Plagiarees: (Real Analsysis, Folland), (SDE, Øksendal), (BM and Stochastic Calc, Karatas & Shreve), Dr. Bodmann, Dr. Blecher, Wikipedia.

Part 1.1 - Bits from measure theory

Image measures:

Image measures:

Given a measure space, $(X, \mathcal{A}, \mu)$, a measurable space, $(Y, \mathcal{B})$, and an $(\mathcal{A}, \mathcal{B})-$measurable function, $f : X \to Y$, we may construct a measure on $(Y, \mathcal{B})$, $\mu_f := \mu \circ f^{-1}$.

proof:

(1) $\phi \in \mathcal{B}$, $\mu_f(\phi) = \mu(F^{-1}(\phi)) = \mu(\phi) = 0$.

(2) $\{B_k\}_{k\in\mathbb{N}} \in \mathcal{B}$, disjoint, $B := \cup_{k\in\mathbb{N}} B_k$. $A := f^{-1}(B) = \cup_{k\in\mathbb{N}} A_k$, $A_k := f^{-1}(B_k)$. Then $A \in \mathcal{A}$ by f being $(\mathcal{A}, \mathcal{B})-$measurable, and $\{A_k\}_{k\in\mathbb{N}}$ is disjoint because, when $E_1 \cap E_2 = \phi$, $E_1, E_2 \in \mathcal{B}$, $\phi = f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$. Then, $\mu_f(\cup_{k\in\mathbb{N}} B_k) = \mu(f^{-1}(\cup_{k\in\mathbb{N}} B_k)) = \mu(\cup_{k\in\mathbb{N}} f^{-1}(B_k)) = \mu(\cup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mu(A_k) = \sum_{k\in\mathbb{N}} (\mu_f)(B_k)$, by countable additivity of μ.

So $(Y, \mathcal{B}, \mu_f)$ is a well defined measure space obtained from inverse images via f.

Simple functions:

$(X, \mathcal{A}, \mu)$ a measuse space, $\chi_E : X \to \{0, 1\}$ is the characteristic function of $E \in \mathcal{P}(X)$ when $\chi_E(x) = 1$ for all $x \in E$. A simple function $f : X \to S$ is a function which can be written as $f(x) = \sum_{k=1}^{n} c_k\, \chi_{E_k}(x)$, with $E_k \in \mathcal{P}(X), c_k \in S$. We can always choose the $\{E_k\}$ to be disjoint and non empty, and the $\{c_k\}$ to be unique and non-zero, this is called the standard representation. f is measureable when the $E_k \in \mathcal{A}$.

Definition of integral:

$(X, \mathcal{A}, \mu)$ a measure space.

1. For f a positive simple function in its standard representation, define $\int_X f\, d\mu = \sum_{k=1}^{n} c_k\, \mu(E_k)$.

2. For f a positive measurable function, define

$$\int_X f\, d\mu = \sup\{\int_X s\, d\mu; s \text{ a standard simple function}, s \leq f\}$$

3. For f a general measurable function, write $f = f^+ - f^-, f^+ \geq 0, f^- \geq 0$, f^+, f^- are always measurable when f is. Then define

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu$$

This integral exists when either of the quantities on the right hand side are finite, as $\infty - \infty$ is undefined. f is integrable when its integral is finite.

4. If f is complex valued, decompose it into real and imaginary parts, check that each is measurable.

A construction of integral:

$(X, \mathcal{A}, \mu)$ a measure space. For $f : X \to [0, \infty]$, measurable, step two of the definition of the integral may be replaced by the following construciton.

Define,

$$E_j = f^{-1}(\,[j2^{-n}, (j+1)2^{-n})\,), \;\; j \in \{0, 1, ..., n2^n - 1\}$$

$$E_j = f^{-1}(\,[n, \infty)\,), \;\; j = n2^n$$

$$s_n = \sum_{j=0}^{n2^n} j2^{-n}\chi_{E_j}$$

$E_j \in \mathcal{A}$ when f is $(\mathcal{A}, \mathcal{B}\ell(\mathbb{R}^+))$ measurabe.

Clearly $s_n(x) \leq n$, so then $s_n(x) \leq s_{n+1}(x)$ for all $x \in X$, $n \in \mathbb{N}$.

If at some $x \in X$, $f(x) = \infty$, then $s_n(x) = n \to \infty$ as $n \to \infty$. If at some $x \in X$, $f(x) < \infty$, then take n large enough so that $f(x) \leq n$, then $|s_n(x) - f(x)| \leq 2^{-n} \to 0$ as $n \to \infty$. So, s_n converges to f pointwise. If f is bounded, then the infinite case does not occur, and this convergence is uniform.

So, $\int_X f\, d\mu = \lim_{n\to\infty} \int_X s_n\, d\mu$ by Lebesgue's monotone convergence theorem.

Part 1.2 - Borel measures in $\mathbb{R}^n$

A Borel measure is one whose domain is a borel sigma algebra.

Given a finite measure space, $(\mathbb{R}, \mathcal{B}\ell(\mathbb{R}), \mu)$, define $F_\mu(x) = \mu((-\infty, x])$.

$y \geq x \Rightarrow (-\infty, x] \subset (-\infty, y] \Rightarrow F_\mu(x) \leq F_\mu(y)$, so F_μ is monotone increasing.

If $x_k \to x$ as $k \to \infty$, and $x_k \geq x$, then $(-\infty, x] = \cap_{k\in\mathbb{N}}(-\infty, x_k]$, so $F_\mu(x) = \mu(\cap_{k\in\mathbb{N}}(-\infty, x_k]) = \lim_{k\to\infty} \mu((-\infty, x_k]) = \lim_{k\to\infty} F_\mu(x_k)$, when some $\mu((-\infty, x_k]) < \infty$, so F_μ is right continuous.
ADD more detail here.

Theorem (Folland 1.16):

If $F : \mathbb{R} \to \mathbb{R}$ is any increasing and right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ iff $F - G$ is constant. Converesely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, and we define

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases}$$

then F_μ is increasing and right continuous and $\mu = \mu_{F_\mu}$. Also, $F_\mu(x) = \mu((-\infty, x]) - \mu((-\infty, 0])$, which makes sense when μ is finite.

proof: see Folland, page 35.

Lebesgue-Stieltjes measure:

$F : \mathbb{R} \to \mathbb{R}$ is any increasing and right continuous function, then

$$\mu(E) = \inf \left\{ \sum_{k\in\mathbb{N}} (F(b_k) - F(a_k)); E \subset \bigcup_{k\in\mathbb{N}} (a_k, b_k] \right\}$$

ADD more detail here.

Part 1.3 - Bits from measure theoretic probability

Main idea:

$(\Omega, \mathcal{F}, P),\ P(\Omega) = 1$ a probability space.

If picking n points, $\{\omega_{n,k}\}_{k=1}^{n}$ "at random" from Ω, so all $\omega_{n,k} \in \Omega$, then the following will be true

$$\lim_{n\to\infty} \frac{\#\{k \in \{1, 2, ..., n\}; \omega_{n,k} \in E\}}{n} = P(E),\ \text{for all } E \in \mathcal{F},$$

where $\#$ is the counting measure.

Nonsense:

$\alpha : \mathbb{N} \to \Omega$, $n \in \mathbb{N}$, onto, but not one to one. Define $\#_n = \frac{1}{n}\#$, $N = \{1, 2, ..., n\}$. Then $(N, \mathcal{P}(N), \#_n)$ is a porobability space. Let $\alpha_n = \alpha|_N$, then this is an Ω valued random varaible. Now we can define

$$P(E) = \lim_{n\to\infty} \#_n \alpha_n^{-1}(E),\ \text{for all } E \in \mathcal{F}$$

Random varaibles:

$(S, \mathcal{S})$ a measurable space, $X : \Omega \to S$ is called a random variable when it is $(\mathcal{F}, \mathcal{S})$-measurable.

Define $P_X : \mathcal{S} \to [0, +\infty]$ by $P_X(E) = P(\{\omega \in \Omega; X(\omega) \in E\})$, this is the image measure by X.

$$P_X(S) = P(\{\omega \in \Omega; X(\omega) \in S\}) = P(\Omega) = 1$$

So the image measure induced by a random variable is a probability measure on its state space.

P_X is called the dirstribution of X.

Define $F_X : \mathbb{R} \to [0, 1] = x \mapsto P_X((-\infty, x])$, this is called the cumulative distribution function.

From wikipedia:
"The probability density function of a random variable is the RadonNikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables)."
ADD many details here

Expectation:

Define the expectation value of X as $E(X) = \int_\Omega X \, dP$, the integral of X.

Suppose X is a simple function, then $X(\omega) = \sum_{k=1}^{n} c_k \chi_{E_k}(\omega)$, $c_k \in S$, unique, and $E_k \in \mathcal{F}$ disjoint.

$$E(X) = \sum_{k=1}^{n} c_k P(E_k)$$

Discrete rv:

A discrete random variable X is one whose state space is countable. In this case there is a bijective map, $\gamma : S \to \mathbb{N}$, and clearly the function $\gamma \circ X$ is $(\mathcal{F}, \mathcal{P}(\mathbb{N}))$-measurable.
We may write $S = \{x_k := \gamma^{-1}(k)\}_{k=1}^{\infty}$, and may define $E_k := X^{-1}(x_k)$, $X(\omega) = \sum_{k=1}^{\infty} x_k \chi_{E_k}$.
If we temporarily adopt the notation "$p(x_k) = P(X = x_k)$"$:= P(E_k)$, then

$$E(X) = \sum_{k \in \mathbb{N}} x_k p(x_k)$$

In this simple case Ω may not really be nescesary, as $(\{x_k\}, \mathcal{P}(\{x_k\}), p)$ is a probability space in it's own right, and note, with $\beta_n := X \circ \alpha_n$, α_n as in the above nonsense,

$$p(x_k) = \lim_{n \to \infty} \#_n \beta_n^{-1}(x_k), \text{ for all } x_k$$

Part 2.2 - Kolmogorov extension

Notation and definitions:

Throughout, $(\Omega, \mathcal{F}, P)$ is a probability space, T is an index set, and $(S, \mathcal{S})$ is a state space. This is cosmetic, really, $T = [0, \infty)$, time, $S = \mathbb{R}^d$, $\mathcal{S} = \mathcal{B}\ell(\mathbb{R}^d)$.

We define $S^T = \{f : T \to S\} = \prod_{t \in T} S$, and $S^{T \times \Omega} = \{f : T \times \Omega \to S\}$.

For A a set, let:
$A^n := \{a = (a_1, a_2, ..., a_n); a_k \in A\}$, for $n \in \mathbb{N}$.
$\tilde{A}^n := \{a \in A^n; a_i \neq a_j \ \forall i \neq j\}$, for $n \in \mathbb{N}$. $\tilde{A}^n \subset A^n$
$\tilde{A} := \{a \in \tilde{A}^n; n \in \mathbb{N}\}$

For A, B sets, $n \in \mathbb{N}$, and $f : A \to B$ let:
$f(a) := (f(a_1), f(a_2), ..., f(a_n)) \in B^n$, for $a \in A^n$.

For $n \in \mathbb{N}$, $t \in T^n$, and $B \in \mathcal{B}\ell(S^n)$ define an n-dimensional cylinder set in S^T as
$C(B, n, t) = \{\omega \in S^T; \omega(t) \in B\}$, and then $\tilde{C} := \{C(B, n, t); B \in \mathcal{B}\ell(S^n), n \in \mathbb{N}, t \in T^n\}$.
Then let $\mathcal{B}(S^T)$ be the sigma algebra generated by $\tilde{C}$.

Consistent family of measures:

Extension Theorem (consistency):

Part 3 - Brownian Motion

Part n - Stochastic Processes

Probability law of a Stochastic process: