Research Documentation - Stochastic Differential Equations, Theory.
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It's ok for this document to be in bad taste.

Plagiarees: (Real Analsysis, Folland), (SDE, Øksendal), (BM and Stochastic Calc, Karatas & Shreve), and Wikipedia.
Wikipedia is a source for keywords, connections and ideas, and is a quick reference. When Ideas are taken from it, they are checked and the details worked out and proved completely.

Part 1 - Preliminaries

<u>Pushforward measure:</u>

Given a measure space, $(X, \mathcal{A}, \mu)$, a measurable space, $(Y, \mathcal{B})$, and an $(\mathcal{A}, \mathcal{B})-$measurable function, $f : X \to Y$, we may construct a measure on $(Y, \mathcal{B})$, $f_*(\mu) := \mu \circ f^{-1}$.

pf: (1) $\phi \in \mathcal{B}$, $f_*(\mu)(\phi) = \mu(F^{-1}(\phi)) = \mu(\phi) = 0$.
(2) $\{B_k\}_{k\in\mathbb{N}} \in \mathcal{B}$, disjoint, $B := \cup_{k\in\mathbb{N}} B_k$. $A := f^{-1}(B) = \cup_{k\in\mathbb{N}} A_k$, $A_k := f^{-1}(B_k)$. Then $A \in \mathcal{A}$ by f being $(\mathcal{A}, \mathcal{B})-$measurable, and $\{A_k\}_{k\in\mathbb{N}}$ is disjoint because, when $E_1 \cap E_2 = \phi$, $E_1, E_2 \in \mathcal{B}$, $\phi = f^{-1}(E_1 \cap E_2) = f^{-1}(E_1) \cap f^{-1}(E_2)$. Then, $f_*(\mu)(\cup_{k\in\mathbb{N}} B_k) = \mu(f^{-1}(\cup_{k\in\mathbb{N}} B_k)) = \mu(\cup_{k\in\mathbb{N}} f^{-1}(B_k)) = \mu(\cup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mu(A_k) = \sum_{k\in\mathbb{N}} (f_*(\mu))(B_k)$, by countable additivity of μ.

So $(Y, \mathcal{B}, f_*(\mu))$ is a well defined measure space obtained by pushing forward μ via f.

Part 2 - Kolmogorov

<u>Notation and definitions:</u>

Throughout, $(\Omega, \mathcal{F}, P)$ is a probability space, T is an index set, and $(S, \mathcal{S})$ is a state space. This is cosmetic, really, $T = [0, \infty)$, time, $S = \mathbb{R}^d$, $\mathcal{S} = \mathcal{B}\ell(\mathbb{R}^d)$.

We define $S^T = \{f : T \to S\} = \prod_{t\in T} S$.

For A a set, let:
$A^n := \{a = (a_1, a_2, ..., a_n); a_k \in A\}$, for $n \in \mathbb{N}$.
$\tilde{A}^n := \{a \in A_n; a_i \neq a_j \ \forall i \neq j\}$, for $n \in \mathbb{N}$.
$\tilde{A} := \{a \in \tilde{A}^n; n \in \mathbb{N}\}$

For A, B sets, $n \in \mathbb{N}$, and $f : A \to B$ let:
$f(a) := (f(a_1), f(a_2), ..., f(a_n)) \in B^n$, for $a \in A^n$.

For $n \in \mathbb{N}$, $t \in T^n$, and $B \in \mathcal{B}\ell(S^n)$ define an n-dimensional cylinder set in S^T as
$C(B, n, t) = \{\omega \in S^T; \omega(t) \in B\}$
Then $\tilde{C} := \{C(B, n, t); B \in \mathcal{B}\ell(S^T), n \in \mathbb{N}, t \in T^n\}$.
Then let $\mathcal{B}\ell(S^T)$ be the sigma algebra generated by $\tilde{C}$.

<u>Stochastic Process:</u>

A stochastic process is a function, $X : T \times \Omega \to S$.

For each $\omega \in \Omega$, we have function $t \mapsto X(t, \omega)$ called the sample path, this is the ω-section of X. Each sample path is an element of S^T.

<u>blah:</u>

Part 3 - Brownian Motion

Part n - Stochastic Processes

<u>Probability law of a Stochastic process:</u>

Given $(\Omega, \mathcal{F}, P)$ a probability space, and T an index set, $(S, \mathcal{A})$ a measurable space. Then $X : T \times \Omega \to S$ is a stochastic process when the $t-$section of X, $X_t(\omega) := X(t, \omega)$ is $(\mathcal{F}, \mathcal{A})-$measurable for all $t \in T$. Let $S^T = \{g : T \to S\}$.
Each stochastic process, X induces a function, $\Phi_X : \Omega \to S^T$ by $\Phi_X(\omega) := t \mapsto X(t, \omega)$, so $\Phi_X(\omega)$ is the $\omega-$section of X. We're interested in defining a measure on a suitable sigma algebra on S^T, by pushing forward P via Φ_X.