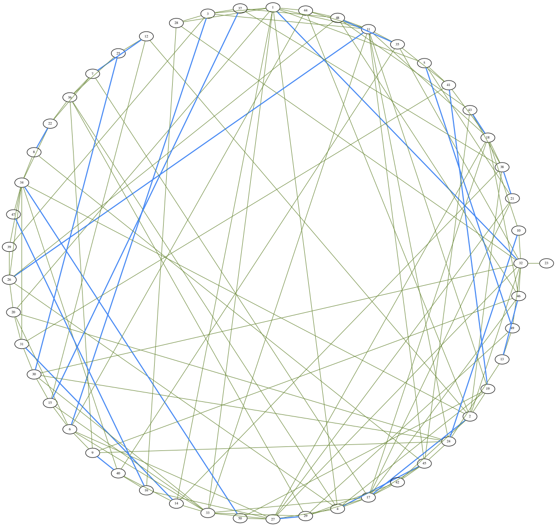
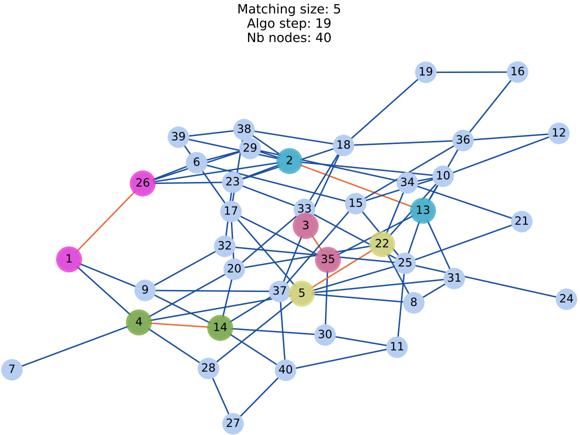
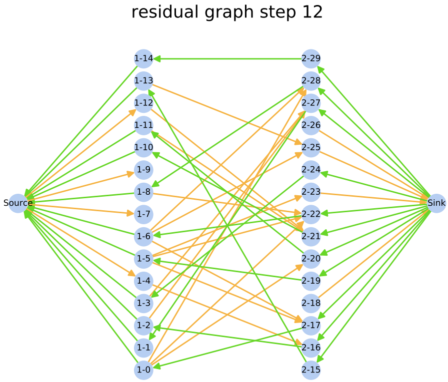
Algorithmic complexity and graphs: the matching problem

September 14, 2024



Matching size: 21
Algo step: 128
Nb nodes: 50


Matching size: 5
Algo step: 19
Nb nodes: 40


residual graph step 12
1-14
1-13
1-12
1-11
1-10
1-9
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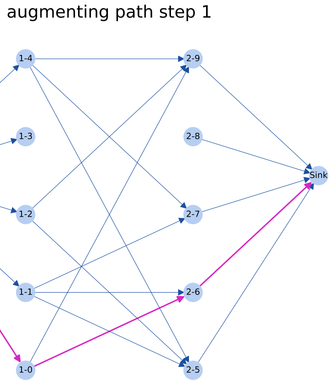
2-15
Source
Sink


augmenting path step 1
1-4
2-9
1-3
2-8
Source
Sink
1-2
2-7
1-1
2-6
1-0
2-5

The matching problem

The matching problem
- Definition of the problem
- Experimental solutions
- Brute force algorithm
- Greedy algorithm

Problem 1 : Max Flow

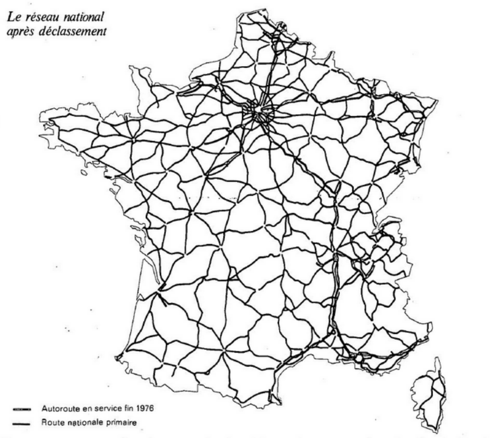


Figure: Problem 1 : transporting merchandise through a network

Problem 2 : Maximum matching (Optimal assignment, problème d'affectation)

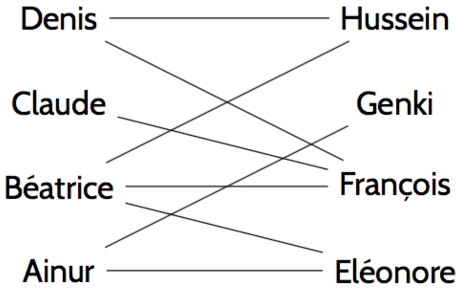


Figure: Problem 2 : Building the largest possible number of teams of 2 persons.

Problem 2

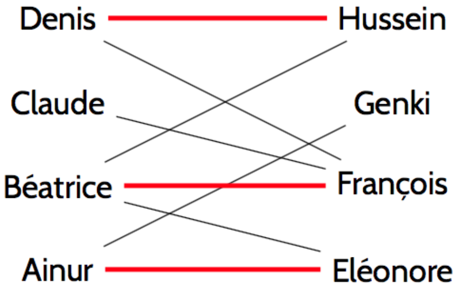


Figure: Problem 2 : not optimal assignment

Problem 2

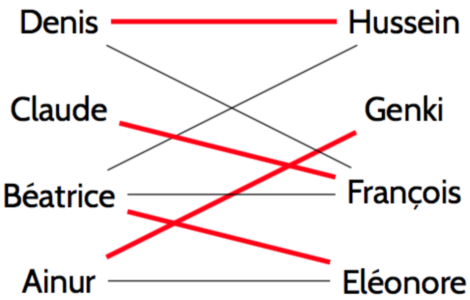


Figure: Problem 2 : optimal assignment

Other examples

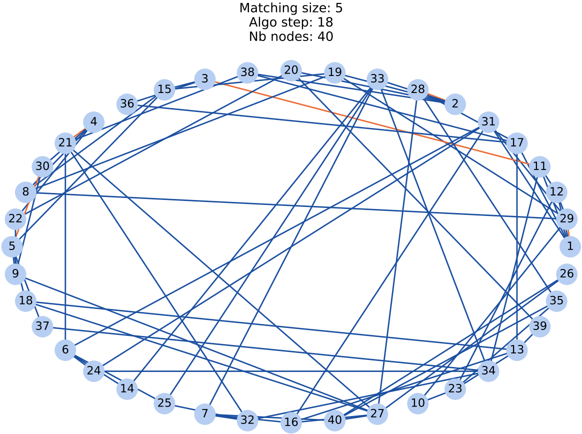
- Assigning students to internships
- Assigning machines to a task

Summary

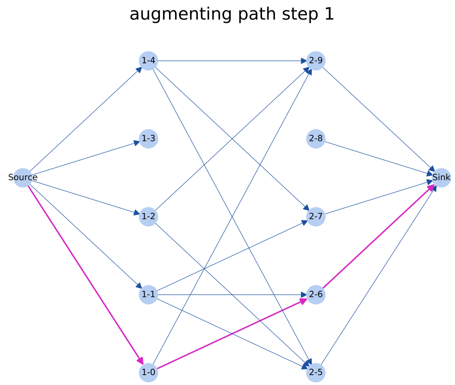
- Today we will work on **connnecting the two problems**.
- In some specific cases, the two problems **equivalent**.

Networkx

We will use networkx.



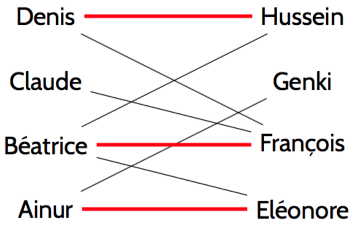
Networkx



Matching problem

Given a **undirected** graph $G = (V, E)$, we want a **matching** M, which means:

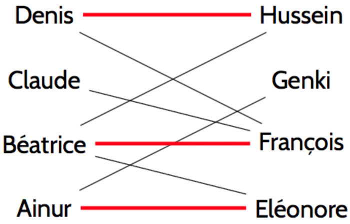
- A subset of edges $M \subset E$



Matching problem

Given a **undirected** graph $G = (V, E)$, we want a **matching**, which means:

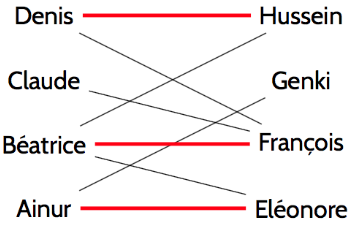
- A subset of edges $M \subset E$
- Such that no pairs of edges of M are incident
- Equivalently, each node in the graph is **at most** in one edge of M.



Matching problem

Given **undirected** a graph $G = (V, E)$, we want a **matching**, which means:

- A subset of edges $M \subset E$
- Equivalently, each node in the graph is **at most** in one edge of M.
- No pairs of edges of M are incident



Maximum matching

- The **size** of a matching is the number of edges it contains.

Maximum matching

- The **size** of a matching is the number of edges it contains.
- We want to find the matching of **largest possible size** in a given graph.

Example 1

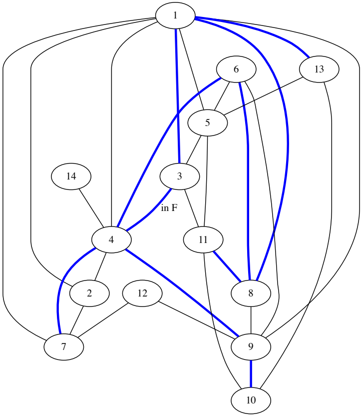


Figure: Is this a matching ?

Example 2

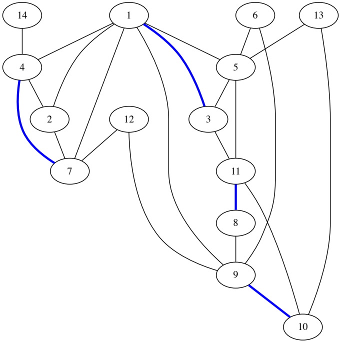


Figure: Is this a matching ?

Example 3

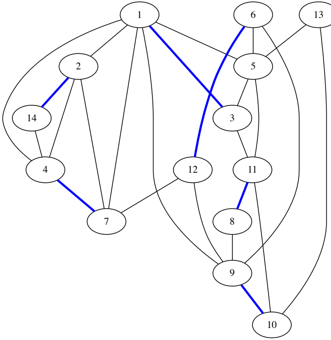


Figure: Is this an optimal matching ?

Example 4

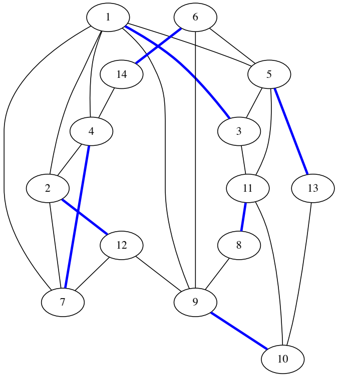


Figure: Is this an optimal matching ?

Example 5

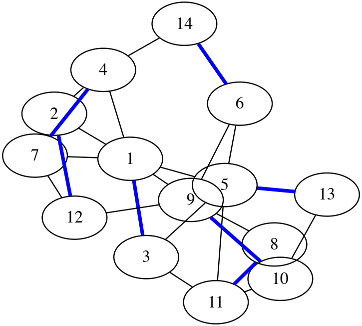


Figure: With neato

Optimal matching

Exercice 1 : What is the maximum size possible for a matching, in a general graph of size n ? (in the sense that no graph with n nodes contains a larger matching)

Optimal matching

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- If n is even : $\frac{n}{2}$
- Else n is odd : $\frac{n-1}{2}$

Hence,

$$\lfloor \frac{n}{2} \rfloor \tag{1}$$

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal

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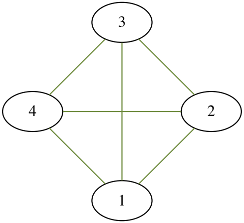

Figure: The complete graph works

Optimal matching

Exercice 1 : Can you think of a graph with n nodes that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

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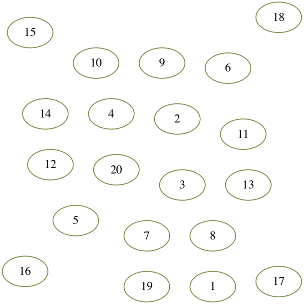


Optimal matching

Exercice 1 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

Optimal matching

Exercice 2 : Can you think of a **non trivial** graph that does **not** contains a matching of size $\frac{n}{2}$? (assuming n is even)

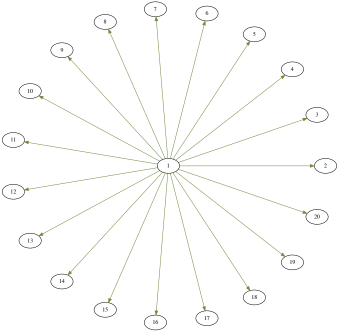


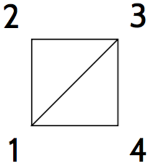
Figure: Star graph

Experiments

Possibilities to code a graph:

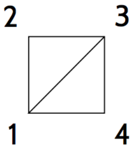
- list of sets of size 2 (for an undirected graph)
- a dictionary of successors (directed of undirected)

Coding a graph : as a list of edges



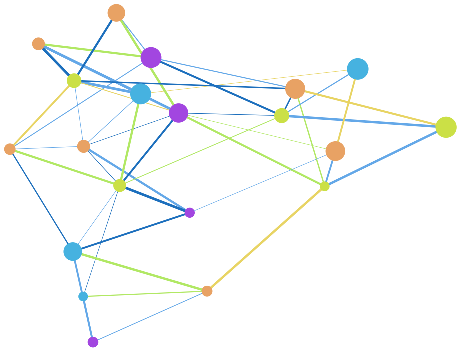
g1 = [{1,2},{1,3},{2,3},{3,4},{1,4}]

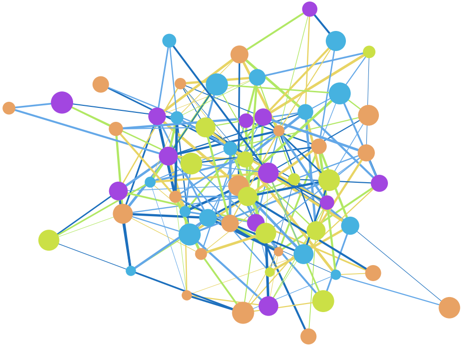
Coding a graph : as a dictionary of neighbors



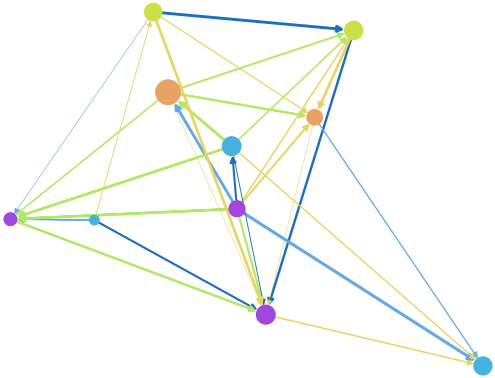

```
g1 = { 1:{2,3,4}, 2:{1,3}, 3:{1,2,4}, 4:{1,3} }
```

Generating graphs with networks.

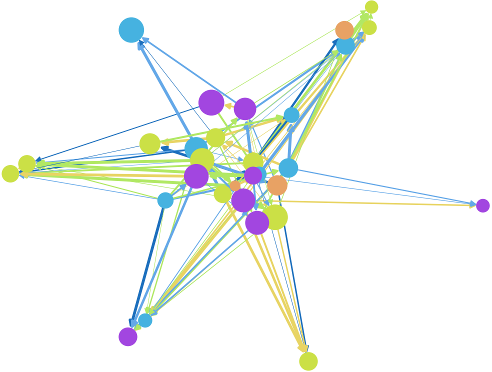




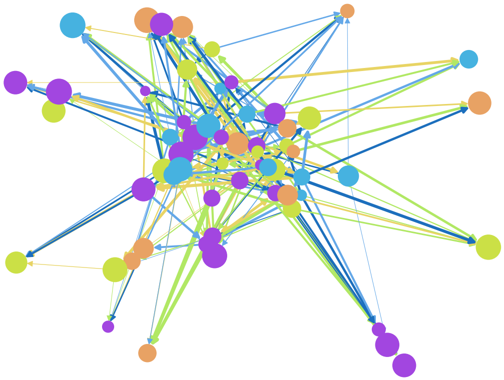
Directed graph



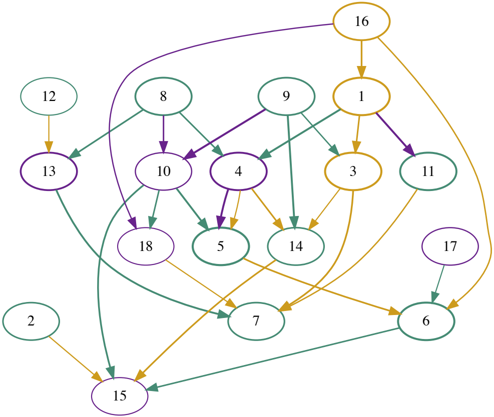
Directed graph II



Directed graph III



Example directed graph



Manual matching

Exercice 3 : Please manually find an **optimal matching** in your **undirected** graph.

Big graph

We could not manually find an optimal matching in this graph :

Summary

- We have defined the matching problem.
- When the size of the problem is large, we can not manually find an optimal matching.

Brute force approach

Exercice 4 : Enumeration

- Given a graph, what would a brute force approach on the matching problem be ?

Brute force approach

Exercice 4 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

Brute force approach

Exercice 4 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
    - 1) Enumerate all possible subsets in the set of the edges.
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If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?

Brute force approach

Exercice 4 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
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If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?
You can give a rough approximation.

Brute force approach

Exercice 4 : Exhaustive search

- Given a graph what would a brute force approach on the matching problem be ?
  - 1) Enumerate all possible subsets in the set of the edges.
  - 2) Check if each subset is a matching.
  - 3) Return the biggest one obtained.

If the graph contains n nodes, and given a subset of edges, what if the number of computations needed to perform step 2 ?
It is a **polynomial** number of computations.

Brute force search

Exercice 5 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?

Brute force search

Exercice 5 : Complexity of brute force

- 1) Enumerate all possible subsets in the set of the edges.
- 2) Check if each subset is a matching.
- 3) Return the biggest one obtained.

What is the complexity of step 1 ?

The number of subsets is $2^{\frac{n(n-1)}{2}}$ (in the worst case), which is exponential. If p is the numberof edges, we can also write it as 2^p.

Brute force search

Exercice 5 : **Complexity of brute force**
Assume that checking a subset requires 1 microsecond. How long should we wait in order to check all possible matchings in a graph with 100 nodes ?

Summary II

- For the matching problem on a large graph, we can neither
  - manually find an optimal matching
  - perform the exhaustive search (brute force algorithm)

Algorithms

- Hence, we need different algorithms to solve the problem.
- Let us first introduce some theoretical notions.

Notion of maximal and maximum matching

We will say that a matching M of cardinality (number of elements) $|M|$ is:

- **Maximum** if is has the maximum possible number of edges (it is thus optimal)

Notion of maximal and maximum matching

We will say that a matching M of cardinality $|M|$ is:

- **Maximum** if it has the maximum possible number of edges (it is thus optimal)
- **Maximal** if the set of edges obtained by adding any edge to it is **not a matching**. This means that $M \cup \{e\}$ is not a matching for any e not in M.
- $\cup$ means union of sets.

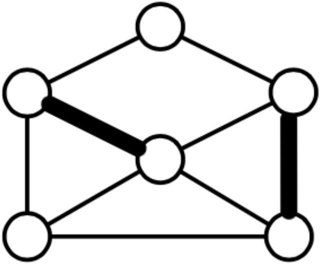
Is being a **maximal** matching the same thing as beeing a **maximum** matching ?

Maximum implies maximal

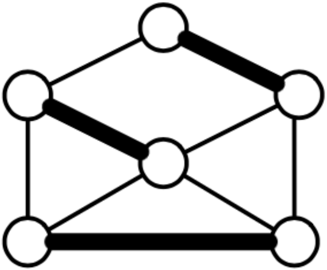
Let us show that a maximum matching is maximal.

Counter Example

However, a matching that is maximal is **not necessary Maximum** (example).



(a) A maximal matching not maximum



(b) A maximum matching

Greedy algorithm

Can you propose a greedy algorithm to address the maximum matching problem ?

Greedy algorithm

Result : Matching M
$M \leftarrow \emptyset$;
for $e \in E$ **do**
  if $M \cup \{e\}$ *is a matching* **then**
    $M \leftarrow M \cup \{e\}$
  end
end
return M

Algorithme 0 : Greedy algorithm to find a matching

Greedy algorithm

- What is the type of matching algorithm returned by this algorithm ?
- What is the complexity of this algorithm ? (as a function of the number of nodes n of the graph)

Greedy algorithm

- The greedy algorithm returns a **maximal** matching (proof)
- Its complexity is smaller than $\mathcal{O}(np)$ (n nodes, p edges) (proof)
- smaller than **cubic** in the number of nodes : $\mathcal{O}(n^3)$

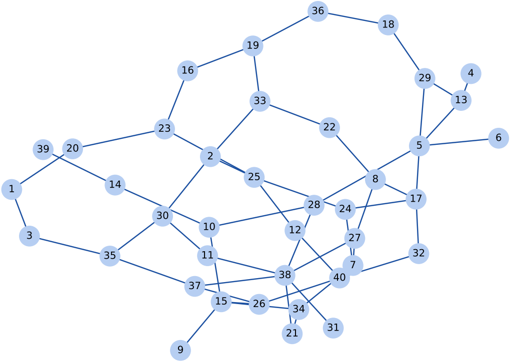
Greedy algorithm

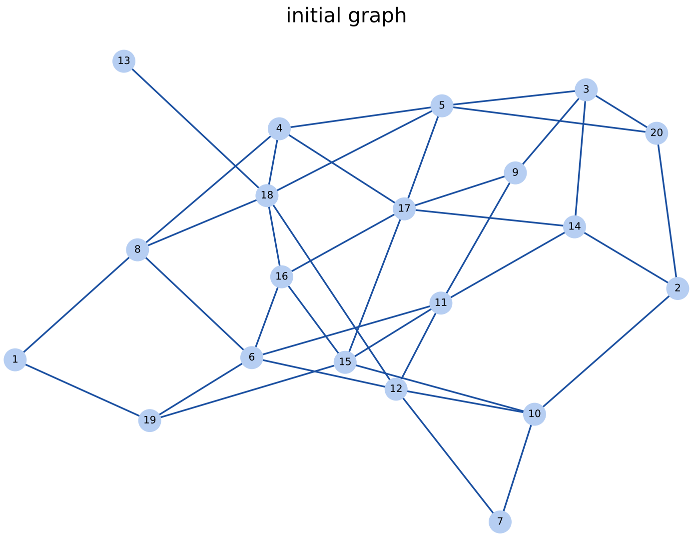
- We will implement the greedy algorithm to find a maximal matching.

Exercice 6 : Implementing the greedy algorithm

Using **main_matching_greedy.py**, you can generate problem instances and apply the greedy algorithm by fixing **matching_greedy/greedy_matching.py**. The images are stored in **matching_greedy/images/**.

initial graph




initial graph
13
5
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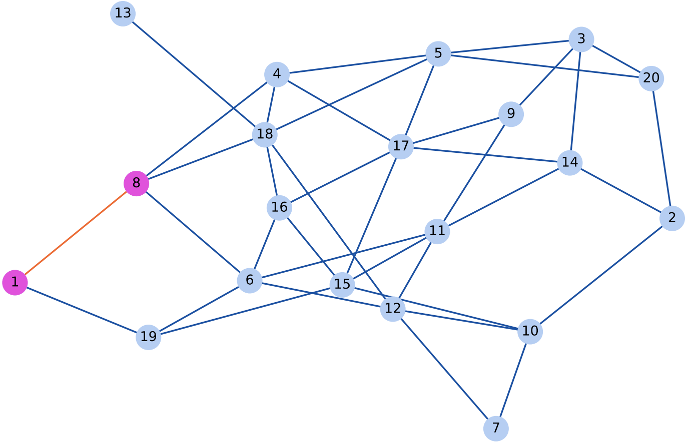
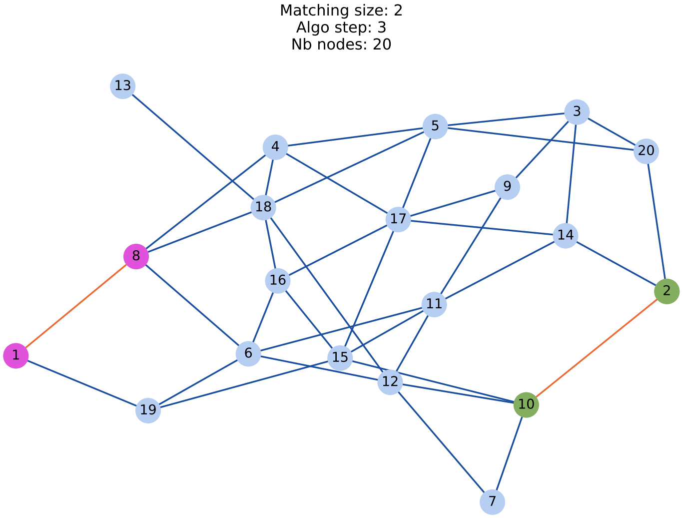
7

Matching size: 1
Algo step: 1
Nb nodes: 20




Matching size: 2
Algo step: 3
Nb nodes: 20
13
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11
1
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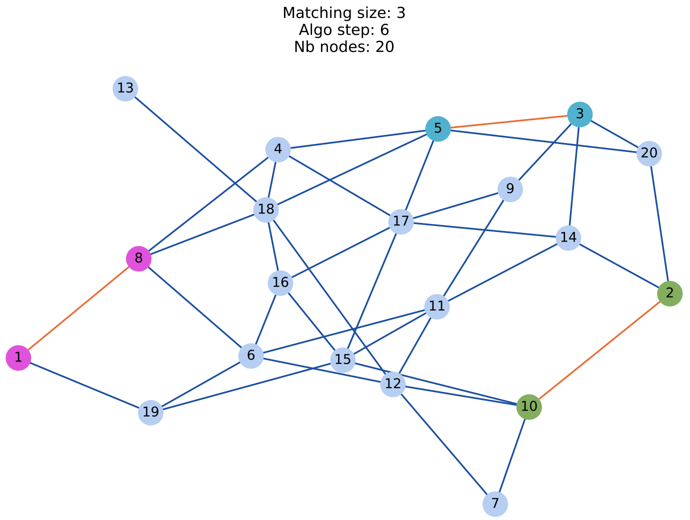
Matching size: 3
Algo step: 6
Nb nodes: 20
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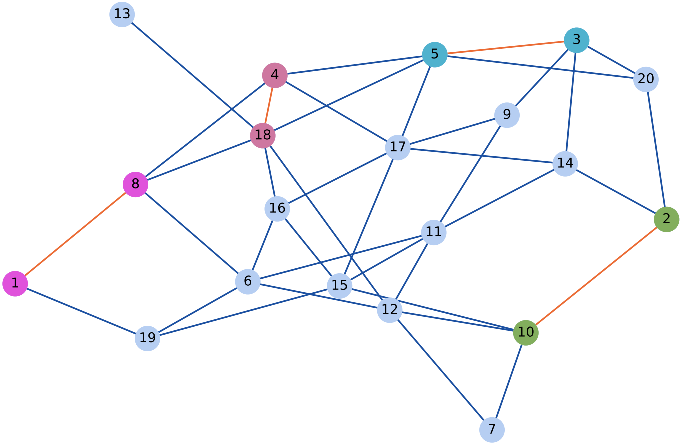
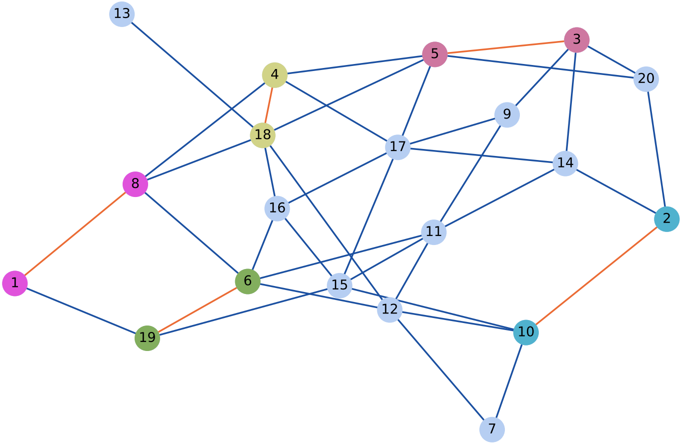
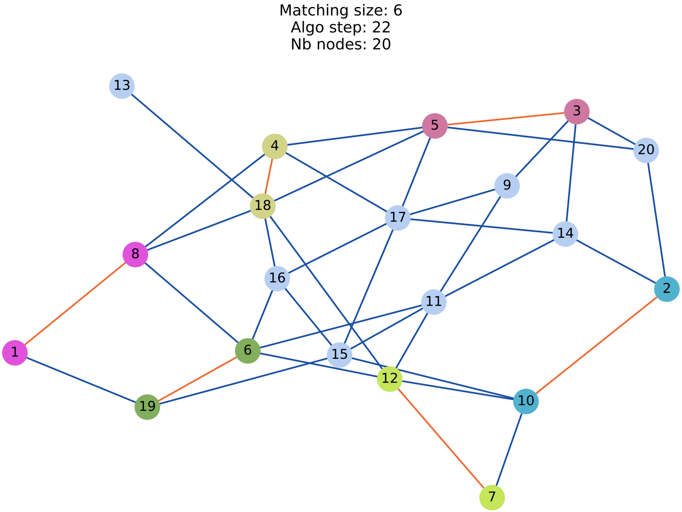
Matching size: 4
Algo step: 11
Nb nodes: 20



Matching size: 5
Algo step: 17
Nb nodes: 20




Matching size: 6
Algo step: 22
Nb nodes: 20
13
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18
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11
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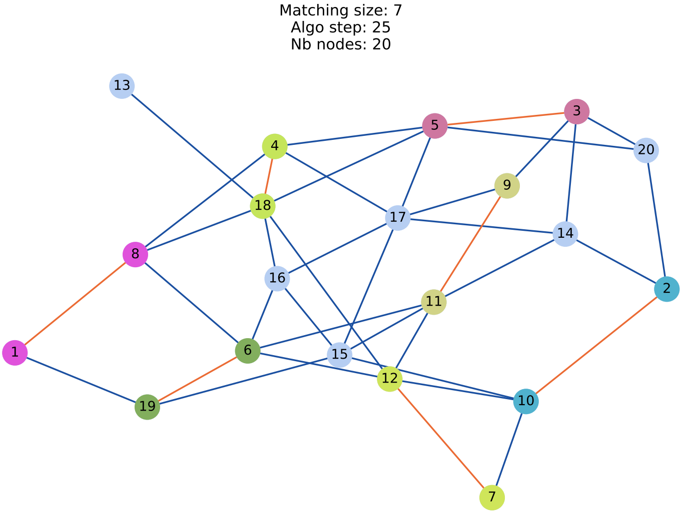
7


Matching size: 7
Algo step: 25
Nb nodes: 20
13
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11
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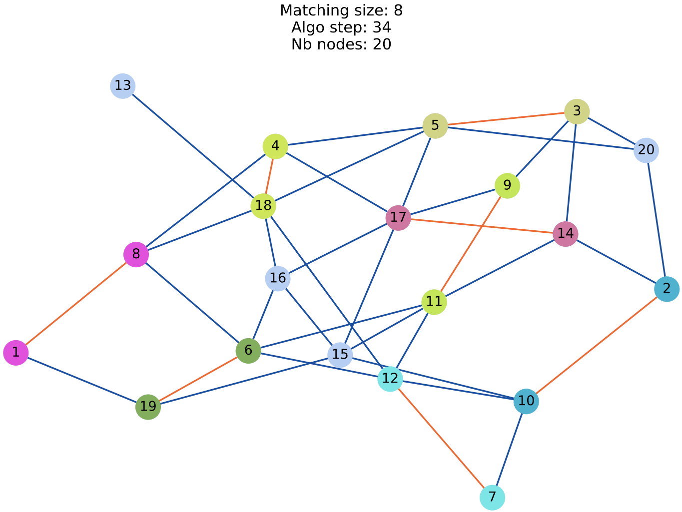
Matching size: 8
Algo step: 34
Nb nodes: 20
13
5
3
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9
18
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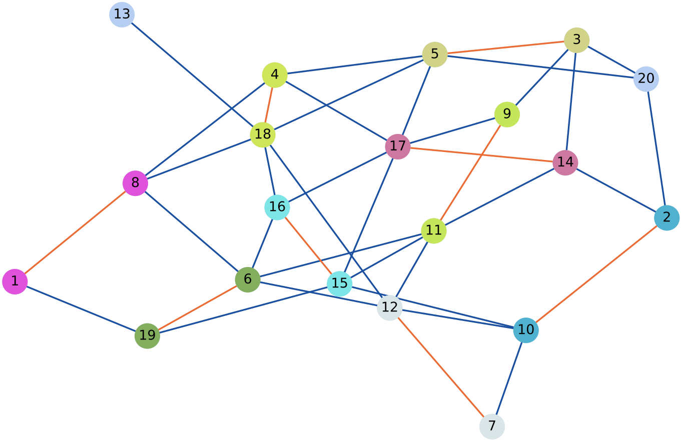
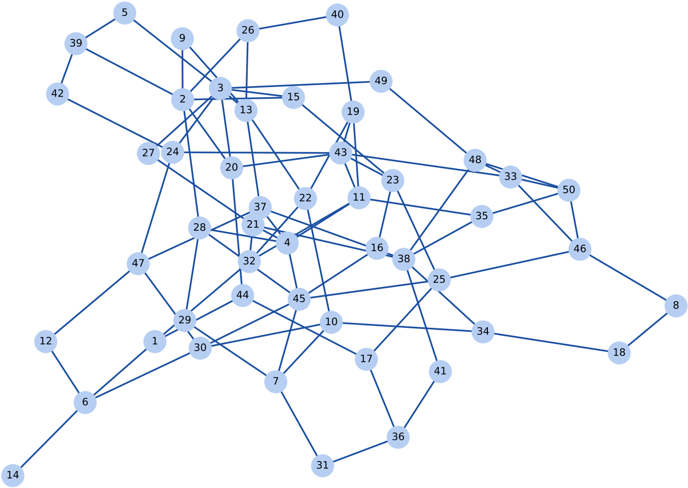
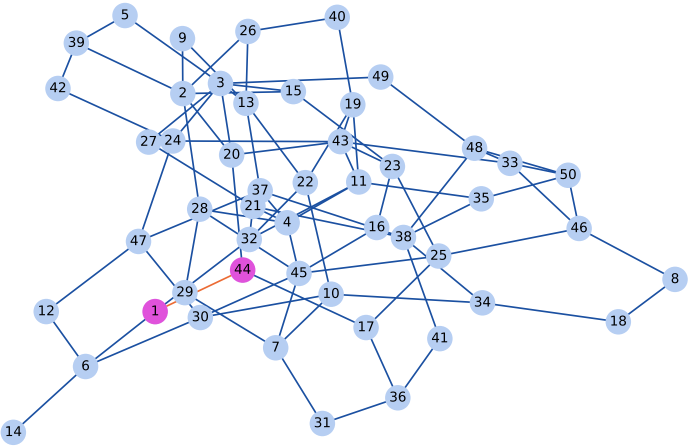
Matching size: 9
Algo step: 36
Nb nodes: 20



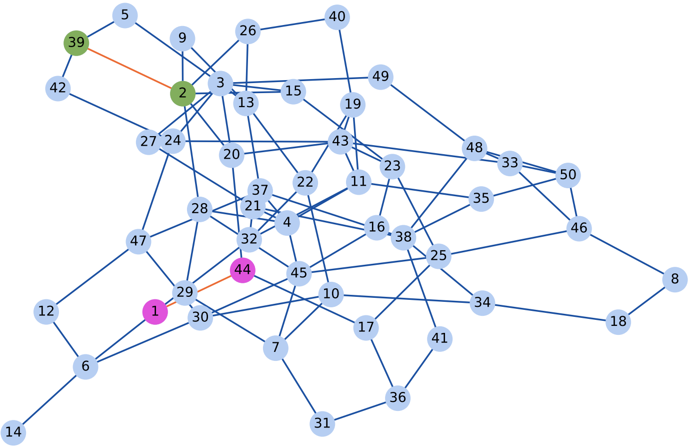
initial graph

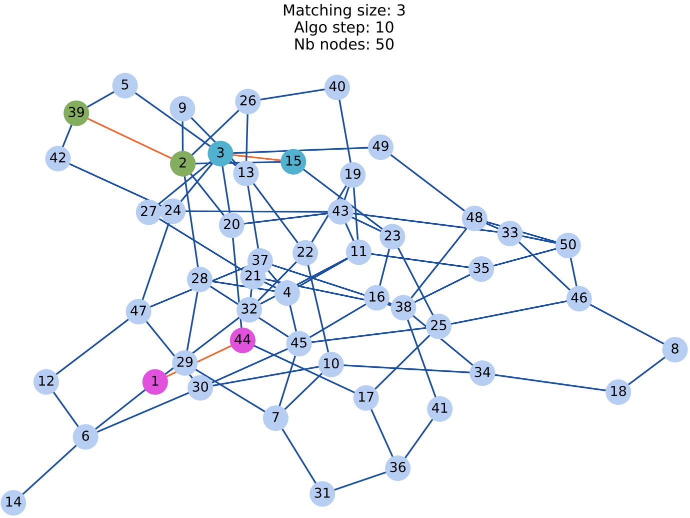


Matching size: 1
Algo step: 1
Nb nodes: 50

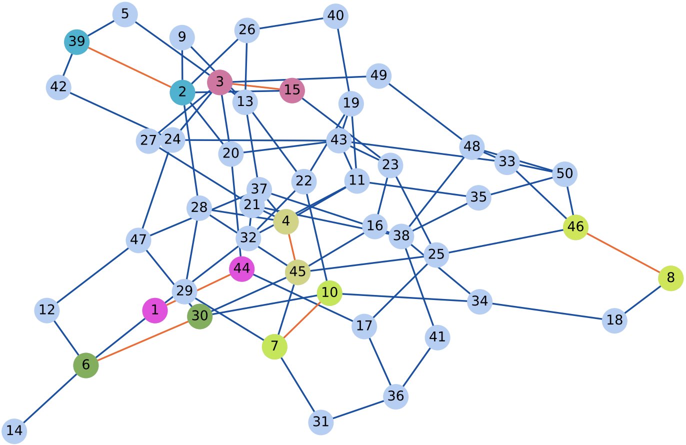


Matching size: 2
Algo step: 4
Nb nodes: 50

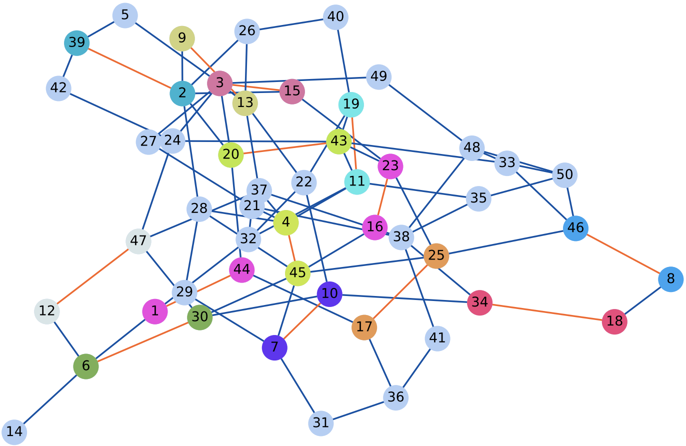



Matching size: 3
Algo step: 10
Nb nodes: 50
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50

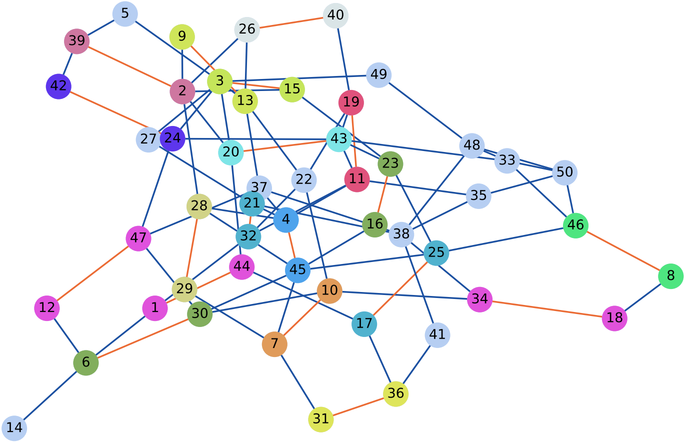
Matching size: 7
Algo step: 30
Nb nodes: 50



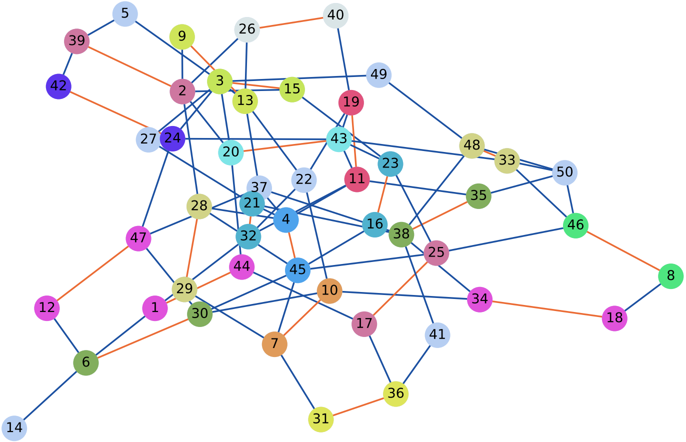
Matching size: 14
Algo step: 54
Nb nodes: 50



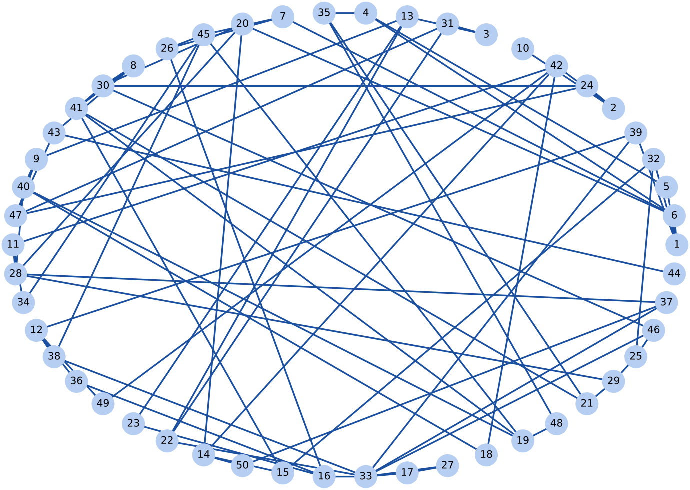
Matching size: 19
Algo step: 72
Nb nodes: 50

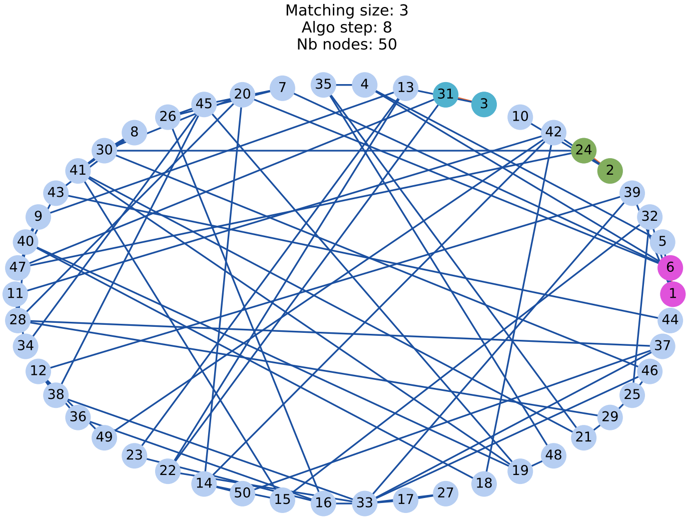


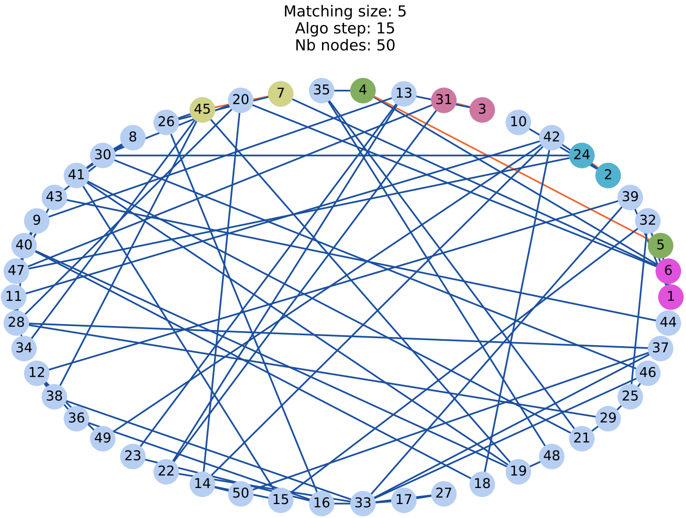
Matching size: 21
Algo step: 78
Nb nodes: 50



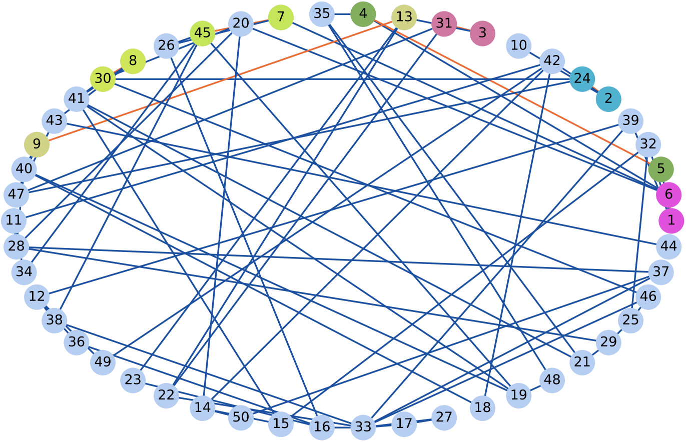
initial graph



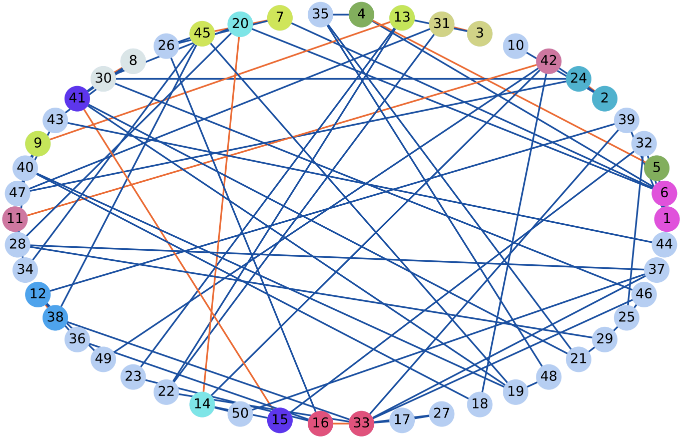

Matching size: 3
Algo step: 8
Nb nodes: 50
35
4
13
31
3
10
42
24
2
39
32
5
6
1
44
37
46
25
29
21
48
19
18
27
17
33
16
15
50
14
22
23
49
36
38
12
34
28
11
47
40
9
43
41
30
8
26
45
20
7

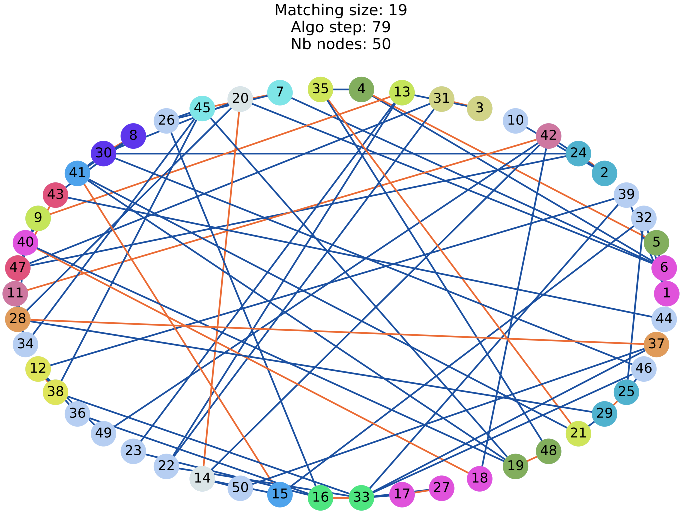

Matching size: 5
Algo step: 15
Nb nodes: 50

Matching size: 7
Algo step: 20
Nb nodes: 50



Matching size: 12
Algo step: 38
Nb nodes: 50




Matching size: 19
Algo step: 79
Nb nodes: 50
35
4
13
31
3
7
20
45
26
8
30
41
43
9
40
47
11
28
34
12
38
36
49
23
22
14
50
15
16
33
17
27
18
19
48
21
29
25
46
37
44
1
6
5
32
39
2
24
42
10

Example

Exercice 6 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

Example

Exercice 6 : Can you think of an example where the greedy algorithm gives a **bad** matching, e.g. of the size **half** the size of an optimal matching ?

Greedy matching

However, is $|M|$ is the cardinality of a matching returned by the greedy algorithm, and if $|M^*|$ is the cardinal of the real optimal matching, we can theoretically show that :

$$|M| \geq \frac{|M^*|}{2} \tag{2}$$

Speed comparison as a function of the data structure

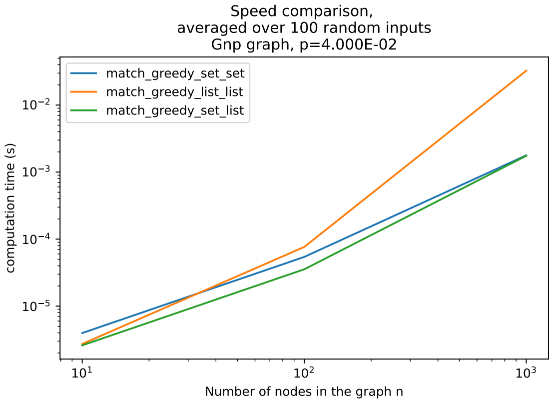


Figure: The functions will be available in code/solutions and shown during the class.

Matchings and vertex covers

Exercice 7 : Show that the nodes of the edges selected in a maximal matching form a **vertex cover**.
https://en.wikipedia.org/wiki/Vertex_cover

Matchings and vertex covers

Exercice 8 : Show that any matching is smaller than any vertex cover.