SE(3) WNOJ Interp Jacobians

$$\begin{aligned}
\hat{\boldsymbol{\gamma}}_k(t_k) &= \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\varpi}_k \\ \dot{\boldsymbol{\varpi}}_k \end{bmatrix} \\
\hat{\boldsymbol{\gamma}}_k(t_{k+1}) &= \begin{bmatrix} \ln(\mathbf{T}_{k+1,k})^\vee \\ \mathcal{J}_{k+1,k}^{-1}\boldsymbol{\varpi}_{k+1} \\ -\frac{1}{2}(\mathcal{J}_{k+1,k}^{-1}\boldsymbol{\varpi}_{k+1})^\curlywedge\boldsymbol{\varpi}_{k+1} + \mathcal{J}_{k+1,k}^{-1}\dot{\boldsymbol{\varpi}}_{k+1} \end{bmatrix}
\end{aligned}$$

$$\hat{\boldsymbol{\gamma}}_k(t_{k+1}) \approx \begin{bmatrix} \ln(\hat{\mathbf{T}}_{\mathrm{op},k+1,k})^\vee + \mathcal{J}_{\mathrm{op},k+1,k}^{-1}(\delta\boldsymbol{\xi}_{k+1} - \mathcal{T}_{\mathrm{op},k+1,k}\delta\boldsymbol{\xi}_k) \\ \left(\mathcal{J}_{\mathrm{op},k+1,k}^{-1} - \frac{1}{2}\left(\mathcal{J}_{\mathrm{op},k+1,k}^{-1}(\delta\boldsymbol{\xi}_{k+1} - \mathcal{T}_{\mathrm{op},k+1,k}\delta\boldsymbol{\xi}_k)\right)^\curlywedge\right)(\boldsymbol{\varpi}_{\mathrm{op},k+1} + \delta\boldsymbol{\varpi}_{k+1}) \\ \mathbf{e}' + \mathcal{J}_{\mathrm{op},k+1,k}^{-1}\dot{\boldsymbol{\varpi}}_{\mathrm{op},k+1} + \frac{1}{2}\dot{\boldsymbol{\varpi}}_{\mathrm{op},k+1}\mathcal{J}_{k+1,k}^{-1}\delta\boldsymbol{\xi}_{k+1,k} + \mathcal{J}_{\mathrm{op},k+1,k}^{-1}\delta\dot{\boldsymbol{\varpi}}_{k+1} \end{bmatrix}$$

From Tim Tang's thesis:

$$\begin{aligned}
\mathbf{e}' &= -\frac{1}{2}(\mathcal{J}_{k+1,k}^{-1}\boldsymbol{\varpi}_{k+1})^\curlywedge\boldsymbol{\varpi}_{k+1} \\
&\approx \bar{\mathbf{e}} - \frac{1}{2}\left((\mathcal{J}_{\mathrm{op},k+1,k}^{-1}\boldsymbol{\varpi}_{\mathrm{op},k+1})^\curlywedge\delta\boldsymbol{\varpi}_{k+1} - \boldsymbol{\varpi}_{\mathrm{op},k+1}^\curlywedge\mathcal{J}_{\mathrm{op},k+1,k}^{-1}\delta\boldsymbol{\varpi}_{k+1} - \frac{1}{2}\boldsymbol{\varpi}_{\mathrm{op},k+1}^\curlywedge\boldsymbol{\varpi}_{\mathrm{op},k+1}^\curlywedge\mathcal{J}_{\mathrm{op},k+1,k}^{-1}\delta\boldsymbol{\xi}_{k+1,k}\right)
\end{aligned}$$

where $\delta\boldsymbol{\xi}_{k+1,k} = \mathcal{J}_{\mathrm{op},k+1,k}^{-1}(\delta\boldsymbol{\xi}_{k+1} - \mathcal{T}_{\mathrm{op},k+1,k}\delta\boldsymbol{\xi}_k)$.

The Jacobians for the prior cost terms are included in Tim Tang's thesis, these can be used to do covariance interpolation as well.

Interpolation Jacobians:

$$\begin{aligned}
\mathbf{T}(\tau) = \exp\Big(\Big(&\Lambda_{12}\boldsymbol{\varpi}_k + \Lambda_{13}\dot{\boldsymbol{\varpi}}_k + \Omega_{11}\ln(\mathbf{T}_{k+1,k})^\vee \\
&+ \Omega_{12}\mathcal{J}_{k+1,k}^{-1}\boldsymbol{\varpi}_{k+1} + \Omega_{13}\Big(-\frac{1}{2}(\mathcal{J}_{k+1,k}^{-1}\boldsymbol{\varpi}_{k+1})^\curlywedge\boldsymbol{\varpi}_{k+1} + \mathcal{J}_{k+1,k}^{-1}\dot{\boldsymbol{\varpi}}_{k+1}\Big)\Big)^\wedge\Big)\mathbf{T}_k
\end{aligned}$$

$$\frac{\partial\mathbf{T}(\tau)}{\partial\mathbf{x}} = \mathcal{J}_{\mathrm{op},\tau}\frac{\partial\boldsymbol{\xi}_\tau}{\partial\mathbf{x}} + \mathcal{T}_{\mathrm{op},\tau}\frac{\partial\mathbf{T}_k}{\partial\mathbf{x}}$$

$$\boldsymbol{\varpi}(\tau) = \mathcal{J}(\boldsymbol{\xi}_\tau)\dot{\boldsymbol{\xi}}_\tau$$

$$\frac{\partial\boldsymbol{\varpi}(\tau)}{\partial\mathbf{x}} = \mathcal{J}_{\mathrm{op},\tau}\frac{\partial\dot{\boldsymbol{\xi}}_\tau}{\partial\mathbf{x}} - \frac{1}{2}\dot{\boldsymbol{\xi}}_{\mathrm{op},\tau}^\curlywedge\frac{\partial\boldsymbol{\xi}_\tau}{\partial\mathbf{x}}$$

$$\dot{\boldsymbol{\varpi}}(\tau) \approx \mathcal{J}(\boldsymbol{\xi}_\tau)(\ddot{\boldsymbol{\xi}}_\tau + \frac{1}{2}\dot{\boldsymbol{\xi}}_\tau^\curlywedge\boldsymbol{\varpi}(\tau))$$

$$\frac{\partial\dot{\boldsymbol{\varpi}}(\tau)}{\partial\mathbf{x}} = \mathcal{J}_{\mathrm{op},\tau}\frac{\partial}{\partial\mathbf{x}}\Big(\ddot{\boldsymbol{\xi}}_\tau + \frac{1}{2}\dot{\boldsymbol{\xi}}_\tau^\curlywedge\boldsymbol{\varpi}(\tau)\Big) - \frac{1}{2}\Big(\ddot{\boldsymbol{\xi}}_{\mathrm{op},\tau} + \frac{1}{2}\dot{\boldsymbol{\xi}}_{\mathrm{op},\tau}^\curlywedge\boldsymbol{\varpi}_{\mathrm{op}}(\tau)\Big)^\curlywedge\frac{\partial\boldsymbol{\xi}_\tau}{\partial\mathbf{x}}$$

$$\begin{aligned}
\frac{\partial}{\partial\mathbf{x}}\Big(\ddot{\boldsymbol{\xi}}_\tau + \frac{1}{2}\dot{\boldsymbol{\xi}}_\tau^\curlywedge\boldsymbol{\varpi}(\tau)\Big) &= \frac{\partial\ddot{\boldsymbol{\xi}}_\tau}{\partial\mathbf{x}} + \frac{1}{2}\frac{\partial}{\partial\mathbf{x}}\dot{\boldsymbol{\xi}}_\tau^\curlywedge\boldsymbol{\varpi}(\tau) \\
&= \frac{\partial\ddot{\boldsymbol{\xi}}_\tau}{\partial\mathbf{x}} - \frac{1}{2}\boldsymbol{\varpi}_{\mathrm{op}}(\tau)^\curlywedge\frac{\partial\dot{\boldsymbol{\xi}}_\tau}{\partial\mathbf{x}} + \frac{1}{2}\dot{\boldsymbol{\xi}}_{\mathrm{op},\tau}^\curlywedge\frac{\partial\boldsymbol{\varpi}(\tau)}{\partial\mathbf{x}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \dot{\boldsymbol{\varpi}}(\tau)}{\partial \mathbf{x}} &= \mathcal{J}_{\mathrm{op},\tau}\left(\frac{\partial \ddot{\boldsymbol{\xi}}_\tau}{\partial \mathbf{x}} - \frac{1}{2}\boldsymbol{\varpi}_{\mathrm{op}}(\tau)^\curlywedge \frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \mathbf{x}} + \frac{1}{2}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\left(\mathcal{J}_{\mathrm{op},\tau}\frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \mathbf{x}} - \frac{1}{2}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\frac{\partial \boldsymbol{\xi}_\tau}{\partial \mathbf{x}}\right)\right) \\
&\quad - \frac{1}{2}\left(\ddot{\boldsymbol{\xi}}_{\mathrm{op},\tau} + \frac{1}{2}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\boldsymbol{\varpi}_{\mathrm{op}}(\tau)\right)^\curlywedge \frac{\partial \boldsymbol{\xi}_\tau}{\partial \mathbf{x}} \\
\frac{\partial \dot{\boldsymbol{\varpi}}(\tau)}{\partial \mathbf{x}} &= \mathcal{J}_{\mathrm{op},\tau}\frac{\partial \ddot{\boldsymbol{\xi}}_\tau}{\partial \mathbf{x}} \\
&\quad + \mathcal{J}_{\mathrm{op},\tau}\left(-\frac{1}{2}\boldsymbol{\varpi}_{\mathrm{op}}(\tau)^\curlywedge + \frac{1}{2}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\mathcal{J}_{\mathrm{op},\tau}\right)\frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \mathbf{x}} \\
&\quad + \left(-\frac{1}{4}\mathcal{J}_{\mathrm{op},\tau}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau} - \frac{1}{2}\left(\ddot{\boldsymbol{\xi}}_{\mathrm{op},\tau} + \frac{1}{2}\dot{\boldsymbol{\xi}}^\curlywedge_{\mathrm{op},\tau}\boldsymbol{\varpi}_{\mathrm{op}}(\tau)\right)^\curlywedge\right)\frac{\partial \boldsymbol{\xi}_\tau}{\partial \mathbf{x}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\xi}_\tau = \boldsymbol{\xi}(\tau) &= \boldsymbol{\Lambda}_{12}(\tau)\boldsymbol{\varpi}_k + \boldsymbol{\Lambda}_{13}(\tau)\dot{\boldsymbol{\varpi}}_k + \boldsymbol{\Omega}_{11}(\tau)\ln(\mathbf{T}_{k+1,k})^\vee + \boldsymbol{\Omega}_{12}(\tau)\mathcal{J}^{-1}_{k+1,k}\boldsymbol{\varpi}_{k+1} \\
&\quad + \boldsymbol{\Omega}_{13}\left(-\frac{1}{2}\left(\mathcal{J}^{-1}_{k+1,k}\boldsymbol{\varpi}_{k+1}\right)^\curlywedge \boldsymbol{\varpi}_{k+1} + \mathcal{J}^{-1}_{k+1,k}\dot{\boldsymbol{\varpi}}_{k+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\xi}_{k+1}} &= \Omega_{11}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} + \frac{1}{2}\Omega_{12}\boldsymbol{\varpi}^\curlywedge_{\mathrm{op},k+1}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} \\
&\quad + \frac{1}{4}\Omega_{13}\boldsymbol{\varpi}^\curlywedge_{\mathrm{op},k+1}\boldsymbol{\varpi}^\curlywedge_{\mathrm{op},k+1}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} + \frac{1}{2}\Omega_{13}\dot{\boldsymbol{\varpi}}^\curlywedge_{\mathrm{op},k+1}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\xi}_k} &= -\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\xi}_{k+1}}\mathcal{T}_{\mathrm{op},k+1,k} \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\varpi}_k} &= \Lambda_{12}\mathbf{1} \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\varpi}_{k+1}} &= \Omega_{12}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} - \frac{1}{2}\Omega_{13}\left(\left(\mathcal{J}^{-1}_{\mathrm{op},k+1,k}\boldsymbol{\varpi}_{\mathrm{op},k+1}\right)^\curlywedge - \boldsymbol{\varpi}^\curlywedge_{\mathrm{op},k+1}\mathcal{J}^{-1}_{\mathrm{op},k+1,k}\right) \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \dot{\boldsymbol{\varpi}}_k} &= \Lambda_{13}\mathbf{1} \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \dot{\boldsymbol{\varpi}}_{k+1}} &= \Omega_{13}\mathcal{J}^{-1}_{\mathrm{op},k+1,k}
\end{aligned}
$$

Note that the Jacobians of $\dot{\boldsymbol{\xi}}_\tau$ and $\ddot{\boldsymbol{\xi}}_\tau$ have the same form as above, except that we use the second and third row of the interpolation matrices respectively. For example, $\frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \delta \boldsymbol{\xi}_{k+1}} = \Omega_{21}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} + \frac{1}{2}\Omega_{22}\boldsymbol{\varpi}^\curlywedge_{\mathrm{op},k+1}\mathcal{J}^{-1}_{\mathrm{op},k+1,k} + \cdots$.

Extrapolation

WNOA:

$$
\begin{aligned}
\boldsymbol{\xi}_k(\tau) &= \ln(\mathbf{T}(t)\mathbf{T}_k^{-1})^\vee \\
\dot{\boldsymbol{\xi}}_k(\tau) &= \mathcal{J}(\boldsymbol{\xi}_k(\tau))^{-1}\boldsymbol{\varpi}(t)
\end{aligned}
$$

Past the end knot: $\boldsymbol{\xi}_k = \mathbf{0}^T$ and so $\mathcal{J}(\boldsymbol{\xi}_k(\tau))^{-1} = \mathbf{1}$, $\dot{\boldsymbol{\xi}}_k = \boldsymbol{\varpi}_k$.

$$
\boldsymbol{\gamma}_\tau = \begin{bmatrix}\mathbf{1} & \Delta t \\ \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}_k \\ \dot{\boldsymbol{\xi}}_k\end{bmatrix} = \begin{bmatrix}\Delta t\boldsymbol{\varpi}_k \\ \boldsymbol{\varpi}_k\end{bmatrix}
$$

WNOJ:

Past the end knot: $\boldsymbol{\xi}_k = \mathbf{0}^T$ and so $\mathcal{J}(\boldsymbol{\xi}_k(\tau))^{-1} = \mathbf{1}$, $\dot{\boldsymbol{\xi}}_k = \boldsymbol{\varpi}_k$.

$$
\begin{aligned}
\ddot{\boldsymbol{\xi}}_k &= -\frac{1}{2}\dot{\boldsymbol{\xi}}_k(t)^{\curlywedge}\boldsymbol{\varpi}(t) + \mathcal{J}(\boldsymbol{\xi}_k(t))^{-1}\dot{\boldsymbol{\varpi}}(t) \\
&= \dot{\boldsymbol{\varpi}}_k
\end{aligned}
$$

So the extrapolation for WNOJ looks like this:

$$
\boldsymbol{\gamma}_\tau = \begin{bmatrix} \boldsymbol{\xi}_\tau \\ \dot{\boldsymbol{\xi}}_\tau \\ \ddot{\boldsymbol{\xi}}_\tau \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{1} & \Delta t \mathbf{1} & \frac{1}{2}\Delta t^2 \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \Delta t \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}}_{\boldsymbol{\Phi}(\tau, t_k)} \underbrace{\begin{bmatrix} \boldsymbol{\xi}_k \\ \dot{\boldsymbol{\xi}}_k \\ \ddot{\boldsymbol{\xi}}_k \end{bmatrix}}_{\boldsymbol{\gamma}_k(t_k)} = \begin{bmatrix} \Delta t \boldsymbol{\varpi}_k + \frac{1}{2}\Delta t^2 \dot{\boldsymbol{\varpi}}_k \\ \boldsymbol{\varpi}_k + \Delta t \dot{\boldsymbol{\varpi}}_k \\ \dot{\boldsymbol{\varpi}}_k \end{bmatrix}
$$

We can use this formula for doing the extrapolation of any GP prior including WNOA, WNOJ, Singer, etc:

$$
\boldsymbol{\gamma}(\tau) = \boldsymbol{\Phi}(\tau, t_k)\boldsymbol{\gamma}_k(t_k)
$$

In order to convert from local to global variables, we do the following:

$$
\begin{aligned}
\mathbf{T}(\tau) &= \exp(\boldsymbol{\xi}_\tau^\wedge)\hat{\mathbf{T}}_k \\
\boldsymbol{\varpi}(\tau) &= \mathcal{J}(\boldsymbol{\xi}_\tau)\dot{\boldsymbol{\xi}}_\tau \\
&\approx \dot{\boldsymbol{\xi}}_\tau \\
\dot{\boldsymbol{\varpi}}(\tau) &\approx \mathcal{J}(\boldsymbol{\xi}_\tau)(\ddot{\boldsymbol{\xi}}_\tau + \frac{1}{2}\dot{\boldsymbol{\xi}}_\tau^{\curlywedge}\boldsymbol{\varpi}(\tau)) \\
&\approx \ddot{\boldsymbol{\xi}}_\tau
\end{aligned}
$$

where we have made the approximation that $\boldsymbol{\xi}_\tau$ is small, and so $\mathcal{J}(\boldsymbol{\xi}_\tau)$ is close to identity. Hence, our extrapolation formulas are approximate and this approximation only holds so long as $\boldsymbol{\xi}_\tau$ is small.

$$
\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{0} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{33} \end{bmatrix}
$$

$\boldsymbol{\Phi}$ is defined differently for the WNOJ and Singer priors.

Next, we can easily define the Jacobians with respect to the perturbations:

$$
\begin{aligned}
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \boldsymbol{\varpi}_k} &= \mathbf{C}_{12} \\
\frac{\partial \boldsymbol{\xi}_\tau}{\partial \delta \dot{\boldsymbol{\varpi}}_k} &= \mathbf{C}_{13} \\
\frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \delta \boldsymbol{\varpi}_k} &= \mathbf{C}_{22} \\
\frac{\partial \dot{\boldsymbol{\xi}}_\tau}{\partial \delta \dot{\boldsymbol{\varpi}}_k} &= \mathbf{C}_{23} \\
\frac{\partial \ddot{\boldsymbol{\xi}}_\tau}{\partial \delta \dot{\boldsymbol{\varpi}}_k} &= \mathbf{C}_{33}
\end{aligned}
$$