Geometry of Horospheres in Right-Angled Coxeter Groups

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Introduction

This project aims to understand and visualize horospheres in hyperbolic right-angled Coxeter groups. Using tools such as finite state machines and fiber products we are able to efficiently generate pieces of the horospheres of right-angled Coxeter Groups. The quick computation allows us to visualize the geometry in a manner not allowed for before. This thus gives us insight into the boundaries of such groups. This is a continued project from Fall 2022, joint with Qianruixi Wang & Kaicheng Xue.

Background

Definition. Let G be a group and $S \subset G$ a generating set for G. The **Cayley Graph** of (G, S) is the graph having one vertex associated with each group element and edges (a, b) whenever $ab^{-1} \in S$.

Definition. For a **defining graph** $\Gamma = (V, E)$, we define the **right-angled Coxeter group (RACG)** W_Γ as the presentation

$$W_\Gamma \coloneqq \langle V \mid v^2 = v_1 v_2 v_1^{-1} v_2^{-1} = 1, v \in V, (v_1, v_2) \in E \rangle.$$

Coxeter groups describe reflections on a space [6]. We were motivated to study these groups because the graphs we chose make the RACGs **hyperbolic groups**, which have interesting geometric properties [2]. RACGs that are hyperbolic are also **automatic groups**, which have well studied algorithms that inspired our programs [3].

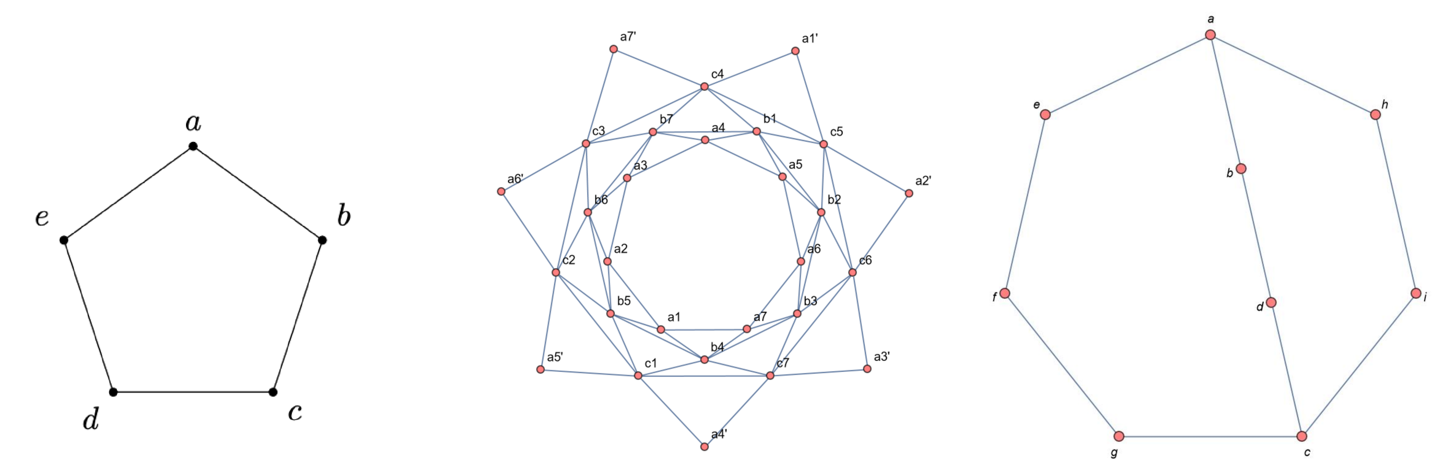

Fig. 1: From Left to Right: Defining graphs of Pentagon, Torus, and Θ graph

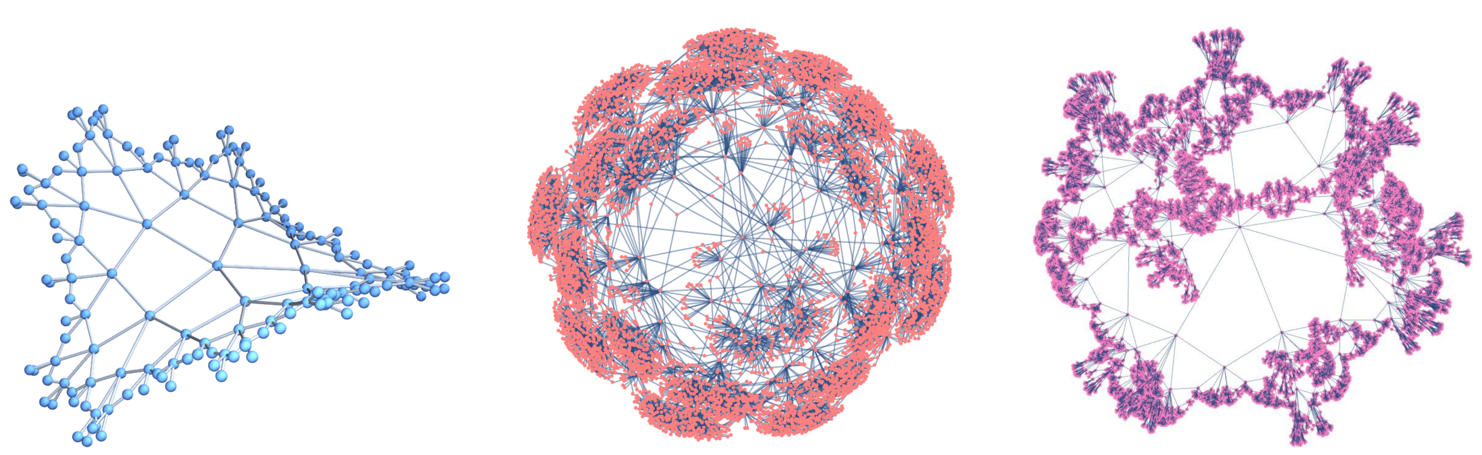

Fig. 2: From Left to Right: Cayley graphs of Pentagon, Torus, and Θ graph

Horospheres formalize the notion of centering a sphere "at infinity" for hyperbolic spaces (see Figure 3). The Cayley graph of our RACGs is hyperbolic, so we examine the horosphere of that graph. Our goal was to visualize a finite subsection of the horosphere for our RACGs.

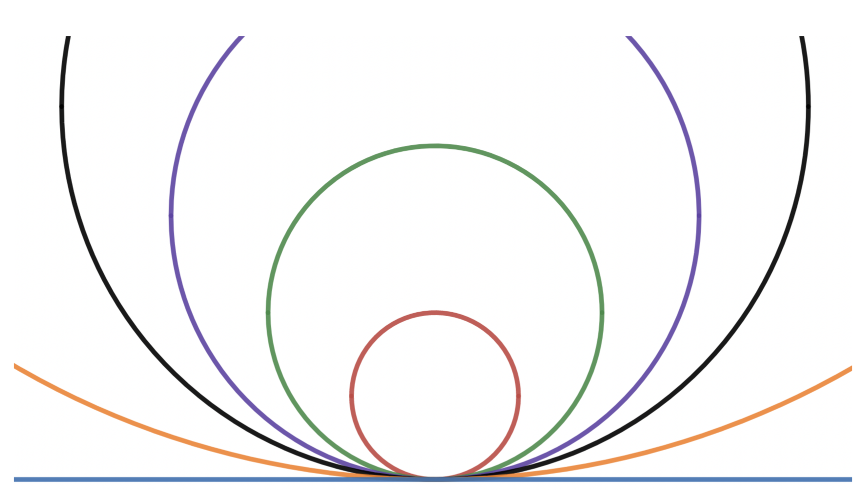

Fig. 3: Horosphere Intuition in $\mathbb{R}^2$

Results

Using our algorithms, we generated the following horospheres:

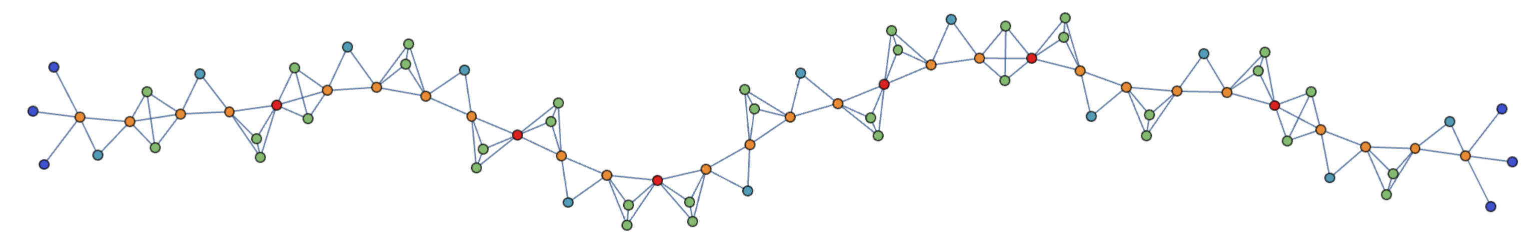

Fig. 4: Length 4 horosphere of the Pentagon defining graph. Visualized in Mathematica [4].

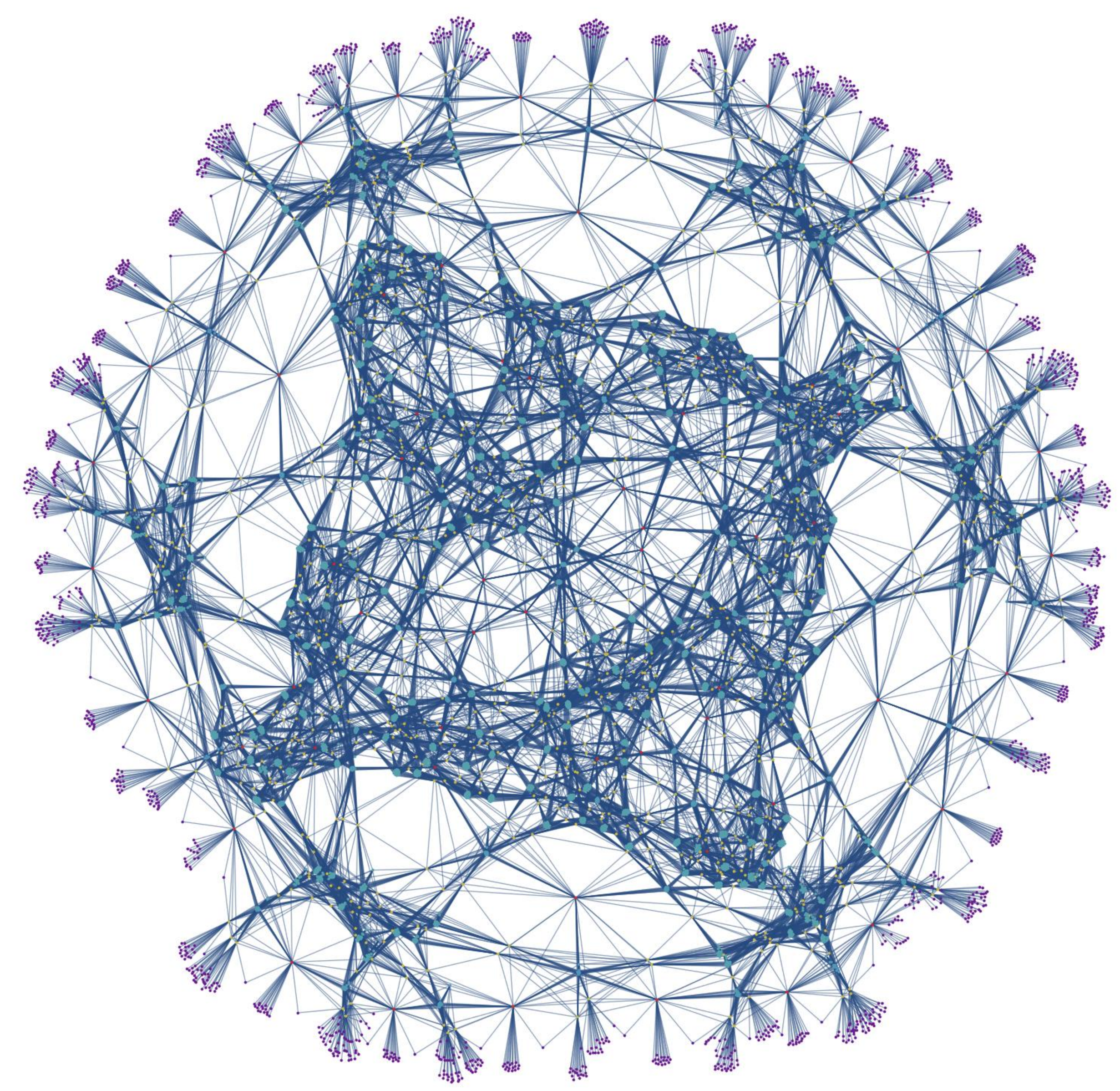

Fig. 5: Length 3 horosphere of the Torus defining graph.

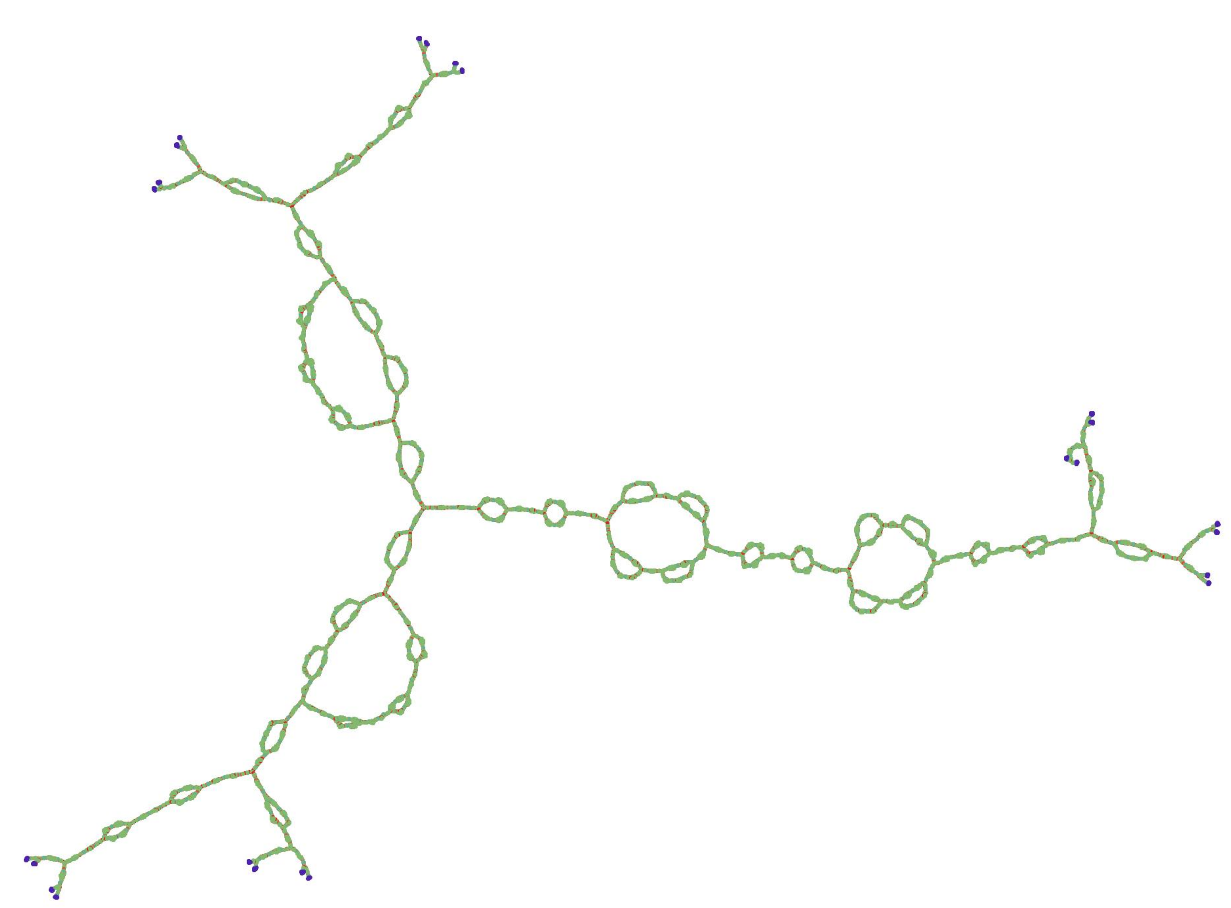

Fig. 6: Side Quest: Length 5 horosphere of the Θ graph. This graph displays fractal-like properties.

Methods

Given a defining graph $\Gamma = (V, E)$, we define an injective map $N : V \longrightarrow \mathbb{N}$ to provide a total order over the vertices of Γ. Thus, for every $u, v \in V$, either $N(u) < N(v)$ or $N(v) < N(u)$. We can then define a word in our group to be *ShortLex* if it is ordered according to N up to commutation.

Using the definition of ShortLex and the Commutations, we generate two Finite State Machines (FSM) to efficiently calculate words on our horospheres. The first FSM is a ShortLex machine and reads in a word and returns an exhaustive set of letter which when appended will keep the word ShortLex. The second FSM is a Last Letter machine and can read a word as input and return the set of letters that can be commuted to be last in the word. This FSM is used to determine word adjacencies in the horosphere.

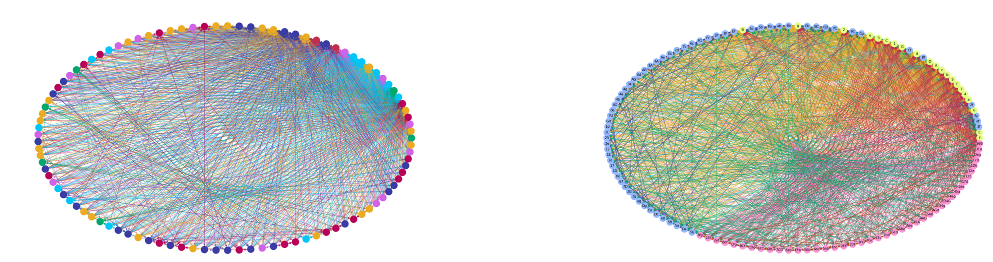

Fig. 7: ShortLex (left) Last Letter (right) FSMs for Torus Defining Graph

Discussion

The **ends** of a graph are the number of unbounded connected components after taking a compact subset out of it. The number of ends of the Cayley graph of a finitely generated group is either 0, 1, 2, or ∞ by a theorem of Freudenthal and Hopf [5]. However, the horosphere is not the entire Cayley graph, and can have more interesting numbers of ends (such as 3 in the Θ graph in Figure 5). In the horosphere of our torus graph, we find that we have only one end (see Figure 4). And in the pentagon graph, we have two ends (see Figure 3). We can prove the number of ends a horosphere is based on our defining graph and which ray we choose, and we now visualize this.

We expect a horosphere of the pentagon graph (see Figure 4) to look like a circle with a point removed at infinity. In other words, we expect a line, which we see in the horosphere.

Our algorithm is general enough to work on a large amount of defining graphs for right-angled Coxeter groups.

Generating a finite part of the horosphere on a group gives us a discrete visualization of the **boundary** of that group. The boundary of a hyperbolic space minus a point looks similar to these finite horospheres. In fact, the boundaries of some of our RACGs can be understood as homeomorphic to one of three objects: a sphere, and two fractal-like surfaces [1].

References

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[5] Nic Koban and John Meier. "Ends of Groups". In: *Office Hours with a Geometric Group Theorist*. Princeton University Press, 2017, pp. 203–218. URL: http://www.jstor.org/stable/j.ctt1vwmg8g.14 (visited on 04/27/2023).

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