Algorithmic complexity and graphs: complexities

14 septembre 2024

Complexity

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- Today we will **quantify** the **complexity** of several problems : how many operations are required to answer a given question, as a function of the size of the input ? Is it possible to **compute** an answer with a computer ?
- Importantly, this is called the **time complexity** of the problem. It does not take the memory usage into account.
- However, we will also (shortly) discuss **space complexity**, that quantifies memory usage.

Complexity

- The answer is that **it depends on the problem**. For some problems, it is very likely that there exists **no exact "fast" (polynomial)** solution (for instance the NP-hard problems)

Average and worst case complexities

- Often, for a given algorithm, the exact number of operations needed will **depend on the instance of the problem**.
- It is possible to compute several complexities given a problem size n :
  - **worst-case** the maximum number of operation needed
  - **average-case** average complexity, averaged over a **distribution** on the input. Thus this distribution is to be known, or assumed.

Measuring complexities

- Let us start by measuring the complexity of some simple programs.
- We can first measure the computing time.

Measuring execution times

Exercice 1 : Linear complexity

- **cd complexity/** and use **linear_complexity.py** to verify that the complexity of a sequence of multiplications is proportionnal to its length.

Measuring execution times

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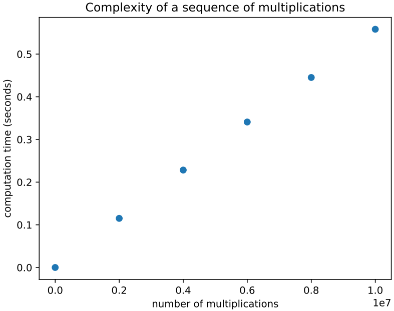

Measuring execution times

Exercice 2 : Non linear complexity

- What happens with matrix multiplication ? Use **matrix_multiplication.py** to estimate the computing time as a function of the size of the matrix.

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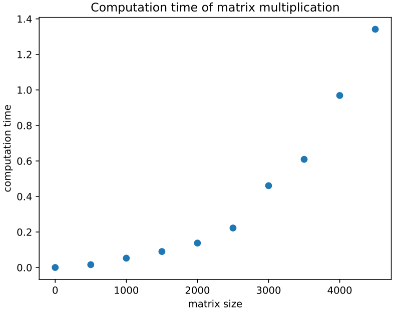
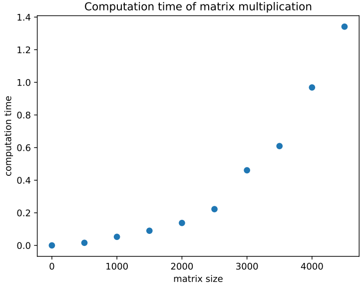
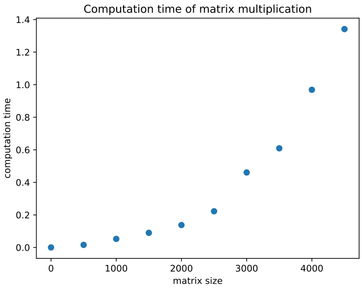


Matrix multiplication

- 

- Let's give a rough approximation of the number of operations as a function of the size n of the matrix.

Matrix multiplication



- Matrix multiplication is of order $\mathcal{O}(n^3)$.
- However, some **sub-cubic** algorithms exists : n^t with $t < 3$. But with very large constants (notion of **galactic algorithms** : `https://en.wikipedia.org/wiki/Galactic_algorithm`)

timeit

For a measurement of the execution time, timeit is also available :
`https://docs.python.org/3/library/timeit.html`

Measuring the time ?

- The measured time is a valid measure of the runtime, and can be useful to compare different algorithms on a given machine.
- However, it directly depends on the machine
- We could count the number of elementary operations instead.

Experimental evaluation

Exercice 3 : Counting the number of elementary operations

- Please use a variable in **exponentiation_complexity.py** to compute the number of operations in fast exponentiation.

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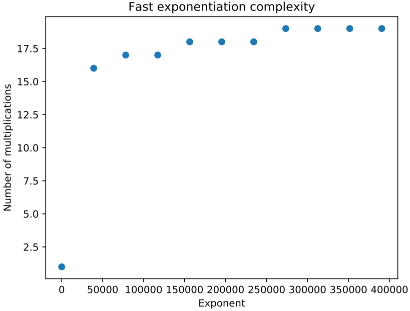
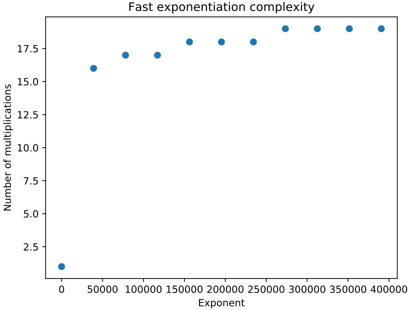


Experimental evaluation

Exercice 3 : Counting the number of elementary operations

- We recognize the **logarithmic complexity** $\mathcal{O}(\log n)$



Asymptotic behavior

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- We study the **asymptotic** behavior, when $n \to \infty$
- This tells if the algorithm **scales** (still works when the instance of the problem is larger)

Asymtptic behavior : $\mathcal{O}$ notation (notation de Landau)

- Mathematically speaking, we say that $f = \mathcal{O}(g)$ if the ratio $\frac{|f(n)|}{|g(n)|}$ is **bounded**.

$$\exists A \geq 0, \forall n \in \mathbb{N}, \ |\frac{f(n)}{g(n)}| \leq A \tag{1}$$

- $||$ means "absolute value"
- intuitively, this means that f is "not bigger" than g

Asymptotic behavior : examples

- $$n^2 + n = \mathcal{O}(\,?) \tag{2}$$

- $$5 \times n^4 + 2178 \times n^3 + \log 3n = \mathcal{O}(?) \tag{3}$$

Asymptotic behavior : examples

- $$n^2 + n = \mathcal{O}(n^2) \tag{4}$$
- $$5 \times n^4 + 2178 \times n^3 + \log 3n = \mathcal{O}(n^4) \tag{5}$$

Asymtptic behavior : o notation

- Mathematically speaking, we say that $f = o(g)$ if the ratio $\frac{|f(n)|}{|g(n)|}$ converges to 0 when $n \to +\infty$

$$\lim_{n \to +\infty} \left|\frac{f(n)}{g(n)}\right| = 0 \tag{6}$$

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$$\lim_{n \to +\infty} \left|\frac{f(n)}{g(n)}\right| = 0 \tag{7}$$

- Please define this limit with quantifiers (quantificateurs) ?

Asymtptic behavior : o notation

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$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0 \tag{8}$$

- $$\forall \epsilon > 0, \exists A \in \mathbb{R}, \forall n \geq A, |\frac{f(n)}{g(n)}| \leq \epsilon \tag{9}$$

Asymptotic behavior : general rules

When $n \to +\infty$:

- if $\alpha > 0$, $\beta \in \mathbb{R}$, $(\log n)^\beta = o(n^\alpha)$
- if $\alpha < \beta$, $n^\alpha = o(n^\beta)$
- if $a > 1$, $n^\alpha = o(a^n)$
- if $0 < a < b$, $a^n = o(b^n)$

Asymptotic behavior : equivalence

- We say that $f(n) \underset{n \to +\infty}{\sim} g(n)$ when

$$f(n) \underset{n \to +\infty}{=} g(n) + o(g(n)) \tag{10}$$

Asymptotic behavior : equivalence

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$$f(n) \underset{n\to+\infty}{=} g(n) + o(g(n)) \tag{11}$$

- When talking about complexities, we will be interested in the **simplest equivalent.**

Equivalence

Exercice 3 : Find equivalents and the limits for the following functions :

- $u_n = 3n^3 - n^2(\sqrt{n} \sin n) + \cos(\sqrt{n})$
- $v_n = -0.2 * n^n + 10 * n^2 * n!$
- Maximum number of edges in a simple directed graph
- $n!$

Examples of algorithms

- Fast exponentiation
- Naive exponentiation
- Merge sort
- Insertion sort
- Matrix multiplication
- Enumeration of subsets, TSP, coloring
- Enumeration of permutations

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- Enumeration of subsets, TSP, coloring $\mathcal{O}(2^n)$
- Enumeration of permutations $\mathcal{O}(n!)$

Orders of magnitude

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Taille	$n \log n$	n^3	2^n
$n = 20$	60	8000	1048576
$n = 50$	196	125000	1125899907000000
$n = 100$	461	1000000	1267650600000000000000000000000

$\Longrightarrow$ Hence the idea of a border between polynomial and exponential algorithms.

Profiling

- Another useful tool to monitor the execution of a program is **profiling**
- From the python docs : "A profile is a set of statistics that describes how often and for how long various parts of the program executed"
- `https://docs.python.org/3.6/library/profile.html`

Profiling

Exercice 4 : Profiling a piece of code

- **cd profiling** and profile some programs that we used before
- However note that when profiling **profiling_demo.py**, the elementary multiplications are not taken into account in the profiling output.

Time complexities in python

`https://wiki.python.org/moin/TimeComplexity`

Computing complexities

We now want to compute some complexities with paper and pen.
Let us focus on some intuitive rules :

- For a sequence of blocks :
- For a loop :

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Computing complexities

We now want to compute some complexities with paper and pen.
Let us focus on some intuitive rules :

- For a sequence of blocks : complexities sum up
- For a loop : complexities of all iterations sum up
- If a loop consists in similar iterations, its complexity is the product of the compexity of one iteration by the size of the loop.

Exercice 5 : Computing a running time I

Please compute the complexity of the following algorithm.

```
result = 0
for i in range(n):
    result += i**2
```

Exercice 6 : Computing a running time II

Please compute the complexity of the following algorithm.

```
for i in range(n):
    for j in range(i):
        l = [i+j+k for k in range(n)]
```

Running times

Exercice 7 : Computing a running time II
Could we have known that is was polynomial without performing the exact computation ?

```
for i in range(n):
    for j in range(i):
        l = [i+j+k for k in range(n)]
```

Some mathematical concepts

- Mathematical induction
- Applications : prime factors decomposition, $\sum_{k=1}^{n} k$
- Optional

$$\sum_{k=1}^{n} k^2 \, ? \tag{12}$$

$$\sum_{k=1}^{n} k^3 \, ? \tag{13}$$

Insertion Sort

- We will study the classic **Insertion sort algorithm**, in order to illustrate the concept of **average-case complexity**.

Insertion Sort

Exercice 8 : **Insertion sort** :
cd insertion_sort/ and fix the function in **insertion_sort.py** in order to perform the algorithm.
A test file **test_insertion_sort.py** is provided in order to check the correctness of the function.

Average-case complexity

- We assume a **uniform distribution** on the integer that we want to sort. All values have the same probability.
- What is the average-case complexity of the algorithm ?

Complexity

Exercice 9 : use the file **complexity.py** in order to check if our theoretical reslut is correct. You will need to fix the function **number_of_operations()**

Python sorting

In python, *sort*() uses a variant of mergesort.

`https://github.com/python/cpython/blob/master/Objects/listsort.txt`

Horner Algorithm

- Let us consider the case of evaluating polynoms
- A polynom is a function of the form
  $f : x \to a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- How many multiplications are involved with the naive method ?

Horner Algorithm

- Let us consider the case of evaluating polynoms
- A polynom is a function of the form
  $f : x \to a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- How many multiplications are involved with the naive method ?
- We look fot an algorithm that is faster than the naive solution.

Horner Algorithm

- Example of Horner algorithm when $P : x \rightarrow 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{14}$$

Horner Algorithm

- Example of Horner algorithm when $P : x \rightarrow 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{15}$$

- How many multiplications are now involved ?

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(a) = (((7a + 2)a + 0)a - 5)a + 1 \tag{16}$$

- How many multiplications are now involved ? $\mathcal{O}(n)$.
- So we went from quadratic to linear.

Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \tag{17}$$

- We input the polynom to the algorithm as the list of the coefficients $[a_n, a_{n-1}, \ldots, a_0]$

Evaluating polynoms

Exercice 9 : Implementation of Horner Algorithm

- Example of Horner algorithm when $P : x \to 7x^4 + 2x^3 - 5x + 1$:

$$P(x) = (((7x + 2)x + 0)x - 5)x + 1 \qquad (18)$$

- We input the polynom to the algorithm as the list of the coefficients $[a_n, a_{n-1}, \ldots, a_0]$
- Please modify **complexity/horner.py** so that it performs the horner algorithm.
- In order to test that our method is correct, we will test it against the method **polyval** from **numpy**.

Horner

- **help(numpy.polyval)** :

```
Horner's scheme [1]_ is used to evaluate the polynomial. Even so,
for polynomials of high degree the values may be inaccurate due to
rounding errors. Use carefully.

References
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.. [1] I. N. Bronshtein, K. A. Semendyayev, and K. A. Hirsch (Eng.
   trans. Ed.), *Handbook of Mathematics*, New York, Van Nostrand
   Reinhold Co., 1985, pg. 720.
```
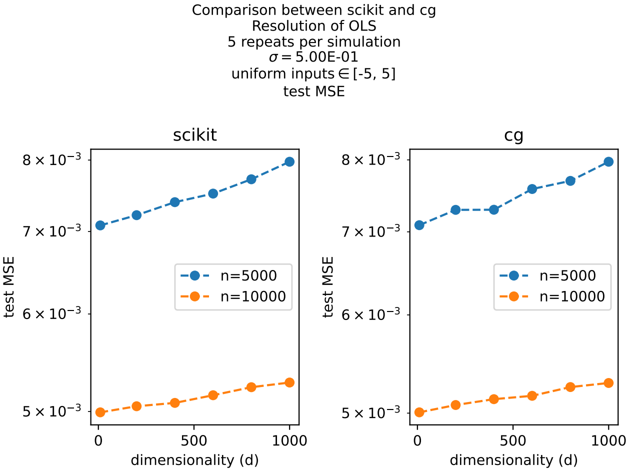

Figure – The Horner algorithm is actually the method used by numpy !

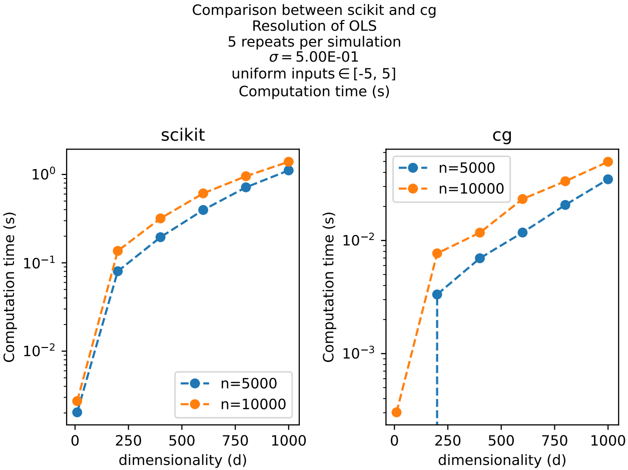
Linear systems

In many applications, we must solve **linear systems** (backboard). To do so, several algorithms exist and the optimal one will depend on the input (similarly to sorting). In the following slides, the **conjugate gradient algorithm (CG)** from scipy is compared to the scikit-learn implemenration which uses numpy.linalg.lstsq.

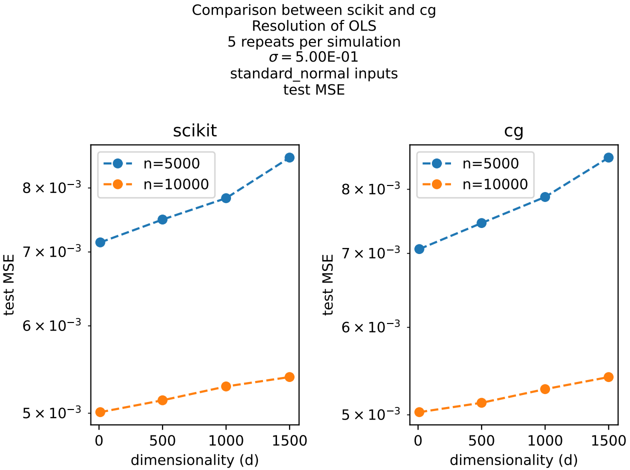
Uniform inputs : result quality (test MSE)



Uniform inputs : speed



Normally distributed inputs : result quality (test MSE)



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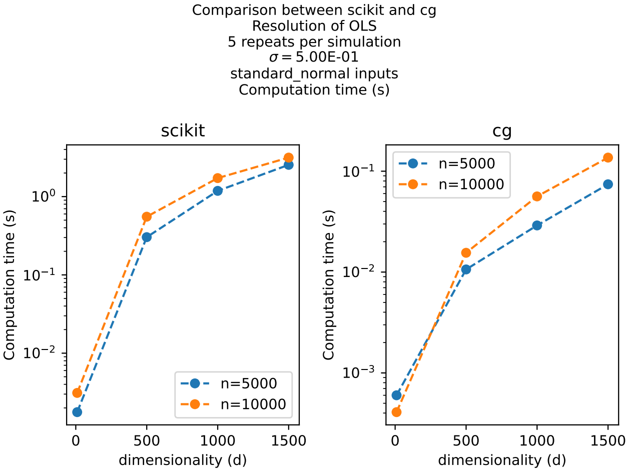
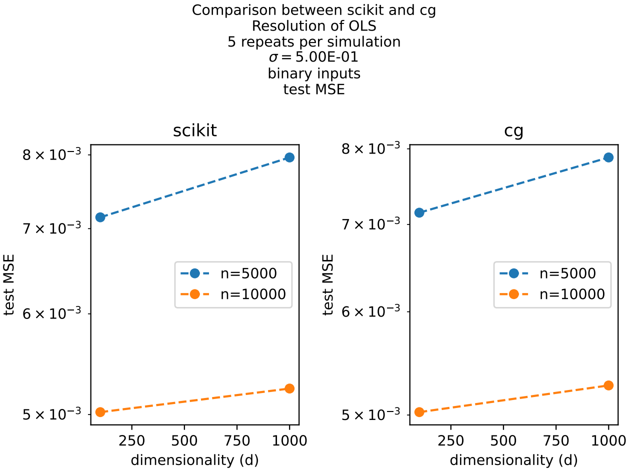


Binary inputs : result quality (test MSE)



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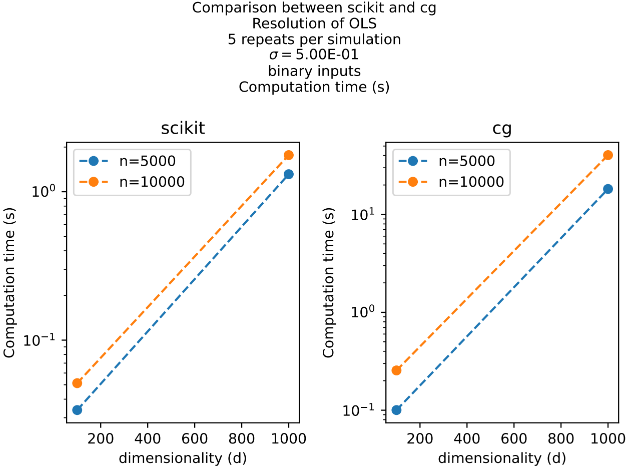

Space complexty

Space complexity is the sum of :

- input space
- auxiliary space : temporary space used during the algorithm