Theory

Theorem 1: (Girsanov) Let $W(t) \in \mathbb{R}^n$ be a Brownian motion with respect to the measure P, and $X(t) \in \mathbb{R}^n$ an Itô process given by

$$dX(t) = a(t,\omega)dt + dW(t), \ \ 0 \leq t \leq T \leq \infty.$$

Then with

$$M(t,\omega) = \exp\left(-\int_0^t a(s,\omega) \cdot dW(s) - \frac{1}{2}\int_0^t a^2(s,\omega)\,ds\right), \ \ 0 \leq t \leq T$$

define the measure Q by

$$dQ(\omega) = M(T,\omega)dP(\omega),$$

then $X(t)$ is an N-dimensional Brownian motion wirh respect to Q.

Theorem 2: Let $D \subset \mathbb{R}^n$, open and connected, ∂D its boundary, and $f : D \to \mathbb{R}$. Let $W(t)$, $X(t)$, $M(t)$, P, as in the last theorem. Define

$$\tau_x(\omega) = \inf\left(\{0 \leq t \leq T; X(t,\omega) \notin D, \}\right),$$

where $X(0) = x$. Then

$$u(x) = \int_\Omega f(X(\tau_x(\omega),\omega))\,M(\tau_x(\omega),\omega)\,dP(\omega) = E_{Q_x}[f(X(\tau_x))],$$

wheres $dQ_x(\omega) = M(\tau_x(\omega),\omega)dP(\omega)$, solves the equation

$$\nabla^2 u(x) = \begin{cases} 0, & x \in D \\ f(x), & x \in \partial D. \end{cases}$$

Remarks: The drift, $a(t,\omega)$, may be chosen to increase the rate of convergence in the above integral.

Experiment