Notes on complex measures, etc.
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For $\mathcal{A}$ a sigma algebra, $E \in \mathcal{A}$, define $\mathcal{P}^*(E, \mathcal{A}) := \{\{E_k \in \mathcal{A}; k \in \mathbb{N}\}; E = \cup_k E_k, E_i \cap E_j = \phi \;\forall i \neq j\}$. Always $\{\phi, E\} \in \mathcal{P}^*(E, \mathcal{A})$, so $\mathcal{P}^*(E, \mathcal{A})$ is never empty, and also $\mathcal{P}^*(\phi, \mathcal{A}) = \{\phi\}$.

A complex or a signed and finite measure on a measurable space $(X, \mathcal{A})$ is a function, ν, from $\mathcal{A}$ to $\mathbb{R}$ or $\mathbb{C}$ such that
1) $\nu(\phi) = 0$
2) $\nu(\cup_{k\in\mathbb{N}} E_k) = \sum_{k\in\mathbb{N}} \nu(E_k)$, $E_k \in A$, disjoint.

Because the union in (2) is independent of the labeling of the $\{E_k\}$, the sum in (2) is rearangement-invariant, which implies that it converges iff it does so to absolutely, and does to the same number.

Alternatively, a complex measure ν on $(X, \mathcal{A})$ is a complex function on $\mathcal{A}$ such that
3) $\nu(E) = \sum_{k\in\mathbb{N}} \nu(E_k)$, for all $\{E_k\} \in \mathcal{P}^*(E, \mathcal{A})$.

$(3 \Rightarrow 1), \phi = E = \cup_k E_k \Rightarrow E_k = \phi. \Rightarrow \nu(\phi) = \sum_{k\in\mathbb{N}} \nu(\phi) \Rightarrow \nu(\phi) = 0$. $(3 \Leftrightarrow 2), E := \cup_k E_k$.

Write $\mathcal{M}(X, \mathcal{A})$ or $\mathcal{M}(\mathcal{A})$ for the set of all complex or signed and finite measures on $\mathcal{A}$. Write $\mathcal{M}^\pm(X, \mathcal{A})$ or $\mathcal{M}^\pm(\mathcal{A})$ for the set of all signed and finite measures on $\mathcal{A}$. Write $\mathcal{M}^+(X, \mathcal{A})$ or $\mathcal{M}^+(\mathcal{A})$ for the set of all positive measures on $\mathcal{A}$. Positive measures need not be finite, so $\mathcal{M}^+(\mathcal{A}) \not\subset \mathcal{M}(\mathcal{A})$.

If $\mu_1, \mu_2 \in \mathcal{M}^+(\mathcal{A})$, then we say that $\mu_1 \leq \mu_2$ iff $\mu_1(E) \leq \mu_2(E)$ for all $E \in \mathcal{A}$.

Given $\nu \in \mathcal{M}(\mathcal{A})$, we wish to find the smallest $\mu \in \mathcal{M}^+(\mathcal{A})$ s.t.

$$\mu(E) \geq |\nu(E)| \text{ for all } E \in \mathcal{A}\ ^{(\dagger_1)},$$

smallest in the sense of the previous point. When $(\dagger_1)$ holds we say that μ dominates ν. Let $\{E_k\} \in \mathcal{P}^*(E, \mathcal{A})$ arbitrarily, we then have that $|\nu(E_k)| \leq \mu(E_k)$ for all E_k by $(\dagger_1)$, summing these gives

$$\mu(E) = \sum_{k\in\mathbb{N}} \mu(E_k) \geq \sum_{k\in\mathbb{N}} |\nu(E_k)| \geq |\nu(E)|, \quad \text{for all } \{E_k\} \in \mathcal{P}^*(E, \mathcal{A}).$$

Thus, for any μ dominating ν, we can find a $\sum_{k\in\mathbb{N}} |\nu(E_k)|$ not strictly between any $\mu(E)$ and $|\nu(E)|$. So the best we could do, in the sense of minimizing $(\dagger_1)$, is $\sum_{k\in\mathbb{N}} |\nu(E_k)|$, for some $\{E_k\} \in \mathcal{P}^*(E, \mathcal{A})$ which minimizes this quantity. This suggests the definition

$$|\nu|(E) := \sup \left\{ \sum_{k\in\mathbb{N}} |\nu(E_k)|;\ \{E_k\} \in \mathcal{P}^*(E, \mathcal{A}) \right\}.$$

Briefly, $(\dagger_1)$ holds because this sup is an upper bound, and the "smallest" criterion holds because the sup is the smallest such upper bound. This quantity is called the total variation measure of ν.

$(X, \mathcal{A})$ measurable, $\nu \in \mathcal{M}(X, \mathcal{A})$, then $|\nu| \in \mathcal{M}^+(X, \mathcal{A})$, and $|\nu| \leq \mu$ for all $\mu \in \mathcal{M}^+(X, \mathcal{A})$ satisfying $\mu(E) \geq |\nu(E)|$ for all $E \in \mathcal{A}\ ^{(\dagger_1)}$.

Proof:

First, for $E \in \mathcal{A}$, let $F = \{\sum_{k\in\mathbb{N}} |\nu(E_k)|;\ \{E_k\} \in \mathcal{P}^*(E,\mathcal{A})\}$ is a well defined set, because $\mathcal{P}^*(E,\mathcal{A}) \subset \mathcal{A}$, so that these sums are well defined. Note that $F \subset \mathbb{R}$, if F is unbounded, then $|\nu|(E) = \infty$, otherwise F is bounded and this $\sup(F) \in \mathbb{R}$ exists.

$\mathcal{P}^*(\phi,\mathcal{A}) = \{\phi\} \Rightarrow |\nu|(\phi) = |\nu(\phi)| = 0$.
For any $\{E_i\} \in \mathcal{P}^*(E,\mathcal{A})$, that $|\nu|(E) = \sum_{i\in\mathbb{N}} |\nu|(E_i)$ follows by "$\leq$" and "$\geq$" cases.

"$\geq$": If $|\nu|(E) = \infty$ then this case always holds, so assume $|\nu|(E) < \infty$. $\{E_i\} \in \mathcal{P}^*(E,\mathcal{A})$ is given. Pick $\{t_i \in \mathbb{R}; i \in \mathbb{N}, t_i \geq 0\}$ such that $t_i < |\nu|(E_i)$, but if $|\nu|(E_i) = 0$, then let $t_i = 0$. Given each t_i we can find a partition of E_i, $\{A_{i,j}\} \in \mathcal{P}^*(E_i,\mathcal{A})$, such that $\sum_{j\in\mathbb{N}} |\nu(A_{i,j})| \geq t_i$. Each E_i has atleast one well defined partition; at a minimum $\{E_i,\phi\} \in \mathcal{P}^*(E_i,\mathcal{A})$. If this is the only partition in $\mathcal{P}^*(E_i,\mathcal{A})$, then $|\nu|(E_i) = |\nu(E_i)|$, and in this case $\sum_{j\in\mathbb{N}} |\nu(A_{i,j})| = |\nu(E_i)| = |\nu|(E_i) > t_i$. Now we have that $\{A_{i,j}; i,j \in \mathbb{N}\} \in \mathcal{P}^*(E,\mathcal{A})$, so that by the sup in the difinition of $|\nu|$, summing over i,

$$|\nu|(E) \geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\nu(A_{i,j})| \geq \sum_{i=1}^{\infty} t_i.$$

Lemma about $\mathbb{R}$, Let $L \in \mathbb{R}, \{a_k \in \mathbb{R}; k \in \mathbb{N}\}$, then

$$\left(\sum_{k=1}^{n} t_k \leq L \text{ for all } \{t_k\} \in \mathbb{R}^n,\ t_k \leq a_k,\ n \in \mathbb{N}\right) \Rightarrow \sum_{k=1}^{\infty} a_k \leq L.$$

If $n = 1$, suppose $a > L$, then can find some $t \in \mathbb{R}$ s.t. $L < t \leq a$, so the statement $(t \leq L\ \forall\, t \in \mathbb{R}, t \leq a)$ contradicts $a > L$, but either $a > L$ or $a \leq L$. Suppose the lemma is true for the case $n \in \mathbb{N}$, fixed, then $\sum_{k=1}^{n+1} t_k \leq L$ for all $\{t_k\} \in \mathbb{R}^{n+1}$, $t_k \leq a_k \Rightarrow \sum_{k=1}^{n} t_k \leq L - t_{k+1}$ for all $\{t_k\} \in \mathbb{R}^n, t_{k+1} \in \mathbb{R}$, $t_k \leq a_k \Rightarrow$ (by statement is true for $n \in \mathbb{N}$) $\sum_{k=1}^{n} a_k \leq L - t_{k+1}, t_{k+1} \in \mathbb{R}$, $t_{k+1} \leq a_{k+1} \Rightarrow t_{k+1} \leq L - \sum_{k=1}^{n} a_k, t_{k+1} \in \mathbb{R}$, $t_{k+1} \leq a_{k+1} \Rightarrow$ (by statement is true for $n = 1$) $a_{k+1} \leq L - \sum_{k=1}^{n} a_k \Rightarrow \sum_{k=1}^{n+1} a_k \leq L$.

Now, using this lemma, with $L = |\nu|(E)$, $a_k = |\nu|(E_k)$, and t_k chosen so that $t_k < |\nu|(E_k)$ as previously, and relying on the result that $\sum_{k=1}^{\infty} t_k \leq |\nu|(E)$, we have that

$$\sum_{k=1}^{\infty} |\nu|(E_k) \leq |\nu|(E).$$

"$\leq$": $\{E_k\} \in \mathcal{P}^*(E,\mathcal{A})$ is given. Then for all $\{A_j\} \in \mathcal{P}^*(E,\mathcal{A})$, $\{A_j \cap E_k; k \in \mathbb{N}\} \in \mathcal{P}^*(A_j,\mathcal{A})$, then

$$\sum_{j\in\mathbb{N}} |\nu(A_j)| = \sum_{j\in\mathbb{N}} |\sum_{k\in\mathbb{N}} \nu(A_j \cap E_k)| \leq \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\nu(A_j \cap E_k)| = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} |\nu(A_j \cap E_k)| \leq \sum_{k\in\mathbb{N}} |\nu|(E_k),$$

by $\{A_j \cap E_k; j \in \mathbb{N}\} \in \mathcal{P}^*(E_k,\mathcal{A})$. This was for all $\{A_j\} \in \mathcal{P}^*(E,\mathcal{A})$, so is true for the sup in the definition of $|\nu|$, so

$$\sum_{k=1}^{\infty} |\nu|(E_k) \geq |\nu|(E).$$

So we have that $|\nu| \in \mathcal{M}^+(X,\mathcal{A})$. That $|\nu|(E) \geq |\nu(E)|$ follows by noting that $\{E,\phi\} \in \mathcal{P}^*(E,\mathcal{A})$ so that $|\nu|(E) \geq |\nu(E)| + |\nu(\phi)| = |\nu(E)|$. Suppose $\mu \in \mathcal{M}^+(\mathcal{A})$ was another positive measure satisfying $(\dagger_1)$, then for $\{E_k\} \in \mathcal{P}^*(E,\mathcal{A})$ arbitrarily, applying $(\dagger_1)$ and summing, $\sum_{k\in\mathbb{N}} |\nu(E_k)| \leq \sum_{k\in\mathbb{N}} \mu(E_k) = \mu(E)$, now by its definition, $|\nu|(E)$ is the sup of numbers of the form on the LHS, and by this inequality, $\mu(E)$ is an upper bound for such numbers, thus $|\nu|(E) \leq \mu(E)$, for all $E \in \mathcal{A}$. $\square$

$\mathcal{M}(X,\mathcal{A})$ is a vector space, with respect to measure addition, $(\nu_1+\nu_2)(E) = \nu_1(E)+\nu_2(E)$, scaling, $\lambda(\nu)(E) = (\lambda\nu)(E)$, the zero measure, $0(E) = 0$ for all $E \in \mathcal{A}$. The details to this are obvious.

$\nu_1, \nu_2 \in \mathcal{M}(X,\mathcal{A})$, $\lambda \in \mathbb{R}$ or $\mathbb{C}$ then $|\nu_1+\nu_2| \leq |\nu_1|+|\nu_2|$, $|\lambda\nu_1| = |\lambda|\,|\nu_1|$.

Proof: For all $E \in \mathcal{A}$, $|(\nu_1+\nu_2)(E)| = |\nu_1(E)+\nu_2(E)| \leq |\nu_1(E)|+|\nu_2(E)| \leq |\nu_1|(E)+|\nu_2|(E) = (|\nu_1|+|\nu_2|)(E)$. Scaling follows by $|\lambda\nu_1(E)| = |\lambda||\nu_1(E)|$, and for any $A \subset \mathbb{R}, a \in \mathbb{R}$, $A \neq \phi, a > 0$ $\sup\{ax : x \in A\} = a \sup A$.

Theorem 6.4 in Rudin: If $\nu \in \mathcal{M}(X,\mathcal{A})$, then $|\nu|(X) < \infty$.

$\mathcal{M}(X,\mathcal{A})$ is a normed space w.r.t. $||\nu|| := \nu(X)$ for all $\nu \in \mathcal{M}(X,\mathcal{A})$.

Proof: $||\nu_1+\nu_2|| = |\nu_1+\nu_2|(X) \leq |\nu_1|(X)+|\nu_2|(X) = ||\nu_1||+||\nu_2||$. $||\lambda\nu|| = |\lambda\nu|(X) = |\lambda||\nu|(X) = |\lambda|||\nu||$. $||\nu||(X) = |\nu|(X) \geq 0$, $||\nu|| = 0 \Rightarrow |\nu|(X) = 0 \Rightarrow 0 = |\nu|(X) \geq |\nu|(E) \geq |\nu(E)|$ for all $E \in \mathcal{A} \Rightarrow \nu = 0$.

$\mathcal{M}(X,\mathcal{A})$ is a complete metric space with respect to the canonical metric induced by the norm: $d(\nu_1,\nu_2) = (\nu_1-\nu_2)(X)$. Thus $\mathcal{M}(X,\mathcal{A})$ is a Banach space.

Proof: ADD

For all $\nu \in \mathcal{M}^{\pm}(X,\mathcal{A})$, , define $\nu^+ = \frac{1}{2}(|\nu|+\nu)$, $\nu^- = \frac{1}{2}(|\nu|-\nu)$. Then $\nu^+,\nu^- \in \mathcal{M}^+(X,\mathcal{A})$, $\nu = \nu^+-\nu^-$, $|\nu| = \nu^+ + \nu^-$. This is the Jordan decomposition of ν, and is unique. Further, if $\nu \in \mathcal{M}(X,\mathcal{A})$, then define $\mathrm{Re}(\nu)(E) = \mathrm{Re}(\nu(E))$, $\mathrm{Im}(\nu)(E) = \mathrm{Im}(\nu(E))$ for all $E \in \mathcal{A}$, then $\mathrm{Re}(\nu), \mathrm{Im}(\nu) \in \mathcal{M}(X,\mathcal{A})$, and so $\nu = \sum_{k=0}^{3} i^k \nu_k$, where each $\nu_k \in \mathcal{M}^+(X,\mathcal{A})$, $i = \sqrt{-1}$.

Proof: ADD

For all $f : X \to \mathbb{C}$, $\mathcal{A}$-measurable, $\nu \in \mathcal{M}(X,\mathcal{A})$, say that f is ν-integrable if it is $|\nu|$-integrable, so $f \in \mathcal{L}(X,|\nu|)$. Write $\nu_0 = \mathrm{Re}(\nu)^+, \nu_1 = \mathrm{Re}(\nu)^-, \nu_2 = \mathrm{Im}(\nu)^+, \nu_3 = \mathrm{Im}(\nu)^-$, then

$$\int_X f d\nu = \sum_{k=0}^{3} i^k \int_X f\, d\nu_k$$

and $f \in \mathcal{L}(X,|\nu|)$ iff $|f| \in \mathcal{L}(X,|\nu|)$ iff $|f| \in \mathcal{L}(X,|\nu|)$ iff $|f| \in \mathcal{L}(X,\nu_k)$ iff $f \in \mathcal{L}(X,\nu_k)$, for all $k \in \{0,1,2,3\}$.

Proof: ADD

For $\nu \in \mathcal{M}(X,\mathcal{A})$, say that ν is concentrated on $A \in \mathcal{A}$ if $\nu(E) = \nu(A \cap E)$ for all $E \in \mathcal{A}$, or equivalently $\nu(E) = 0$ for all $E \in \mathcal{A}, E \subset A^c$. Not equivalently that $\nu(A^c) = 0$.

For $\nu \in \mathcal{M}(X,\mathcal{A})$, $\mu \in \mathcal{M}^+(X,\mathcal{A})$, say that ν is absolutely continuous w.r.t. μ if $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \mathcal{A}$, and write $\nu << \mu$.

For $\nu_1, \nu_2 \in \mathcal{M}(X,\mathcal{A})$, say ν_1 and ν_2 are mutually singular if they are concentrated on disjoint sets, and write $\nu_1 \perp \nu_2$.

For $\mu \in \mathcal{M}^+(X, \mathcal{A})$, $\nu, \nu_1, \nu_2 \in \mathcal{M}(X, \mathcal{A})$,

a) ν concentrated on $A \Rightarrow |\nu|$ concentrated on $\mathcal{A}$.
b) $\nu_1 \perp \nu_2 \Rightarrow |\nu_1| \perp |\nu_2|$.
c) $\nu_1 \perp \mu$, $\nu_2 \perp \mu \Rightarrow \nu_1 + \nu_2 \perp \mu$.
d) $\nu_1 << \mu, \nu_2 << \mu \Rightarrow \nu_1 + \nu_2 << \mu$.
e) $\nu << \mu \Rightarrow |\nu| << \mu$.
f) $\nu_1 << \mu, \nu_2 \perp \mu \Rightarrow \nu_1 \perp \nu_2$.
g) $\nu << \mu, \nu \perp \mu \Rightarrow \nu = 0$.

Proof: ADD

For $\mu \in \mathcal{M}^+(X, \mathcal{A})$, σ-finite, then there is a function $w \in L^1(\mu)$ s.t. $w(x) \in (0, 1)$ for all $x \in X$. Thus $\tilde{\mu} := w\,\mu \in \mathcal{M}^+(X, \mathcal{A})$, and $\mu(E) = 0 \Leftrightarrow \tilde{\mu}(E) = 0$ and $\tilde{\mu}(E) < \infty$ for all $E \in \mathcal{A}$.

Proof: $X = \cup_k E_k, E_k \in \mathcal{A}, \mu(E_k) < \infty$. Let $w_k(x) = 0$ if $x \in E_k^c$, $w_k(x) = 2^{-k}/(1 + \mu(E_k))$, else. Then on $E_k, 1 + \mu(E_k) \in (1, \infty), 1/(1 + \mu(E_k)) \in (0, 1)$. Then $w(x) = \sum_{k=1}^{\infty} w_k(x) \in (0, 1)$.
Proof: ADD

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