Linear Wave Propagation

Nick Maxwell, Qingqing Liao, Dr. Kouri

April 30, 2010

First approach (1 of 3)

Starting with,

$$\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = C^2(\mathbf{x}) \nabla^2 u(\mathbf{x}, t),$$

we note that the action of $C^2(\mathbf{x}) \nabla^2$ is independent of t, so in a sloppy sense, we take $z = \sqrt{C^2(\mathbf{x}) \nabla^2}$, $g(t) = u(\mathbf{x}, t)$. Then we solve,

$$\frac{d^2}{d t^2} g(t) = z^2 g(t),$$

the general solution to which is a linear combination of $exp(\pm z\, t)$.

First approach (2 of 3)

So using $\cosh(x) = \frac{1}{2}(e^{+x} + e^{-x})$, $\sinh(x) = \frac{1}{2}(e^{+x} - e^{-x})$, we can pick the solutions,

$$g(t) = \cosh(z\,t)\,A + \sinh(z\,t)\,B, \; g(0) = A,$$

$$g'(t) = \sinh(z\,t)\,z\,A + \cosh(z\,t)\,z\,B, \; g'(0) = z\,B \Rightarrow B = z^{-1}g'(0).$$

Then,

$$g(t) = \cosh(z\,t)\,g(0) + \sinh(z\,t)\,z^{-1}g'(0),$$

$$g'(t) = \sinh(z\,t)\,z\,g(0) + \cosh(z\,t)\,z^{-1}\,z\,g'(0),$$

rewriting,

$$\left(\begin{array}{c} g(t) \\ g'(t) \end{array} \right) = \left(\begin{array}{cc} \cosh(z\,t) & \sinh(z\,t)\,z^{-1} \\ \sinh(z\,t)\,z & \cosh(z\,t) \end{array} \right) \left(\begin{array}{c} g(0) \\ g'(0) \end{array} \right).$$

First approach (3 of 3)

Substituting $u(\mathbf{x}, t)$ for $g(t)$, $v(\mathbf{x}, t)$ for $g'(t)$,

$$\left(\begin{array}{c} u(\mathbf{x}, t + \delta) \\ v(\mathbf{x}, t + \delta) \end{array} \right) = \mathrm{P}(\delta) \left(\begin{array}{c} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{array} \right),$$

$$\mathrm{P}(\delta) = \left(\begin{array}{cc} cosh(\delta \, z) & sinh(\delta \, z) \, z^{-1} \\ sinh(\delta \, z) \, z & cosh(\delta \, z) \end{array} \right).$$

Second approach (1 of 5)

Starting with,

$$\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = C^2(\mathbf{x}) \nabla^2 u(\mathbf{x}, t) + f(\mathbf{x}, t),$$

and the notation,

$$\mathrm{M} = \left(\begin{array}{cc} 0 & 1 \\ C^2(\mathbf{x}) \nabla^2 & 0 \end{array} \right), \; \mathbf{w}(t) = \left(\begin{array}{c} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{array} \right), \mathbf{f}(\mathbf{x}, t) = \left(\begin{array}{c} 0 \\ f(\mathbf{x}, t) \end{array} \right),$$

$$\frac{\partial}{\partial t} \mathbf{w}(t) = \mathrm{M} \, \mathbf{w}(t) + \mathbf{f}(\mathbf{x}, t).$$

Second approach (2 of 5)

We intruduce an integrating factor, $\exp\left(-t\,\mathrm{M}\right)$

$$e^{-t\,\mathrm{M}}\,\frac{\partial}{\partial t}\,\mathbf{w}(t) - e^{-t\,\mathrm{M}}\,\mathrm{M}\,\mathbf{w}(t) = e^{-t\,\mathrm{M}}\,\mathbf{f} \Rightarrow$$

$$\frac{\partial}{\partial t}\left(e^{-t\,\mathrm{M}}\,\mathbf{w}(t)\right) = e^{-t\,\mathrm{M}}\,\mathbf{f} \Rightarrow d\left(e^{-t\,\mathrm{M}}\,\mathbf{w}(t)\right) = e^{-t\,\mathrm{M}}\,\mathbf{f}\,dt,$$

$$\int_t^{t+\delta} d\left(e^{-s\,\mathrm{M}}\,\mathbf{w}(s)\right) = \int_t^{t+\delta} e^{-s\,\mathrm{M}}\,\mathbf{f}\,ds$$

$$e^{-(t+\delta)\,\mathrm{M}}\,\mathbf{w}(t+\delta) = e^{-t\,\mathrm{M}}\,\mathbf{w}(t) + \int_t^{t+\delta} e^{-s\,\mathrm{M}}\,\mathbf{f}\,ds.$$

Then we apply $e^{+(t+\delta)\,\mathrm{M}}$,

$$\mathbf{w}(t+\delta) = e^{\delta\,\mathrm{M}}\,\mathbf{w}(t) + \int_t^{t+\delta} e^{+(\delta+t-s)\,\mathrm{M}}\,\mathbf{f}\,ds.$$

Second approach (3 of 5)

So to propagate from t_1 to t_2, $\delta = t_2 - t_1$,

$$\left(\begin{array}{c} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{array} \right) = e^{\delta \, \mathrm{M}} \left(\begin{array}{c} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{array} \right) + \int_{t_1}^{t_2} e^{+(t_2 - t) \, \mathrm{M}} \, \mathbf{f}(t) \, dt.$$

Now, what is $e^{t \, \mathrm{M}}$? Well, again writing $z = \sqrt{C^2(\mathbf{x}) \, \nabla^2}$, and

$$\mathrm{M}^2 = \left(\begin{array}{cc} 0 & 1 \\ z^2 & 0 \end{array} \right)^2 = \left(\begin{array}{cc} z^2 & 0 \\ 0 & z^2 \end{array} \right),$$

so $\mathrm{M}^{2k} = z^{2k} \, \mathrm{I}$, then $\mathrm{M}^{2k+1} = z^{2k} \, \mathrm{M}$. We split the taylor expansion of $e^{t \, \mathrm{M}}$ by parity,

$$e^{t \, \mathrm{M}} = \sum_{k=0}^{\infty} (t \, \mathrm{M})^k / k! = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \, z^{2k} \mathrm{I} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \, z^{2k} \mathrm{M}.$$

Second approach (4 of 5)

$$e^{t\,M} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \begin{pmatrix} z^{2k} & 0 \\ 0 & z^{2k} \end{pmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{2k+1}\,z^{-1} \\ z^{2k+1}\,z^{+1} & 0 \end{pmatrix}$$

Now, $\cosh\,(t\,z) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}z^{2k+1}$, $\sinh\,(t\,z) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}z^{2k}$, so we recognise,

$$e^{t\,M} = \begin{pmatrix} cosh(t\,z) & sinh(t\,z)\,z^{-1} \\ sinh(t\,z)\,z & cosh(t\,z) \end{pmatrix}.$$

Second approach (5 of 5)

Then we have,

$$\left(\begin{array}{c} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{array} \right) = \mathrm{P}(t_2 - t_1) \left(\begin{array}{c} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{array} \right) + \int_{t_1}^{t_2} \mathrm{P}(t_2 - t) \, \mathbf{f}(\mathbf{x}, t) \, dt,$$

$$\left(\begin{array}{c} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{array} \right) = \mathrm{P}(t_2 - t_1) \left(\begin{array}{c} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{array} \right) + \int_{0}^{t_2 - t_1} \mathrm{P}(s) \, \mathbf{f}(\mathbf{x}, t_2 - s) \, ds.$$

Anisotropic Case (1 of 1)

We can addapt this approach to solve the anisotropic wave equation,

$$\frac{\partial^2}{\partial t^2}\, u_i(\mathbf{x}, t) - \frac{1}{\rho}\, C_{ijkl}(\mathbf{x})\, \frac{\partial^2}{\partial x_j\, \partial x_l}\, u_k(\mathbf{x}, t) = \frac{1}{\rho}\, f_i(\mathbf{x}, t),$$

then

$$\frac{\partial^2}{\partial t^2}\, u_i(\mathbf{x}, t) - \sum_k M_{ik}\, u_k(\mathbf{x}, t) = \frac{1}{\rho}\, f_i(\mathbf{x}, t),$$

with

$$M_{ik} = \sum_{j,l} \frac{1}{\rho}\, C_{ijkl}(\mathbf{x})\, \frac{\partial^2}{\partial x_j\, \partial x_l}$$