Notes on an acoustic wave propagator

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We present and discuss an operator which propagates solutions of the acoustic wave equation.

I. INTRODUCTION

We consider the equation

$$\ddot{u}(\mathbf{x}, t) = C^2(\mathbf{x})\, \nabla^2\, u(\mathbf{x}, t), \tag{I.1}$$

This equation is of the form

$$\frac{d^2}{dt^2}\, f(t) = m^2 f(t), \tag{I.2}$$

because C depends only on $\mathbf{x}$. If $f(0) = f_0$, and $\dot{f}(0) = \dot{f}_0$, then this is an initial value problem, and it is standard to derive the particular solution as follows.

$$\begin{aligned} f(t) &= \cosh{(m\,t)}\, c_1 + \sinh{(m\,t)}\, c_2, \\ \dot{f}(t) &= \sinh{(m\,t)}\, m\, c_1 + \cosh{(m\,t)}\, m\, c_2, \\ f(0) &= c_1 \Rightarrow c_1 = f_0, \\ \dot{f}(0) &= m\, c_2 \Rightarrow c_2 = \dot{f}_0/m. \end{aligned} \tag{I.3}$$

Thus, writing $z := \sqrt{C^2 \nabla^2}$, we assert that the exact solution of equation (I.1) is

$$\begin{aligned} u(\mathbf{x}, t) &= \cosh{(t\,z)}\, A + \sinh{(t\,z)}\, B, \\ \dot{u}(\mathbf{x}, t) &= \sinh{(t\,z)}\, z\, A + \cosh{(t\,z)}\, z\, B. \end{aligned} \tag{I.4}$$

And following the same argument,

$$\begin{aligned} A &= u_0, \\ B &= z^{-1}\, \dot{u}_0. \end{aligned} \tag{I.5}$$

Then equation (I.4) becomes

$$\begin{aligned} u(\mathbf{x}, t) &= \cosh{(t\,z)}\, u_0 + \sinh{(t\,z)}\, z^{-1}\, \dot{u}_0, \\ \dot{u}(\mathbf{x}, t) &= \sinh{(t\,z)}\, z\, u_0 + \cosh{(t\,z)}\, \dot{u}_0, \end{aligned} \tag{I.6}$$

which can be rewritten as

$$\begin{pmatrix} u(\mathbf{x}, t) \\ \dot{u}(\mathbf{x}, t) \end{pmatrix} = \mathrm{P}(t; C) \begin{pmatrix} u(\mathbf{x}, 0) \\ \dot{u}(\mathbf{x}, 0) \end{pmatrix}, \quad \mathrm{P}(t; C) = \begin{pmatrix} \cosh{(t\,z)} & \sinh{(t\,z)}/z \\ \sinh{(t\,z)}\, z & \cosh{(t\,z)} \end{pmatrix}. \tag{I.7}$$

II. INVESTIGATION

Using the identities

$$\begin{aligned}\sinh(t)\cosh(s)+\cosh(t)\sinh(s)&=\sinh(t+s),\\ \cosh(t)\cosh(s)+\sinh(t)\sinh(s)&=\cosh(t+s),\end{aligned} \tag{II.1}$$

and assuming that $\cosh(t\,z)$ commutes with $\sinh(t\,z)$, and both commute with z and z^{-1}, it is straight-forward to show that

$$\mathrm{P}(t;C)\,\mathrm{P}(s;C)=\mathrm{P}(t+s;C),\quad \mathrm{P}^n(t;C)=\mathrm{P}(n\,t;C). \tag{II.2}$$

We interpret $\cosh(t\sqrt{C^2\nabla^2})$ and $\sinh(t\sqrt{C^2\nabla^2})$ by their power series expansions. The the Taylor series expansions are given by $\cosh(x)=\sum_{n=0}^{\infty}x^{2n}/(2n)!$, and $\sinh(x)=\sum_{n=0}^{\infty}x^{2n+1}/(2n+1)!$, so

$$\cosh(t\sqrt{C^2\nabla^2})=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \tag{II.3}$$

$$\sinh(t\sqrt{C^2\nabla^2})=\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n\sqrt{C^2\nabla^2}. \tag{II.4}$$

Now, it is complicated to evaluate $\sqrt{C^2\nabla^2}$, however, examining the form of P, we see that the odd power of $\sqrt{C^2\nabla^2}$ in the expansion is absorbed, and we can write

$$\mathrm{P}(t;C)=\sum_{n=0}^{\infty}\begin{pmatrix}\frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n & \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n\\ \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^{n+1} & \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n\end{pmatrix} \tag{II.5}$$

$$\mathrm{P}(t;C)=\sum_{n=0}^{\infty}\frac{t^{2n}(C^2\nabla^2)^n}{(2n+1)!}\begin{pmatrix}2n+1 & t\\ t\,C^2\nabla^2 & 2n+1\end{pmatrix}. \tag{II.6}$$

To a first order,

$$\mathrm{P}(\tau;C)=\begin{pmatrix}1 & \tau\\ \tau\,C^2\nabla^2 & 1\end{pmatrix}+... \tag{II.7}$$

It is standard to write equation (I.1) as

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ C^2\nabla^2 & 0\end{pmatrix}\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}, \tag{II.8}$$

d then, for small τ,

$$\frac{d}{dt}\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}\approx\frac{1}{\tau}\left(\begin{pmatrix}u(t+\tau)\\ \dot u(t+\tau)\end{pmatrix}-\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}\right)\Rightarrow\begin{pmatrix}u(t+\tau)\\ \dot u(t+\tau)\end{pmatrix}\approx\left(\begin{pmatrix}0 & 1\\ C^2\nabla^2 & 0\end{pmatrix}\tau+1\right)\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}$$

$$\begin{pmatrix}u(t+\tau)\\ \dot u(t+\tau)\end{pmatrix}=\begin{pmatrix}1 & \tau\\ \tau\,C^2\nabla^2 & 1\end{pmatrix}\begin{pmatrix}u(t)\\ \dot u(t)\end{pmatrix}. \tag{II.9}$$

So we can see that truncating the Taylor series of P to the first term is exactly equivalent to Euler's method for the solution of (I.1).

III. APPLICATION

Depending on the application, $\dot{u}(t)$ may be unwanted information, so we might try to apply P in such a way that we avoid its computation. If we know that $\dot{u}(0) = 0$, we can take from equation (I.7) that

$$u(T) = \cosh\left(T\sqrt{C^2\nabla^2}\right) u(0). \tag{III.1}$$

This let's us compute $u(T)$ from $u(0)$, while avoiding $\dot{u}(T)$. We can't then compute $u(T+s)$, as we would again require $\dot{u}(T)$ for that additional step, so we take T as the final step to which we wish to propagate u.

Another, less obvious approach is to use the the binomial theorem to show that

$$(\cosh x)^{2n} = \frac{2}{4^n}\sum_{k=1}^{n}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(2k\,x) + \frac{(2n)!}{4^n(n!)^2}. \tag{III.2}$$

then, for $n = 0$,

$$\cosh(n\,t\,x) = 1, \tag{III.3}$$

for $n = 1$,

$$\cosh(n\,t\,x) = \cosh(t\,x), \tag{III.4}$$

for $n \geq 2$,

$$\cosh(n\,t\,x) = \frac{4^n}{2}(\cosh(t\,x/2))^{2n} - \sum_{k=1}^{n-1}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(k\,t\,x) - \frac{1}{2}\frac{(2n)!}{(n!)^2}. \tag{III.5}$$

$$\cosh(n\,t\,x) = 2^{n-1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(\cosh(t\,x))^k - \sum_{k=1}^{n-1}\frac{(2n)!}{(n+k)!(n-k)!}\cosh(k\,t\,x) - \frac{1}{2}\frac{(2n)!}{(n!)^2}. \tag{III.6}$$

IV. DAMPING.

We modify (I.1) to

$$\ddot{u}(\mathbf{x},t) = C^2(\mathbf{x})\,\nabla^2\,u(\mathbf{x},t) \;-\; 2\,D(\mathbf{x})\,\dot{u}(\mathbf{x},t) \;-\; D^2(\mathbf{x})\,u(\mathbf{x},t), \tag{IV.1}$$

$$\ddot{u}(t) = z^2\,u(t) \;-\; 2\,D\,\dot{u}(t) \;-\; D^2\,u(t), \tag{IV.2}$$

$$u(t) = e^{-Dt}\cosh(t\,z)\,A + e^{-Dt}\sinh(t\,z)\,B, \tag{IV.3}$$

$$\dot{u}(t) = -D\,u(t) + z\,e^{-Dt}\sinh(t\,z)\,A + z\,e^{-Dt}\cosh(t\,z)\,B, \tag{IV.4}$$

$$u(0) = A = u_0,\, \dot{u}(0) = -D\,u_0 + z\,B = \dot{u}_0, \tag{IV.5}$$

$$A = u_0, B = (\dot{u}_0 + D\,u_0)/z, \tag{IV.6}$$

$$u(t) = e^{-Dt}\cosh(t\,z)\,u_0 + e^{-Dt}\sinh(t\,z)\,\dot{u}_0/z + e^{-Dt}\sinh(t\,z)\,D\,u_0/z, \tag{IV.7}$$

$$\dot{u}(t) = -D\,e^{-Dt}\cosh(t\,z)\,u_0 + -D\,e^{-Dt}\sinh(t\,z)\,\dot{u}_0/z + -D\,e^{-Dt}\sinh(t\,z)\,D\,u_0/z, + z\,e^{-Dt}\sinh(t\,z)\,A + z\,e^{-Dt}\cosh(t\,z)\,B, \tag{IV.8}$$

...

$$\frac{\partial}{\partial t}\,u(t,x) = -c\,\frac{\partial}{\partial x}\,u(t,x), \tag{IV.9}$$

$$\frac{\partial}{\partial t}\,\hat{u}(t,k) = -c\,ik\,\hat{u}(t,k), \tag{IV.10}$$

$$\hat{u}(t,k) = \exp(-c\,i\,k\,t)\,\hat{u}(0,k), \tag{IV.11}$$

$$u(t,x) = u(0,x-ct), \tag{IV.12}$$

V. TAYLOR EXPANSION.

The the Taylor series expansions are given by $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, and $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$, so

$$\cosh(t\sqrt{C^2\nabla^2}) = \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \tag{V.1}$$

$$\sinh(t\sqrt{C^2\nabla^2}) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n\sqrt{C^2\nabla^2}, \tag{V.2}$$

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty}\begin{pmatrix} \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n & \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^n \\ \frac{t^{2n+1}}{(2n+1)!}(C^2\nabla^2)^{n+1} & \frac{t^{2n}}{(2n)!}(C^2\nabla^2)^n \end{pmatrix} \tag{V.3}$$

$$\mathrm{P}(t;C) = \sum_{n=0}^{\infty}\frac{t^{2n}(C^2\nabla^2)^n}{(2n)!}\begin{pmatrix} 1 & t/(2n+1) \\ t\,C^2\nabla^2/(2n+1) & 1 \end{pmatrix}. \tag{V.4}$$

So, to compute $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \mathrm{P}(t;C)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, truncated to M term in the expansion, we can generate the sequences $(U_n)_{n=0}^{M+1}$ and $(V_n)_{n=0}^{M}$, as

$$U_0 = u_0,\quad U_n = t^2\,C^2\nabla^2\,U_{n-1}/(4n^2-2n), \tag{V.5}$$

$$V_0 = v_0,\quad V_n = t^2\,C^2\nabla^2\,V_{n-1}/(4n^2-2n) \tag{V.6}$$

$$u_1 = \sum_{n=0}^{M} U_n + t\,V_n/(2n+1), \tag{V.7}$$

$$v_1 = \sum_{n=0}^{M}\frac{1}{t}\,(4n^2+6n+2)U_{n+1}/(2n+1) + V_n, \tag{V.8}$$

$$\tag{V.9}$$

Or, to compute $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \mathrm{P}(t;C) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, truncated to M term in the expansion, we can generate the sequences $(U_n)_{n=0}^{M+1}$ and $(V_n)_{n=0}^{M}$, as

$$U_0 = u_0, \;\; U_n = \lambda^{-2}\, C^2 \nabla^2\, U_{n-1}/(4n^2 - 2n), \tag{V.10}$$

$$V_0 = v_0, \;\; V_n = \lambda^{-2}\, C^2 \nabla^2\, V_{n-1}/(4n^2 - 2n) \tag{V.11}$$

$$u_1 = \sum_{n=0}^{M} (\lambda\, t)^{2n}\, U_n + (\lambda\, t)^{2n}\, t\, V_n/(2n+1), \tag{V.12}$$

$$v_1 = \sum_{n=0}^{M} (\lambda\, t)^{2n}\, \frac{1}{t}\, (4n^2 + 6n + 2) U_{n+1}/(2n+1) + (\lambda\, t)^{2n}\, V_n, \tag{V.13}$$

$$\tag{V.14}$$

where λ is a real number which serves to shift magnitude from the sequences $(U_n), (V_n)$.

VI. HERMITE EXPANSION.

using the expansion,

$$\exp\,(-i\,t\,x) = \sum_{m=0}^{\infty} \frac{(-i\,t\,\lambda)^m}{m!\,2^m} \exp\left(-\frac{(t\,\lambda)^2}{4}\right) H_m(x/\lambda), \tag{VI.1}$$

$$\cosh\,(t\,x) = \sum_{m=0}^{\infty} c_m(t\,\lambda) H_{2m}(x/\lambda), \;\; c_m(t\,\lambda) = \frac{(t\,\lambda)^{2m}}{(2m)!\,4^m} \exp\left(\frac{(t\,\lambda)^2}{4}\right). \tag{VI.2}$$

Then, we can use the recurrence relation

$$H_{2m}(x) = (4x^2 - 8m + 6) H_{2m-2}(x) - 8(m-1)(2m-3) H_{2m-4}(x), \tag{VI.3}$$

$$H_0(x) = 1, \; H_2(x) = 4x^2 - 2.$$

VII. LAGUERRE EXPANSION.

We wish to expand $\exp\,(i\,t\,x)$ into a power series,

$$e^{i\,t\,x} = \sum_{n=0}^{\infty} \tilde{c}_n^{(\alpha)}(t) L_n^{(\alpha)}(x), \tag{VII.1}$$

where $L_n^{(\alpha)}$ is the n^{th} generalized Laguerre polynomial, which have α as a parameter. If $\alpha = \pm\frac{1}{2}$, then (IV.1) becomes a Hermite polynomial expansion. It is non-trivial to show that

$$\tilde{c}_n^{(\alpha)}(t) = (-i\,t)^n (1 - i\,t)^{-1-n-\alpha}, \tag{VII.2}$$

which we've been unable to simplify to a more enlightening form.

Then, the expansions are

$$\cosh\,(t\,x) = \sum_{n=0}^{\infty} c_n^{(\alpha)}(t) \frac{1}{2} \left(L_n^{(\alpha)}(x) + L_n^{(\alpha)}(-x) \right), \tag{VII.3}$$

$$\sinh{(t\,x)} = \sum_{n=0}^{\infty} c_n^{(\alpha)}(t) \frac{1}{2} \left(L_n^{(\alpha)}(x) - L_n^{(\alpha)}(-x) \right), \tag{VII.4}$$

where $c_n^{(\alpha)}(t) = \tilde{c}_n^{(\alpha)}(i\,t) = (t)^n (1+\,t)^{-1-n-\alpha}$.

Using that

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(-x)^m}{m!} \binom{n+\alpha}{n-m}, \tag{VII.5}$$

we truncate these expansions to $2N$ terms and switch the sums,

$$\cosh{(t\,x)} \approx \sum_{m=0}^{N} \frac{a_m^{(\alpha,N)}(t)}{(2m)!} x^{2m},\ a_m^{(\alpha,N)}(t) = \sum_{n=2m}^{2N} c_n^{(\alpha)}(t) \binom{n+\alpha}{n-2m} \tag{VII.6}$$

$$\sinh{(t\,x)} \approx \sum_{m=0}^{N} \frac{b_m^{(\alpha,N)}(t)}{(2m+1)!} x^{2m+1},\ b_m^{(\alpha,N)}(t) = \sum_{n=2m+1}^{2N} c_n^{(\alpha)}(t) \binom{n+\alpha}{n-2m-1}. \tag{VII.7}$$

With $\binom{z}{n} = \prod_{k=1}^{n} \frac{z-n+k}{k}$,

$$a_m^{(\alpha,N)}(t) = \sum_{n=2m}^{2N} \exp\left(n\log t - (+1+n+\alpha)\log(1+t) + \sum_{k=1}^{n-2m} (\log(k+\alpha+2m) - \log k) \right) \tag{VII.8}$$

$$b_m^{(\alpha,N)}(t) = \sum_{n=2m+1}^{2N} \exp\left(n\log t - (+1+n+\alpha)\log(1+t) + \sum_{k=1}^{n-2m-1} (\log(k+\alpha+2m+1) - \log k) \right) \tag{VII.9}$$

VIII. APPLICATION OF ∇^2.

∇^2 is diagonal in the Fourier basis. Letting $\hat{u}(\mathbf{k}) = \mathcal{F}(u(\mathbf{x}))$, with $\mathcal{F}$ being the Fourier transform, mapping functions over the $\mathbf{x}$ space to functions over the $\mathbf{k}$ space, we may apply ∇^2 as

$$\nabla^2\,u(\mathbf{x}) = \mathcal{F}^{-1}(-\mathbf{k}\cdot\mathbf{k}\;\hat{u}(\mathbf{k})),\ \nabla^2\,u(\mathbf{x}) = \mathcal{F}^{-1}(\hat{\nabla}^2(\mathbf{k})\;\hat{u}(\mathbf{k})), \tag{VIII.1}$$

$$\hat{\nabla}^{2n}(\mathbf{k}) = (-1)^n\,(\mathbf{k}\cdot\mathbf{k})^n \tag{VIII.2}$$

This is a valid approach, however expansions of operators involving ∇^2 may require n to be large, and further, in the discretization of $u(\mathbf{x})$, many grid points may be required, implying large magnitudes of $\mathbf{k}$, so that $(\mathbf{k}\cdot\mathbf{k})^n$ will be very large. In practice, $\hat{u}(\mathbf{k})$ may be essentially zero outside of some region of the $\mathbf{k}$ domain, but may not be identically zero. In such cases the large values of $(\mathbf{k}\cdot\mathbf{k})^n$ will multiply those small values of $\hat{u}(\mathbf{k})$ and yield large numerical error. The solution is to apply a low-pass filter, $\mathrm{f}(\mathbf{k})$, to $\hat{\nabla}^2$,

$$\nabla^2\,u(\mathbf{x}) = \mathcal{F}^{-1}((-\mathrm{f}(\mathbf{k})\,\mathbf{k}\cdot\mathbf{k})\;\hat{u}(\mathbf{k})). \tag{VIII.3}$$

As analyzed in [1], one such filter is the system function of the Hermite Distributed Approximating Functionals(HDAFs). If we define $Z_m(\xi) = e^{-\xi}\sum_{n=0}^{m} \xi^n/n!$, then

$$\hat{\delta}(k;\,\sigma,m) = Z_m\left(\frac{k^2\,\sigma^2}{2}\right), \tag{VIII.4}$$

is the Fourier transform of the HDAF $\delta(x;\,\sigma,m)$. It is shown in [1] that the inflection point of $\hat{\delta}$ is given by $k_c = \sqrt{2m+1}/\sigma$, and the distance between the inflection points of the first derivative of $\hat{\delta}$, Δk is $\Delta k = \sqrt{4m+3-4\sqrt{m^2+m/2}}/\sigma \approx \sqrt{2}/\sigma$. So we may interpret k_c as the cutoff frequency, and Δk as the length of the transition region from 1 to 0, of $\hat{\delta}$. Then we take $\hat{\delta}(k;\,\sigma,m) = \hat{\delta}(k;\,k_0,\Delta k)$, with $\sigma = \sqrt{2}/(\Delta k)$, and $m = \mathrm{ceil}\Big(\big(\frac{k_c}{\Delta k}\big)^2 - \frac{1}{2}\Big)$. It is practical to evaluate $\hat{\delta}(k;\,\sigma,m)$ by making use of logarithms, $Z_m(\xi) = \sum_{n=0}^{m} \exp\left(-x + n\log(\xi) - \log(n!)\right)$.

In the three-dimensional case, we have several options for defining the low-pass filter, $\mathrm{f}(\mathbf{k})$. Most straight-forwardly we could take

$$\mathrm{f}(\mathbf{k}) = \hat{\delta}(k_x;\,\sigma_x,m_x)\hat{\delta}(k_y;\,\sigma_y,m_y)\hat{\delta}(k_z;\,\sigma_z,m_z), \tag{VIII.5}$$

where $\mathbf{k} = (k_x,k_y,k_z)$. Another approach would be to take

$$\mathrm{f}(\mathbf{k}) = \hat{\delta}(g(\mathbf{k});\,\sigma,m), \tag{VIII.6}$$

where $g:\mathbb{R}^3 \to \mathbb{R}$, is a geometrical function, thus defining a low-pass region in the $\mathbf{k}$ domain.

We first consider the one-dimensional case, $\mathrm{f}(k) = \hat{\delta}(k;\,\sigma,m)$. In order to construct a scheme for choosing values of k_c to use, we define, $||\hat{u}(k)||_{(K)} = \int_{-K}^{K}|u(k)|\,dk$, so $||\cdot||_{(K)}$ is an L1-norm on $[-K,K]$. Then, we define $S(K;\,\hat{u}) = ||\hat{u}(k)||_{(K)}/||\hat{u}(k)||_{(\infty)}$. S has the properties $S(0)=0$, $\lim_{k\to\infty} S(k) = 1$, $S(k)\in[0,1]$, and S is a monotonically increasing function, so $k1 \le k2 \Rightarrow S(k1) \le S(k2)$. Thus, we may say that for any $\epsilon \in (0,1)$, there exists a k_c such that $1 - S(k_c) \le \epsilon$, valid for any absolutely integrable function.

We discretize $u(x)$ as $u_i = u(x_i)$, $x_i = i\,h_x$, $h_x = L/N$, $N\in\mathbb{N}$, $L = x_f - x_0$. Then, $k_{nyq.} = 1/(2h_x)$ is the Nyquist frequency in the discretization, $\hat{u}(k)$ is discretized as $\hat{u}_j = \hat{u}(k_j)$, $k_j = j\,h_k$, $h_k = 1/L$, which can be found from $(N/2)h_k = k_{nyq.} = 1/(2(L/N))$. Because $u(x)$ is a real-valued function, we have that $\hat{u}(k) = (\hat{u}(-k))^*$, and $|\hat{u}(k)| = |\hat{u}(-k)|$.

With these properties we have $||\hat{u}(k)||_{(K)} = 2\int_0^K |u(k)|\,dk$, and by the assumption of having sampled $u(x)$ with N points, $||\hat{u}(k)||_{(\infty)} = ||\hat{u}(k)||_{(k_{nyq.})}$. It is sufficient to evaluate the interal as

$$s_m := \frac{1}{h_k}\int_0^{m\,h_k} |u(k)|\,dk = \sum_{j=0}^{m} |\hat{u}_j|. \tag{VIII.7}$$

Then, defining $S_m = s_m/s_{N/2}$, we have $S_0 = 0$, $S_{N/2} = 1$, $S_m \in [0,1]$, S is a monotonically increasing sequence, $S_m = S_{m-1} + |\hat{u}_m|/s_{N/2}$. So, to find the cutoff frequency, k_c, we specify an $\epsilon\in(0,1)$, and step through S_m untill $1 - S_c \le \epsilon$; a $c\in\mathbb{N}$, $c \le N/2$ is guaranteed to exist. Now the filter $\mathrm{f}(k)$ is parameterized by Δk, ϵ.

For a more practical approach, we redefine σ so that we specify $\hat{\delta}(k;\,\sigma\sqrt{2m+1},m)$, then, k_c is given by $1/\sigma$. Now σ linearly scales the filter in k, and m sets the sharpness of the filter. We then consider, with $\Delta k = \sqrt{2}/(\sigma\sqrt{2m+1})$, $M(\mu;\,m) = \hat{\delta}(k_c - (\Delta k)\mu/2;\,\sigma\sqrt{2m+1},m)$; $M(1;\,m)$ gives the value of $\hat{\delta}(k;\,\sigma\sqrt{2m+1},m)$, at the first inflection point of the first derivative of $\hat{\delta}(k;\,\sigma\sqrt{2m+1},m)$, further, M is independent of σ.

We can see from table I that, $M(\mu;\,m)$ tends to converge in m, for a given μ, quickly. The strategy will then be this: specify $\epsilon_1,\epsilon_2\in(0,1)$, and find $c_1,c_2,c_1<c_2$ so that $1-S_{c_1}<\epsilon_1, 1-S_{c_2}<\epsilon_2$. So, if $\epsilon_1 = 0.05, \epsilon_2 = 0.01$, then $\{\hat{u}_j : j \le c_1\}$ will represent the first 95 % of the magnitude of $\hat{u}$, and $\{\hat{u}_j : j\le c_2\}$, the first 99 %. Then, use k_{c_2} as the cutoff frequency, $\sigma = 1/k_{c_2}$, and $m = \mathrm{ceil}\big(\frac{1}{4}\mu^2/(1-k_{c_1}/k_{c_2})^2 - 1/2\big)$, so that k_{c_1} is the frequency at which the response is $M(\mu;\,m)$

[] Bodmann, Hoffman, Kouri, Papadakis, *Hermite Distributed Approximating Functionals as Almost-Ideal Low-Pass Filters* , (Sampling Theory in Signal And Image Processing, 2008 Vol. 7, No. 1)

TABLE I: $M(\mu;\, m)$ for several values of m, μ

m	μ =0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
1	0.64	0.72	0.79	0.85	0.9	0.94	0.97	0.99	0.99	1
2	0.62	0.7	0.77	0.83	0.89	0.93	0.96	0.98	0.99	1
3	0.62	0.69	0.76	0.83	0.88	0.92	0.95	0.97	0.98	0.99
4	0.61	0.69	0.76	0.82	0.87	0.91	0.95	0.97	0.98	0.99
5	0.61	0.69	0.76	0.82	0.87	0.91	0.94	0.97	0.98	0.99
6	0.61	0.68	0.75	0.82	0.87	0.91	0.94	0.96	0.98	0.99
7	0.6	0.68	0.75	0.81	0.87	0.91	0.94	0.96	0.98	0.99
8	0.6	0.68	0.75	0.81	0.86	0.91	0.94	0.96	0.98	0.99
9	0.6	0.68	0.75	0.81	0.86	0.9	0.94	0.96	0.98	0.99
10	0.6	0.68	0.75	0.81	0.86	0.9	0.94	0.96	0.98	0.99
20	0.59	0.67	0.74	0.8	0.86	0.9	0.93	0.96	0.97	0.98
40	0.59	0.67	0.74	0.8	0.85	0.89	0.93	0.95	0.97	0.98
60	0.59	0.66	0.73	0.8	0.85	0.89	0.93	0.95	0.97	0.98
80	0.59	0.66	0.73	0.8	0.85	0.89	0.92	0.95	0.97	0.98
100	0.59	0.66	0.73	0.79	0.85	0.89	0.92	0.95	0.97	0.98
2000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
4000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
6000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
8000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
10000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
12000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
14000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
16000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
18000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98
20000	0.58	0.66	0.73	0.79	0.84	0.89	0.92	0.95	0.96	0.98