

Linear Wave Propagation

Nick Maxwell, Qingqing Liao, Dr. Kouri

October 7, 2009

First approach (1 of 3)

Starting with,

$$\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = C^2(\mathbf{x}) \nabla^2 u(\mathbf{x}, t),$$

we note that the action of $C^2(\mathbf{x}) \nabla^2$ is independent of t , so in a sloppy sense, we take $z = \sqrt{C^2(\mathbf{x}) \nabla^2}$, $g(t) = u(\mathbf{x}, t)$. Then we solve,

$$\frac{d^2}{dt^2} g(t) = z^2 g(t),$$

the general solution to which is a linear combination of $\exp(\pm z t)$.

First approach (2 of 3)

So using $\cosh(x) = \frac{1}{2}(e^{+x} + e^{-x})$, $\sinh(x) = \frac{1}{2}(e^{+x} - e^{-x})$, we can pick the solutions,

$$g(t) = \cosh(z t) A + \sinh(z t) B, \quad g(0) = A,$$

$$g'(t) = \sinh(z t) z A + \cosh(z t) z B, \quad g'(0) = z B \Rightarrow B = z^{-1} g'(0).$$

Then,

$$g(t) = \cosh(z t) g(0) + \sinh(z t) z^{-1} g'(0),$$

$$g'(t) = \sinh(z t) z g(0) + \cosh(z t) z^{-1} z g'(0),$$

rewriting,

$$\begin{pmatrix} g(t) \\ g'(t) \end{pmatrix} = \begin{pmatrix} \cosh(z t) & \sinh(z t) z^{-1} \\ \sinh(z t) z & \cosh(z t) \end{pmatrix} \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}.$$

First approach (3 of 3)

Substituting $u(\mathbf{x}, t)$ for $g(t)$, $v(\mathbf{x}, t)$ for $g'(t)$,

$$\begin{pmatrix} u(\mathbf{x}, t + \delta) \\ v(\mathbf{x}, t + \delta) \end{pmatrix} = P(\delta) \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{pmatrix},$$

$$P(\delta) = \begin{pmatrix} \cosh(\delta z) & \sinh(\delta z) z^{-1} \\ \sinh(\delta z) z & \cosh(\delta z) \end{pmatrix}.$$

Second approach (1 of 5)

Starting with,

$$\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = C^2(\mathbf{x}) \nabla^2 u(\mathbf{x}, t) + f(\mathbf{x}, t),$$

and the notation,

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ C^2(\mathbf{x}) \nabla^2 & 0 \end{pmatrix}, \quad \mathbf{w}(t) = \begin{pmatrix} u(\mathbf{x}, t) \\ v(\mathbf{x}, t) \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}, t) = \begin{pmatrix} 0 \\ f(\mathbf{x}, t) \end{pmatrix},$$
$$\frac{\partial}{\partial t} \mathbf{w}(t) = \mathbf{M} \mathbf{w}(t) + \mathbf{f}(\mathbf{x}, t).$$

Second approach (2 of 5)

We introduce an integrating factor, $\exp(-t M)$

$$e^{-tM} \frac{\partial}{\partial t} \mathbf{w}(t) - e^{-tM} M \mathbf{w}(t) = e^{-tM} \mathbf{f} \Rightarrow$$

$$\frac{\partial}{\partial t} (e^{-tM} \mathbf{w}(t)) = e^{-tM} \mathbf{f} \Rightarrow d(e^{-tM} \mathbf{w}(t)) = e^{-tM} \mathbf{f} dt,$$

$$\int_t^{t+\delta} d(e^{-sM} \mathbf{w}(s)) = \int_t^{t+\delta} e^{-sM} \mathbf{f} ds$$

$$e^{-(t+\delta)M} \mathbf{w}(t+\delta) = e^{-tM} \mathbf{w}(t) + \int_t^{t+\delta} e^{-sM} \mathbf{f} ds.$$

Then we apply $e^{+(t+\delta)M}$,

$$\mathbf{w}(t+\delta) = e^{\delta M} \mathbf{w}(t) + \int_t^{t+\delta} e^{+(\delta+t-s)M} \mathbf{f} ds.$$

Second approach (3 of 5)

So to propagate from t_1 to t_2 , $\delta = t_2 - t_1$,

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = e^{\delta M} \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \int_{t_1}^{t_2} e^{+(t_2-t)M} \mathbf{f}(t) dt.$$

Now, what is e^{tM} ? Well, again writing $z = \sqrt{C^2(\mathbf{x}) \nabla^2}$, and

$$M^2 = \begin{pmatrix} 0 & 1 \\ z^2 & 0 \end{pmatrix}^2 = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix},$$

so $M^{2k} = z^{2k} I$, then $M^{2k+1} = z^{2k} M$. We split the Taylor expansion of e^{tM} by parity,

$$e^{tM} = \sum_{k=0}^{\infty} (tM)^k / k! = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} z^{2k} I + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} z^{2k} M.$$

Second approach (4 of 5)

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \begin{pmatrix} z^{2k} & 0 \\ 0 & z^{2k} \end{pmatrix} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & z^{2k} \\ z^{2k+2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^{2k+1} z^{-1} \\ z^{2k+1} z^{+1} & 0 \end{pmatrix}$$

Now, $\cosh(tz) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} z^{2k}$, $\sinh(tz) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} z^{2k+1}$,
so we recognise,

$$e^{tM} = \begin{pmatrix} \cosh(tz) & \sinh(tz) z^{-1} \\ \sinh(tz) z & \cosh(tz) \end{pmatrix}.$$

Second approach (5 of 5)

Then we have,

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = P(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \int_{t_1}^{t_2} P(t_2 - t) \mathbf{f}(\mathbf{x}, t) dt.$$

Then we have,

$$\begin{aligned} \begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} &= P(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \\ &\int_{t_1}^{t_2} \left[\frac{\sinh((t_2 - t) \sqrt{C^2(\mathbf{x}) \nabla^2})}{\sqrt{C^2(\mathbf{x}) \nabla^2}} f(\mathbf{x}, t) \right] + \\ &\left[\cosh((t_2 - t) \sqrt{C^2(\mathbf{x}) \nabla^2}) f(\mathbf{x}, t) \right] dt. \end{aligned}$$

Second approach (6 of 5)

$$\begin{pmatrix} u(\mathbf{x}, t_2) \\ v(\mathbf{x}, t_2) \end{pmatrix} = P(t_2 - t_1) \begin{pmatrix} u(\mathbf{x}, t_1) \\ v(\mathbf{x}, t_1) \end{pmatrix} + \int_{t_1}^{t_2} \sum_{k=0}^{\infty} \left(\frac{(t_2 - t)^{2k}}{(2k+1)!} (t_2 - t + 2k + 1) \right) \left((C^2(\mathbf{x}) \nabla^2)^k f(\mathbf{x}, t) \right) dt.$$

Anisotropic Case (1 of 1)

We can adapt this approach to solve the anisotropic wave equation, by letting

$$u(\mathbf{x}, t) = \begin{pmatrix} u_x(\mathbf{x}, t) \\ u_y(\mathbf{x}, t) \\ u_z(\mathbf{x}, t) \end{pmatrix},$$

$$\mathbf{z}^2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_9 & c_9 \end{pmatrix} \begin{pmatrix} \frac{\partial^2}{\partial x \partial x} & \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial z \partial x} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y \partial y} & \frac{\partial^2}{\partial z \partial y} \\ \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial z \partial z} \end{pmatrix}$$