subset size: 6
Algo step: 6



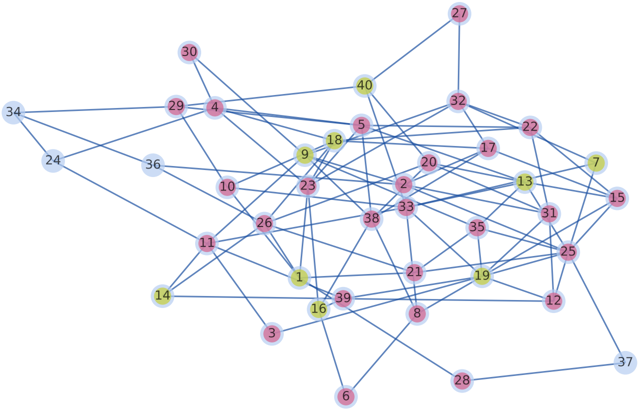
Subset size: 7
Algo step: 7



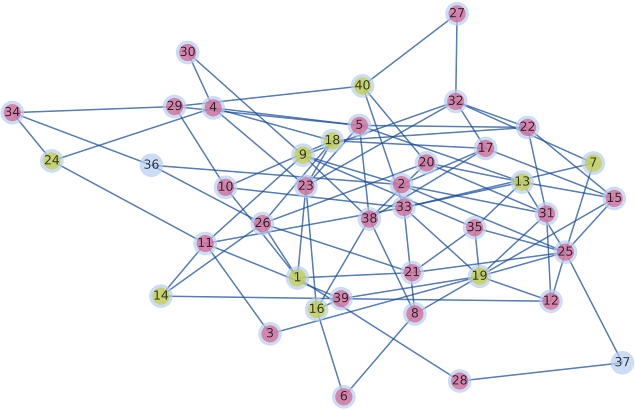
Subset size: 8
Algo step: 8



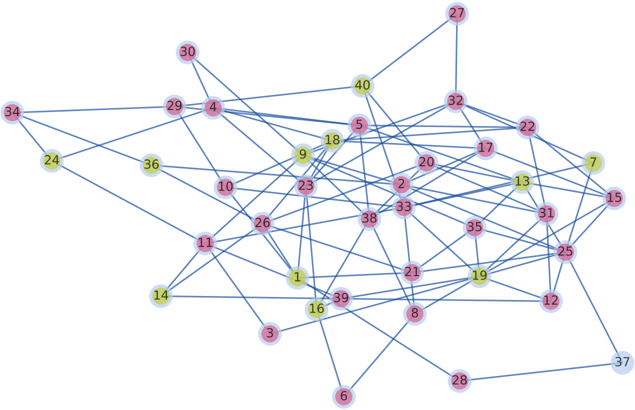
Subset size: 9
Algo step: 9



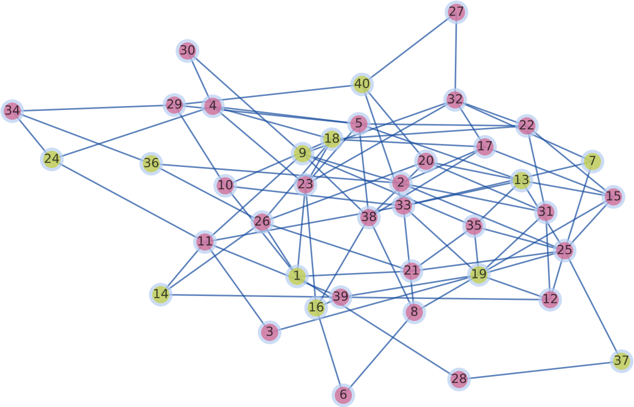
Subset size: 10
Algo step: 10



Subset size: 11
Algo step: 11



Subset size: 12
Algo step: 12



dominating set

Exercice 4 : Greedy algorithm implementation
Generate new instances of the problem using **generate_problem_instance.py** and apply the algorithm to them.
You can use the file **params.txt**.

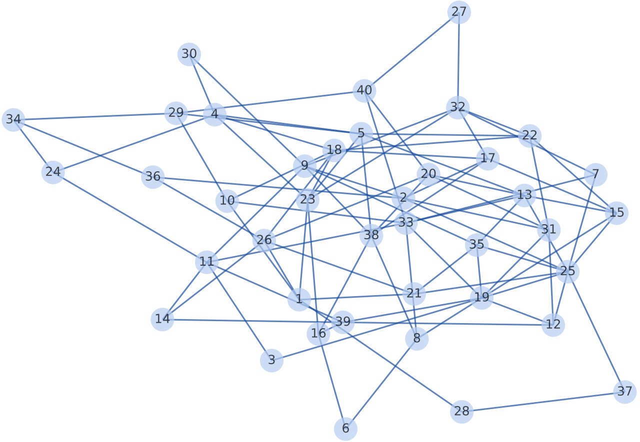
Complexity

Exercice 5 : What is the complexity of the greedy algorithm ?

Variant

Exercice 6 : Try to see what happens using a variant of the heurstic, where we can add nodes that are already dominated to the set of selected nodes. Which method is faster ?
You can use **greedy_bis.py**

Initial graph



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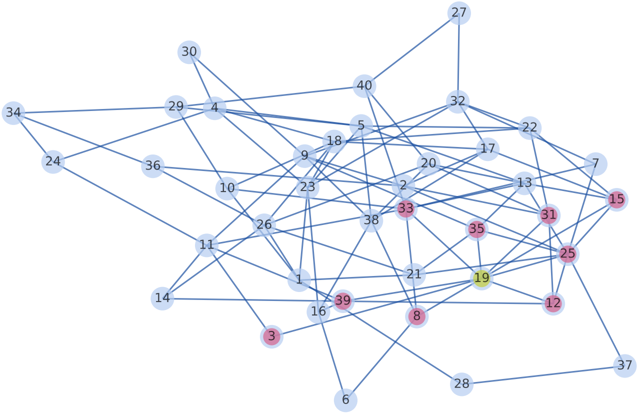
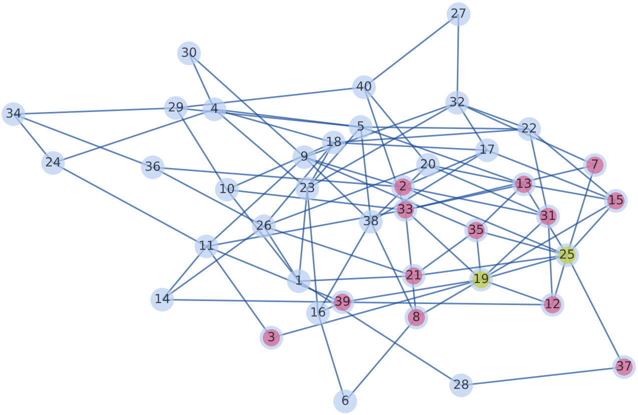
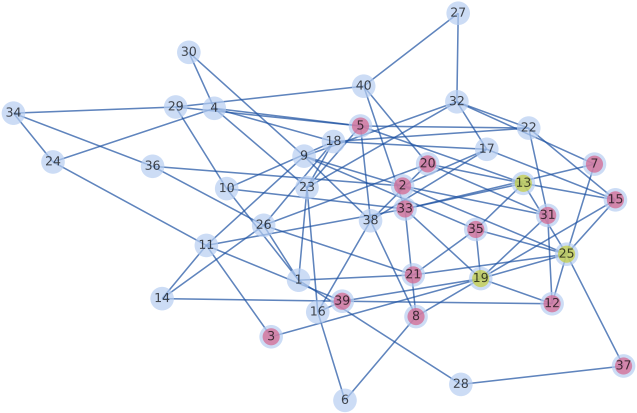


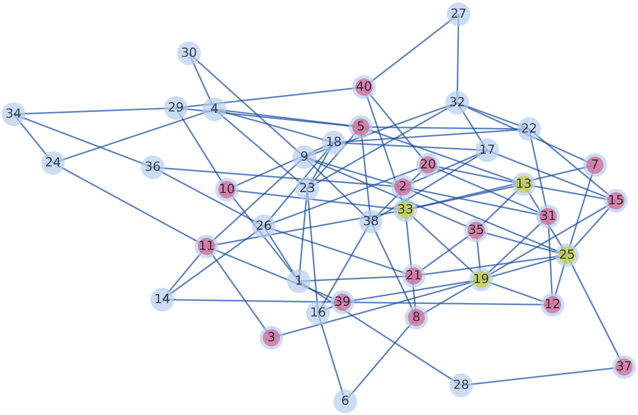
Subset size: 2
Algo step: 2



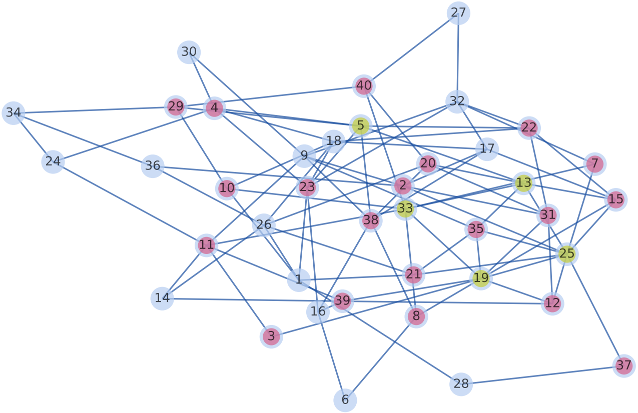
Subset size: 3
Algo step: 3



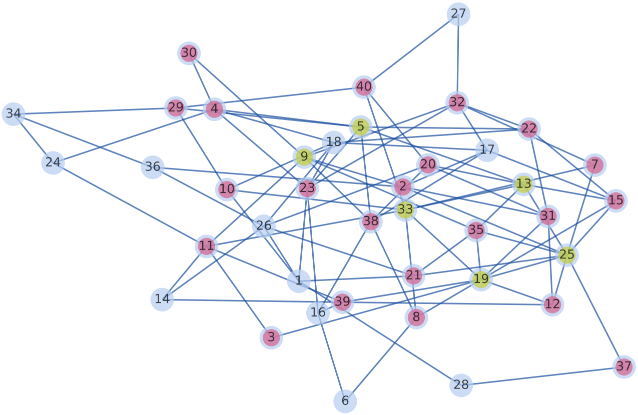
Subset size: 4
Algo step: 4



Subset size: 5
Algo step: 5



Subset size: 6
Algo step: 6



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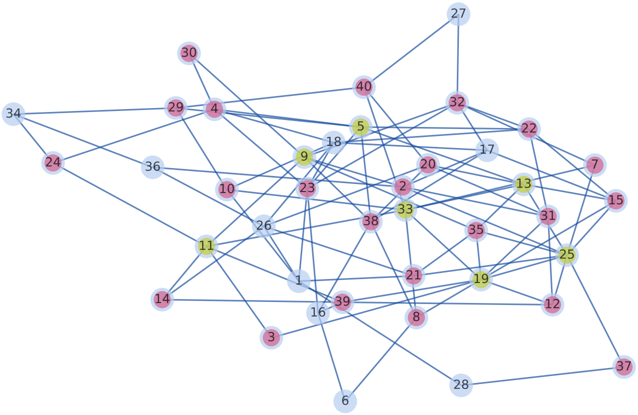

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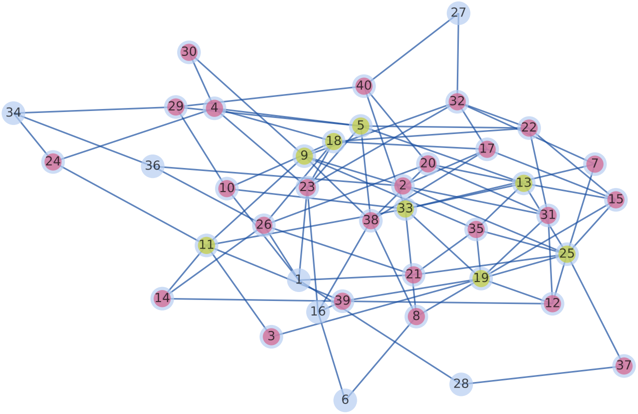

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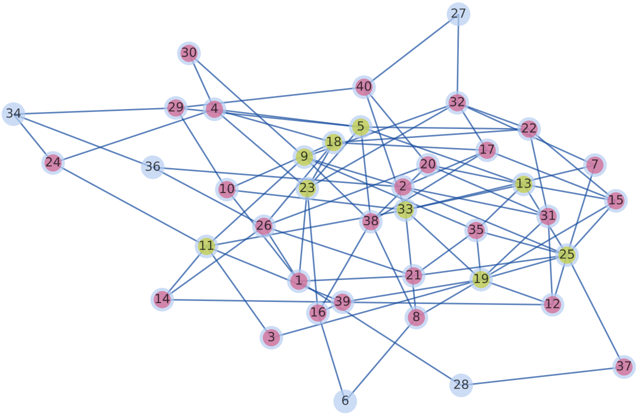
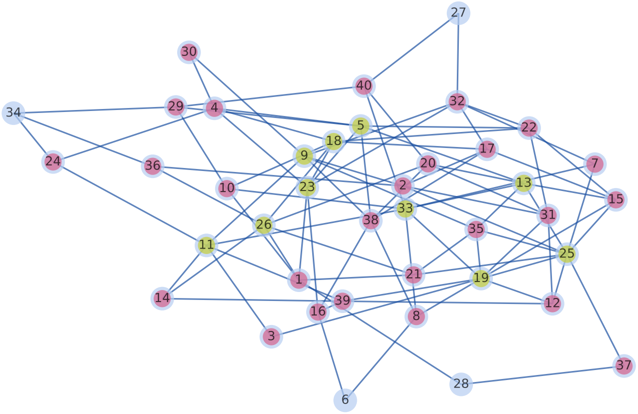
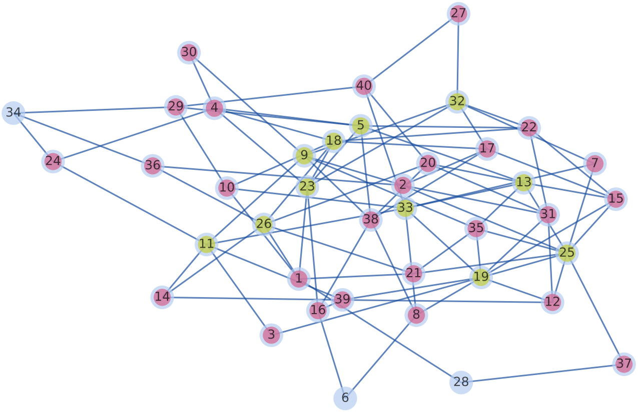
Algo step: 9



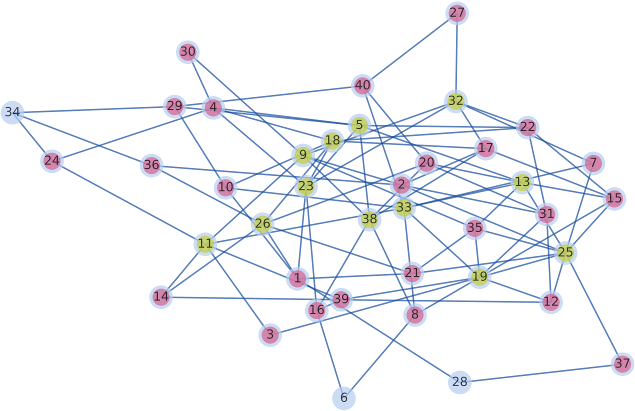
Subset size: 10
Algo step: 10



Subset size: 11
Algo step: 11



Subset size: 12
Algo step: 12



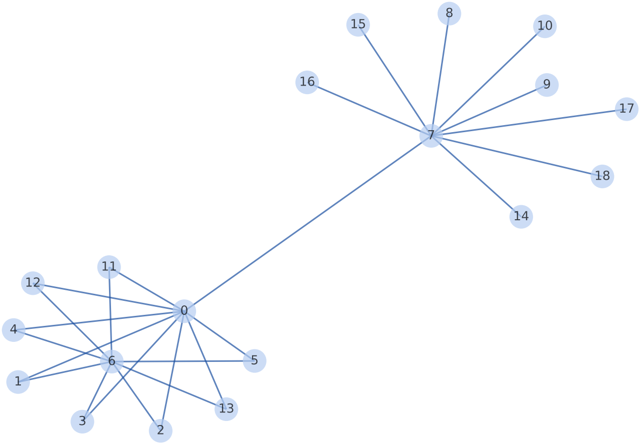
Variant 2

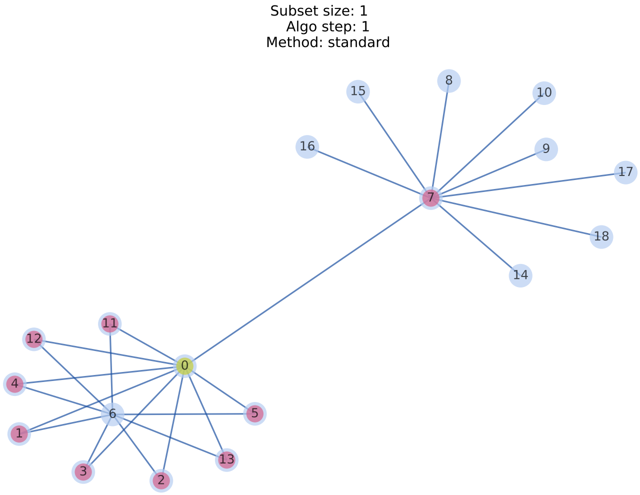
Exercice 7 : Implement of another variant where the degrees of the nodes are recomputed after each algorithm step.
You can use **greedy_ter.py**

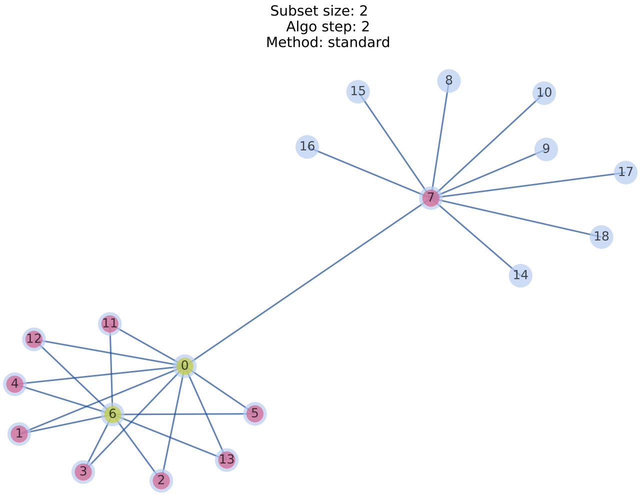
Different performances

We have 3 variants of the algorithm, it seems that on most random cases "ter" works better (gives a smaller dominating set).
Exercice 8 : Can you find graph for which "standard" and "ter" are beaten by "bis" ?

Initial graph




Subset size: 1
Algo step: 1
Method: standard
15
8
10
16
9
17
7
18
14
11
12
0
4
6
5
1
13
3
2


Subset size: 2
Algo step: 2
Method: standard
15
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10
16
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7
18
14
11
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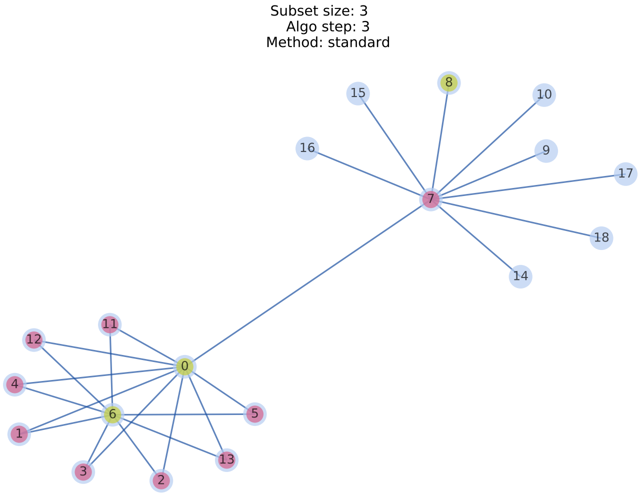
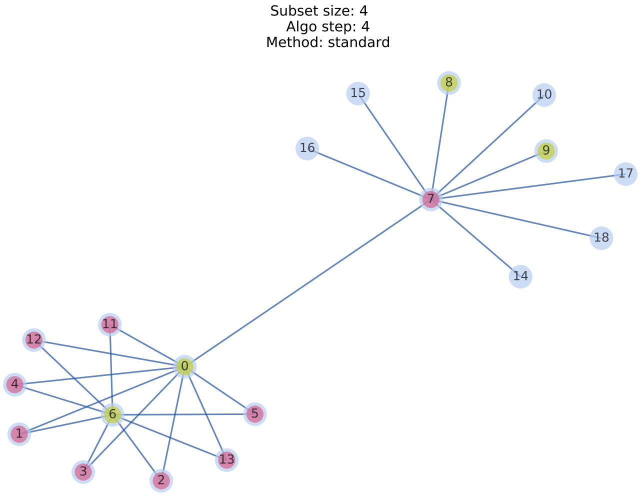
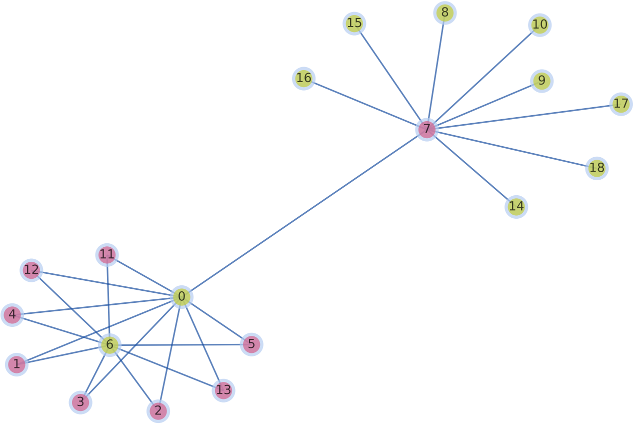
4
6
5
1
13
3
2

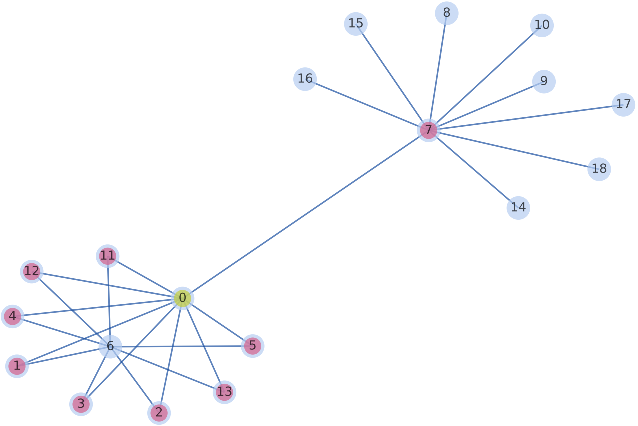

Subset size: 3
Algo step: 3
Method: standard
15
8
10
16
9
17
7
18
14
11
12
0
4
6
5
1
13
3
2

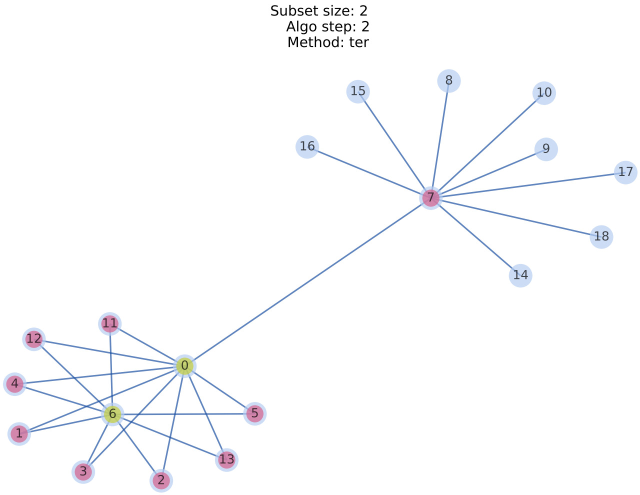

Subset size: 4
Algo step: 4
Method: standard
15
8
10
16
9
17
7
18
14
11
12
0
4
6
5
1
13
3
2

Subset size: 10
Algo step: 10
Method: standard



Subset size: 1
Algo step: 1
Method: ter




Subset size: 2
Algo step: 2
Method: ter
15
8
10
16
9
17
7
18
14
11
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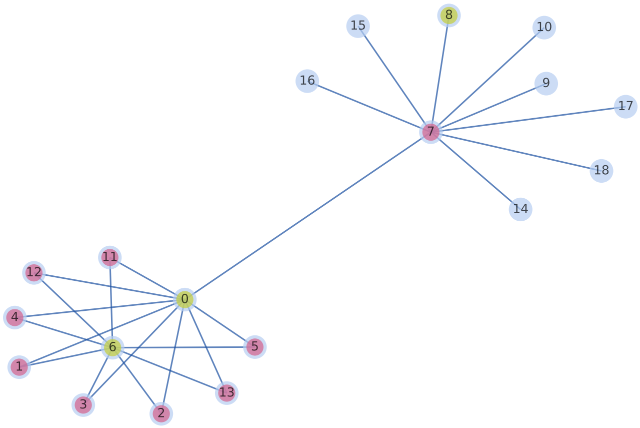
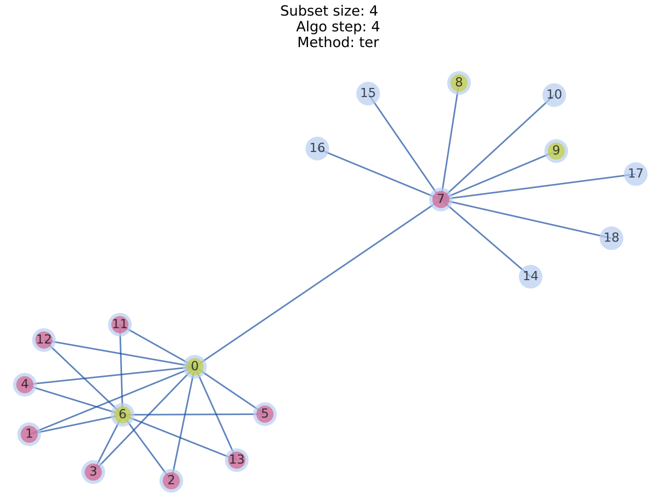
0
4
6
5
1
13
3
2

Subset size: 3
Algo step: 3
Method: ter




Subset size: 4
Algo step: 4
Method: ter
15
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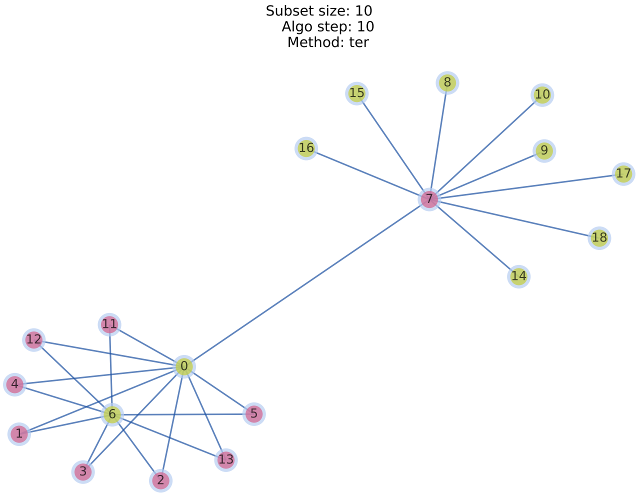
Subset size: 10
Algo step: 10
Method: ter
15
8
10
16
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0
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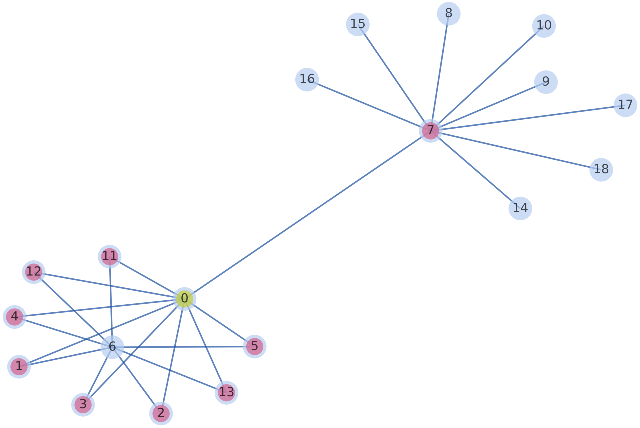
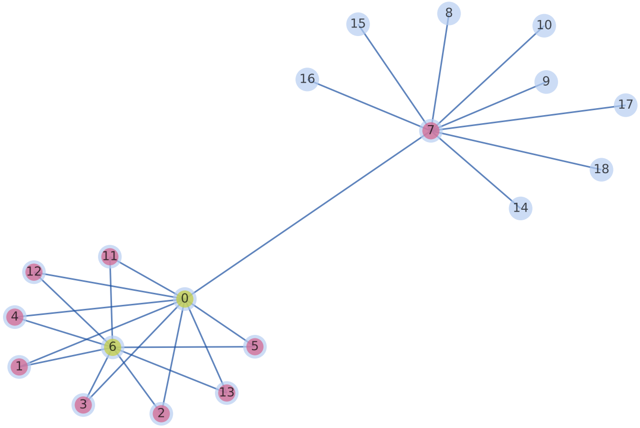
6
5
1
13
3
2

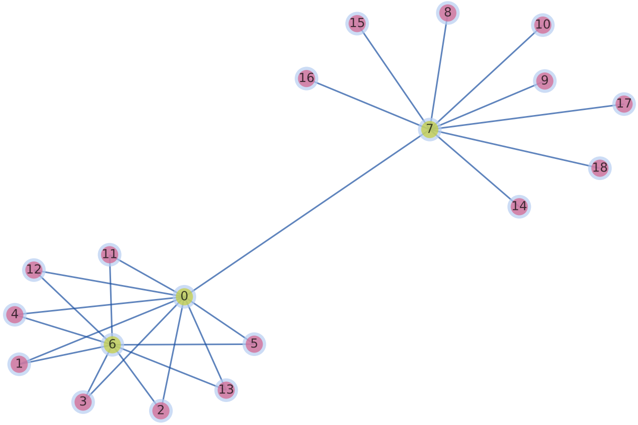
Subset size: 1
Algo step: 1
Method: bis



Subset size: 2
Algo step: 2
Method: bis



Subset size: 3
Algo step: 3
Method: bis



Non optimal greedy algorithm

Exercice 9 : Find a graph for which "standard" gives a very bad solution.

Non optimal greedy algorithm

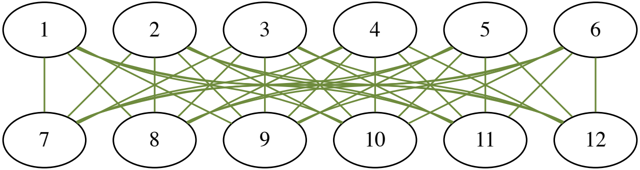


Figure – Complete bipartie graph

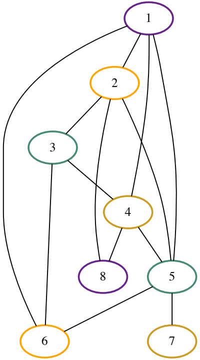
Networkx

- The library networkx has some functions that implement many graph algorithms.
- `https://networkx.org/documentation/stable/reference/algorithms/generated/networkx.algorithms.dominating.dominating_set.html`
- See **dominating_nx.py**.

The coloring problem

Say you have a map with different countries. You need to assign a color to each country, so that two countries that have a common border are filled with a different color. We assume that we would like to use a small number of colors (the smaller, the better).
Exercice 10 : How would you formalize this problem with a graph ?

The coloring problem

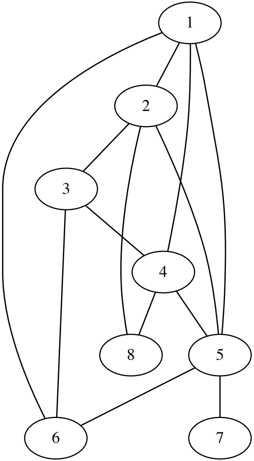


We want to find the smallest number of **fully disconnected subgraphs** in a graph.

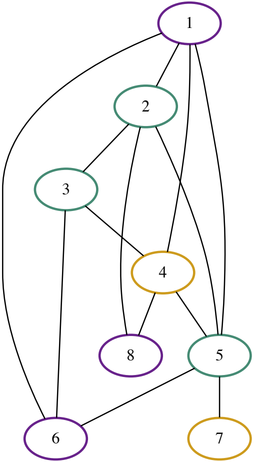
The coloring problem

We want to find the smallest number of **fully disconnected subgraph** in a graph.
Each subgraph will be associated with a color.

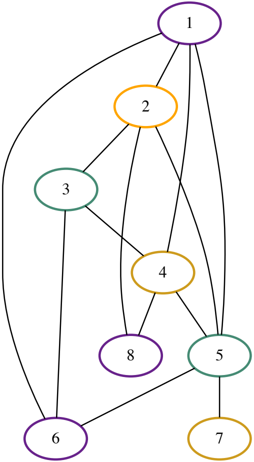
Coloring



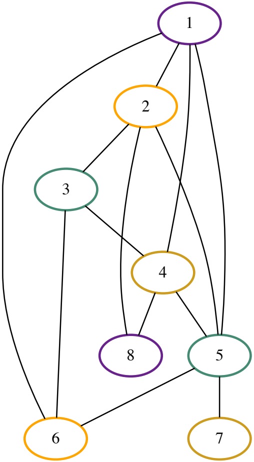
Is this a coloring ?



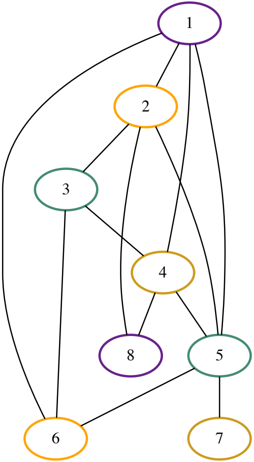
Is this a coloring ?



Is this a coloring ? yes



Could we have used only 3 colors ?



Coloring

- What would be a trivial coloring ?

Coloring

- What would be a trivial coloring ? assign a color to each node (very bad solution)
- Could you think of a heuristic ?

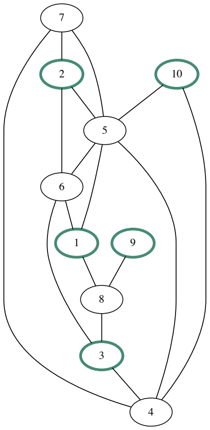
Other applications

- Planning activities (color : time in the day)
- Assigning frequencies (color : frequency)

Independent set

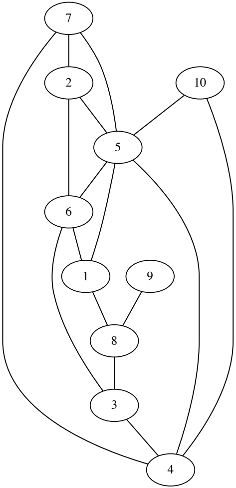
You have a group of people. Some people cannot work with each other. You want to build to largest possible team of people.
Exercice 11 : How would you formalize this with a graph ?

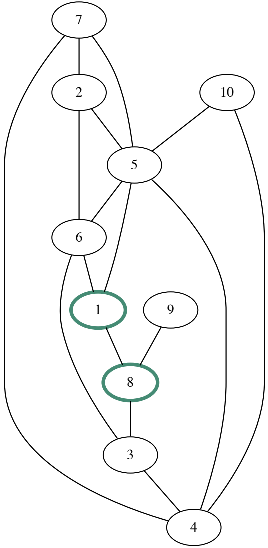
Independent Set



Assuming that an edge represents the fact that two persons cannot work with each other, we want to find **the largest disconnected subgraph**.

Independent set : what is a trivial independent set ?

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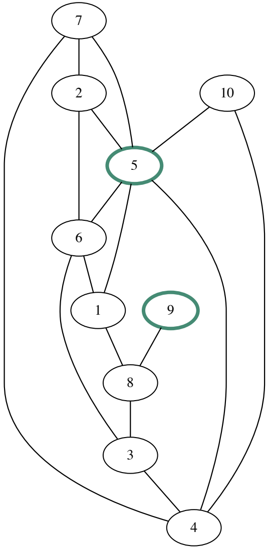
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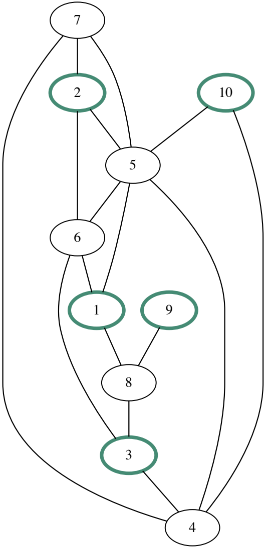
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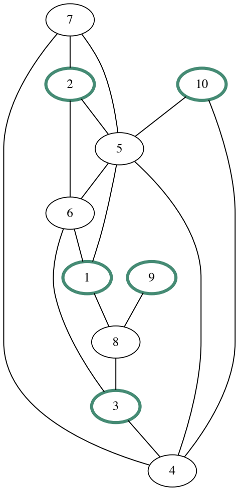
Maximal vs maximum independent set



Complexity

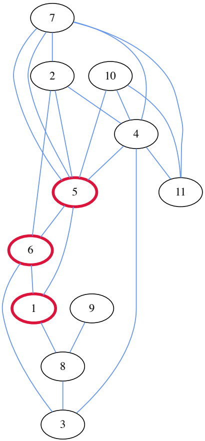
- The running time T of an algorithm A is its running time on the worst possible input (instance I) it can get (for a given size)
- The complexity $T(P)$ of a problem P is the running time of the best possible algorithm for that problem.

$$T(P) = \min_{A} \max_{I} T(P, A, I) \tag{2}$$

Equivalence between problems

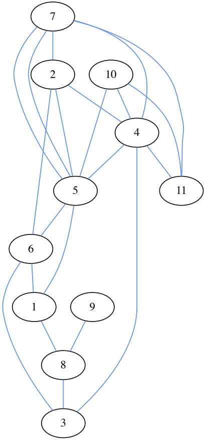
- Some problems have the same difficulty because they are "equivalent", in a specific way (polynomial reduction).
- On the contrary, some problems are strictly more complex than others.
- Hard problems : Maximum independent set, minimum coloring, smallest dominating set, TSP, etc.
- Easier problem : Shortest Path

Maximum clique problem

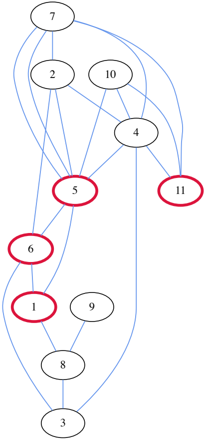


The **maxium clique** problem consists in finding the largest completely connected subgraph (the induced subgraph is **complete**)

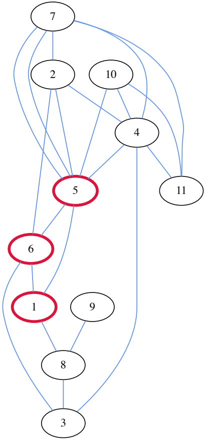
Maximum clique problem



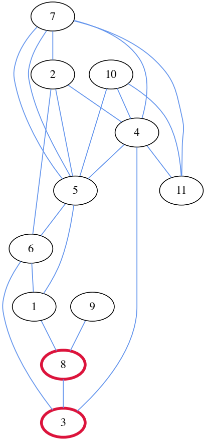
Maximum clique problem



Maximum clique problem



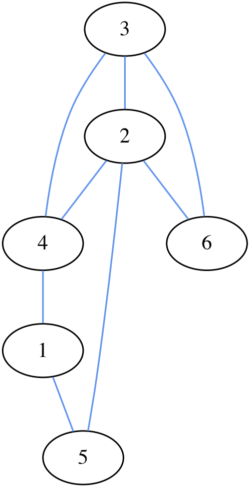
Maximum clique problem



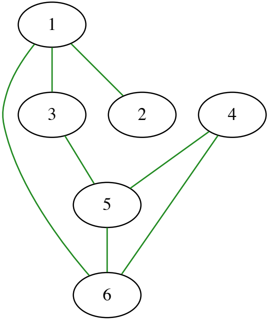
Equivalence between problems

Exercice 12 : Can you relate the maximum clique problem to another problem we saw before ?

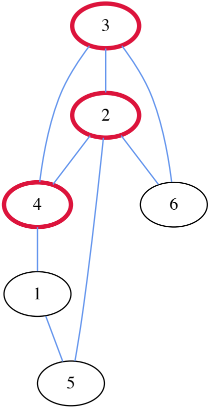
Linking problems



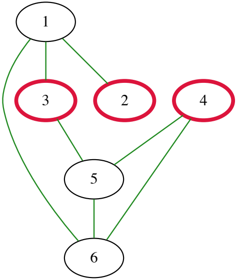
Linking problems



Linking problems



Linking problems



Polynomial-time reduction

To study a problem, it is sometimes useful to transform it into another.

Exercice 13 : Transformation

cd clique/ and use **complement_graph.py** in order to transform a graph into its **complement graph**.

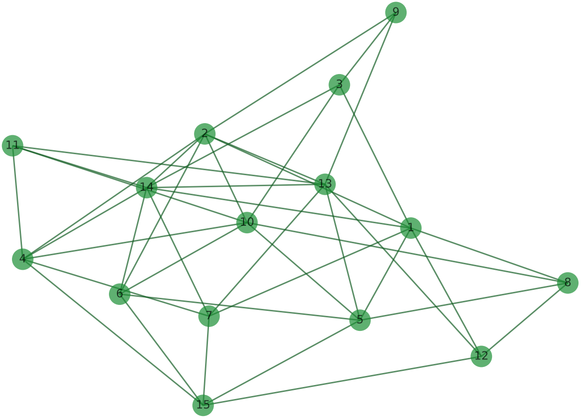
Exercice 13 : Transformation

cd clique/ and use **complement_graph.py** in order to transform a graph into its **complement graph**.

What is the complexity of this operation ?

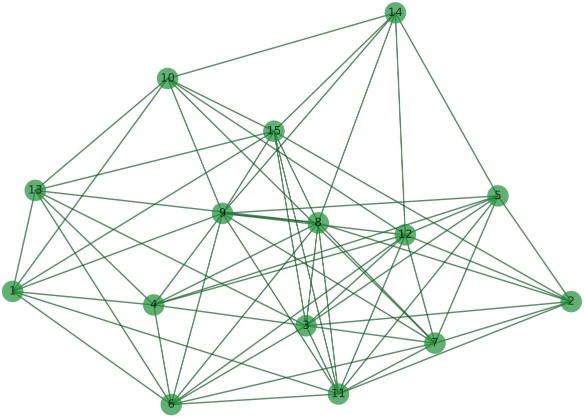
Complement graph

images/initial graph.pdf



Complement graph

images/complement graph.pdf



Dominating set to set covering

- This is another example of two problems that are equivalent.

Problems that are not equivalent

- Eulerian paths and hamiltonian paths

Classes of complexity

- Problems have been gathered under **classes of complexity**
- **P** : we can obtain a solution with polynomial complexity
- **NP** : we can verify a solution in polynomial time (doesn't mean we can find a solution in polynomial time)
- **NP hard** : if it is in P, all NP problems are in P.
- **NP complete** : NP and NP hard

P=NP ?