Algorithmic complexity and graphs: problem statement

14 septembre 2024

Need for speed

Speed is important for most applications of computer science, e.g. :

- games
- apps
- scientifc calculus, machine learning
- artistic software (Digital Audio Workstations, graphic software, etc.)

Speed bottleneck

The speed at runtime depends on the complete processing chain :

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- 2) low-level implementation (programming language, data structure optimization, ressource management)
- 3) algorithms used

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Remarks I :

- the separation between these 3 aspects is somewhat artificial, as the low-level implementation also depends on the hardware and might be considered as part of the algorithm.
- this separation is nonetheless reasonable and it also makes sense to study each aspect separately, in the right context.

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Remarks II :

- 1) between an average machine and and an excellent machine, we most of the time obtain a speedup that is ≤ 10 (depending on the task).
- 2) better libraries, languages, protocols often yield important speedups, however it is not possible to give general figures here. But most of the time, for a given task, the speedup is ≤ 100.
- 3) some algorithms are fundamentally unusable because of the time necessary to run them on some problems.

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Remarks III :

- quantum computing **might** behave differently (faster) than classical algorithms in the future, on some tasks, and change the landscape in some fields. However, it is hard to know what the extent of this change will be (in research and industry). Right now, it is still limited by the hardware possibilities, although some companies exist in cryptography.

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Remarks IV :

- our topic is mostly 3), but we also will work a little bit on 2)

What is an algorithm ?

- **Proposed definition** "A method to solve a problem based on a sequence of elementary operations, arranged in a determined order"

First example

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- Please propose the simplest possible algorithm to complete this task (without worrying about the formalization yet)

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- most intuitive solution : check each folder one by one, in their order (**linear seach**)
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- We could split the folders stack in two parts, then check the first folder of the lower part of the stack (**dichotomic search**)

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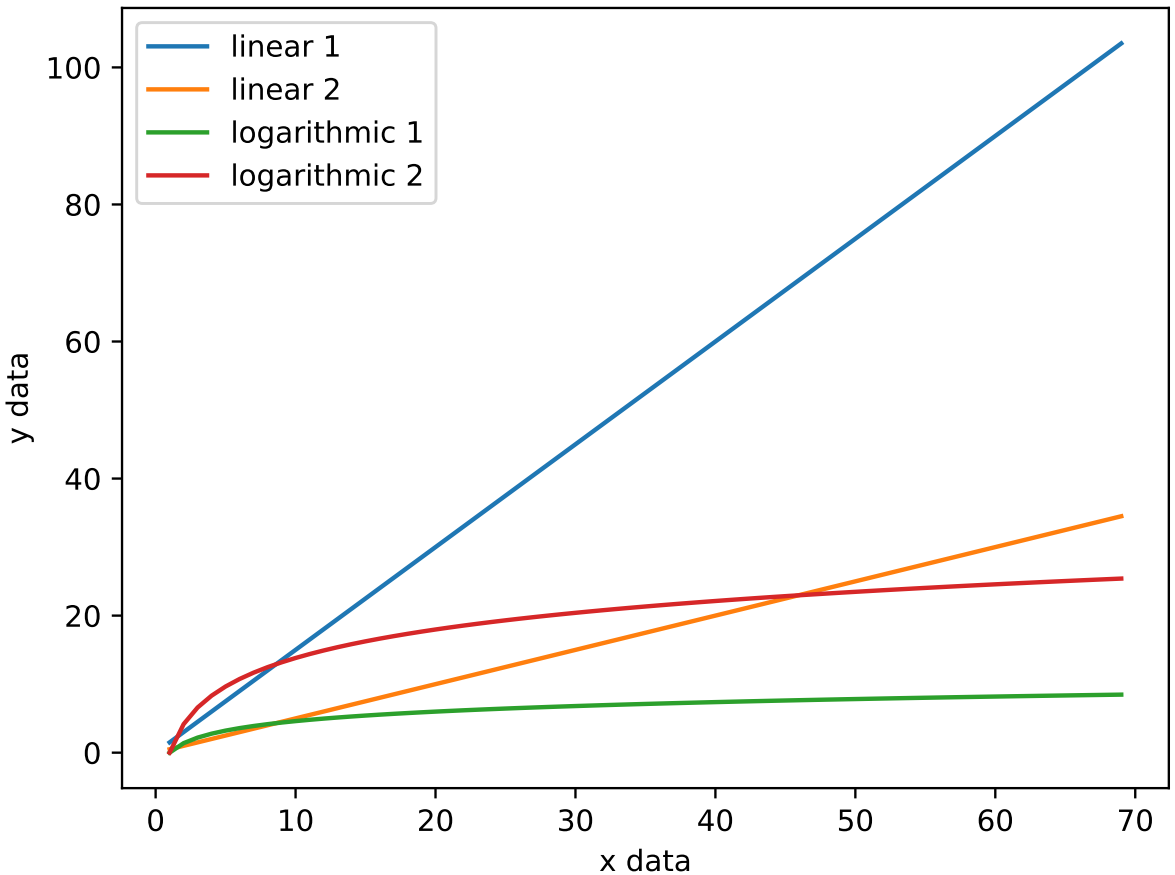
- We need at most $\log_2 n$ checks (backboard)

Logarithms and linear functions



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- How many checks does the **linear search need** at most ? (n is still the initial number of folders)
- It needs n checks.
- So we have to algorithms that perform the same task, but one of them is faster ($\log n$ versus n)

Simple search example

Exercice 1 : **Comparison between linear and dichotomic search**
If the stack contains 10 folders, how faster is the dichotomic search compared to the linear seach ?

Simple search example

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And if the stack contains $50, 100, 1000, 10000$ folders ?

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- We say that they have a different **complexity**, which will be the topic of the course
- The complexity is an **approximation** : we are often interested in **orders of magnitude**

Time Complexity

- We will study the complexity of algorithms.
- More specifically we will focus on the **time complexity** of algorithms. It is an order of magnitude of the number elementary operations required to solve a problem, given a method (ie : given an algorithm)
- Several computations are possible : for instance worst-case complexity, average-case complexity.

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  - $2 \times n^2$ is of the same order of magnitude as $100 \times n^2$
  - 10×2^n is of the same order of magnitude as 500×2^n
  - but 3^n is **NOT** the same order of magnitude as 2^n
- But the **constants** (e.g. 2 in 2^n) are also important !

What is the size of the input ?

What we mean by size of the input will depend on the context. Actually, this size will often be represented by several numbers !

- in machine learning, we always express the complexity as a function of **at least** the number of samples n and the dimension of each sample d.
- in graph theory, the complexity is often expressed as function of the number of nodes n and edges p of the graph.

Space complexity

- There is another time of complexity : the **space complexity**
- It is a measure of the memory used by the algorithm in order to run.

Dependency to the language

- When considering an actual implementation, both space and time complexity depend on the programming language.
- In this course, our focus is slightly more mathematical / abstract, than language specific.
- Python makes it easier to illustrate this aspect, compared to a lower level language.

Formalization

- Let us define how we should **specify** an algorithm. It needs :
  - inputs
  - outputs
  - preconditions
  - postconditions

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  - preconditions : the folders stack is sorted in the alphebetical order
  - postconditions :

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  - inputs : stack of folder
  - outputs : boolean (**True** if the stack contains the given name X, **False** otherwise)
  - preconditions : the folders stack is sorted in the alphebetical order
  - postconditions : the output is **True** if and only if the folders stack contains a folder whose name is X.

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- the termination : the algorithm should end after a **finite number** of computation steps

Theoretical notions

In order to study algorithms from a mathematical point of view and give the intuitive notion a solid grounding, several **axiomatic systems** (plusieurs "axiomatiques") have been built.

- Turing Machines (Turing, 1936)
- Lambda calculus (Alonzo Church, 1930s)
- Recursive functions (Kurt Gödel, 1930s)

Importantly, it was shown that these axiomatic systems are **equivalent**, in the sense that they describe the same **computable functions** (fonctions calculables).

Speedups unrelated to complexity

Before diving into algorithmic complexity, let us look more precisely at some example program speedups that are only related to **implementation** and language-specific features.

First example : vectorization of computations

Using vectorization instead of for loops can bring a significant speedup, especially in python. Typically, in scientific computing, C libraries such as **numpy** and **scipy** are used.
`https://en.wikipedia.org/wiki/Array_programming`

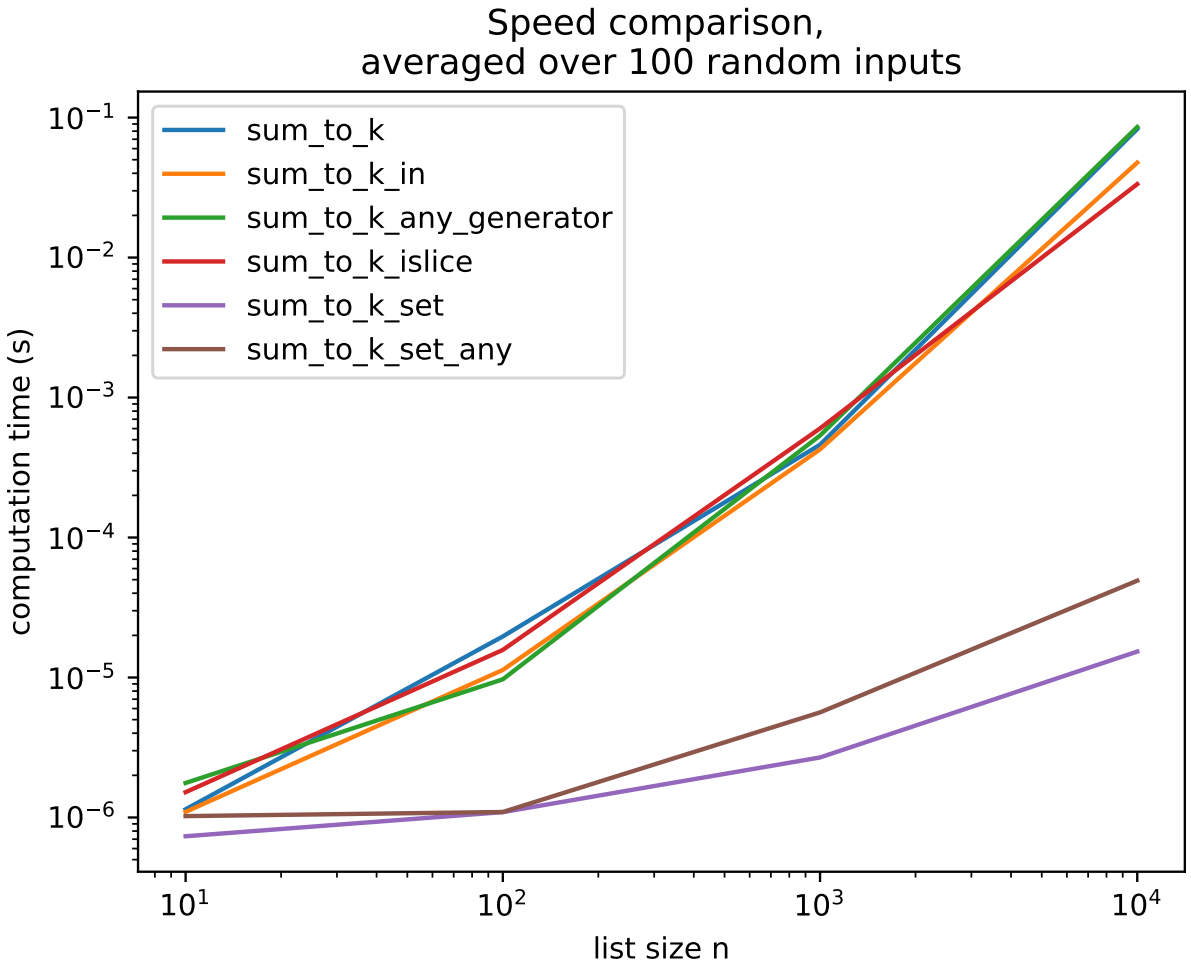
In **vectorization/law_of_large_numbers.py**, the function **empirical_average_loop()** computes the average of a number of random samples drawn from a uniform distribution between 1 and 2 and the squared (see numpy demo).
We would like to perform the same computation using vectorization and to study the speedup. We will also profile and display a **flamegraph** with **snakeviz**. In order to profile, simply run "snakeviz profile.prof" (after running the python file first).
Discussion on "tottime" vs "cumtime".

Exercice 1 :

Fix the function **empirical_average_array()** in order to perform the same computation (samples generation + averaging) and observe the speedup.

Second example : Python specific implementation hacks

Similarly to most languages, python itsself can be used in a more or less optimal way.
See **misc/sum_to_k.py** for some examples, tests, and benchmarking (more details on many of these aspects later in the course).
This will not be the core focus of the course !


Speed comparison,
averaged over 100 random inputs
sum_to_k
sum_to_k_in
sum_to_k_any_generator
sum_to_k_islice
sum_to_k_set
sum_to_k_set_any
computation time (s)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{1}
10^{2}
10^{3}
10^{4}
list size n

Algorithms are not always that important

- In some areas, choosing the correct data structure and having a readable code will be more important or useful than the algorithm chosen.
  Rob Pike's rule 3 : "Fancy algorithms are slow when n is small, and n is usually small"
  `https://users.ece.utexas.edu/~adnan/pike.html`
- In other areas (scientific computing, machine learning, etc) the choice of the algorithm might be crucial, both for speed **and** the quality of the results.