Algorithmic complexity and graphs: complexities

29 septembre 2022

Complexity

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- Importantly, this is called the **time complexity** of the problem. It does not take the memory usage into account.

Complexity

- Today we will **quantify** the **complexity** of several problems : how many operations are required to answer a given question, as a function of the size of the input ? Is it possible to **compute** an answer with a computer ?
- Importantly, this is called the **time complexity** of the problem. It does not take the memory usage into account.
- However, we will also (shortly) discuss **space complexity**, that quantifies memory usage.

Complexity

- The answer is that **it depends on the problem**. For some problems, it is very likely that there exists **no exact fast (polynomial)** solution (for instance the NP-hard problems)

Average and worst case complexities

- Often, for a given algorithm, the exact number of operations needed will **depend on the instance of the problem**.
- It is possible to compute several complexities given a problem size n :
  - **worst-case** the maximum number of operation needed
  - **average-case** average complexity, averaged over a **distribution** on the input. Thus this distribution is to be known, or assumed.

Measuring complexities

- Let us measure the time complexity of some simple programs. How ?

Measuring complexities

- Let us start by measuring the complexity of some simple programs.
- We can first measure the computing time.

Measuring execution times

Exercice 1 : Linear complexity

- **cd complexity/** and use **linear_complexity.py** to verify that the complexity of a sequence of multiplications is proportionnal to its length.

Measuring execution times

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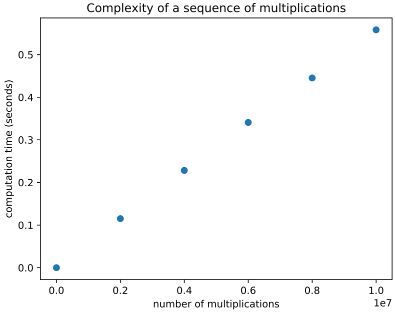
- It should look like this :



Measuring execution times

Exercice 2 : Non linear complexity

- What happens with matrix multiplication ? Use **matrix_multiplication.py** to estimate the computing time as a function of the size of the matrix.

Measuring execution times

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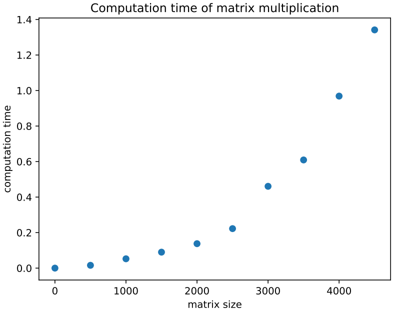
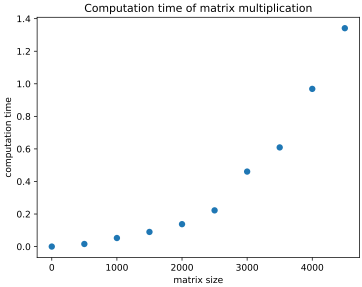
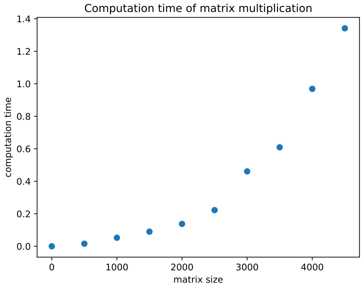


Matrix multiplication

- 

- Let's give a rough approximation of the number of operations as a function of the size n of the matrix.

Matrix multiplication

▶ 

- Let's give a rough approximation of the number of operations as a function of the size n of the matrix.
- It should then be of order $\mathcal{O}(n^3)$. Remark : However, some **sub-cubic** algorithms exists : faster than n^3

Measuring the time ?

- Why is **time** maybe not the best tool to evaluate the complexity of an algorithm ?

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- It depends on the machine

Measuring the time ?

- Why is **time** maybe not the best tool to evaluate the complexity of an algorithm ?
- It depends on the machine
- We could count the number of elementary operations instead.

Experimental evaluation

Exercice 3 : Counting the number of elementary operations

- Please use a variable in **exponentiation_complexity.py** to compute the number of operations in fast exponentiation and normal exponentiation.

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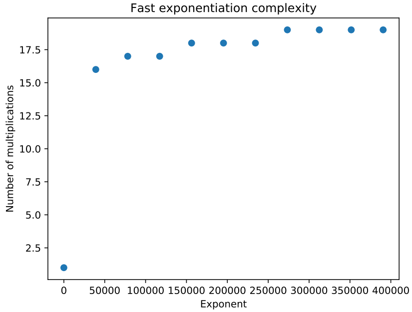
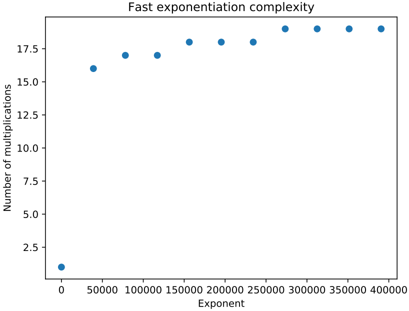


Experimental evaluation

Exercice 3 : Counting the number of elementary operations

- We recognize the **logarithmic complexity** $\mathcal{O}(\log n)$



Asymptotic behavior

- We study the **asymptotic** behavior, when $n \to \infty$

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- We study the **asymptotic** behavior, when $n \to \infty$
- This tells if the algorithm **scales** (still works when the instance of the problem is larger)

Asymtptic behavior : $\mathcal{O}$ notation (notation de Landau)

- Mathematically speaking, we say that $f = \mathcal{O}(g)$ if the ratio $\frac{|f(n)|}{|g(n)|}$ is **bounded**.

$$\exists A \geq 0, \forall n \in \mathbb{N}, \ |\frac{f(n)}{g(n)}| \leq A \qquad (1)$$

- $||$ means "absolute value"
- intuitively, this means that f is "not bigger" than g

Asymptotic behavior : examples

- $$n^2 + n = \mathcal{O}(\,?) \tag{2}$$
- $$5 \times n^4 + 2178 \times n^3 + \log 3n = \mathcal{O}(?) \tag{3}$$

Asymptotic behavior : examples

- $$n^2 + n = \mathcal{O}(n^2) \tag{4}$$
- $$5 \times n^4 + 2178 \times n^3 + \log 3n = \mathcal{O}(n^4) \tag{5}$$

Asymtptic behavior : o notation

- Mathematically speaking, we say that $f = o(g)$ if the ratio $\frac{|f(n)|}{|g(n)|}$ converges to 0 when $n \to +\infty$

$$\lim_{n \to +\infty} |\frac{f(n)}{g(n)}| = 0 \tag{6}$$

- intuitively, this means that f is "smaller" than g

Asymtptic behavior : o notation

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$$\lim_{n \to +\infty} |\frac{f(n)}{g(n)}| = 0 \tag{7}$$

- Please define this limit mathematically ?

Asymtptic behavior : o notation

- Mathematically speaking, we say that $f = o(g)$ if the ratio $\frac{|f(n)|}{|g(n)|}$ converges to 0 when $n \rightarrow +\infty$

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0 \qquad (8)$$

- $$\forall \epsilon > 0, \exists A \in \mathbb{R}, \forall n \geq A, \left|\frac{f(n)}{g(n)}\right| \leq \epsilon \qquad (9)$$

Asymptotic behavior : general rules

When $n \to +\infty$:

- if $\alpha < \beta$, $n^\alpha = o(n^\beta)$
- if $0 < a < b$, $a^n = o(b^n)$
- if $\alpha > 0$, $\beta \in \mathbb{R}$, $(\log n)^\beta = o(n^\alpha)$
- if $a > 1$, $n^\alpha = o(a^n)$

Asymptotic behavior : equivalence

- We say that $f(n) \underset{n \to +\infty}{\sim} g(n)$ when

$$f(n) \underset{n \to +\infty}{=} g(n) + o(g(n)) \tag{10}$$

Asymptotic behavior : equivalence

- We say that $f(n) \underset{n \to +\infty}{\sim} g(n)$ when

$$f(n) \underset{n \to +\infty}{=} g(n) + o(g(n)) \tag{11}$$

- When talking about complexities, we will be interested in the **simplest equivalent.**

Equivalence

Exercice 3 : Find equivalents and the limits for the following functions :

- $u_n = 3n^3 - n^2(\sqrt{n} \sin n) + \cos(\sqrt{n})$
- $v_n = -0.2 * n^n + 10 * n^2 * n!$
- Maximum number of edges in a simple directed graph
- $n!$

Examples of algorithms

- Fast exponentiation
- Naive exponentiation
- Merge sort
- Insertion sort
- Matrix multiplication
- Enumeration of subsets, TSP, coloring
- Enumeration of permutations

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- Naive exponentiation $\mathcal{O}(n)$
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- Insertion sort $\mathcal{O}(n^2)$
- Matrix multiplication $\mathcal{O}(n^{2.37})$
- Enumeration of subsets, TSP, coloring $\mathcal{O}(2^n)$
- Enumeration of permutations $\mathcal{O}(n!)$

Orders of magnitude

Orders of magnitude

Taille	$n \log n$	n^3	2^n
$n = 20$	60	8000	1048576
$n = 50$	196	125000	1125899907000000
$n = 100$	461	1000000	1267650600000000000000000000000

$\Longrightarrow$ Hence the idea of a border between polynomial and exponential algorithms.

Profiling

- Another useful tool to monitor the execution of a program is **profiling**
- From the python docs : "A profile is a set of statistics that describes how often and for how long various parts of the program executed"
- `https://docs.python.org/3.6/library/profile.html`

Profiling

Exercice 4 : Profiling a piece of code

- **cd profiling** and profile some programs that we used before

Profiling

Exercice 4 : Profiling a piece of code

- **cd profiling** and profile some programs that we used before
- However note that when profiling **profiling_demo.py**, the elementary multiplications are not taken into account in the profiling output.

Computing complexities

We now want to compute some complexities with paper and pen.
Let us focus on some intuitive rules :

- For a sequence of blocks :
- For a loop :

Computing complexities

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Computing complexities

We now want to compute some complexities with paper and pen.
Let us focus on some intuitive rules :

- For a sequence of blocks : complexities sum up
- For a loop : complexities of all iterations sum up
- If a loop consists in similar iterations, its complexity is the product of the compexity of one iteration by the size of the loop.

Running time

Exercice 5 : Computing a running time I

Please compute the running time and give the complexity of the following algorithm.

```
result = 0
for i in range(n):
    result += i**2
```

Running times

Exercice 6 : Computing a running time II

Please compute the running time and give the complexity of the following algorithm.

```
for i in range(n):
    for j in range(i):
        l = [i+j+k for k in range(n)]
```

Running times

Exercice 7 : Computing a running time II
Could we have known that is was polynomial without performing the exact computation ?

```
for i in range(n):
    for j in range(i):
        l = [i+j+k for k in range(n)]
```

Some mathematical concepts

- Mathematical induction
- Applications : prime factors decomposition, $\sum_{k=1}^{n} k$
- Optional

$$\sum_{k=1}^{n} k^2\ ? \tag{12}$$

$$\sum_{k=1}^{n} k^3\ ? \tag{13}$$

Insertion Sort

- We will study the classic **Insertion sort algorithm**, in order to illustrate the concept of **average-case complexity**.

Insertion Sort

Exercice 8 : **Insertion sort** :
cd insertion_sort/ and fix the function in **insertion_sort.py** in order to perform the algorithm.

Average-case complexity

- We assume a **uniform distribution** on the integer that we want to sort. All values have the same probability.
- What is the average-case complexity of the algorithm ?

Complexity

Exercice 9 : use the file **complexity.py** in order to check if our theoretical reslut is correct. You will need to fix the function **number_of_operations()**

Python sorting

In python, *sort*() uses a variant of mergesort. `https://github.com/python/cpython/blob/master/Objects/listsort.txt`

Horner Algorithm

- Let us consider the case of evaluating polynoms
- A polynom is a function of the form
  $f : x \to a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- How many multiplications are involved with the naive method ?

Horner Algorithm

- Let us consider the case of evaluating polynoms
- A polynom is a function of the form
  $f : x \rightarrow a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- How many multiplications are involved with the naive method ?
- We look fot an algorithm that is faster than the naive solution.

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{14}$$

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{15}$$

- How many multiplications are now involved ?

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(a) = (((7a + 2)a + 0)a - 5)a + 1 \tag{16}$$

- How many multiplications are now involved ? $\mathcal{O}(n)$.
- So we went from quadratic to linear.

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{17}$$

- We input the polynom to the algorithm as the list of the coefficients $[a_n, a_{n-1}, \dots, a_0]$

Evaluating polynoms

Exercice 9 : Implementation of Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{18}$$

- We input the polynom to the algorithm as the list of the coefficients $[a_n, a_{n-1}, \ldots, a_0]$
- Please modify **complexity/horner.py** so that it performs the horner algorithm.
- In order to test that our method is correct, we will test it against the method **polyval** from **numpy**.

Horner

- What do you see if you write **help(numpy.polyval)** inside python ?

Horner

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```
Horner's scheme [1]_ is used to evaluate the polynomial. Even so,
for polynomials of high degree the values may be inaccurate due to
rounding errors. Use carefully.

References
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.. [1] I. N. Bronshtein, K. A. Semendyayev, and K. A. Hirsch (Eng.
   trans. Ed.), *Handbook of Mathematics*, New York, Van Nostrand
   Reinhold Co., 1985, pg. 720.
```

Figure – The Horner algorithm is actually the method used by numpy !

Space complexty

Space complexity is the sum of :

- input space
- auxiliary space : temporary space used during the algorithm