Linear Wave Propagation

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First approach (1 of 3)

Starting with,

$$\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = C^2(\mathbf{x}) \nabla^2 u(\mathbf{x}, t),$$

we note that the action of $C^2(\mathbf{x}) \nabla^2$ is independent of t, so in a sloppy sense, we take $z = \sqrt{C^2(\mathbf{x}) \nabla^2}$, $g(t) = u(\mathbf{x}, t)$. Then we solve,

$$\frac{d^2}{d t^2} g(t) = z^2 g(t),$$

the general solution to which is a linear combination of $exp(\pm z\, t)$.

First approach (2 of 3)

So using $\cosh(x) = \frac{1}{2}(e^{+x} + e^{-x})$, $\sinh(x) = \frac{1}{2}(e^{+x} - e^{-x})$, we can pick the solutions,

$$g(t) = cosh(z\,t)\,A + sinh(z\,t)\,B,\ g(0) = A,$$

$$g'(t) = sinh(z\,t)\,z\,A + cosh(z\,t)\,z\,B,\ g'(0) = z\,B \Rightarrow B = z^{-1}g'(0).$$

Then,

$$g(t) = cosh(z\,t)\,g(0) + sinh(z\,t)\,z^{-1}g'(0),$$
$$g'(t) = sinh(z\,t)\,z\,g(0) + cosh(z\,t)\,z^{-1}\,z\,g'(0),$$

rewriting,

$$\begin{pmatrix} g(t) \\ g'(t) \end{pmatrix} = \begin{pmatrix} cosh(z\,t) & sinh(z\,t)\,z^{-1} \\ sinh(z\,t)\,z & cosh(z\,t) \end{pmatrix} \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}.$$

First approach (3 of 3)

Substituting $u(\mathbf{x}, t)$ for $g(t)$, $v(\mathbf{x}, t)$ for $g'(t)$,

$$\begin{pmatrix} u(\mathbf{x}, t+\delta) \\ v(\mathbf{x}, t+\delta) \end{pmatrix} = \mathrm{P}(\delta) \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{pmatrix},$$

$$\mathrm{P}(\delta) = \begin{pmatrix} cosh(\delta\, z) & sinh(\delta\, z)\, z^{-1} \\ sinh(\delta\, z)\, z & cosh(\delta\, z) \end{pmatrix}.$$

Second approach (1 of 5)

Starting with,

$$\frac{\partial^2}{\partial t^2}\, u(\mathbf{x}, t) = C^2(\mathbf{x})\, \nabla^2\, u(\mathbf{x}, t) + f(\mathbf{x}, t),$$

and the notation,

$$\mathrm{M} = \begin{pmatrix} 0 & 1 \\ C^2(\mathbf{x})\, \nabla^2 & 0 \end{pmatrix},\ \mathbf{w}(t) = \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{pmatrix}, \mathbf{f}(\mathbf{x}, t) = \begin{pmatrix} 0 \\ f(\mathbf{x}, t) \end{pmatrix},$$

$$\frac{\partial}{\partial t}\, \mathbf{w}(t) = \mathrm{M}\, \mathbf{w}(t) + \mathbf{f}(\mathbf{x}, t).$$

Second approach (2 of 5)

We intruduce an integrating factor, $\exp\left(-t\,\mathrm{M}\right)$

$$e^{-t\,\mathrm{M}}\,\frac{\partial}{\partial\,t}\,\mathbf{w}(t) - e^{-t\,\mathrm{M}}\,\mathrm{M}\,\mathbf{w}(t) = e^{-t\,\mathrm{M}}\,\mathbf{f} \Rightarrow$$

$$\frac{\partial}{\partial\,t}\,\left(e^{-t\,\mathrm{M}}\,\mathbf{w}(t)\right) = e^{-t\,\mathrm{M}}\,\mathbf{f} \Rightarrow d\left(e^{-t\,\mathrm{M}}\,\mathbf{w}(t)\right) = e^{-t\,\mathrm{M}}\,\mathbf{f}\,dt,$$

$$\int_t^{t+\delta} d\left(e^{-s\,\mathrm{M}}\,\mathbf{w}(s)\right) = \int_t^{t+\delta} e^{-s\,\mathrm{M}}\,\mathbf{f}\,ds$$

$$e^{-(t+\delta)\,\mathrm{M}}\,\mathbf{w}(t+\delta) = e^{-t\,\mathrm{M}}\,\mathbf{w}(t) + \int_t^{t+\delta} e^{-s\,\mathrm{M}}\,\mathbf{f}\,ds.$$

Then we apply $e^{+(t+\delta)\,\mathrm{M}}$,

$$\mathbf{w}(t+\delta) = e^{\delta\,\mathrm{M}}\,\mathbf{w}(t) + \int_t^{t+\delta} e^{+(\delta+t-s)\,\mathrm{M}}\,\mathbf{f}\,ds.$$

Second approach (3 of 5)

So to propagate from t_1 to t_2, $\delta = t_2 - t_1$,

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = e^{\delta\,\mathrm{M}} \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \int_{t_1}^{t_2} e^{+(t_2 - t)\,\mathrm{M}}\,\mathbf{f}(t)\,dt.$$

Now, what is $e^{t\,\mathrm{M}}$? Well, again writing $z = \sqrt{C^2(\mathbf{x})\,\nabla^2}$, and

$$\mathrm{M}^2 = \begin{pmatrix} 0 & 1 \\ z^2 & 0 \end{pmatrix}^2 = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix},$$

so $\mathrm{M}^{2k} = z^{2k}\,\mathrm{I}$, then $\mathrm{M}^{2k+1} = z^{2k}\,\mathrm{M}$. We split the taylor expansion of $e^{t\,\mathrm{M}}$ by parity,

$$e^{t\,\mathrm{M}} = \sum_{k=0}^{\infty} (t\,\mathrm{M})^k/k! = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\,z^{2k}\mathrm{I} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\,z^{2k}\mathrm{M}.$$

Second approach (4 of 5)

$$e^{t\,\mathrm{M}} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \begin{pmatrix} z^{2k} & 0 \\ 0 & z^{2k} \end{pmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{2k+1}\,z^{-1} \\ z^{2k+1}\,z^{+1} & 0 \end{pmatrix}$$

Now, $\cosh(t\,z) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} z^{2k+1}$, $\sinh(t\,z) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} z^{2k}$, so we recognise,

$$e^{t\,\mathrm{M}} = \begin{pmatrix} cosh(t\,z) & sinh(t\,z)\,z^{-1} \\ sinh(t\,z)\,z & cosh(t\,z) \end{pmatrix}.$$

Second approach (5 of 5)

Then we have,

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = \mathrm{P}(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \int_{t_1}^{t_2} \mathrm{P}(t_2 - t)\, \mathbf{f}(\mathbf{x}, t)\, dt.$$

Then we have,

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = \mathrm{P}(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} +$$

$$\int_{t_1}^{t_2} \left[\frac{\sinh((t_2 - t)\sqrt{C^2(\mathbf{x})\,\nabla^2})}{\sqrt{C^2(\mathbf{x})\,\nabla^2}}\, f(\mathbf{x}, t) \right] +$$

$$\left[\cosh((t_2 - t)\sqrt{C^2(\mathbf{x})\,\nabla^2})\, f(\mathbf{x}, t) \right]\, dt.$$

Second approach (6 of 5)

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = \mathrm{P}(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} +$$

$$\int_{t_1}^{t_2} \sum_{k=0}^{\infty} \left(\frac{(t_2 - t)^{2k}}{(2k+1)!} (t_2 - t + 2k + 1) \right) \left((C^2(\mathbf{x}) \, \nabla^2)^k \, f(\mathbf{x}, t) \right) \, dt.$$

Anisotropic Case (1 of 1)

We can addapt this approach to solve the anisotropic wave equation,

$$\frac{\partial^2}{\partial t^2} u_i(\mathbf{x}, t) - \frac{1}{\rho} C_{ijkl}(\mathbf{x}) \frac{\partial^2}{\partial x_j \partial x_l} u_k(\mathbf{x}, t) = \frac{1}{\rho} f_i(\mathbf{x}, t),$$

then

$$\frac{\partial^2}{\partial t^2} u_i(\mathbf{x}, t) - \sum_k M_{ik} u_k(\mathbf{x}, t) = \frac{1}{\rho} f_i(\mathbf{x}, t),$$

with

$$M_{ik} = \sum_{j,l} \frac{1}{\rho} C_{ijkl}(\mathbf{x}) \frac{\partial^2}{\partial x_j \partial x_l}$$