Notes on an acoustic wave propagator

Donald J. Kouri, Nicholas Maxwell, and Thomas Markovich
Departments of Mathematics and Physics
University of Houston
Houston, TX 77204-5006
kouri@uh.edu, nicholas.maxwell@gmail.com

We present and discuss an operator which propagates solutions of the acoustic wave equation.

I. INTRODUCTION

We consider the equation

$$\ddot{u}(\mathbf{x},t) = C^2(\mathbf{x})\,\nabla^2\,u(\mathbf{x},t), \tag{I.1}$$

This equation is of the form

$$\frac{d^2}{dt^2}\,f(t) = m^2 f(t), \tag{I.2}$$

because C depends only on $\mathbf{x}$. If $f(0) = f_0$, and $\dot{f}(0) = \dot{f}_0$, then this is an initial value problem, and it is standard to derive the particular solution as follows.

$$\begin{aligned} f(t) &= \cosh{(m\,t)}\,c_1 + \sinh{(m\,t)}\,c_2, \\ \dot{f}(t) &= \sinh{(m\,t)}\,m\,c_1 + \cosh{(m\,t)}\,m\,c_2, \\ f(0) &= c_1 \Rightarrow c_1 = f_0, \\ \dot{f}(0) &= m\,c_2 \Rightarrow c_2 = \dot{f}_0/m. \end{aligned} \tag{I.3}$$

Thus, writing $z := \sqrt{C^2\nabla^2}$, we assert that the exact solution of equation (I.1) is

$$\begin{aligned} u(\mathbf{x},t) &= \cosh{(t\,z)}\,A + \sinh{(t\,z)}\,B, \\ \dot{u}(\mathbf{x},t) &= \sinh{(t\,z)}\,z\,A + \cosh{(t\,z)}\,z\,B. \end{aligned} \tag{I.4}$$

And following the same argument,

$$\begin{aligned} A &= u_0, \\ B &= z^{-1}\,\dot{u}_0. \end{aligned} \tag{I.5}$$

Then equation (I.4) becomes

$$\begin{aligned} u(\mathbf{x},t) &= \cosh{(t\,z)}\,u_0 + \sinh{(t\,z)}\,z^{-1}\,\dot{u}_0, \\ \dot{u}(\mathbf{x},t) &= \sinh{(t\,z)}\,z\,u_0 + \cosh{(t\,z)}\,\dot{u}_0, \end{aligned} \tag{I.6}$$

which can be rewritten as

$$\begin{pmatrix} u(\mathbf{x},t) \\ \dot{u}(\mathbf{x},t) \end{pmatrix} = \mathrm{P}(t;C) \begin{pmatrix} u(\mathbf{x},0) \\ \dot{u}(\mathbf{x},0) \end{pmatrix}, \quad \mathrm{P}(t;C) = \begin{pmatrix} \cosh{(t\,z)} & \sinh{(t\,z)}/z \\ \sinh{(t\,z)}\,z & \cosh{(t\,z)} \end{pmatrix}. \tag{I.7}$$

II. INVESTIGATION

Using the identities

$$\begin{aligned} \sinh(t)\cosh(s) + \cosh(t)\sinh(s) &= \sinh(t+s), \\ \cosh(t)\cosh(s) + \sinh(t)\sinh(s) &= \cosh(t+s), \end{aligned} \tag{II.1}$$

and assuming that $\cosh(t\,z)$ commutes with $\sinh(t\,z)$, and both commute with z and z^{-1}, it is straight-forward to show that

$$\mathrm{P}(t;C)\,\mathrm{P}(s;C) = \mathrm{P}(t+s;C), \quad \mathrm{P}^n(t;C) = \mathrm{P}(n\,t;C). \tag{II.2}$$

We interpret $\cosh(t\sqrt{C^2\nabla^2})$ and $\sinh(t\sqrt{C^2\nabla^2})$ by their power series expansions. The the Taylor series expansions are given by $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, and $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$, so

$$\cosh(t\sqrt{C^2\nabla^2}) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \tag{II.3}$$

$$\sinh(t\sqrt{C^2\nabla^2}) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n\sqrt{C^2\nabla^2}. \tag{II.4}$$

Now, it is complicated to evaluate $\sqrt{C^2\nabla^2}$, however, examining the form of P, we see that the odd power of $\sqrt{C^2\nabla^2}$ in the expansion is absorbed, and we can write

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n & \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n \\ \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^{n+1} & \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \end{pmatrix} \tag{II.5}$$

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty} \frac{t^{2n}(C^2\nabla^2)^n}{(2n+1)!} \begin{pmatrix} 2n+1 & t \\ t\,C^2\nabla^2 & 2n+1 \end{pmatrix}. \tag{II.6}$$

To a first order,

$$\mathrm{P}(\tau;C) = \begin{pmatrix} 1 & \tau \\ \tau\,C^2\nabla^2 & 1 \end{pmatrix} + \ldots \tag{II.7}$$

It is standard to write equation (I.1) as

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ C^2\nabla^2 & 0 \end{pmatrix}\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}, \tag{II.8}$$

d then, for small τ,

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} \approx \frac{1}{\tau}\left(\begin{pmatrix} u(t+\tau) \\ \dot{u}(t+\tau) \end{pmatrix} - \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}\right) \Rightarrow \begin{pmatrix} u(t+\tau) \\ \dot{u}(t+\tau) \end{pmatrix} \approx \left(\begin{pmatrix} 0 & 1 \\ C^2\nabla^2 & 0 \end{pmatrix}\tau + 1\right)\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}$$

$$\begin{pmatrix} u(t+\tau) \\ \dot{u}(t+\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ \tau\,C^2\nabla^2 & 1 \end{pmatrix}\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}. \tag{II.9}$$

So we can see that truncating the Taylor series of P to the first term is exactly equivalent to Euler's method for the solution of (I.1).

III. APPLICATION

Depending on the application, $\dot{u}(t)$ may be unwanted information, so we might try to apply P in such a way that we avoid its computation. If we know that $\dot{u}(0) = 0$, we can take from equation (I.7) that

$$u(T) = \cosh\left(T\sqrt{C^2\nabla^2}\right) u(0). \tag{III.1}$$

This let's us compute $u(T)$ from $u(0)$, while avoiding $\dot{u}(T)$. We can't then compute $u(T+s)$, as we would again require $\dot{u}(T)$ for that additional step, so we take T as the final step to which we wish to propagate u.

Another, less obvious approach is to use the the binomial theorem to show that

$$(\cosh x)^{2n} = \frac{2}{4^n}\sum_{k=1}^{n}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(2k\,x) + \frac{(2n)!}{4^n(n!)^2}. \tag{III.2}$$

then, for $n = 0$,

$$\cosh(n\,t\,x) = 1, \tag{III.3}$$

for $n = 1$,

$$\cosh(n\,t\,x) = \cosh(t\,x), \tag{III.4}$$

for $n \geq 2$,

$$\cosh(n\,t\,x) = \frac{4^n}{2}(\cosh(t\,x/2))^{2n} - \sum_{k=1}^{n-1}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(k\,t\,x) - \frac{1}{2}\frac{(2n)!}{(n!)^2}. \tag{III.5}$$

$$\cosh(n\,t\,x) = 2^{n-1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(\cosh(t\,x))^k - \sum_{k=1}^{n-1}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(k\,t\,x) - \frac{1}{2}\frac{(2n)!}{(n!)^2}. \tag{III.6}$$

IV. DAMPING.

V. TAYLOR EXPANSION.

The the Taylor series expansions are given by $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, and $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$, so

$$\cosh\left(t\sqrt{C^2\nabla^2}\right) = \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \tag{V.1}$$

$$\sinh\left(t\sqrt{C^2\nabla^2}\right) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n\sqrt{C^2\nabla^2}, \tag{V.2}$$

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty}\begin{pmatrix} \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n & \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n \\ \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^{n+1} & \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \end{pmatrix} \tag{V.3}$$

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty}\frac{t^{2n}(C^2\nabla^2)^n}{(2n)!}\begin{pmatrix} 1 & t/(2n+1) \\ t\,C^2\nabla^2/(2n+1) & 1 \end{pmatrix}. \tag{V.4}$$

So, to compute $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \mathrm{P}(t;C) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, truncated to M term in the expansion, we can generate the sequences $(U_n)_{n=0}^{M+1}$ and $(V_n)_{n=0}^{M}$, as

$$U_0 = u_0,\ \ U_n = t^2\, C^2 \nabla^2\, U_{n-1}/(4n^2 - 2n), \tag{V.5}$$
$$V_0 = v_0,\ \ V_n = t^2\, C^2 \nabla^2\, V_{n-1}/(4n^2 - 2n) \tag{V.6}$$

$$u_1 = \sum_{n=0}^{M} U_n + t\, V_n/(2n+1), \tag{V.7}$$

$$v_1 = \sum_{n=0}^{M} \frac{1}{t}\,(4n^2 + 6n + 2) U_{n+1}/(2n+1) + V_n, \tag{V.8}$$

$$\tag{V.9}$$

VI. HERMITE EXPANSION.

using the expansion,

$$\exp\left(-i\,t\,x\right) = \sum_{m=0}^{\infty} \frac{(-i\,t\,\lambda)^m}{m!\,2^m} \exp\left(-\frac{(t\,\lambda)^2}{4}\right) H_m(x/\lambda), \tag{VI.1}$$

$$\cosh\left(t\,x\right) = \sum_{m=0}^{\infty} c_m(t\,\lambda) H_{2m}(x/\lambda),\ \ c_m(t\,\lambda) = \frac{(t\,\lambda)^{2m}}{(2m)!\,4^m} \exp\left(\frac{(t\,\lambda)^2}{4}\right). \tag{VI.2}$$

Then, we can use the recurrence relation

$$H_{2m}(x) = (4x^2 - 8m + 6) H_{2m-2}(x) - 8(m-1)(2m-3) H_{2m-4}(x), \tag{VI.3}$$

$$H_0(x) = 1,\ H_2(x) = 4x^2 - 2.$$

VII. LAGUERRE EXPANSION.

We wish to expand $\exp\left(i\,t\,x\right)$ into a power series,

$$e^{i\,t\,x} = \sum_{n=0}^{\infty} \tilde{c}_n^{(\alpha)}(t) L_n^{(\alpha)}(x), \tag{VII.1}$$

where $L_n^{(\alpha)}$ is the n^{th} generalized Laguerre polynomial, which have α as a parameter. If $\alpha = \pm\frac{1}{2}$, then (IV.1) becomes a Hermite polynomial expansion. It is non-trivial to show that

$$\tilde{c}_n^{(\alpha)}(t) = (-i\,t)^n (1 - i\,t)^{-1-n-\alpha}, \tag{VII.2}$$

which we've been unable to simplify to a more enlightening form.

Then, the expansions are

$$\cosh\left(t\,x\right) = \sum_{n=0}^{\infty} c_n^{(\alpha)}(t) \frac{1}{2} \left(L_n^{(\alpha)}(x) + L_n^{(\alpha)}(-x)\right), \tag{VII.3}$$

$$\sinh(t\,x) = \sum_{n=0}^{\infty} c_n^{(\alpha)}(t)\frac{1}{2}\left(L_n^{(\alpha)}(x) - L_n^{(\alpha)}(-x)\right), \tag{VII.4}$$

where $c_n^{(\alpha)}(t) = \tilde{c}_n^{(\alpha)}(i\,t) = (t)^n(1+\,t)^{-1-n-\alpha}$.
Using that

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-x)^m}{m!}\binom{n+\alpha}{n-m}, \tag{VII.5}$$

we truncate these expansions to $2N$ terms and switch the sums,

$$\cosh(t\,x) \approx \sum_{m=0}^{N} \frac{a_m^{(\alpha,N)}(t)}{(2m)!}x^{2m},\;\; a_m^{(\alpha,N)}(t) = \sum_{n=2m}^{2N} c_n^{(\alpha)}(t)\binom{n+\alpha}{n-2m} \tag{VII.6}$$

$$\sinh(t\,x) \approx \sum_{m=0}^{N} \frac{b_m^{(\alpha,N)}(t)}{(2m+1)!}x^{2m+1},\;\; b_m^{(\alpha,N)}(t) = \sum_{n=2m+1}^{2N} c_n^{(\alpha)}(t)\binom{n+\alpha}{n-2m-1}. \tag{VII.7}$$

With $\binom{z}{n} = \prod_{k=1}^{n} \frac{z-n+k}{k}$,

$$a_m^{(\alpha,N)}(t) = \sum_{n=2m}^{2N} \exp\left(n\log t - (+1+n+\alpha)\log(1+t) + \sum_{k=1}^{n-2m}(\log(k+\alpha+2m) - \log k)\right) \tag{VII.8}$$

$$b_m^{(\alpha,N)}(t) = \sum_{n=2m+1}^{2N} \exp\left(n\log t - (+1+n+\alpha)\log(1+t) + \sum_{k=1}^{n-2m-1}(\log(k+\alpha+2m+1) - \log k)\right) \tag{VII.9}$$

[1] Bodmann, Hoffman, Kouri, Papadakis, *Hermite Distributed Approximating Functionals as Almost-Ideal Low-Pass Filters* , (Sampling Theory in Signal And Image Processing, 2008 Vol. 7, No. 1)