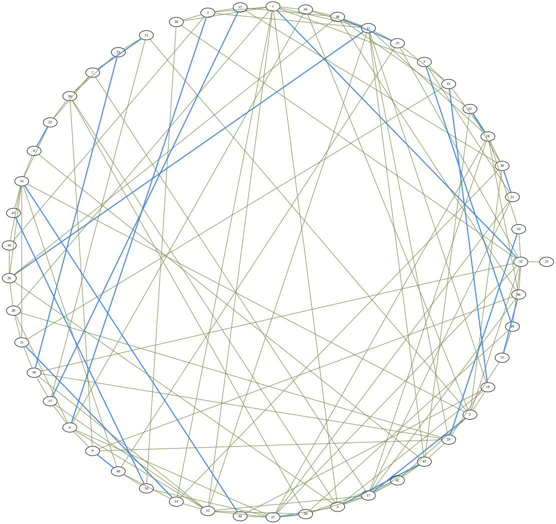
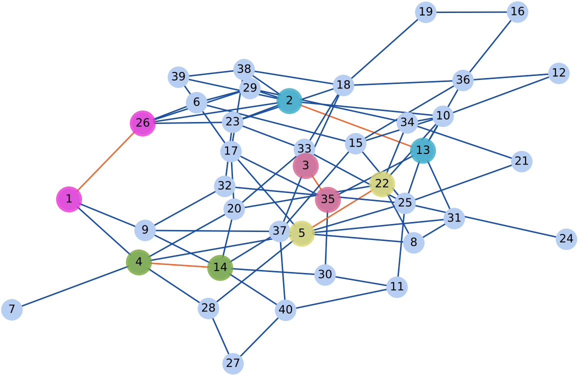
Algorithmic complexity and graphs: the matching problem

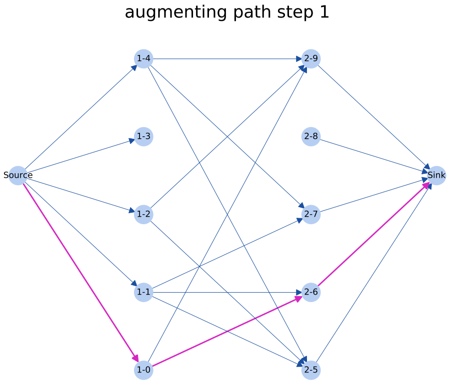
September 22, 2024

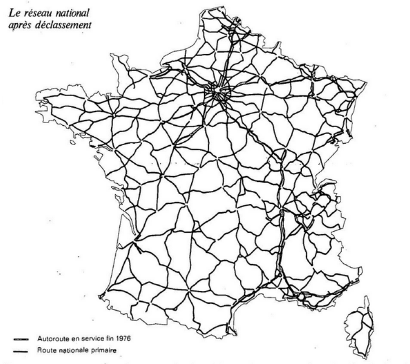


Matching size: 21
Algo step: 128
Nb nodes: 50

Matching size: 5
Algo step: 19
Nb nodes: 40




augmenting path step 1
1-4
2-9
1-3
2-8
Source
Sink
1-2
2-7
1-1
2-6
1-0
2-5


Le réseau national
après déclassement
Autoroute en service fin 1976
Route nationale primaire

The matching problem

Maximum matching (Optimal assignment, problème d'affectation)

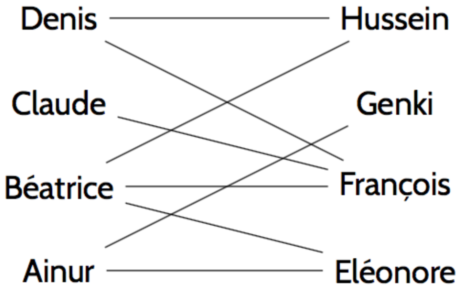


Figure: Problem: Building the largest possible number of teams of 2 persons.

Matching problem

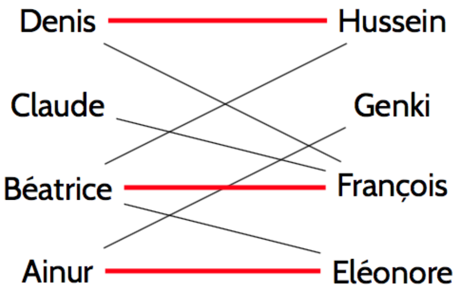


Figure: Problem: not optimal assignment

Matching problem

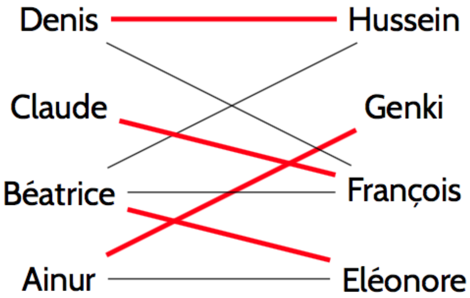


Figure: Problem: optimal assignment

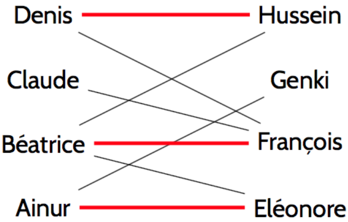
Other examples

- Assigning students to internships
- Assigning machines to a task

Matching problem: formal definition

Given a **undirected** graph $G = (V, E)$, we want a **matching**, which means:

- A subset of edges $M \subset E$
- Such that no pairs of edges of M are incident
- Equivalently, each node in the graph is **at most** in one edge of M.



Maximum matching

- The **size** of a matching is the number of edges it contains.
- We want to find the matching of **largest possible size** in a given graph.

Example 1

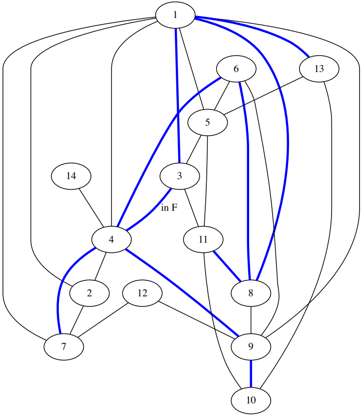


Figure: Is this a matching ?

Example 2

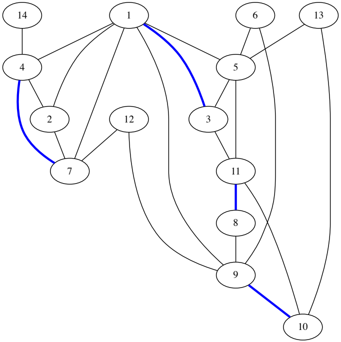


Figure: Is this a matching ?

Example 3

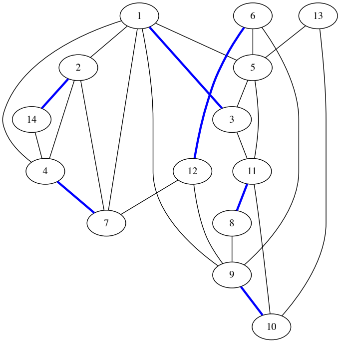


Figure: Is this an optimal matching ?

Example 4

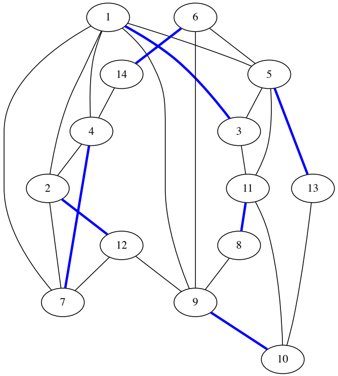


Figure: Is this an optimal matching ?

Example 5

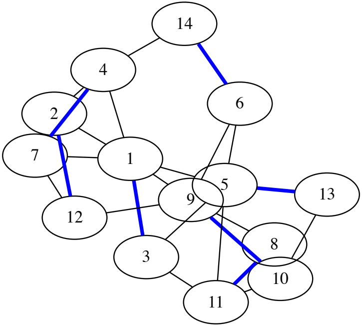


Figure: With neato

Optimal matching

Exercice 1 : What is the maximum size possible for a matching, in a general graph of size n ? (in the sense that no graph with n nodes contains a larger matching)

Optimal matching

Exercice 1 : What is the maximum size possible for a matching, in a general graph of size n ? (in the sense that no graph with n nodes contains a larger matching)

- If n is even : $\frac{n}{2}$
- Else n is odd : $\frac{n-1}{2}$

Hence,

$$\lfloor \frac{n}{2} \rfloor \tag{1}$$

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal

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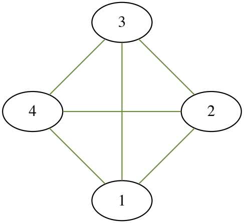

Figure: The complete graph works

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

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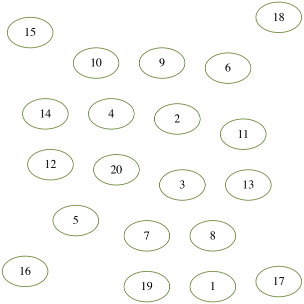


Optimal matching

Exercice 1 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

Exercice 2 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

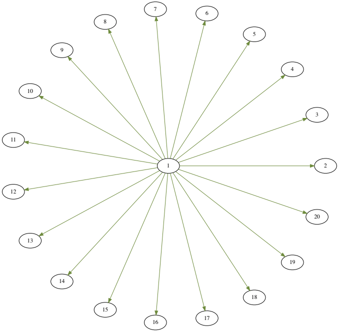


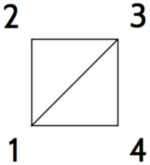
Figure: Star graph

Experiments

Possibilities to code a graph:

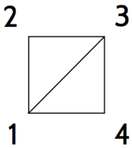
- list of sets of size 2 (for an undirected graph)
- a dictionary of successors (directed of undirected)

Coding a graph : as a list of edges



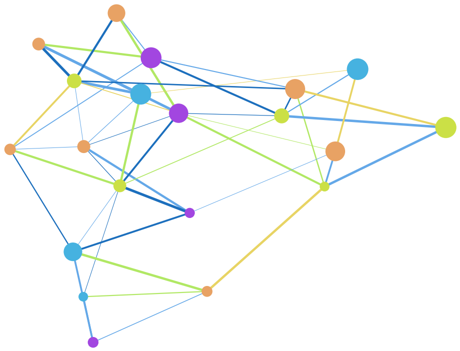

```
g1 = [{1,2},{1,3},{2,3},{3,4},{1,4}]
```

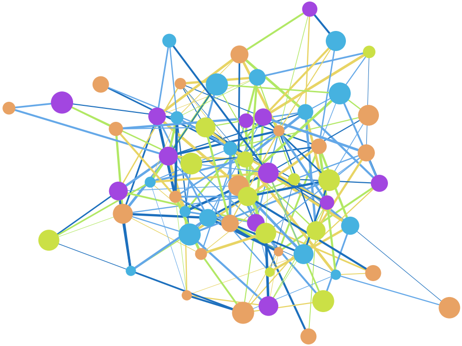
Coding a graph : as a dictionary of neighbors



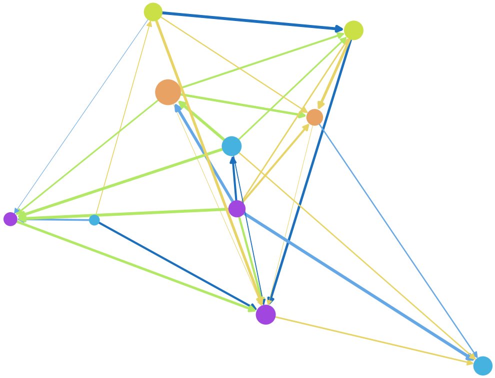

```
g1 = { 1:{2,3,4}, 2:{1,3}, 3:{1,2,4}, 4:{1,3} }
```

Generating graphs with networks.

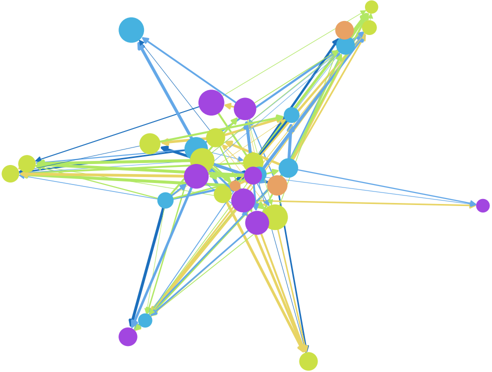




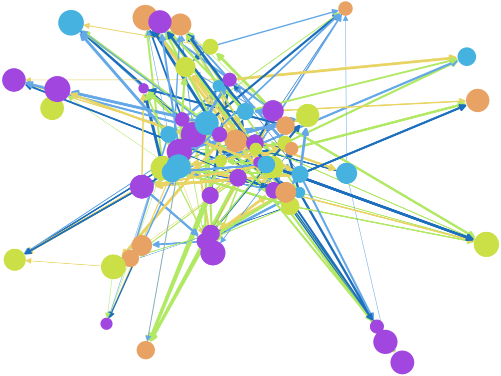
Directed graph



Directed graph II



Directed graph III



Big graph

We could not manually find an optimal matching in this graph :

Summary

- We have defined the matching problem.
- When the size of the problem is large, we can not manually find an optimal matching.

Brute force approach

Exercice 3 : Enumeration

- Given a graph, what would a brute force approach on the matching problem be ?

Brute force approach

Exercice 3 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

If the graph contains n nodes, and p edges:

- what is the complexity of step 1 ?
- what is the complexity of step 2 ?

Brute force approach

Exercice 3 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

If the graph contains n nodes, and p edges:

- what is the complexity of step 1 ? $\mathcal{O}(2^p)$ exponential
- what is the complexity of step 2 ? $\mathcal{O}(np)$: polynomial

Hence, as expected, brute force is not possible on graphs that are not very small.

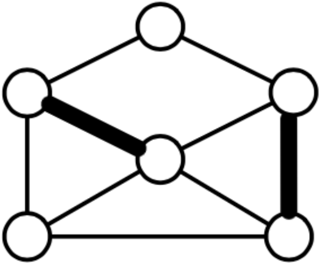
Notion of maximal and maximum matching

We will say that a matching M of cardinality $|M|$ is:

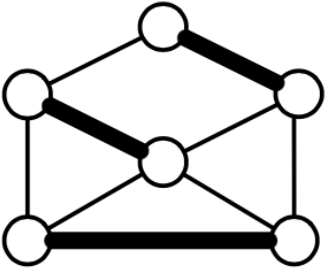
- **Maximum** if it has the maximum possible number of edges (it is thus optimal)
- **Maximal** if the set of edges obtained by adding any edge to it is **not a matching**. This means that $M \cup \{e\}$ is not a matching for any e not in M.
- $\cup$ means union of sets.

Is being a **maximal** matching the same thing as beeing a **maximum** matching ?

A matching that is maximal is **not necessary Maximum** (example).



(a) A maximal matching not maximum



(b) A maximum matching

Greedy algorithm

Can you propose a greedy algorithm to address the maximum matching problem ?

Greedy algorithm

Result : Matching M

$M \leftarrow \emptyset$;

for $e \in E$ **do**

  if $M \cup \{e\}$ *is a matching* **then**

    $M \leftarrow M \cup \{e\}$

  end

end

return M

Algorithme 0 : Greedy algorithm to find a matching

Greedy algorithm

- What is the type of matching algorithm returned by this algorithm ?
- What is the complexity of this algorithm ? (as a function of the number of nodes n of the graph)

Greedy algorithm

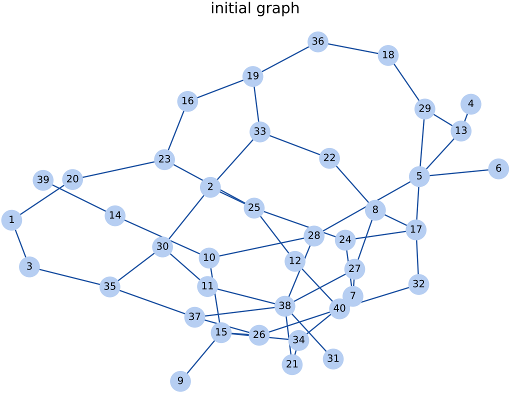
- The greedy algorithm returns a **maximal** matching (proof)
- Its complexity is smaller than $\mathcal{O}(np)$ (n nodes, p edges) (proof)
- smaller than **cubic** in the number of nodes : $\mathcal{O}(n^3)$

Greedy algorithm

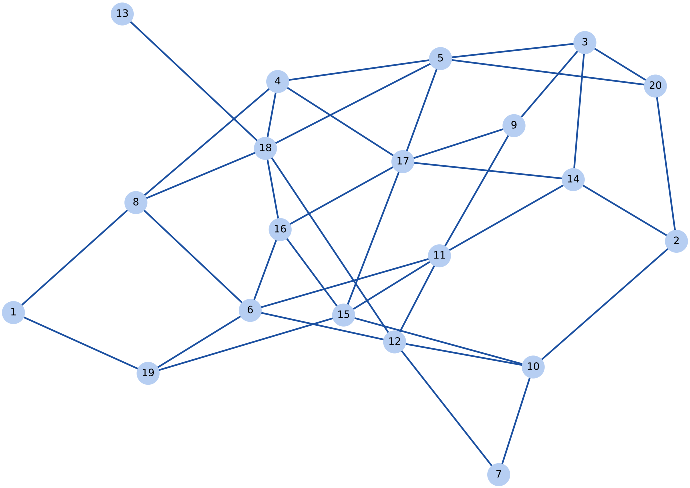
- We will implement the greedy algorithm to find a maximal matching.

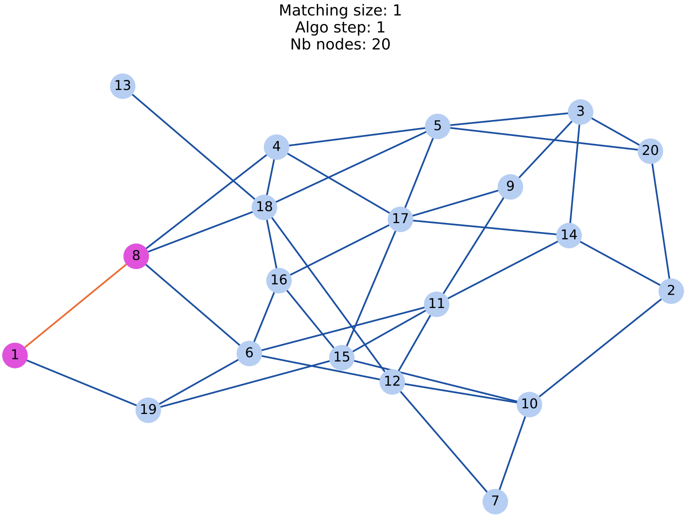
Exercice 3 : Implementing the greedy algorithm

Using **main_matching_greedy.py**, you can generate problem instances and apply the greedy algorithm by fixing **matching_greedy/greedy_matching.py**. The images are stored in **matching_greedy/images/**.



initial graph




Matching size: 1
Algo step: 1
Nb nodes: 20
1
2
3
4
5
6
7
8
9
10
11
12
13
14
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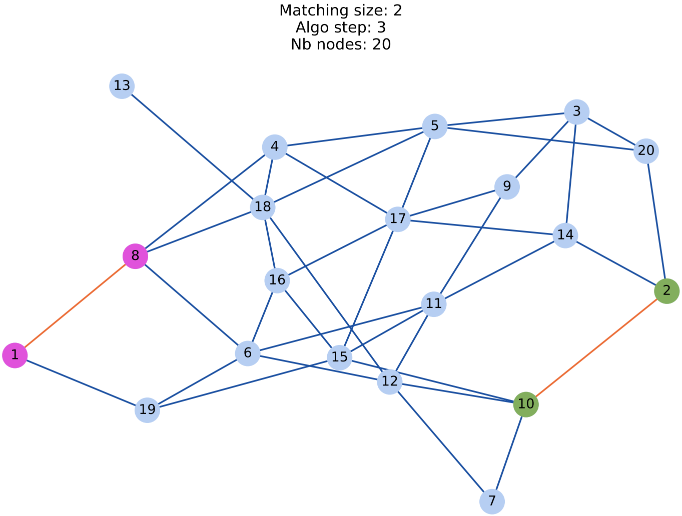
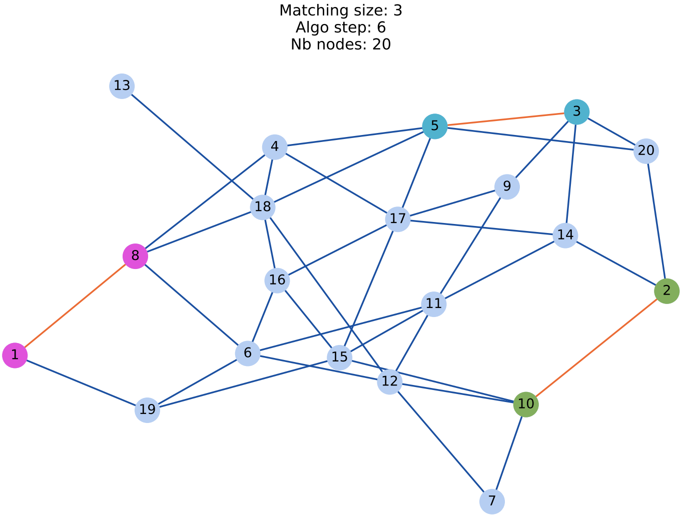
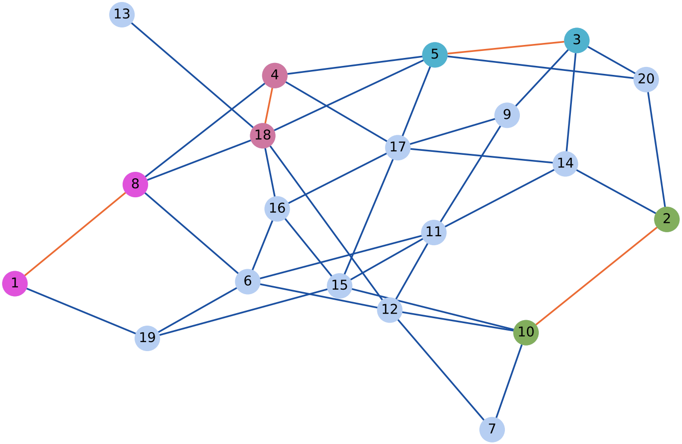
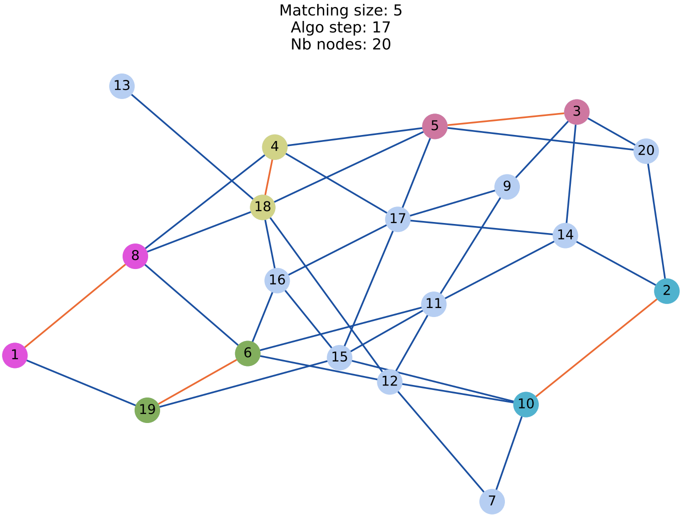
16
17
18
19
20


Matching size: 2
Algo step: 3
Nb nodes: 20
13
3
5
4
20
9
18
17
14
8
16
2
11
1
6
15
12
10
19
7

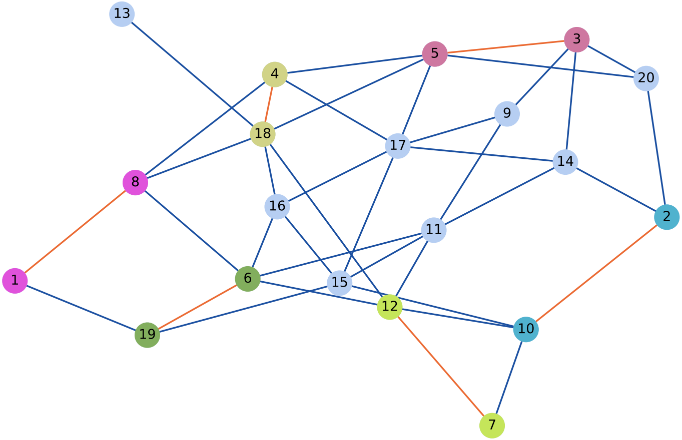

Matching size: 3
Algo step: 6
Nb nodes: 20
13
3
5
4
20
9
18
17
14
8
16
2
11
1
6
15
12
10
19
7

Matching size: 4
Algo step: 11
Nb nodes: 20

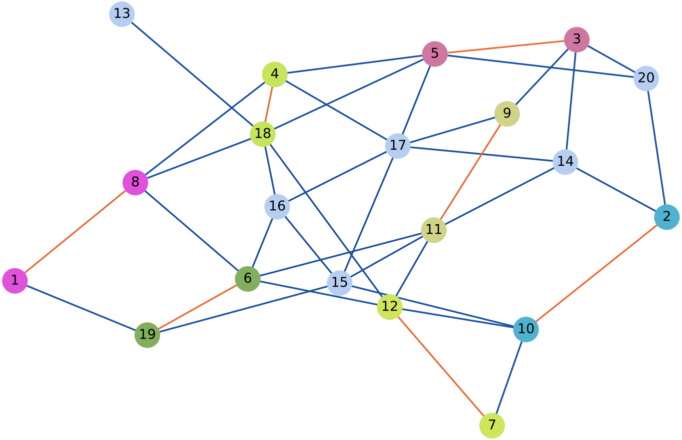



Matching size: 5
Algo step: 17
Nb nodes: 20
13
3
5
4
20
9
18
17
14
8
16
2
11
1
6
15
12
10
19
7

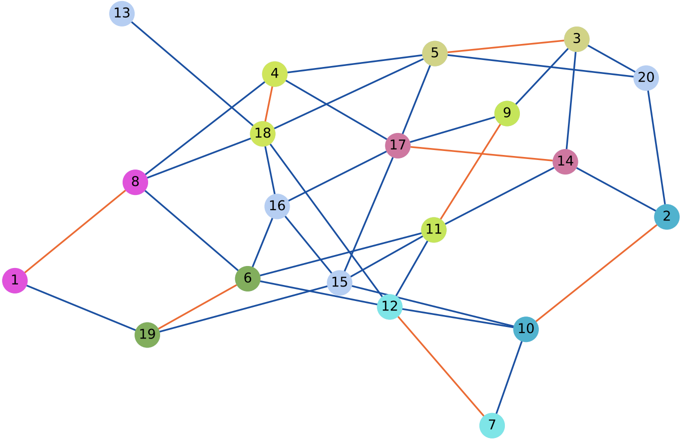
Matching size: 6
Algo step: 22
Nb nodes: 20



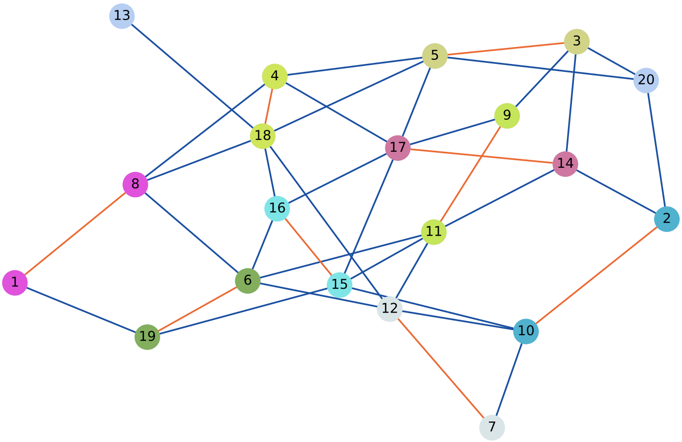
Matching size: 7
Algo step: 25
Nb nodes: 20



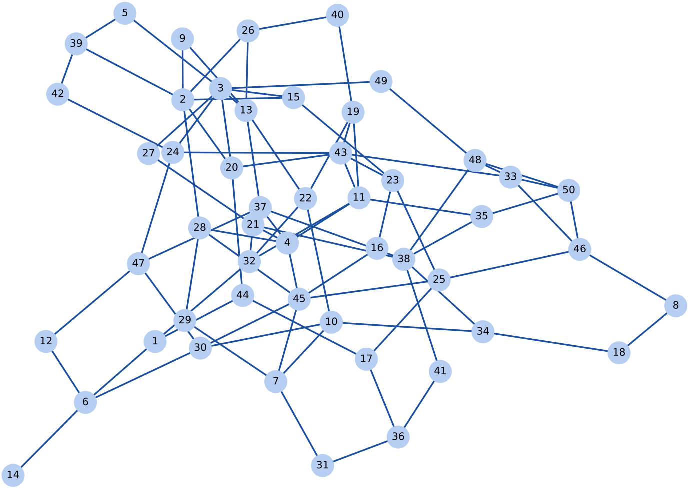
Matching size: 8
Algo step: 34
Nb nodes: 20



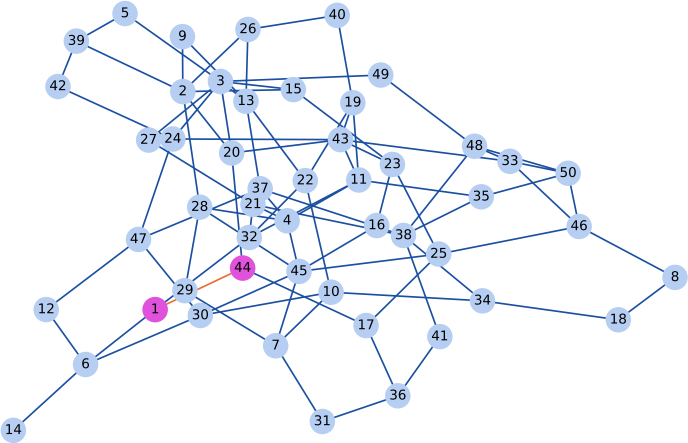
Matching size: 9
Algo step: 36
Nb nodes: 20



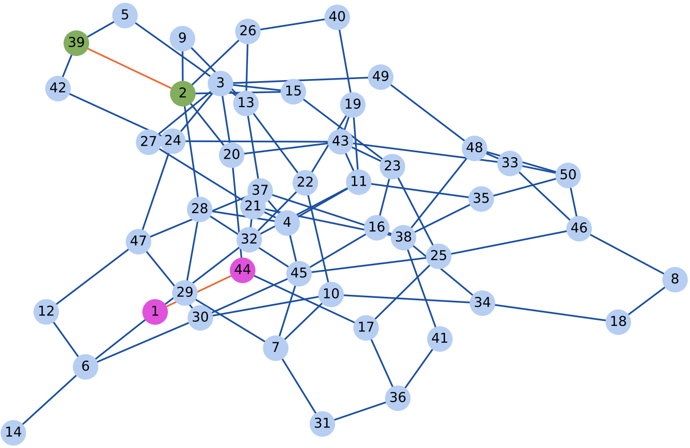
initial graph



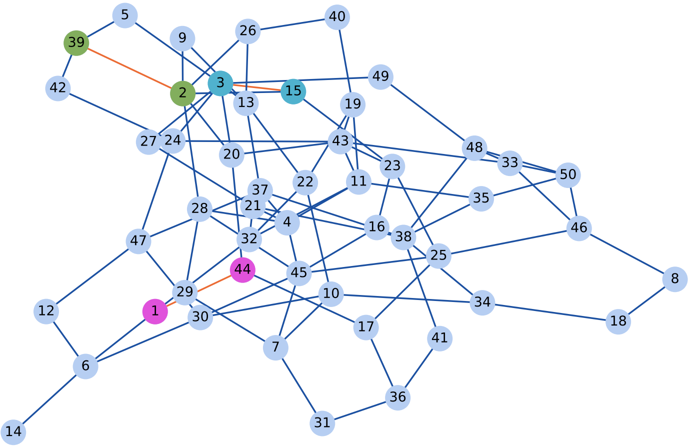
Matching size: 1
Algo step: 1
Nb nodes: 50



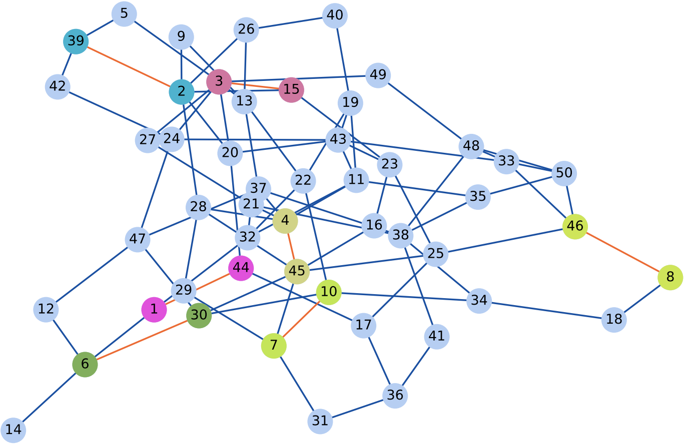
Matching size: 2
Algo step: 4
Nb nodes: 50



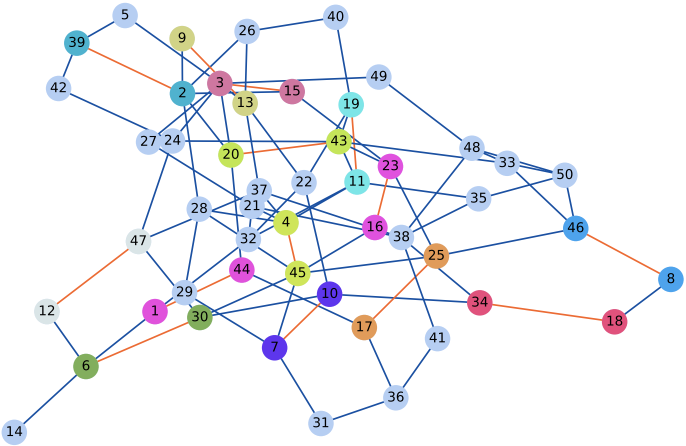
Matching size: 3
Algo step: 10
Nb nodes: 50



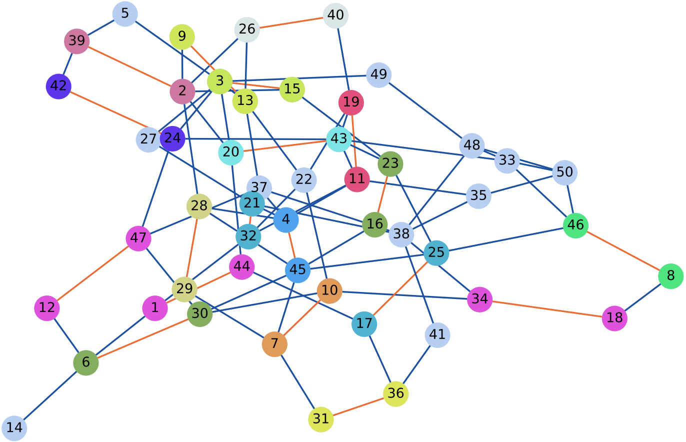
Matching size: 7
Algo step: 30
Nb nodes: 50



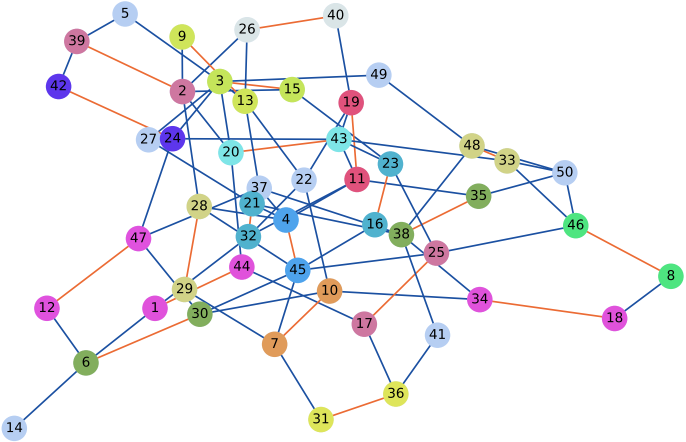
Matching size: 14
Algo step: 54
Nb nodes: 50



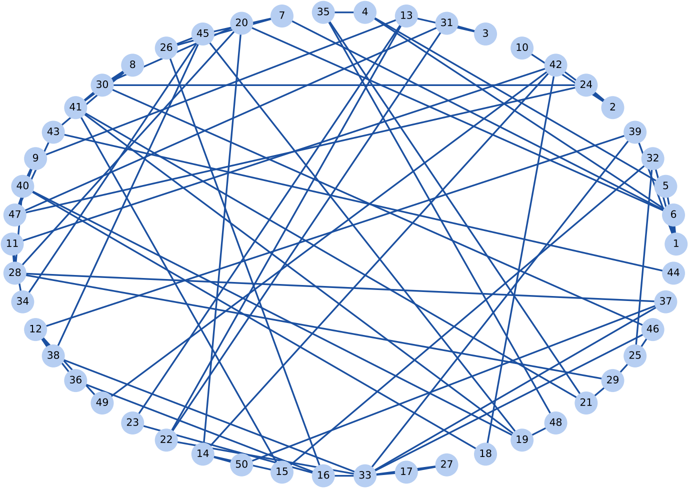
Matching size: 19
Algo step: 72
Nb nodes: 50



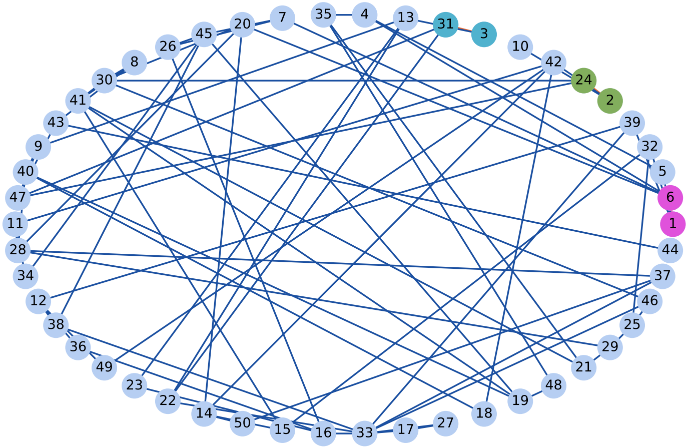
Matching size: 21
Algo step: 78
Nb nodes: 50

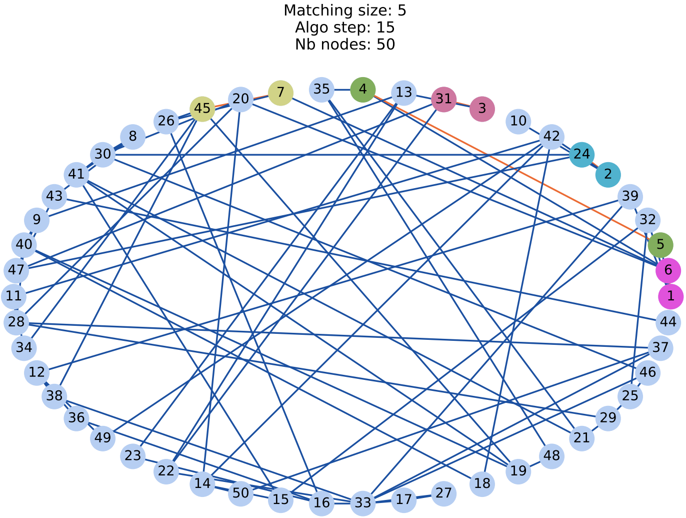


initial graph

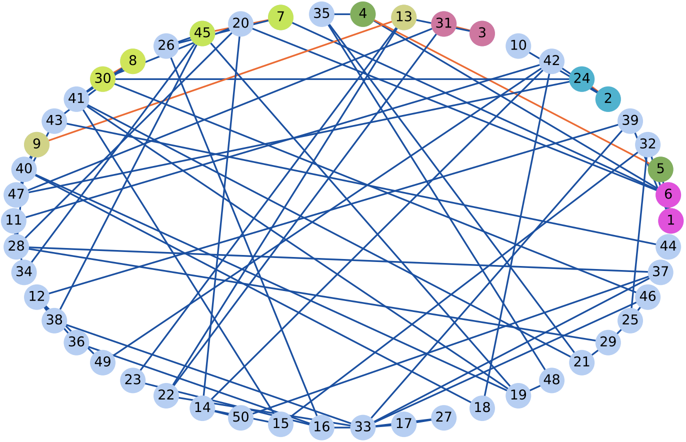


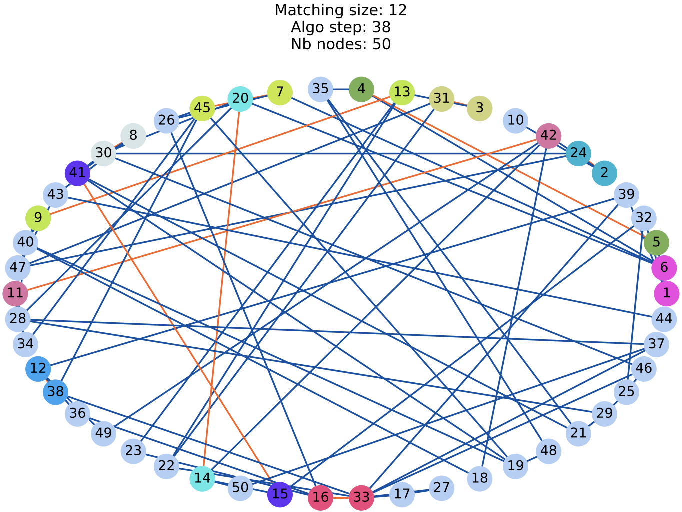
Matching size: 3
Algo step: 8
Nb nodes: 50




Matching size: 5
Algo step: 15
Nb nodes: 50

Matching size: 7
Algo step: 20
Nb nodes: 50




Matching size: 12
Algo step: 38
Nb nodes: 50
35
4
13
31
3
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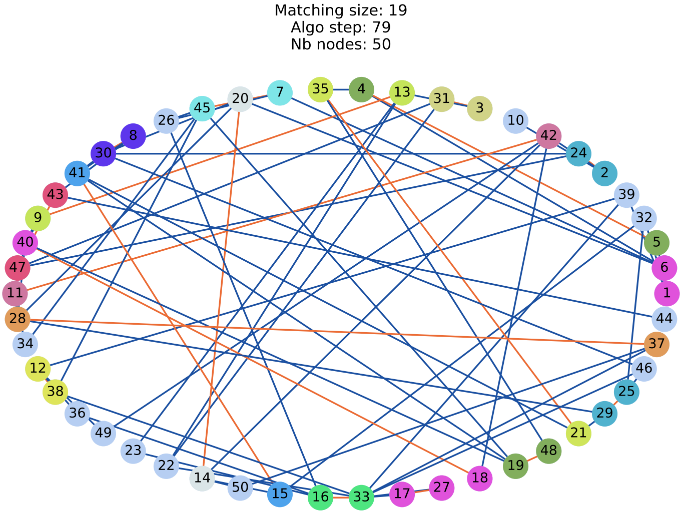
Matching size: 19
Algo step: 79
Nb nodes: 50
35
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10

Example

Exercice 3 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

Example

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Greedy matching

However, is $|M|$ is the cardinality of a matching returned by the greedy algorithm, and if $|M^*|$ is the cardinal of the real optimal matching, we can theoretically show that :

$$|M| \geq \frac{|M^*|}{2} \tag{2}$$

Speed comparison as a function of the data structure

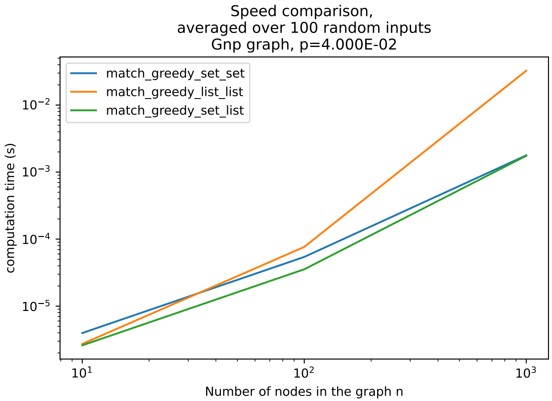

Figure: The functions will be available in code/solutions and shown during the class.

Matchings and vertex covers

Exercice 4 : Show that the nodes of the edges selected in a maximal matching form a **vertex cover**.
https://en.wikipedia.org/wiki/Vertex_cover

Matchings and vertex covers

Exercice 5 : Show that any matching is smaller than any vertex cover.